PLANE AND SOLID GEOMETRY

SLAUGHT
AND
LENNES

PLANE AND SOLID
GEOMETRY
WITH
PROBLEMS AND APPLICATIONS

BY

H. E. SLAUGHT, PH.D.

ASSOCIATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CHICAGO

AND

N. J. LENNES, PH.D.

INSTRUCTOR IN MATHEMATICS IN COLUMBIA UNIVERSITY

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PREFACE.

IN writing this book the authors have been guided by two main purposes:

(*a*) That pupils may gain by gradual and natural processes the power and the habit of deductive reasoning.

(*b*) That pupils may learn to know the essential facts of elementary geometry as properties of the space in which they live, and not merely as statements in a book.

The important features by which the Plane Geometry seeks to accomplish these purposes are:

1. *The simplification of the first five chapters by the exclusion of many theorems found in current books.* These five chapters correspond to the usual five books, and the most important omissions are the formal treatment of the theory of limits, the incommensurable cases, maxima and minima, and numerous other theorems, together with the deduction of complicated algebraic formulæ.

Chapter VI contains a graphic representation of certain important theorems and an informal presentation of incommensurable cases and limits. The treatment of limits is based upon the graph, since the visual or graphic method appeals more directly to the intuition than the usual abstract processes. Chapter VII is devoted to advanced work and to a review of the preceding chapters.

2. *The subject has been enriched by including many applications of special interest to pupils.* Here an effort has been made to include only such concrete problems as come fairly within

the observation and comprehension of the average pupil. This led to the omission, for example, of problems relating to machinery and technical industries, which might appeal to an exceptional boy, but which are entirely inappropriate for the average student. On the other hand, free use is made of certain sources of problems which may be easily comprehended without extended explanation and which involve varied and simple combinations of geometric forms. Such problems pertain to decoration, ornamental designs, and architectural forms. They are found in tile patterns, parquet floors, linoleums, wall papers, steel ceilings, grill work, ornamental windows, etc., and they furnish a large variety of simple exercises both for geometric construction and proofs and for algebraic computation. They are not of the puzzle type, but require a thorough acquaintance with geometric facts and develop the power to use mathematics.

These problems form an *entirely new type of exercises*, and while they require more space in the text-book than the more difficult "originals" stated in the usual abstract terms, they excel the latter in interest for the pupil and in helping to train his mathematical common sense. Many of these exercises are simple enough to be solved at sight, and such solution should be encouraged whenever possible. All the designs are taken from photographs or from actual commercial patterns now in use.

3. *Persistent effort is made to vitalize the content of the definitions and theorems.* It is well known that pupils often study and recite definitions and theorems without really comprehending their meaning. It is sought to check this tendency by giving definitions only when they are to be used, and by immediately verifying both definitions and theorems in concrete cases. The figure on page 4 is the basis for a large number of questions of this type. For example, see § 25, Ex. 3; § 30, Ex. 1; § 34, Exs. 1, 2; § 36, Ex. 3; §§ 322, 324.

In this connection special attention is called to the emphasis placed upon those theorems which are of fundamental importance both in the logical chain and in their immediate use in effecting constructions and indirect measurements otherwise difficult or impossible. For example, see the theorems on congruence of triangles, §§ 31–43, the constructions of §§ 44–58, and the theorems on proportional segments, §§ 243–254. Compare especially § 34, Ex. 5, § 244, Ex. 2, and § 254, Exs. 4, 5.

The summaries at the close of the chapters, which are to be made by the pupil himself, will vitalize the theorems as no made-to-order summaries can possibly do.

4. *The student is made to approach the formal logic of geometry by natural and gradual processes.* He is expected to *grow* into this new, and to him unusual, way of thinking. The treatment is at the start informal, leading through the congruence theorems directly to concrete applications and geometric constructions. The formal development then follows gradually and is characterized by a *judicious guidance* of the student, by questions, outlines, and other devices, into an attitude of mental independence and an appreciation of clear reasoning.

There are certain terms whose general meaning in mathematics is inconvenient in some parts of a book like the present. Thus we say that two circles (meaning by circle the curved line) meet in two points, and it is also convenient to say that the base of a certain cylinder is a circle. In such cases the generally accepted mathematical meanings of the terms have been given in the definitions, while notes have been added that different meanings are sometimes to be understood, depending on the context to make the meaning clear. See §§ 523, 595. This avoids learning definitions which must be discarded later, and also the clumsiness resulting from a strict adherence to the same meaning throughout. Experience has shown that no confusion whatever results from this, while the resulting simplicity of statement is of considerable importance.

By its arrangement the Plane Geometry is adapted to three different courses:

(*a*) *A minimum course,* consisting of Chapters I to VI, without the problems and applications at the end of each chapter. This would provide about as much material, theorems, constructions, and originals as is found in the briefest books now in use.

(*b*) *A medium course*, consisting of Chapters I to VI, including a reasonable number, say one half or two thirds, of the applications at the end of each chapter. This would fully cover the college entrance requirements.

(*c*) *An extended course*, including Chapter VII, which contains a complete review, together with many additional theorems and a large number of further applications. This would provide ample work for the strongest high schools, and for normal schools in which more mature students are found or more time can be given to the subject.

Chapter VII gives a complete treatment of the incommensurable cases, though not based on the formal theory of limits. It is believed that for high school pupils the notion of a limit is best studied as a process of *approximation,* and that the best preparation for the later understanding of the theory is by a preliminary study of what is meant by "approaches," such as is given in Chapters III and IV.

The important features by which the Solid Geometry seeks to accomplish the two main purposes stated with reference to the Plane Geometry are:

1. *The concepts of three-dimensional space are made clear by many simple illustrations and exercises.* At best the average pupil comes but slowly and gradually to a full comprehension of space forms and relations. Pages 282, 283, 286, 288, 289, 295, 296, etc., will show how the pupil is aided in understanding and appreciating these forms and relations at the outset. In this connection, Chapter XIII on Graphic Representation

is of fundamental importance, since it exhibits to the eye the functional relations among varying geometric forms.

2. *Interesting concrete applications are interspersed throughout the text.* It is the aim profoundly to impress upon the mind of the pupil the practical significance of certain fundamental theorems in solid geometry. For instance, how many pupils in the ordinary study of the theorems on the ratios of surfaces and volumes of similar figures realize that there is in them any connection with the possibility of successfully launching a steamship a mile long? See the exercises on pages 390, 391, 444, 445. For illustrations of other interesting and useful applications, see pages 337, 344, 345, 434, 435.

3. *The logical structure is made more complete and more prominent than in the Plane Geometry.* Solid Geometry is studied by more mature pupils who have been led by gradual stages in the Plane Geometry to a knowledge and appreciation of deductive reasoning. Hence the axioms are stated and applied in strictly scientific form and at the precise points where they are to be used. For instance, see §§ 460, 462, 464, 538, 560, 562, 593, 612, etc.

Note also the consistent and scientific definitions of all solids, not as bounded *portions of space*, but as *configurations in space*, the uniform conception in all higher mathematical usage. See §§ 68, 70, 99, 132, 149, 200, 523, 525, 554, 587, 604, 655.

4. *Throughout both parts of this Geometry a consistent scheme has been followed in the presentation of incommensurables and the theory of limits.* In Chapters I to VI the idea of "approach" is made clear by many concrete illustrations, and the theorems involving this idea are shown to hold for *all possible approximations.* In Chapter VII rigorous proofs of these theorems are given and in far simpler terminology than is found in current text-books. This latter method is followed throughout

Chapters VIII to XIII, thus giving a complete and scientific treatment up to that point. In Chapter XIV the theory of limits is presented in such a way as to leave nothing to be unlearned or compromised in later mathematical work. This chapter may be omitted without affecting the logical completeness of the book.

Acknowledgment is due to Miss Mabel Sykes, of Chicago, for the use in the Plane Geometry of a large number of drawings and designs from her extensive collection; also to numerous commercial and manufacturing houses, both in this country and in Europe, through whose courtesy many of the patterns were obtained.

H. E. SLAUGHT.
N. J. LENNES.

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PLANE GEOMETRY.

CHAPTER I.

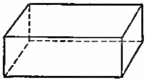
RECTILINEAR FIGURES.

INTRODUCTION.

1. Elementary geometry is a science which deals with the space in which we live. It begins with the consideration of certain elements of this space which are called points, lines, planes, solids, angles, triangles, etc.

Some of these terms, such as **point**, **line**, **plane**, are here used without being defined in a strictly logical sense. Their meaning is made clear by description and by concrete illustrations like the following.

2. Certain portions of space are occupied by objects which we call physical solids, as, for instance, an ordinary brick. That which separates a solid from the surrounding space is called its **surface**. This may be rough or smooth. If a surface is smooth and flat, we call it a **plane surface**.



A pressed brick has six plane surfaces called **faces**. Two adjoining faces meet in an **edge**. Three edges meet in a **corner**.

The brick is bounded by its six faces. Each face is bounded by four edges, and each edge is bounded by two corners.

3. If instead of the brick we think merely of its form and magnitude, we get a notion of a **geometrical solid**, which has the three dimensions, **length**, **breadth**, and **thickness**.

The faces of this ideal solid are called **planes**. These are flat and have length and breadth, but no thickness.

The edges of this solid are called **lines**. They are straight and have length, but neither breadth nor thickness. The corners of this solid are called **points**. They have position, but neither length, breadth, nor thickness; that is, they have no magnitude.

4. It is possible to think of these concepts quite independently of any physical solid. Thus we speak of the *line of sight* from one point to another; and we say that light travels in a straight line.

The term *straight line* is doubtless connected with the idea of a *stretched string*. Of all the lines which may be conceived as passing through two fixed points that one is said to be straight between these points which corresponds most nearly to a stretched string.

Likewise a plane may be thought of as *straight* or *stretched* in every direction, so that a straight line passing through any two of its points lies wholly in the plane.

5. If one of two intersecting straight lines turns about their common point as a pivot, the lines will continue to have only one point in common until *all at once* they will coincide throughout their whole length. Hence,

Two straight lines cannot have more than one point in common unless they coincide and are the same line; that is, two points determine a straight line.

This would not be so if the lines had width, as may be seen by examining the figures.

6. **EXERCISES.**

1. How does a carpenter use a straight-edge to determine whether a surface is a plane? Do you know of any surface to which this test will apply in *one* direction but not in *all* directions?

2. What tool does a carpenter use in reducing an uneven surface to a plane surface? Why is the tool so named?

3. If two points of a straight line lie in a plane, what can be said of the whole line?

4. How many points of a straight line can lie in a plane if it contains at least one point not in the plane?

5. If two straight lines coincide in more than one point, what can you say of them throughout their whole length? Do you know of any lines other than straight lines of which this must be true?

6. How do the material points and lines made by crayon or pencil differ in magnitude from the ideal points and lines of geometry?

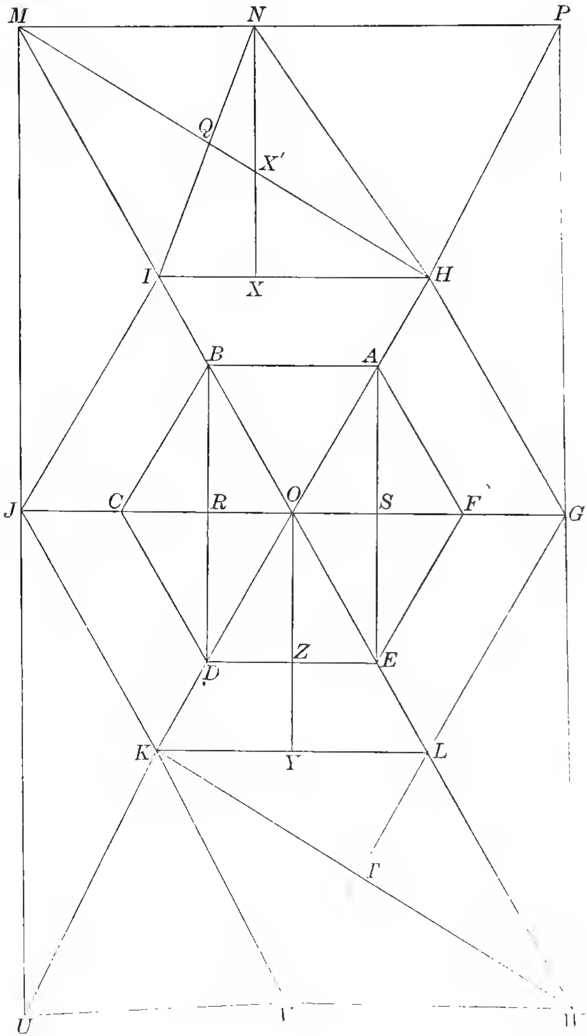
7. A machine has been made which rules 20,000 distinct lines side by side within the space of one inch. Do such lines have width? Are they geometrical lines?

8. Of all the lines, straight or curved, through two points, on which one is the shortest distance measured between the two points? See the figure of § 4.

Historical Note. The Egyptians appear to have been the first people to accumulate any considerable body of exact geometrical facts. The building of the great pyramids (before 3000 B.C.) required not a little knowledge of geometric relations. They also used geometry in surveying land. Thus it is known that Rameses II (about 1400 B.C.) appointed surveyors to measure the amount of land washed away by the Nile, so that the taxes might be equalized.

The Greeks, however, were the first to study geometry from a logical point of view. Between 600 B.C., when Thales, a Greek from Asia Minor, learned geometry from the Egyptians, and 300 B.C., when Euclid, a Greek residing in Alexandria, Egypt, wrote his Elements of Geometry, the crude, practical geometric information of the Egyptians was transformed into a well-nigh perfect logical system.

Euclid's "Elements" contains the essential facts of every text-book on elementary geometry that has been written since his time.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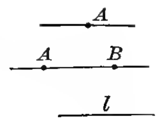
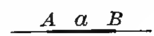
NOTATION FOR POINTS AND LINES.

7. A point is denoted by a capital letter. A straight line is denoted by two capital letters marking two of its points or by one small letter. The word *line* alone usually means straight line.



Thus, the point A, the line AB, or the line l.

8. A straight line is usually understood to be unlimited in length in both directions, while that part of it which lies between two of its points is called a **line-segment**, or simply a **segment**. These points are called the **end-points** of the segment.



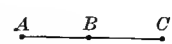
Thus, the segment AB or the segment a.

Two segments with the same end-points are coincident.

9. A part of a straight line, called a **ray** or **half-line**, may be thought of as generated by a point starting from a fixed position and moving indefinitely in one direction. The starting point is called the **end-point** or **origin** of the ray.

If A is the origin of a ray and B any other point on it, then it is read *the ray* AB, not *the ray* BA.

10. Two line-segments are said to be **added** if they are placed end to end so as to form a single segment.



Thus, segment AC = segment AB + segment BC, or $AC = AB + BC$.

If $AC = AB + BC$, then AC is **greater** than either AB or BC, and this is written $AC > AB$ and $AC > BC$.

A segment may also be **subtracted** from a greater or from an equal segment.

Thus, if $AC = AB + BC$, then $AB = AC - BC$ and $BC = AC - AB$.

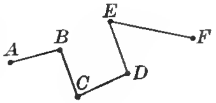
A segment is **multiplied** by an integer n by taking the sum of n such segments.

Thus, if AC is the sum of n segments each AB, then $AC = n \cdot AB$.

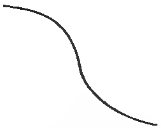
If AC is n times AB, then AC may be **divided** by n.

Thus, $AB = AC \div n$ or $AB = \frac{1}{n} \cdot AC$.

11. A **broken line** is composed of connected line-segments not all lying in the same straight line.

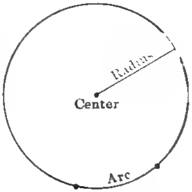


A **curved** line, or simply a **curve**, is a line no part of which is straight.



A curved line or a broken line may inclose a portion of a plane, while a straight line cannot.

12. A **circle** is a plane curve containing all points equally distant from a fixed point in the plane, and no other points.



The fixed point is called the **center** of the circle. Any line-segment joining the center to a point on the circle is a **radius** of the circle.

Any portion of a circle lying between two of its points is called an **arc**.

Evidently all radii of the same circle are equal.

Any combination of points, segments, lines, or curves in a plane is called a **plane geometric figure**.

Plane Geometry deals with plane geometric figures.

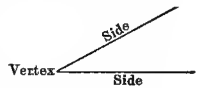
13. EXERCISES.

1. How many end-points has a straight line? How many has a line-segment? How many has a ray? A circle?

2. Can you inclose a portion of a plane with two line-segments? With three? With four?

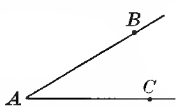
ANGLES AND THEIR NOTATION.

14. An **angle** is a figure formed by two rays proceeding from the same point. The point is the **vertex** of the angle and the rays are its **sides**. The angle formed by two rays is said to be the angle *between* them or simply their angle.



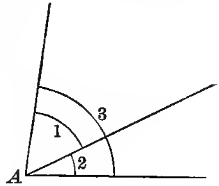
Two line-segments having a common end-point also form an angle, namely, the angle of the rays on which the segments lie. An angle is determined entirely by the relative *directions* of its rays and not by the *lengths* of the segments laid off on them.

15. An angle is denoted by three letters, one at its vertex and one marking a point on each of its sides. The one at the vertex is read between the other two, as the angle CAB, or the angle BAC, not ABC. The one letter at the vertex is also used alone to denote an angle in case no other angle has the same vertex, as, for instance, the angle A.

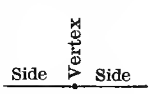


In case several angles have the same vertex, a small letter or figure placed within each angle, together with an arc connecting its sides, is a convenient notation. The sign $\angle$ is used for the word *angle*.

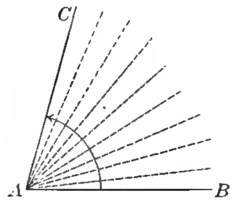
Thus in the figure we have $\angle 1$, $\angle 2$, $\angle 3$, read, *angle one, angle two, angle three.*



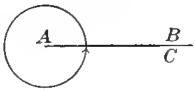
16. If two rays lie in the same straight line and extend from the same end-point in *opposite* directions they are said to form a **straight angle.** If they extend in the *same* direction, they coincide and form a **zero angle.**



17. An angle may be thought of as *generated* by a ray turning about its end-point as a pivot.



Thus $\angle BAC$ is generated by a ray rotating from the position AB to the position AC. The rotating ray is usually conceived as moving in the direction *opposite* to the hands of a clock and the sides of the angle should usually be read in this order. Thus $\angle BAC$, not $\angle CAB$.



If the ray continues to rotate until it lies in a direction exactly *opposite* to its original position, it generates a *straight angle*, as the straight angle *BAC*.

If the ray rotates until it reaches its original position, the angle generated is called **a perigon**, that is, an angle of complete rotation.

The position from which the rotating ray starts is called the **origin** of the angle. From this point of view two rays from the same point form *two* angles according as one or the other of the rays is regarded as the origin. In elementary geometry only angles less than or equal to a straight angle are usually considered.

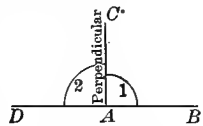
The units of measure for angles are one three-hundred-sixtieth of a perigon which is called a **degree**, one sixtieth of a degree called a **minute**, and one sixtieth of a minute called a **second**. These are denoted respectively by the symbols $^\circ$, $'$, $''$. Thus, an angle of $20^\circ\ 45'\ 30''$. A straight angle is therefore an angle of 180° and a perigon is an angle of 360°.

18. Angles are measured by means of an instrument called a **protractor**, which consists of a semicircular scale with degrees from 0° to 180° marked upon it.

An inexpensive protractor made of cardboard or brass may be had at any stationery store. See the figure of § 33.

19. Two angles are said to be equal if they can be made to **coincide** without changing the form of either.

If a ray is drawn from a point in a straight line so that the two angles thus formed are equal, each angle is called a **right angle**, and the ray is said to be **perpendicular** to the line.



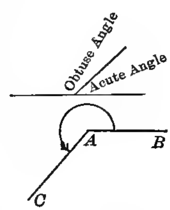
Thus, if $\angle 1 = \angle 2$, each angle is a right angle and AC is then said to be perpendicular to BD.

Since the *straight angle* BAD is composed of $\angle 1$ and $\angle 2$, each of which is a right angle, it appears that *a straight angle equals two right angles.*

See § 39 for the addition of angles in general.

An **acute angle** is less than a right angle.

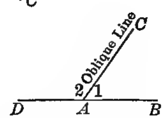
An **obtuse angle** is greater than a right angle and less than a straight angle.



Acute and obtuse angles are called **oblique** angles.

A **reflex angle** is greater than a straight angle and less than a perigon.

One line is **oblique** to another if the angles between them are oblique.



A ray which divides an angle into two equal angles is called its **bisector**. Thus a perpendicular is the bisector of a straight angle.

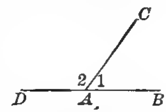
The angles considered in this book are greater than the zero angle and less than or equal to a straight angle.

20. EXERCISES.

1. Since we can always place two straight angles so as to make them coincide, what can we say as to whether or not they are equal? What of two right angles?

2. What part of a straight angle is a right angle?

3. Suppose that in the figure the ray AC rotates about the point A from the position AB to the position AD. What change takes place in $\angle 1$? What in $\angle 2$? Can there be more than one position of AC for which $\angle 1 = \angle 2$? In this way it may be made clear that any angle has one and only one bisector.



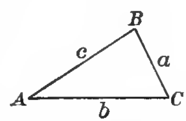
4. How many rays perpendicular to BD at the point A can be drawn on the same side of BD? Does the answer to this question depend upon the answers to the questions in Ex. 4? How?

5. Pick out three acute angles, three right angles, and three obtuse angles in the figure on page 4.

TRIANGLES AND THEIR NOTATION.

21. If the points A, B, C do not lie in the same straight line, the figure formed by the three segments, AB, BC, and CA, is called a **triangle.**

The segments are the **sides** of the triangle, and the points are its **vertices.** The symbol $\triangle$ is used for the word *triangle.*



Each angle of a triangle has one side **opposite** and two sides **adjacent** to it. Similarly each side of a triangle has one angle opposite and two angles adjacent to it. The side opposite an angle is often denoted by the corresponding small letter.

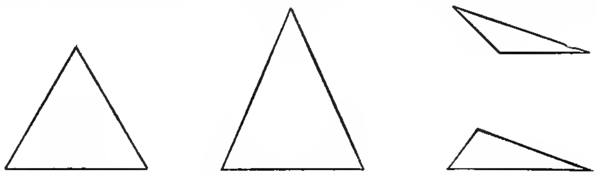
Thus, in the figure the side a and the angle A are *opposite* parts, as are angle B and side b and angle C and side c. The three sides and three angles of a triangle are called the parts of the triangle.

These six parts are considered as lying in order around the figure, as $\angle A$, side b, $\angle C$, side a, etc.

An angle of a triangle is said to be *included* between its two adjacent sides, and a side is said to be *included* between the two angles adjacent to it.

Thus, in the figure the side a is included between $\angle B$ and $\angle C$, and $\angle A$ is included between the sides b and c.

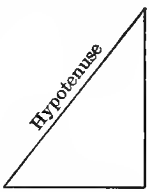
22. A triangle is called **equilateral** if it has its three sides equal, **isosceles** if it has at least two sides equal,



scalene if it has no two sides equal, **equiangular** if it has its three angles equal.

Select each kind from the figures on this page.

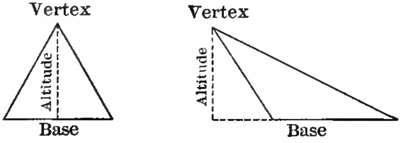
23. A triangle is called a **right triangle** if it has one right angle, an **obtuse triangle** if it has one obtuse angle, an **acute triangle** if all its angles are acute.



Select each kind from the figures on this page.

The side of a right triangle opposite the right angle is called the **hypotenuse** in distinction from the other two sides, which are sometimes called its **legs.**

24. The side of a triangle on which it is supposed to stand is called its **base.** The angle opposite the base is called the **vertex angle**, and its vertex is the **vertex** of the triangle.



The **altitude** of a triangle is the perpendicular from the vertex to the base or the base produced. Evidently any side may be taken as the base, and hence a triangle has three different altitudes.

25. EXERCISES.

1. Is every equilateral triangle also isosceles? Is every isosceles triangle also equilateral?

2. Is a right triangle ever isosceles? Is an obtuse triangle ever isosceles? Draw figures to illustrate your answers.

3. In the figure on page 4 determine by measuring sides which of the triangles *HNP, LKW, IHN, MIJ, KVU, OKJ, LVW*, are isosceles, which are equilateral, and which are scalene.

4. Determine whether *J, K, V* of the same figure may be the vertices of a triangle; also whether *J, O, G* may be.

5. Pick out ten obtuse triangles in this figure; also ten acute triangles.

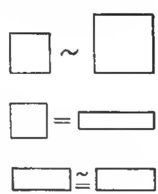
CONGRUENCE OF GEOMETRIC FIGURES.

26. In comparing geometric figures it is assumed that *they may be moved about at will, either in the same plane or out of it, without changing their shape or size.*

27. Two figures are said to be **similar** if they have the same shape. This is denoted by the symbol $\sim$, read *is similar to.*

For a more precise definition see §§ 255, 256.

Two figures are said to be **equivalent** or simply **equal** if they have the same size or magnitude. This is denoted by the symbol $=$, read *is equivalent to* or *is equal to.*



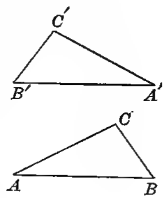
Two figures are said to be **congruent** if, without changing the shape or size of either, they may be so placed as to coincide throughout. This is denoted by the symbol $\cong$, read *is congruent to.*

In the case of line-segments and angles, congruence is determined by *size* alone. Hence in these cases we use the symbol $=$ to denote congruence, and read it *equals* or *is equal to.*

28. It is clear that *if each of two figures is congruent to the same figure they are congruent to each other.*

Hence if we make a pattern of a figure, say on tracing paper, and then make a second figure from this pattern, the two figures are congruent to each other.

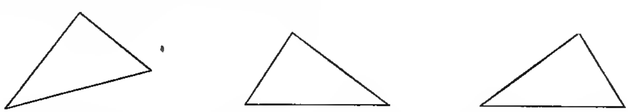
29. If $\triangle ABC \cong \triangle A'B'C'$, the notation of the triangles may be so arranged that $AB = A'B'$, $BC = B'C'$, $CA = C'A'$, $\angle A = \angle A'$, $\angle B = \angle B'$ and $\angle C = \angle C'$. In this case AB is said to **correspond** to $A'B'$, BC to $B'C'$, CA to $C'A'$, $\angle A$ to $\angle A'$, etc.



Hence, we say that **corresponding parts** of congruent triangles are equal.

30. EXERCISES.

1. Using tracing paper, draw triangles congruent to the triangles *MIN*, *NHP*, *OAB*, *OFE*, *OKL*, *UKV*, *OGL* on page 4, and by applying the pattern of each triangle to each of the others determine whether any two are congruent.



2. Find as in § 28 whether any two of three accompanying triangles are congruent, and if so arrange the notation so as to show the corresponding parts.

3. Give examples of figures which are similar, equal, or congruent, different from those in § 27.

4. If two figures are congruent, does it follow that they are equal? Similar?

5. If two figures are similar, does it follow that they are equal? Congruent?

6. If two figures are equal, are they similar? Congruent?

TESTS FOR CONGRUENCE OF TRIANGLES.

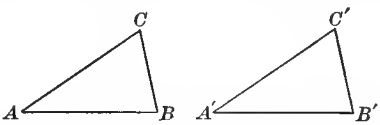
31. The method of determining whether two triangles are congruent by making a pattern of one and applying it to the other is often inconvenient or impossible. There are other methods in which it is necessary only to determine whether certain sides and angles are equal.

These methods are based upon three important tests for congruence of triangles.

32. First Test for Congruence of Triangles.

If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.

This may be shown by the following argument:



Let ABC **and** $A'B'C'$ **be two triangles in which** $AB = A'B'$, $AC = A'C'$, **and** $\angle A = \angle A'$.

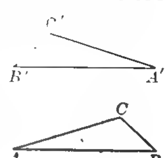
We are to show that $\triangle ABC \cong \triangle A'B'C'$.

Place $\triangle ABC$ upon $\triangle A'B'C'$ so that $\angle A$ coincides with $\angle A'$, which can be done since it is given that $\angle A = \angle A'$.

Then point B will coincide with B' and C with C', since it is given that $AB = A'B'$ and $AC = A'C'$.

Hence, side BC will coincide with $B'C'$ (§ 8).

Thus, the two triangles coincide throughout and hence are congruent (§ 27).



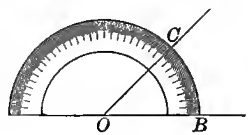
The process just used is called **superposition**. It may sometimes be necessary to move a figure out of its plane in order to superpose it upon another, as in the case of the accompanying triangles.

33. The equality of short line-segments is conveniently tested by means of the **dividers** or **compasses**.

Place the divider points on the end-points of one segment AB and then see whether they will also coincide with the end-points of the other segment $A'B'$. If so, the two segments are equal.

The equality of two angles may be tested by means of the **protractor**.

Place the protractor on one angle BOC as shown in the figure and read the scale where OC crosses it. Then place the protractor on the other angle $B'O'C'$ and see whether $O'C'$ crosses the scale at the same point. If so, the two angles are equal.



34. EXERCISES.

1. Using the protractor determine which pairs of the following angles on page 4 are equal:

$$HPG,\ LGW,\ GWL,\ AOB,\ VLW,\ LVW.$$

2. By the test of § 32 determine whether, on page 4,

$$\triangle JKU \cong \triangle GWL, \text{ also whether } \triangle MIH \cong \triangle KVW.$$

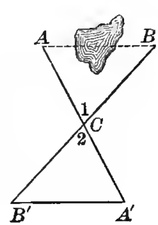
First find whether two sides of one are equal respectively to two sides of the other, and if so compare the included angles.

3. Could two sides of one triangle be equal respectively to two sides of another and still the triangles not be congruent? Illustrate by constructing two such triangles.

4. Show by the test of § 32 that two right triangles are congruent if the legs of one are equal respectively to the legs of the other. Can this be shown directly by superposition?

5. Find the distance AB when, on account of some obstruction, it cannot be measured directly.

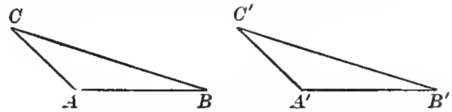
Solution. To some convenient point C measure the distances AC and BC. Continuing in the direction AC lay off $CA' = AC$, and in the direction BC lay off $CB' = BC$. Then $\angle 1 = \angle 2$ (see § 74). Test this with the protractor. Show that the length AB is found by measuring $A'B'$.



35. Second Test for Congruence of Triangles.

If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent.

This is shown by the following argument:



Let ABC and $A'B'C'$ be two triangles in which $\angle A = \angle A'$, $\angle B = \angle B'$, and $AB = A'B'$.

We are to show that $\triangle ABC \cong \triangle A'B'C'$.

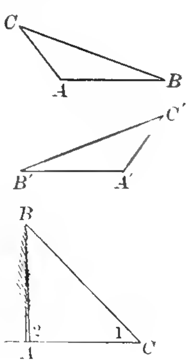
Place $\triangle ABC$ upon $\triangle A'B'C'$ so that AB coincides with its equal $A'B'$, making C fall on the same side of $A'B'$ as C'.

Then AC will take the direction of $A'C'$, since $\angle A = \angle A'$, and the point C must fall somewhere on the ray $A'C'$.

Also BC will take the direction of $B'C'$ (Why?), and hence C must lie on the ray $B'C'$.

Since the point C lies on both of the rays $A'C'$ and $B'C'$, it must lie at their point of intersection C' (§ 5). Hence, the triangles coincide and are, therefore, congruent (§ 27).

36. EXERCISES.



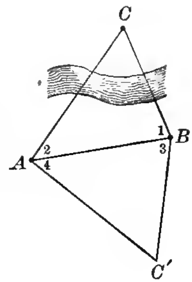
1. In the figure of § 35 is it necessary to move $\triangle ABC$ out of the plane in which the triangles lie? Is it necessary in the figure here given?

2. Show how to measure the height of a tree by using the second test for congruence.

SUGGESTION. Lay out a triangle on the ground which is congruent to $\triangle ABC$, using § 35.

3. By the second test determine whether $\triangle OHG \cong \triangle OJK$ on page 4.

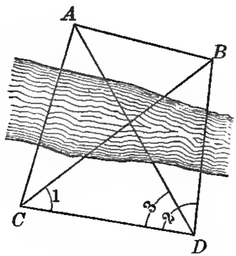
4. Draw any triangle. Construct another triangle congruent to it. Use § 35 and also § 32. Use the protractor to construct the angles.



5. Find the distance AC, when C is inaccessible.

Let B be a convenient point from which A and C are visible. Lay out a triangle ABC' making $\angle 3 = \angle 1$ and $\angle 4 = \angle 2$. Show that the distance AC may be found by measuring AC'.

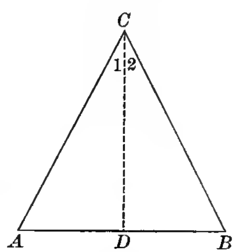
6. Show how to find the distance between two inaccessible points A and B.



Solution. Suppose that both A and B are visible from C and D. (1) Using the triangle CDA, find the length of AD as in Ex. 5 above. (2) Using the triangle CBD, find DB in the same manner. (3) Using the triangle DBA, find AB as in Ex. 5, § 34.

37. The proof of the third test for congruence of triangles involves the following:

The angles opposite the equal sides of an isosceles triangle are equal.



Let ABC be an isosceles triangle having $AC = BC$.

We are to show that $\angle A = \angle B$.

Suppose CD divides $\angle ACB$ so that $\angle 1 = \angle 2$.

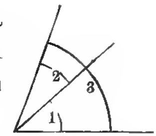
By means of § 32 show that $\triangle ACD \cong \triangle BCD$.

Then $\angle A = \angle B$ by § 29.

The theorems § 35 and § 37 are due to Thales. It is said he used § 35 in calculating the distance from the shore to a ship at sea.

38. EXERCISE.

On page 4 pick out as many pairs of angles as possible which may be shown to be equal by § 37. Test these by using the protractor.



39. Definitions. Two angles which have a common vertex and a common side are said to be **adjacent** if neither angle lies within the other.

Thus, $\angle 1$ and $\angle 2$ are adjacent, while $\angle 1$ and $\angle 3$ are not adjacent.

The **sum of two angles** is the angle formed by the sides not common when the two angles are placed adjacent.

Thus, $\angle 3 = \angle 1 + \angle 2$.

If $\angle 3 = \angle 1 + \angle 2$, then we say that $\angle 3$ is greater than either $\angle 1$ or $\angle 2$. This is written $\angle 3 > \angle 1$ and $\angle 3 > \angle 2$.

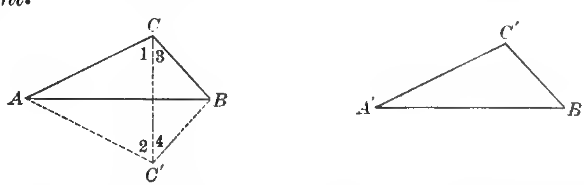
An angle may also be **subtracted** from a greater or equal angle. Thus if $\angle 3 = \angle 1 + \angle 2$, then $\angle 3 - \angle 1 = \angle 2$ and $\angle 3 - \angle 2 = \angle 1$. It is clear that:

If equal angles are added to equal angles, the sums are equal angles.

Angles may be **multiplied** or **divided** by a positive integer as in the case of line-segments. See § 10.

40. We may now prove the **third test for congruence of triangles,** namely:

If two triangles have three sides of one equal respectively to three sides of the other, the triangles are congruent.



Let ABC **and** $A'B'C'$ **be two triangles in which** $AB = A'B'$, $BC = B'C'$, $CA = C'A'$.

We are to show that $\triangle ABC \cong \triangle A'B'C'$.

Place $\triangle A'B'C'$ so that $A'B'$ coincides with AB and so that C' falls on the side of AB which is opposite C.

(Why is it possible to make $A'B'$ coincide with AB?)

Draw the segment CC'. From the data given, how can § 37 be used to show that in $\triangle ACC'$ $\angle 1 = \angle 2$?

Use the same argument to show that $\angle 3 = \angle 4$.

But if $\angle 1 = \angle 2$

and $\angle 3 = \angle 4$,

then $\angle 1 + \angle 3 = \angle 2 + \angle 4$. (§ 39)

That is, $\angle ACB = \angle BC'A$.

How does it now follow that $\triangle ABC \cong \triangle ABC'$? (§ 32)

But $\triangle ABC' \cong \triangle A'B'C'$. (§ 26)

Hence, $\triangle ABC \cong \triangle A'B'C'$. (§ 28)

Make an outline of the steps in the above argument, and see that each step is needed in deriving the next.

41. Definition. If one triangle is congruent to another because certain parts of one are equal to the corresponding parts of the other, then these parts are said to **determine the triangle.** That is, *any other triangle* constructed with these given parts will be congruent to the given triangle.

42. EXERCISES.

1. In § 37 show that CD is perpendicular to AB and that $AD = DB$. State this fully in words.

2. Using § 40, determine which of the following triangles on page 4 are congruent: *OJK*, *HNP*, *OIH*, *PHG*, *JKU*.

3. Do two sides *determine* a triangle? Three sides? Two angles? Three angles? Illustrate by figures.

4. A segment drawn from the vertex of an isosceles triangle to the middle point of the base bisects the vertex angle and is perpendicular to the base.

5. What parts of a triangle have been found sufficient to determine it? In each case how many parts are needed?

43. The three tests for congruence of triangles, §§ 32, 35, 40, lie at the foundation of the mathematics used in land surveying. The fact that certain parts of a triangle *determine* it shows that it *may* be possible to compute the other parts when these parts are known. Rules for doing this are found in Chapter III.

CONSTRUCTION OF GEOMETRIC FIGURES.

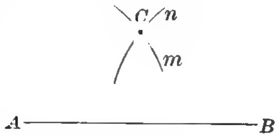
44. The straight-edge ruler and the compasses are the instruments most commonly used in the construction of geometric figures.

By means of the ruler straight lines are drawn, and the compasses are used in laying off equal line-segments and also in constructing arcs of circles (§ 12).

Other common instruments are the protractor (§ 33) and the triangular ruler with one square corner or right angle.

The three tests for congruence of two triangles are of constant use in geometrical constructions.

45. Problem. *To find a point whose distances from the extremities of a given segment are specified.*



Solution. Let *AB* be the given segment and let it be required to find a point *C* which shall be one inch from each extremity of *AB*.

Set the points of the compasses one inch apart. With *A* as a center draw an arc *m*, and with *B* as a center draw an arc *n* meeting the arc *m* in the point *C*. Then every point in the arc *m* is one inch from *A* and every point in the arc *n* is one inch from *B* (§ 12).

Hence *C*, which lies on both *m* and *n*, is one inch from *A* and also from *B*.

46. EXERCISES.

1. In the preceding problem is there any other point in the plane besides C which is one inch distant from both A and B? If so, show how to find it.

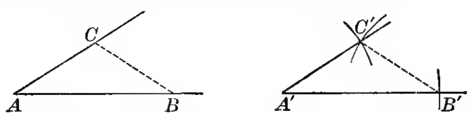
2. Could AB be given of such length as to make the construction in § 45 impossible?

3. Is there any condition under which one point only could be found in the above construction? If so, what would be the length of AB?

4. Find a point one inch from A and two inches from B and discuss all possibilities as above.

5. Given three segments a, b, c, construct a triangle having its sides equal to these segments. Discuss all possibilities depending upon the relative lengths of the given segments.

47. PROBLEM. *To construct an angle equal to a given angle, without using the protractor.*



SOLUTION. **Given the angle A.**

Lay off any distance AB on one of its sides and any distance AC on the other.

Draw the segment BC forming the triangle ABC.

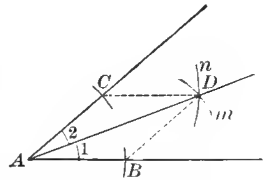
As in Ex. 5 above, construct a triangle $A'B'C'$ so that $A'B' = AB$, $B'C' = BC$, $A'C' = AC$.

Show that $\triangle ABC \cong \triangle A'B'C'$ by one of the tests, and hence that $\angle A = \angle A'$, being corresponding angles of congruent triangles, § 29.

In the above construction, would it be *wrong* to make $AB = AC$? Is it *necessary* to do so?

48. PROBLEM. *To construct the ray dividing a given angle into two equal angles, that is, to bisect the angle.*

SOLUTION. **Given the angle A.**



To construct the ray bisecting it.

On the sides of the angle lay off segments AB and AC so that $AB = AC$.

With B and C as centers and with equal radii construct arcs m and n meeting at D.

Draw the segments CD, BD, and AD.

Now show that one of the tests for congruence is applicable to make $\triangle ACD \cong \triangle ABD$.

Does it follow that $\angle 1 = \angle 2$? Why?

49. EXERCISES.

1. Is it necessary in § 48 to make $AB = AC$? In this respect compare with the construction in § 47.

2. Is any restriction necessary in choosing the radii for the arcs m and n? Is it possible to so construct the arcs m and n, still using equal radii for both, that the point D shall not lie within the angle BAC? In that case does the ray AD bisect $\angle BAC$?

3. By means of § 48 bisect a straight angle. What is the ray called which bisects a straight angle? In this case what restriction is necessary on the radii used for the arcs m and n?

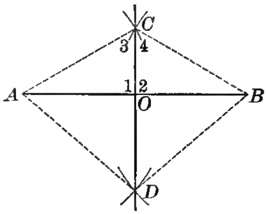
4. By Ex. 3 construct a perpendicular to a line at a given point in it.

5. Construct a perpendicular to a segment at one end of it without prolonging the segment and without using the square ruler.

SUGGESTION. Let AB be the given segment. Construct a right angle $A'B'C'$ as in Ex. 4. Then as in § 47 construct $\angle ABC = \angle A'B'C'$.

50. Definition. A line which is perpendicular to a line-segment at its middle point is called the **perpendicular bisector** of the segment.

51. PROBLEM. *To construct the perpendicular bisector of a given line-segment.*



SOLUTION. **Let *AB* be the given segment.**

As in § 45, locate two points, C and D, each of which is equally distant from A and B.

Draw the segment CD meeting AB in O.

Then CD is the required perpendicular bisector of AB.

To prove this, show that $\triangle ACD \cong \triangle BCD$.

Hence $\angle 3 = \angle 4$. (Why?)

By what test can it now be shown that

$$\triangle AOC \cong \triangle BOC?$$

Hence $\angle 1 = \angle 2$. (Why?)

Therefore CO (or CD) is perpendicular to AB (Why?) and also $AO = OB$ (Why?).

It has thus been shown that CD is perpendicular to AB and bisects it, as was required.

52. The steps proved in the above argument are:

(*a*) $\triangle ACD \cong \triangle BCD$. (*b*) $\angle 3 = \angle 4$. (*c*) $\triangle AOC \cong \triangle BOC$. (*d*) $\angle 1 = \angle 2$, and $AO = BO$.

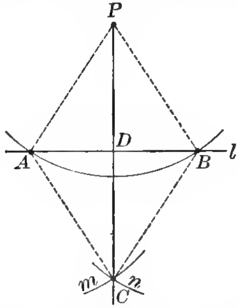
Study this outline with care. What is wanted is the last result (*d*). Notice that (*d*) is obtained from (*c*), (*c*) from (*b*), and (*b*) from (*a*). Thus each step depends on the one preceding, and would be impossible without it. To understand clearly the order of the steps in a proof as shown by such an outline is of great importance in mastering it.

53. EXERCISES.

1. In the construction of § 51, is it necessary to use the same radius in locating the points *C* and *D*?

2. Name the isosceles triangles in the figure § 51: (*a*) if the *same* radius is used for locating *C* and *D*, (*b*) if *different* radii are used.

54. PROBLEM. *To construct a perpendicular to a given straight line from a given point outside the line.*



SOLUTION. **Let l be the given line, and P the given point outside it.**

With P as a center draw an arc cutting the line l in two points, A and B.

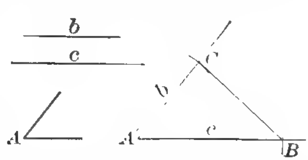
With A and B as centers, and with equal radii, draw the arcs m and n intersecting in C. Draw the line PC cutting l in the point D.

Then the line PC is the perpendicular sought.

To prove this, draw the segments PA, PB, CA, CB.

Complete the proof by showing that PC is the perpendicular bisector of AB, and hence is perpendicular to l from the point P.

55. PROBLEM. *To construct a triangle when two sides and the included angle are given.*



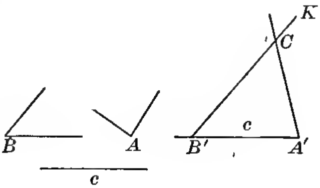
SOLUTION. **Let b and c be the given sides, and A the given angle.**

As in § 47, construct an angle A' equal to $\angle A$. On the sides of $\angle A'$ lay off $A'B = c$ and $A'C = b$. Connect B and C.

Then $A'BC$ is the required triangle. (Why?)

56. PROBLEM. *Construct a triangle when two angles and the included side are given.*

SOLUTION. Let $\angle A$ and $\angle B$ be the given angles, and c the given side.



Construct $\angle A' = \angle A$.

On one side of $\angle A'$ lay off $A'B' = c$.

At B' construct $\angle A'B'K$ equal to $\angle B$.

Let $B'K$ meet the other side of $\angle A'$ at C.

Then $A'B'C$ is the required triangle. (Why?)

57. EXERCISES.

1. If in the preceding problem two different triangles are constructed, each having the required properties, how will these triangles be related? Why?

2. If in the problem of § 55, two different triangles are constructed, each having the required properties, how will these triangles be related? Why?

3. If two triangles are constructed so that the angles of one are equal respectively to the angles of the other, will the triangles necessarily be congruent?

4. If two different triangles are constructed with the same sides, how will they be related? Why?

5. Construct an equilateral triangle. Use § 37 to show that it is also equiangular.

58. We have now seen that the three tests for the congruence of triangles are useful in making *indirect measurements* of heights and distances when direct measurement is inconvenient or impossible, and also in making numerous geometric constructions. It will be found, as we proceed, that these tests are of increasing usefulness and importance.

THEOREMS AND DEMONSTRATIONS.

59. A **geometric proposition** is a statement affirming certain properties of geometric figures.

Thus: "Two points determine a straight line" and "The base angles of an isosceles triangle are equal" are geometric propositions.

A proposition is **proved** or **demonstrated** when it is shown to follow from other known propositions.

A **theorem** is a proposition which is to be proved. The argument used in establishing a theorem is called a **proof.**

60. In every mathematical science some propositions *must be left unproved,* since every proof depends upon other propositions which in turn require proof. Propositions which for this reason are left unproved are called **axioms.**

While axioms for geometry may be chosen in many different ways, it is customary to select such simple propositions as are *evident on mere statement.*

61. Among the axioms thus far used are the following:

Axioms. I. *A figure may be moved about in space without changing its shape or size.* See § 26.

II. *Through two points one and only one straight line can be drawn.* See §§ 8, 32.

III. *The shortest distance between two points is measured along the straight line-segment connecting them.*

Thus one side of a triangle is less than the sum of the other two.

IV. *If each of two figures is congruent to the same figure, they are congruent to each other.* See §§ 28, 40.

V. *If a, b, c, d are line-segments (or angles) such that $a=b$ and $c=d$, then $a+c=b+d$ and $a-c=b-d$.*

In the latter case we suppose $a>c$, $b>d$. See §§ 10, 39.

VI. *If a and b are line-segments (or angles) such that $a = b$, then $a \times n = b \times n$ and $a \div n = b \div n$; and if $a > b$, then $a \times n > b \times n$ and $a \div n > b \div n$, n being a positive integer.* See §§ 10, 39.

NOTE. An equality or an inequality may be read from left to right or from right to left. Thus, $a > b$ may also be read $b < a$.

Other axioms are given in §§ 82, 96, 119, and in Chapter VII.

Certain other simple propositions may be assumed at present without ***detailed*** proof. These are called **preliminary theorems.**

PRELIMINARY THEOREMS.

62. *Two distinct lines can meet in only one point.*

For if they have two points in common, then by Ax. II they are the same line.

63. *All straight angles are equal.* § 20, Ex. 1.

64. *All right angles are equal.* See Ax. VI.

65. *Every line-segment has one and only one middle point.*

See § 51, where the middle point is found by construction.

66. *Every angle has one and only one bisector.*

See § 48, where the bisector is constructed.

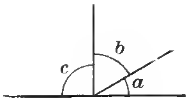
67. *One and only one perpendicular can be drawn to a line through a point whether that point is on the line or not.* See § 20, Exs. 4, 5; § 49, Ex. 4; § 54.

68. *The sum of all the angles about a point in a straight line and on one side of it is two right angles.*

69. *The sum of all the angles about a point in a plane is four right angles.*

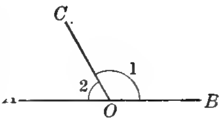
In §§ 68, 69 no side of one angle is to lie inside another.

70. Definitions. Two angles are said to be **complementary** if their sum is one right angle. Each is then called the complement of the other.



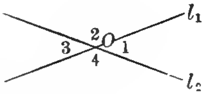
Thus, $\angle a$ and $\angle b$ are complementary angles.

Two angles are said to be **supplementary** if their sum is two right angles. Each is then said to be the supplement of the other.



Thus, $\angle 1$ and $\angle 2$ are supplementary angles.

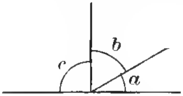
Two angles are called **vertical angles** if the sides of one are prolongations of the sides of the other.

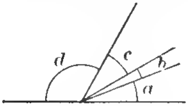


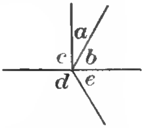
Thus, $\angle 1$ and $\angle 3$ are vertical angles, and also $\angle 2$ and $\angle 4$.

71. EXERCISES.

1. What is the complement of 45°? the supplement?
2. If the supplement of an angle is 140°, find its complement.
3. If the complement of an angle is 21°, find its supplement.
4. Find the supplement of the complement of 30°.
5. Find the angle whose supplement is five times its complement.
6. Find the angle whose supplement is n times its complement.
7. Find an angle whose complement plus its supplement is 110°.







8. If in the first figure $\angle b = 2\angle a$, and $\angle c = \angle a + \angle b$, find each angle.

9. If in the second figure $\angle b = \frac{1}{2}\angle a$, $\angle c = \angle a + \angle b$, and $\angle d = 6\angle a$, find each angle.

10. If in the third figure $\angle b = \angle e$, $\angle c = \angle a + \angle b$, $\angle d = 2\angle b$, and $\angle e = \frac{1}{2}\angle d$, find each angle.

PRELIMINARY THEOREMS.

72. *Angles which are complements of the same angle or of equal angles are equal.*

For they are the remainders when the given equal angles are subtracted from equal right angles. Ax. V.

73. *Angles which are supplements of the same angle or of equal angles are equal.*

For they are the remainders when the given equal angles are subtracted from equal straight angles.

74. *Vertical angles are equal.*

They are supplements of the same angle.

75. *If two adjacent angles are supplementary, their exterior sides are in the same straight line.*

For the two angles together form a straight angle.

76. *If two adjacent angles have their exterior sides in the same straight line, they are supplementary.*

For a straight angle is equal to two right angles.

77. EXERCISES.

1. Prove that if one of the four angles formed by two intersecting straight lines is a right angle, then all are right angles.

2. Show that the rays bisecting two complementary adjacent angles form an angle of 45°.

3. Find the angle formed by the rays bisecting two supplementary adjacent angles. Prove.

4. Find the angle formed by the rays bisecting two vertical angles. Prove.

5. The sum of two adjacent angles is 74°. Find the angle formed by their bisectors.

6. The angle formed by the bisectors of two adjacent angles is 37° 18′. Find the sum of the adjacent angles.

ON THE NATURE OF A DEMONSTRATION.

78. A theorem consists of two distinct parts, **hypothesis** and **conclusion**.

In a geometrical theorem, the hypothesis specifies certain properties which the figures in question are *assumed to possess.* The conclusion *asserts* that certain other properties belong to the figures whenever the assumed properties are present.

The hypothesis and conclusion are often intermingled in a single statement, in which case they should be explicitly separated before making the proof.

For example, in the theorem of § 37, *The angles opposite the equal sides of an isosceles triangle are equal,* the hypothesis is, "Two sides of a triangle are equal," and the conclusion is, "The angles opposite them are equal."

79. If the hypothesis consists of several parts, these should be tabulated and then checked off as the demonstration proceeds. If the theorem is properly stated, each part of the hypothesis will be used in the proof.

For instance, in the theorem of § 32, the hypothesis is: $AB = A'B'$, $AC = A'C'$, and $\angle A = \angle A'$; and the conclusion is: $\triangle ABC \cong \triangle A'B'C'$.

It will be found on examining the proof that each part of the hypothesis is *needed and used* in the course of the demonstration.

If the conclusion could be proved without using every part of the hypothesis, then the parts not used should be omitted from the hypothesis in the statement of the theorem.

80. In the proof of a theorem no conclusion should be taken for granted simply from the *appearance* of the figure. Each step in a proof should be based upon a definition, an axiom, or a theorem previously proved.

It will then follow that the theorem is *as certainly true* as are the simple, unproved propositions with which we start, and upon which our argument is based.

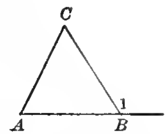
Historical Note. The Egyptians showed no knowledge of a logical demonstration, nor did the Arabians, who studied geometry quite extensively. The Greeks developed the process of demonstration to a high state of perfection. They were fully aware, moreover, that certain propositions must be admitted without proof (see § 60). Thus Aristotle (384–322 B.C.) says: "Every demonstration must start from undemonstrable principles. Otherwise the steps of a demonstration would be endless." Euclid divided unproved propositions into two classes: *axioms*, or "common notions," which are true of all things, such as, "If things are equal to the same thing they are equal to each other"; and *postulates*, which apply only to geometry, such as, "Two points determine a line." The best usage in modern mathematics is to adopt the one word *axiom* for both of these, as in § 60.

Much practice is needed in writing demonstrations in full detail. This should be done in the shortest possible sentences, usually giving a separate line to each statement, followed by the definition, axiom, or theorem on which it depends.

For this purpose the following symbols and abbreviations are convenient:

∠, ∠s, angle, angles.
△, △s, triangle, triangles.
▱, ▱s, parallelogram, parallelograms.
□, □s, rectangle, rectangles.
rt. ∠, rt. ∠s, right angle, right angles.
st. ∠, st. ∠s, straight angle, straight angles.
rt. △, rt. △s, right triangle, right triangles.
⊙, ⊙s, circle, circles.
⌒, ⌒s, arc, arcs.
=, is equal, or equivalent, to.
∼, is similar to.
≅, is congruent to.
>, is greater than.
<, is less than.
≦, is less than or equal to.
≧, is greater than or equal to.
∥, parallel, or is parallel to.
⊥, perpendicular, or is perpendicular to.
∥s, parallels.
⊥s, perpendiculars.
∴, therefore *or* hence.
ax., axiom.
th., theorem.
def., definition.
cor., corollary.
alt., alternate.
ext., exterior.
int., interior.
hyp., by hypothesis.

INEQUALITIES OF PARTS OF TRIANGLES.



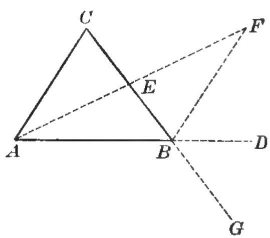
81. Definition. If one side of a triangle is produced, the angle thus formed is called an **exterior angle of the triangle.**

Thus, $\angle 1$ is an exterior angle of the triangle ABC.

82. Axiom VII. *If a, b, c are line-segments (or angles) such that $a > b$ and $b \geqq c$, or such that $a \geqq b$ and $b > c$, then $a > c$.*

The proof of the following theorem is shown in full detail as it should be written by the pupil or given orally, except that the numbers of paragraphs should not be required.

83. THEOREM. *An exterior angle of a triangle is greater than either of the opposite interior angles.*



Given the $\triangle ABC$ with the exterior angle DBC formed by producing the side AB.

To prove that $\angle DBC > \angle C$ and also $\angle DBC > \angle A$.

Proof: Let E be the middle point of BC.

Find E by the construction for bisecting a line-segment (§ 51).

Draw AE and prolong it, making $EF = AE$, and draw BF.

In the two $\triangle s$ ACE and FBE, we have by construction

$$CE = EB \text{ and } AE = EF.$$

Also $\angle CEA = \angle BEF.$

(Vertical angles are equal, § 74.)

$\therefore \triangle ACE \cong \triangle FBE.$

(Two triangles which have two sides and the included angle of the one equal respectively to two sides and the included angle of the other are congruent, § 32.)

$\therefore \angle C = \angle FBE.$

(Being angles opposite equal sides in congruent triangles, § 29.)

But $\angle DBC > \angle FBE.$

(If an angle is the sum of two angles it is greater than either of them, § 39.)

$\therefore \angle DBC > \angle C.$

(Since $\angle DBC > \angle FBE$ and $\angle FBE = \angle C$, Ax. VII, § 82.)

In order to prove $\angle DBC > \angle A$, prolong *CB* to some point *G*.

Then $\angle ABG = \angle DBC.$

(Vertical angles are equal, § 74.)

Now bisect *AB*, and in the same manner as before we may prove $\angle ABG > \angle A.$

$\therefore \angle DBC > \angle A.$

(Since $\angle DBC = \angle ABG$ and $\angle ABG > \angle A$, Ax. VII, § 82.)

For the second part of the proof let *H* be the middle point of *AB*. Draw *CH* and prolong it to *K*, making $CH = HK$.

Let the student draw the figure for the second part of the proof and give it in full.

Hereafter more and more of the details of the proofs will be left for the student to fill in.

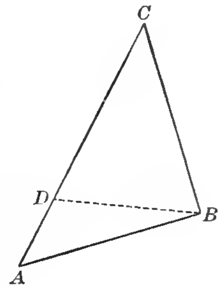
When reference is made to a paragraph in the text or when the reason for a step is called for, the complete statement of the definition, axiom, or theorem should be given by the student.

84. THEOREM. *If two sides of a triangle are unequal, the angles opposite these sides are unequal, the greater angle being opposite the greater side.*

Given $\triangle ABC$ **in which** $AC > BC$.

To prove that $\angle ABC > \angle A$.

Proof: Lay off $CD = CB$ and draw BD.



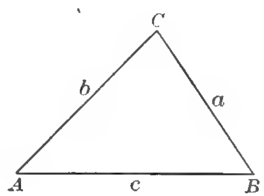
Now give the reasons for the following steps:

(1)	$\angle ABC > \angle DBC$.	(§ 39)
(2)	$\angle DBC = \angle CDB$.	(§ 37)
(3)	$\angle CDB > \angle A$.	(§ 83)
(4)	$\therefore \angle ABC > \angle A$.	(§ 82)

85. THEOREM. *If two angles of a triangle are unequal, the sides opposite them are unequal, the greater side being opposite the greater angle.*

Given $\triangle ABC$ **in which** $\angle B > \angle A$.

To prove that $b > a$.



Proof: One of the following three statements must be true:

(1) $b = a$, (2) $b < a$, (3) $b > a$.

But it cannot be true that $b = a$, for in that case $\angle B = \angle A$, (§ 37)
contrary to the hypothesis that $\angle B > \angle A$.

And it cannot be true that $b < a$, for in that case

$$\angle B < \angle A, \quad (\S\ 84)$$

contrary to the hypothesis.

Hence it follows that $b > a$.

86. The above argument is called **proof by exclusion.** Its success depends upon being able to enumerate *all the possible cases*, and then to exclude all but *one* of them by showing that each in turn leads to some *contradiction.*

87. **EXERCISES.**

1. The hypotenuse of a right triangle is greater than either leg.

2. Show that not more than two equal line-segments can be drawn from a point to a straight line.

SUGGESTION. Suppose a third drawn. Then apply §§ 37, 83, 84.

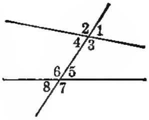
3. Show by joining the vertex A of the triangle ABC to any point of the side BC that $\angle B + \angle C < 2$ rt. $\angle$s. Use § 83.

4. *If two angles of a triangle are equal, the sides opposite them are equal.* Use §§ 84, 86.

5. Either leg of an isosceles triangle is greater than half of the base.

6. Show that an equiangular triangle is equilateral, and conversely.

THEOREMS ON PARALLEL LINES.



88. A straight line which cuts two straight lines is called a **transversal.** The various angles formed are named as follows:

$\angle 4$ and $\angle 5$ are **alternate-interior** angles; also $\angle 3$ and $\angle 6$.

$\angle 2$ and $\angle 7$ are **alternate-exterior** angles; also $\angle 1$ and $\angle 8$.

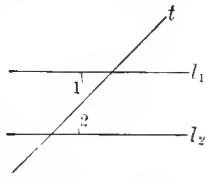
$\angle 1$ and $\angle 5$ are **corresponding** angles; also $\angle 3$ and $\angle 7$, $\angle 2$ and $\angle 6$, $\angle 4$ and $\angle 8$.

$\angle 3$ and $\angle 5$ are **interior angles** on the same side of the transversal; also $\angle 4$ and $\angle 6$.

89. **Definition.** Two complete lines which lie in the same plane and which do not meet are said to be **parallel.**

Two line-segments are parallel if they lie on parallel lines.

90. THEOREM. *If two lines cut by a transversal have equal alternate interior angles, the lines are parallel.*



Given the lines l_1 and l_2 cut by t so that $\angle 1 = \angle 2$.

To prove that $l_1 \parallel l_2$.

Proof: Suppose the lines l_1 and l_2 were to meet on the *right* of the transversal. Then a triangle would be formed of which $\angle 1$ is an exterior angle and $\angle 2$ an opposite interior angle.

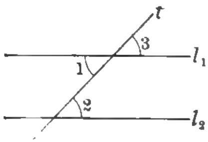
This gives an exterior angle of a triangle equal to an opposite interior angle, which is impossible. (Why?)

Repeat this argument, supposing l_1 and l_2 to meet on the *left* side of the transversal.

Hence l_1 and l_2 cannot meet and are parallel (§ 89).

91. The type of proof used here is called an **indirect proof.** It consists in showing that something impossible or contradictory results if the theorem is supposed *not* true.

92. THEOREM. *If two lines cut by a transversal have equal corresponding angles, the lines are parallel.*



Given the lines l_1 and l_2 cut by t so that $\angle 2 = \angle 3$.

To prove that $l_1 \parallel l_2$.

Proof: Quote the authority for each of the following steps:

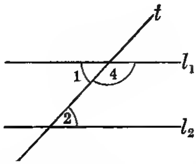
$\angle 3 = \angle 2.$

$\angle 3 = \angle 1.$ (§ 74)

$\therefore \angle 1 = \angle 2.$ (Ax. IV)

$\therefore l_1 \parallel l_2.$ (§ 90)

93. THEOREM. *If two lines cut by a transversal have the sum of the interior angles on one side of the transversal equal to two right angles, the lines are parallel.*



Given l_1 **and** l_2 **cut by** t **so that** $\angle 4 + \angle 2 = 2$ **rt.** $\angle$s.

To prove that $l_1 \parallel l_2$.

Proof: $\angle 4$ is supplementary to $\angle 1$ and also to $\angle 2$. (Why?)

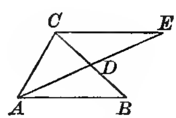
$\therefore \angle 1 = \angle 2.$ (Why?)

$\therefore l_1 \parallel l_2.$ (Why?)

94. EXERCISES.

1. *Show that if each of two lines is perpendicular to the same line, they are parallel to each other.*

2. Let ABC be any triangle. Bisect BC at D. Draw AD and prolong it to make $DE = AD$. Draw CE. Prove $CE \parallel AB$.

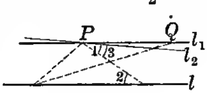


3. Use Ex. 2 to construct a line through a given point parallel to a given line.

SUGGESTION. Let AB be a segment of the given line and let C be the given point. Draw CA and CB and proceed as in Ex. 2.

95. Exs. 2 and 3 above show that through a point P, not on a line l, *at least one line* l_1 can be drawn parallel to l.

It seems reasonable to suppose that no other line l_2 can be drawn through P parallel to l, although this cannot be proved from the preceding theorems. See § 60.

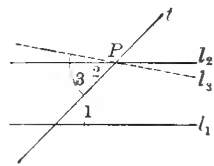


Hence we assume the following:

96. Axiom VIII. *Through a point not on a line only one straight line can be drawn parallel to that line.*

HISTORICAL NOTE. This so-called axiom of parallels has attracted more attention than any other proposition in geometry. Until the year 1829 persistent attempts were made by the world's most eminent mathematicians to prove it by means of the other axioms of geometry. In that year, however, a Russian, Lobachevsky, showed this to be impossible and hence it must forever remain an axiom unless some other equivalent proposition is assumed.

97. THEOREM. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Given $l_1 \parallel l_2$ **and cut by** t.

To prove that $\angle 1 = \angle 2$.

Proof: Suppose $\angle 2$ not equal to $\angle 1$.
Through P draw l_3, making $\angle 3 = \angle 1$.

$\therefore l_3 \parallel l_1$. (Why?)

But by hypothesis $l_2 \parallel l_1$ and thus we have through P two lines parallel to l_1, which is contrary to Ax. VIII.

Therefore, the supposition that $\angle 2$ is not equal to $\angle 1$ leads to a contradiction, and hence $\angle 1 = \angle 2$.

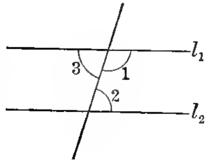
98. Compare this theorem with that of § 90. The hypothesis of either is seen to be the conclusion of the other.

When two theorems are thus related, each is said to be the **converse** of the other. Other pairs of converse theorems thus far are those in § 37 and Ex. 4, § 87; §§ 75 and 76, and §§ 84 and 85.

The converse of a theorem is never to be taken for granted without proof, since it does not follow that a statement is true because its converse is true.

Thus, it is true that *if a triangle is equilateral, it is also isosceles*, but the converse, *if a triangle is isosceles, it is also equilateral*, is not true.

99. THEOREM. *If two parallel lines are cut by a transversal, the sum of the interior angles on one side of the transversal is two right angles.*



Suggestion. Make use of the preceding theorem and give the proof in full. Of what theorem is this the converse?

100. **EXERCISES.**

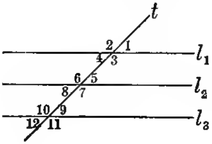
1. State and prove the converse of the theorem in § 92.

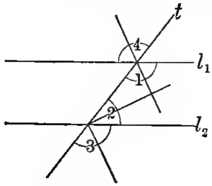
2. Prove that if two parallel lines are cut by a transversal the alternate exterior angles are equal. Draw the figure.

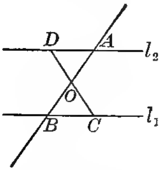
3. State and prove the converse of the theorem in Ex. 2.

4. If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

5. Two straight lines in the same plane parallel to a third line are parallel to each other. Suppose they meet and then use § 96.



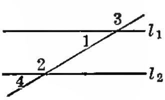




6. If $l_1 \parallel l_2 \parallel l_3$ and if $\angle 1 = 30°$, find the other angles in the first figure.

7. If $l_1 \parallel l_2$, how are the bisectors of $\angle 1$ and $\angle 3$ related? Of $\angle 3$ and $\angle 4$? Of $\angle 1$ and $\angle 2$? Use § 102 for the last case.

8. If $l_1 \parallel l_2$, and $AO = OB$, show that $DO = OC$. State this theorem fully and prove it.



9. If $l_1 \parallel l_2$ and $\angle 2 = 5\angle 1$, find $\angle 4$ and $\angle 3$.

10. If two parallel lines are cut by a transversal, the sum of the exterior angles on one side of the transversal is two right angles.

11. State and prove the converse of the preceding theorem.

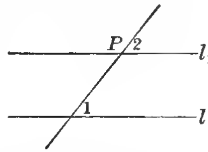
APPLICATIONS OF THEOREMS ON PARALLELS.

101. PROBLEM. *Through a given point to construct a line parallel to a given line.*

Given the line l and the point P outside of it.

To construct a line l_1 through $P \parallel$ to l.

Construction. Through P draw any line making a convenient angle, as $\angle 1$ with l.

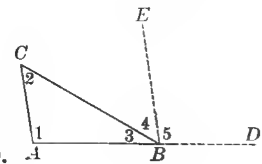


Through P draw the line l_1, making $\angle 2 = \angle 1$ (§ 47). Then $l_1 \parallel l$.

Proof: Use the theorem, § 92.

Hereafter all constructions should be described fully as above, followed by a proof that the construction gives the required figure.

102. THEOREM. *The sum of the angles of a triangle is equal to two right angles.*



Given $\triangle ABC$ **with** $\angle 1$, $\angle 2$, $\angle 3$.

To prove that

$$\angle 1 + \angle 2 + \angle 3 = 2\text{rt}\,\measuredangle s.$$

Proof: Prolong AB to some point D.

Through B draw $BE \parallel AC$.

Then $\quad \angle 5 + \angle 4 + \angle 3 = 2\text{ rt}\,\measuredangle s.$ (Why?)

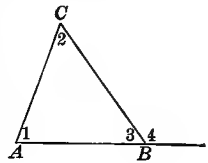
But $\quad \angle 4 = \angle 2$ and $\angle 5 = \angle 1$. (Why?)

Hence, replacing $\angle 5$ and $\angle 4$ by their equals, $\angle 1$ and $\angle 2$, we have $\angle 1 + \angle 2 + \angle 3 = 2\text{ rt}\,\measuredangle s.$

HISTORICAL NOTE. This is one of the famous theorems of geometry. It was known by Pythagoras (500 B.C.), but special cases were known much earlier. The figure used here is the one given by Aristotle and Euclid. As is apparent, the proof depends upon the theorem, § 97, and thus indirectly upon Axiom VIII. The interdependence of these two propositions has been studied extensively during the last two centuries.

103. THEOREM. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

The proof is left to the student. Compare this theorem with that of § 83.



104. Definition. A theorem which follows very easily from another theorem is called a **corollary** of that theorem. *E.g.* the theorem in § 103 is a corollary of that in § 102.

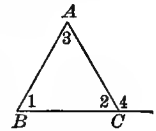
105. EXERCISES.

1. Find each angle of an equiangular triangle.

2. If one angle of an equiangular triangle is bisected, find all the angles in the two triangles thus formed.

3. If in a $\triangle ABC$, $AB = AC$ and $\angle A = \angle B + \angle C$, find each angle.

4. If in the figure $AB = AC$ and $\angle 4 = 120°$, find $\angle 1$, $\angle 2$, $\angle 3$.

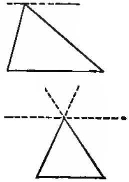


5. If one acute angle of a right triangle is 30°, what is the other acute angle? If one is 41° 23′?

6. If in two right triangles an acute angle of one is equal to an acute angle of the other, what can be said of the remaining acute angles. What axiom is involved?

7. *If in two right triangles the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other, the triangles are congruent.* Prove in full.

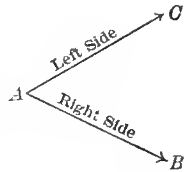
8. Can a triangle have two right angles? Two obtuse angles? Can the sum of two angles of a triangle be two right angles? What is the sum of the acute angles of a right triangle?



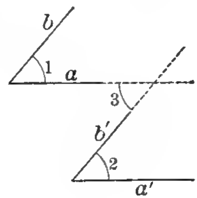
9. If two angles of a triangle are given how can the third be found? If the sum of two angles of one triangle is equal to the sum of two angles of another, how do the third angles compare?

10. Prove the theorem of § 102, using each of the figures in the margin. The first of these figures was used by Pythagoras.

106. Definition. An angle viewed from the vertex has a **right side** and a **left side.**



107. THEOREM. *If two angles have their sides respectively parallel, right side to right side, and left side to left side, the angles are equal.*



Given $\angle 1$ **and** $\angle 2$ **such that** $a \parallel a'$ **and** $b \parallel b'$.

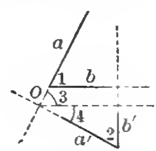
To prove that $\angle 1 = \angle 2$.

Proof: Produce a and b' till they meet, forming $\angle 3$. Complete the proof.

Why do a and b' meet when produced?

Make a proof also by producing b and a' till they meet.

108. THEOREM. *If two angles have their sides respectively perpendicular, right side to right side, and left side to left side, the angles are equal.*



Given $\angle 1$ **and** $\angle 2$ **such that** $a \perp a'$ **and** $b \perp b'$.

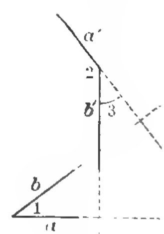
To prove that $\angle 1 = \angle 2$.

Proof: Produce a and a' till they meet in **O**.

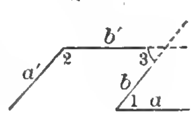
Through **O** draw a line parallel to b.

Now show that $\angle 2 = \angle 3$ since each is the complement of $\angle 4$. Complete the proof.

109. EXERCISES.



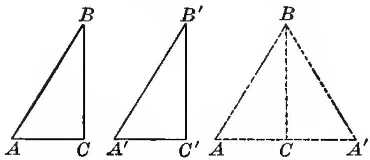
If two angles have their sides respectively parallel, or perpendicular, right side to left side, and left side to right side, the angles are supplementary. In each figure $\angle 1 = \angle 3$. (Why?)



HISTORICAL NOTE. The theorems of §§ 107, 108, 109 are not found in Euclid's Elements.

OTHER THEOREMS ON TRIANGLES.

110. THEOREM. *If in two right triangles the hypotenuse and one side of one are equal respectively to the hypotenuse and one side of the other, the triangles are congruent.*



Given the right △ ABC and $A'B'C'$, having $AB = A'B'$ and $BC = B'C'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof: Place the triangles so that BC and $B'C'$ coincide, and so that A and A' are on opposite sides of BC.

Then AC and CA' lie in a straight line (Why?), and $\triangle ABA'$ isosceles (Why?).

Hence, show that $\angle A = \angle A'$ and complete the proof.

111. COROLLARY. *If from a point in a perpendicular to a straight line equal oblique segments are drawn to the line, these cut off equal distances from the foot of the perpendicular, and make equal angles with the perpendicular and with the given line.*

Give the proof in full, as of an independent theorem.

112. THEOREM. *If from a point in a perpendicular to a line oblique segments are drawn, cutting off equal distances from the foot of the perpendicular, these segments are equal, and make equal angles with the perpendicular and with the given line.*

Give the proof in full, using $\triangle ABA'$ of § 110.

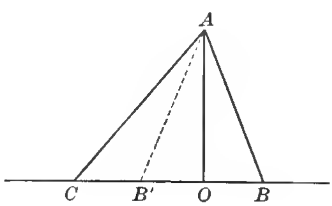
113. THEOREM. *The perpendicular is shorter than any oblique segment from a point to a line.*

Suggestion. Show by § 102 that the angle opposite the oblique segment in the triangle formed is greater than either of the other angles, and then make use of § 85.

114. The **distance** from a point to a line means the ***shortest distance***, and hence is measured on the perpendicular.

115. THEOREM. *If from a point in a perpendicular to a line segments are drawn cutting off unequal distances from the foot of the perpendicular, the segments are unequal, that segment being the greater which cuts off the greater distance.*

Given $AO \perp BC$ **and** $OC > OB$.



To prove that $AC > AB$.

Proof: Let $OB' = OB$ and draw AB'.

Then, $\angle OB'A <$ rt. $\angle$, and $\therefore \angle AB'C >$ rt. $\angle$.

Also $\angle OCA <$ rt. $\angle$.

$\therefore AC > AB'$.

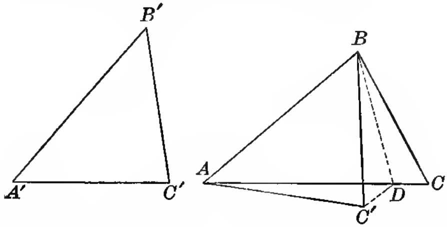
$\therefore AC > AB$ since $AB' = AB$. Give reasons for each step.

116. THEOREM. *If from a point in a perpendicular to a line unequal segments are drawn, these cut off unequal distances from the foot of the perpendicular, the greater segment cutting off the greater distance.*

Suggestion. Using the figure of § 115 and the hypothesis that $AC > AB$, show that two of the following statements are impossible:

(1) $OC = OB$; (2) $OC < OB$; (3) $OC > OB$. See § 86.

117. THEOREM. *If in two triangles two sides of the one are equal respectively to two sides of the other, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given $\triangle ABC$ **and** $A'B'C'$ **in which** $AB = A'B'$, $BC = B'C'$ **and** $\angle B > \angle B'$.

To prove that $AC > A'C'$.

Proof: Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'B'$ coincides with its equal AB and C' is on the same side of AB as C.

Let BD bisect $\angle C'BC$, meeting AC in D. Draw DC'.

Then $\triangle BDC' \cong \triangle BDC$. (Why?)

$\therefore DC' = DC$ and $AD + DC' = AC$. (Why?)

But $AD + DC' > AC'$. (Ax. III, § 61)

$\therefore AC > AC'$. (Ax. VII, § 82)

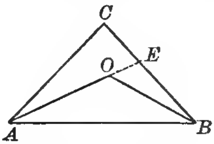
118. THEOREM. *If in two triangles two sides of the one are equal to two sides of the other but the third side of the first is greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*

Suggestion. Using the figure of § 117 and the hypothesis that $AC > A'C'$, show that two of the three following statements are impossible:

(1) $\angle B = \angle B'$; (2) $\angle B < \angle B'$; (3) $\angle B > \angle B'$. See § 86.

119. **Axiom IX.** *If a, b, c, d are line-segments (or angles) such that $a > b$ and $c = d$ or such that $a > b$ and $c > d$, then $a + c > b + d$. Also if $a > b$ and $c \leqq d$, then $a - c > b - d$, provided $a > c$ and $b > d$.* (*See* §§ 10, 39.)

120. THEOREM. *The sum of the segments drawn from a point within a triangle to the extremities of one side is less than the sum of the other two sides.*



Given the point O within the $\triangle ABC$.

To prove that $AO + OB < AC + CB$.

Proof: $AO + OE < AC + CE$. (Why?)

And $OB < OE + EB$. (Why?)

$\therefore AO + OE + OB < AC + CE + OE + EB$. (Ax. IX)

Subtracting OE from both members,

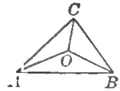
$AO + OB < AC + CE + EB$. (Ax. IX)

That is, $AO + OB < AC + CB$.

121. EXERCISES.

1. Show that any side of a triangle is greater than the difference between the other two sides.

2. Show that the sum of the distances from any point within a triangle to the vertices is greater than one-half the sum of the sides of the triangle.



3. Show that the segment joining the vertex of an isosceles triangle to any point in the base is less than either of the equal sides.

4. Show that any altitude of an equilateral triangle bisects the vertex angle from which it is drawn and also bisects the base.

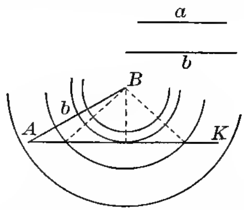
5. State and prove the converse of the theorem in Ex. 4.

6. Construct angles of 60°, 120°, and 30°.

7. Construct angles of 45° and 135°.

SUGGESTION. Bisect a right angle and extend one side.

122. PROBLEM. *Given two sides of a triangle and an angle opposite one of them, to construct the triangle.*



SOLUTION. **Let ∠ A and the segments a and b be the given parts.**

On one side of ∠ A lay off $AB = b$, and let the other side be extended to some point K.

With B as a center and a radius equal to a, construct arcs of circles as shown in the figure.

The following cases are possible:

(1) If a equals the perpendicular distance from B to AK, the arc will meet AK in but one point, and a right triangle is the solution.

(2) If $a < b$ and greater than the perpendicular, the arc cuts AK in two points, and there are then two triangles containing the given parts, as shown by the dotted lines in the figure.

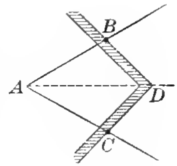
(3) If $a > b$, the arc will cut AK only once on the right of A, and hence only one triangle will be found.

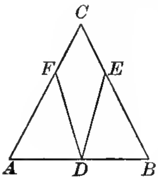
Repeat this construction, making a separate figure for each case.

Make the construction when ∠ A is a right angle. Are all three cases possible then? Make the construction when ∠ A is obtuse. What cases are possible then?

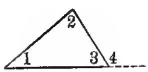
123. **EXERCISES.**

1. A carpenter bisects an angle A as follows: Lay off $AB = AC$. Place a steel square so that $BD = CD$ as shown in the figure. Draw the line AD. Is this method correct? Give proof. Would this method be correct if the square were not right-angled at D?

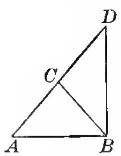


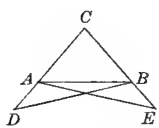


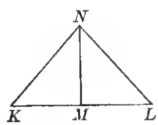
2. In the triangle ABC, $AC = BC$. The points D, E, F are so placed that $AD = BD$ and $AF = BE$. Compare DE and DF. Prove your conclusion.

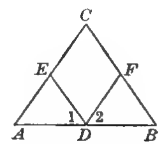


3. If in the figure $\angle 2 - \angle 1 = 15°$, and $\angle 4 = 120$, find each angle of the triangle.





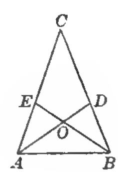




4. If in $\triangle ABC$, $AC = BC$, and if AC is extended to D so that $AC = CD$, prove that DB is perpendicular to AB.

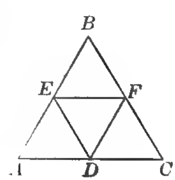
5. In $\triangle ABC$, $CA = CB$, $AD = BE$. Prove $\triangle ADB \cong \triangle ABE$.

6. In the triangle KLN, NM is perpendicular to KL, and $KM = MN = ML$. Prove that KLN is an isosceles right triangle.

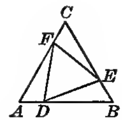


7. If in the isosceles $\triangle ABC$ a point D lies in the base, and $\angle 1 = \angle 2$, determine whether there is any position for D such that $DE = DF$.

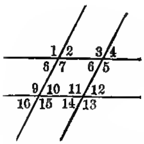
8. If the bisectors AD and BE of the base angles of an isosceles triangle ABC meet in O, what pairs of equal angles are formed? What pairs of equal segments? Of congruent triangles?



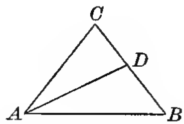
9. If the middle points of the sides of an equilateral triangle are connected as shown in the figure, compare the resulting four triangles.



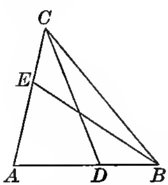
10. Triangle ABC is equilateral. $AD = BE = CF$. Compare the triangles DBE, ECF, FAD.



11. Two railway tracks cross as indicated in the figure. What angles are equal and what pairs of angles are supplementary? State a theorem involved in each case.



12. In a $\triangle ABC$ does the bisector of $\angle A$ also bisect the side BC (1) if $AC = BC$ but $AC < AB$, (2) if $AC = AB$?

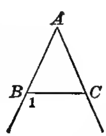


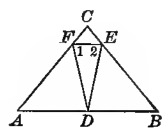
13. If in the triangle ABC, $AB = AC$ and if E is any point on AC, find D on AB so that

$$\triangle ECB \cong \triangle CDB.$$

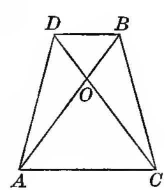
14. If in the figure $AB = AC$, find $\angle 1$ if $\angle A = 60°$; if $\angle A = 40°$. Show that whatever the value of $\angle A$, $\angle 1 = \frac{1}{2}\angle A + \text{rt.}\angle$.

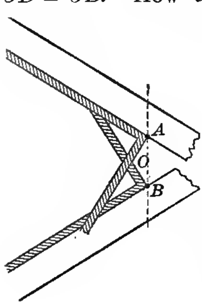
15. If in the isosceles triangle ABC the middle point of AB is D and $AF = BE$, find the relation between $\angle 1$ and $\angle 2$.





16. In the figure $AO = OC$ and $OD = OB$. How are the segments AC and DB related? How are AD and CB related? Prove.

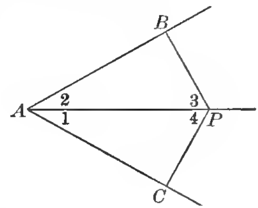




17. To cut two converging timbers by a line AB which shall make equal angles with them, a carpenter proceeds as follows: Place two squares against the timbers, as shown in the figure, so that $AO = BO$. Show that AB is the required line.

DETERMINATION OF LOCI.

124. THEOREM. *Every point on the bisector of an angle is equidistant from the sides of the angle.*



Given P, any point on the bisector of the angle A, and PC and PB perpendicular to the sides.

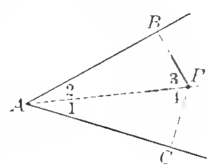
To prove that $PC = PB$.

Proof: By the hypothesis

$$\angle 1 = \angle 2.$$

Show that $\angle 3 = \angle 4$ and complete the proof.

125. THEOREM. *If a point is equally distant from the sides of an angle, it lies on the bisector of the angle.*



Given an angle A, any point P and the perpendiculars PB and PC equal.

To prove that PA bisects the angle A.

Proof: Give the argument in full to show that $\triangle ABP \cong \triangle APC$ and thus show that $\angle 1 = \angle 2$.

Hence AP is the bisector of the angle A.

126. The two preceding theorems enable us to assert the following:

(1) *Every point in the bisector of an angle is equidistant from its sides.*

(2) *Every point equidistant from the sides of an angle lies in its bisector.*

For these reasons the bisector of an angle is called the **locus** of all points equidistant from its sides.

The word *locus* means *place* or *position*. It gives the location of all points having a given property.

127. All points in a plane which satisfy some specified condition, as in the case preceding, will in general be restricted to a certain geometric figure.

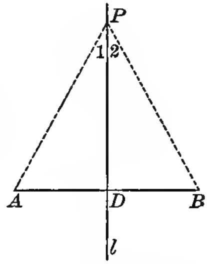
This figure is called the **locus of the points** satisfying the required condition, provided:

(1) *Every point in the figure possesses the required property.*

(2) *Every point in the plane which possesses the required property lies in the figure.*

128. THEOREM. *The locus of all points equidistant from the extremities of a given line-segment is the perpendicular bisector of the segment.*

Given the perpendicular bisector l meeting the segment AB at D.



To prove that (*a*) every point in l is equidistant from A and B; (*b*) every point which is equidistant from A and B lies in l.

Proof: (*a*) Let P be any point in l. Draw PA and PB.

Then $PA = PB$. (§ 112)

(*b*) Let P be any point in the plane such that $PA = PB$.

Bisect $\angle APB$ with the line PD.

Now show that $\triangle ADP \cong \triangle BPD$.

And hence that $AD = BD$ and $\angle ADP = \angle BDP$.

Hence PD is the perpendicular bisector of AB, and since there is only one such, this is the line l (§ 67).

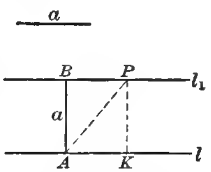
Thus the perpendicular bisector of the segment fulfills the two requirements for the locus in question.

Steps (*a*) and (*b*) together show that a point not on the line l is unequally distant from A and B.

129. PROBLEM. *To find the locus of all points at a given distance from a given straight line.*

SOLUTION. **Given the line l and the segment a.**

Construct a perpendicular to l at some point A and lay off $AB = a$.



Through B draw $l_1 \parallel l$.

Then l_1 is a *part* of the locus required.

Proof: (1) To show that any point P in l_1 is at the distance a from l.

Draw PA and also let fall $PK \perp l$.

Now $\triangle ABP \cong \triangle AKP$ and $PK = AB = a$. (Why?)

(2) To show that any point P in the plane *above* l whose distance from l is a lies in l_1.

Draw a $\perp$ from P to l, meeting l and l_1 in K and P' respectively. Then by (1) $KP' = a$. But $KP = a$. Hence P and P' coincide and P lies on l_1.

$\therefore$ l_1 is a part of the locus sought.

Let the student find another line which is also a part of the locus.

NOTE. In (1) and (2) above, great care is needed in keeping the hypothesis clearly in mind.

130. EXERCISES.

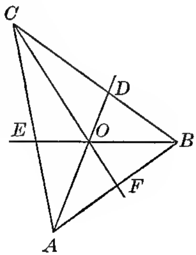
1. Find the locus of all points in the plane equally distant from two parallel lines.

2. Find the complete locus of all points in the plane equally distant from two intersecting lines.

3. Find the locus of all points in the plane equally distant from a fixed point.

4. Find the locus of all points in the plane equally distant from two fixed points.

131. THEOREM. *The bisectors of the three angles of a triangle meet in a point.*



Given ***AD, BE,*** **and** ***CF*** **bisecting** $\angle A$, $\angle B$, $\angle C$ **respectively of the triangle** ***ABC.***

To prove that *AD*, *BE*, and *CF* meet in a common point.

Proof: Let *AD* and *BE* meet in some point, as *O*.

Then *O* is equidistant from *AB* and *AC*. Why?

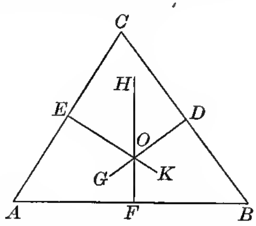
Also *O* is equidistant from *AB* and *BC*. (Why?)

$\therefore$ *O* is equidistant from *AC* and *BC*. (Why?)

Then *O* lies on the bisector of $\angle C$. (Why?)

That is, *CF* passes through the point *O*, and thus the three bisectors meet in a common point.

132. THEOREM. *The three perpendicular bisectors of the sides of a triangle meet in a point.*



Given ***FH, DG,*** **and** ***EK*** **perpendicular bisectors of the sides** ***AB, BC,*** **and** ***CA*** **of** $\triangle$ ***ABC.***

To prove that *FH*, *DG*, and *EK* meet in a point.

Proof: The proof is exactly similar to that of § 131.

133. **EXERCISES.**

1. In § 131 how do we know that *AD* and *BE* meet?

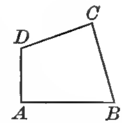
SUGGESTION. Show that *AD* and *BE* cannot be parallel.

2. In § 132 how do we know that *FH* and *GD* meet?

3. Do the bisectors of the angles always meet *inside* the triangle?

4. Do the perpendicular bisectors of the sides always meet *inside* the triangle? Draw figures to illustrate the various cases.

THEOREMS ON QUADRILATERALS.

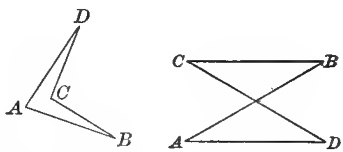


134. Definitions. If no three of the points *A*, *B*, *C*, *D* lie in the same straight line, the figure formed by the four segments *AB*, *BC*, *CD*, *DA*, is called a **quadrilateral.**

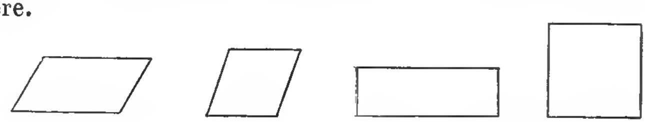
The segments are the **sides** of the quadrilateral and the points are its **vertices.**

Two sides are **adjacent** if they meet in a vertex, as *AB* and *BC*. Otherwise they are **opposite,** as *AB* and *CD*. Two vertices are **adjacent** if they lie on the same side, as *A* and *B*. Otherwise they are **opposite,** as *A* and *C*.

A **diagonal** of a quadrilateral is a segment joining two opposite vertices, as *AC* and *BD*.



Quadrilaterals which have a *reëntrant* angle, such as ∠*BCD*, and those in which two sides *intersect*, such as *AB* and *CD*, are not considered here.



Rhomboid Rhombus Rectangle Square

135. A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.

A **rhomboid** is a parallelogram whose angles are oblique. A **rhombus** is an *equilateral* rhomboid.

A **rectangle** is a parallelogram whose angles are right angles. A **square** is an equilateral rectangle.

The side on which a parallelogram is supposed to stand is called its **lower base,** and the side opposite is its **upper base.**

136. A **trapezoid** is a quadrilateral having only one pair of opposite sides parallel.

An **isosceles** trapezoid is one in which the two non-parallel sides are equal.

In a trapezoid the two parallel sides are the upper and lower bases.

137. The **altitude** of a parallelogram or a trapezoid is the perpendicular distance between its bases, and its **diameter** is the segment joining the middle points of the other sides.

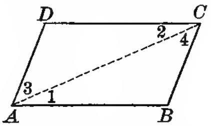
138. EXERCISES.

1. Name each of the following quadrilaterals on page 4: *IHPN*, *IHGO*, *AEDB*, *COED*, *JMPG*, *RSED*, *KLWV*, *JIBC*. To determine whether opposite sides of these figures are parallel, use the protractor for measuring the necessary angles.

2. Is every rectangle a parallelogram? Is the converse true?

3. Is every rectangle a square? Is the converse true?

139. THEOREM. *Opposite sides of a parallelogram are equal.*



Given ***ABCD*, a parallelogram; that is, $AB \parallel CD$ and $AD \parallel BC$.**

To prove that $AB = CD$ and $BC = AD$.

Proof: Draw the diagonal AC.

Prove $\triangle ABC \cong \triangle ACD$ and compare corresponding sides.

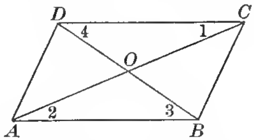
What determines which are corresponding sides?

140. EXERCISES.

1. Show that a diagonal of a parallelogram divides it into two congruent triangles.

2. Give the proof in § 139, using the diagonal *BD*.

141. THEOREM. *The diagonals of a parallelogram bisect each other.*



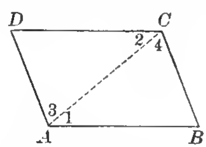
Given ▱ *ABCD* **with its diagonals meeting at the point** *O*.

To prove that $OC = OA$ and $OB = OD$.

Proof: In the triangles *AOB* and *COD*, determine whether sufficient parts are equal to make them congruent, and if so compare corresponding parts. Give all the details of the proof.

Can the proof be given by using the ⩟ *AOD* and *BOC*? If so, give it.

142. THEOREM. *If a quadrilateral has both pairs of opposite sides equal, the figure is a parallelogram.*



Given a quadrilateral *ABCD* **in which** $AB = CD$ **and** $AD = BC$.

To prove that $AB \parallel CD$ and $AD \parallel BC$.

Proof: Draw the diagonal *AC*.

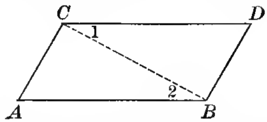
In the ⩟ *ABC* and *ADC* determine whether any test for congruence applies, and if so compare corresponding angles.

143. EXERCISES.

1. By use of the last theorem the question of Ex. 1, § 138, can be answered by measuring *sides* instead of *angles*. Verify the results by this process.

2. Which two of the theorems §§ 139, 141, 142 are converse? State in detail the hypothesis and conclusion of each.

144. THEOREM. *If a quadrilateral has one pair of opposite sides equal and parallel, the figure is a parallelogram.*



Given. **(State all the items given in the hypothesis.)**

To prove. (State what needs to be proved in order to show that $ABCD$ is a parallelogram.)

Proof: From the data given prove that

$$\triangle ABC \cong \triangle BDC.$$

Use corresponding angles of these congruent triangles to show that the *other* two opposite sides are parallel.

Hence show that the figure is a parallelogram.

Write out this demonstration in full.

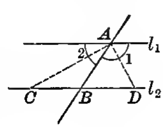
Could the theorem be proved equally well by drawing the other diagonal? If so, draw it and give the proof.

145. **EXERCISES.**

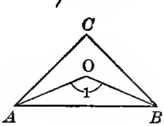
1. Prove that if the diagonals of a quadrilateral bisect each other it is a parallelogram. What is the converse of this proposition?

2. Show that if two intersecting line-segments bisect each other, the lines joining their extremities are parallel.

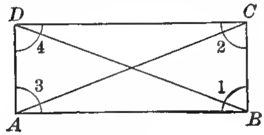
3. The parallel lines l_1 and l_2 are cut by a transversal AB. AC and AD bisect $\angle 2$ and $\angle 1$ respectively. Prove that $\triangle$s CBA and DBA are isosceles. Compare the segments CB and BD.



4. If in an isosceles right triangle ABC the bisectors of the acute angles meet at O, find how many degrees in the $\angle 1$ thus formed.



146. THEOREM. *If the two diagonals of a parallelogram are equal, the figure is a rectangle.*



Given ▱ $ABCD$ **with** $AC = BD$.

To prove that $\angle 1 = \angle 2 = \angle 3 = \angle 4 =$ rt. $\angle$.

Proof: Show that $\triangle ABD \cong \triangle ABC$.

$\therefore \angle 1 = \angle 3.$ (Why?)

But $\angle 1 + \angle 3 = 2$ rt. $\angle$s. (Why?)

$\therefore \angle 1 = \angle 3 =$ rt. $\angle$. (Why?)

In like manner prove that $\angle 2 = \angle 4 =$ rt. $\angle$.
Hence the figure is a rectangle (§ 135).

147. EXERCISES.

1. State and prove the converse of the theorem, § 146.

2. Show that a diameter of a parallelogram passes through the intersection of its diagonals. See § 137.

SUGGESTION. Show that it bisects each diagonal.

3. Prove that the diagonals of a square are perpendicular to each other.

4. Prove that the diagonals of a rhombus are perpendicular to each other.

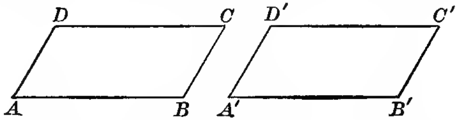
5. Does the same proof apply to Exs. 3 and 4?

6. Are the diagonals of a square equal? Is this true of a rhombus? Prove each answer correct.

7. Do the diagonals of a square bisect each other? Is this true of a rhombus? Of a trapezoid?

8. Show that if two adjacent angles of a parallelogram are equal, the figure is a rectangle.

148. THEOREM. *Two parallelograms having an angle and the two adjacent sides of one equal respectively to an angle and the two adjacent sides of the other are congruent.*



Given ▱s $ABCD$ **and** $A'B'C'D'$ **such that** $AB = A'B'$, $AD = A'D'$ **and** $\angle A = A'$.

To prove that ▱ $ABCD \cong$ ▱ $A'B'C'D'$.

Proof: Apply ▱ $ABCD$ to ▱ $A'B'C'D'$ so as to make $\angle A$ coincide with $\angle A'$, AB falling on $A'B'$, and AD on $A'D'$.

Then BC takes the direction $B'C'$, since $\angle B = \angle B'$, being supplements of the equal angles A and A'. (Why?)

C falls on C', since $BC = B'C'$, being segments equal to the equal segments AD and $A'D'$. (Why?)

Hence CD coincides with $C'D'$. (Why?)

Therefore ▱ $ABCD \cong$ ▱ $A'B'C'D'$, since they coincide throughout.

149. EXERCISES.

1. Are two parallelograms congruent if they have a side and two adjacent angles of the one equal respectively to a side and two adjacent angles of the other? Draw figures to illustrate your answer.

2. Are two parallelograms congruent if they have four sides of one equal to four sides of the other? Show why, and draw figures to illustrate.

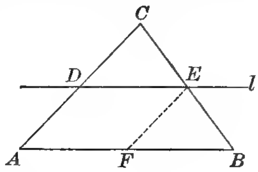
3. Compare the theorem of § 148 and Exs. 1 and 2 preceding with the tests for congruence of triangles.

4. Prove that the opposite angles of a parallelogram are equal.

5. State and prove the converse of Ex. 4.

OTHER THEOREMS APPLYING PARALLELS

150. THEOREM. *If a line bisects one side of a triangle and is parallel to the base, then it bisects the other side and the included segment is equal to one half the base.*



Given a line $l \parallel AB$ in $\triangle ABC$ such that $AD = DC$.

To prove that $BE = EC$ and $DE = \frac{1}{2} AB$.

Proof: Draw EF through E parallel to CA.

Now show that $AFED$ is a parallelogram,

and that $\triangle DEC \cong \triangle FBE$,

from which $CE = EB$, $DE = FB$, $DE = AF$,

and $AB = AF + FB = 2\, DE$,

or $DE = \frac{1}{2} AB$.

State the reasons for each step.

151. THEOREM. *A segment connecting the middle points of two sides of a triangle is parallel to the third side and equal to one half of it.*

Given $\triangle ABC$ **and the segment** DE **such that** $AD = DC$ **and** $CE = EB$. See figure in § 150.

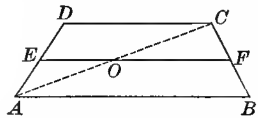
To prove that $DE \parallel AB$ and $DE = \frac{1}{2} AB$.

Proof: Since each of the sides AC and BC has but **one** middle point (§ 65), it follows that there is but **one** segment DE bisecting both these sides. But by § 150 a certain segment parallel to the base fulfills this condition.

Hence, DE is parallel to the base AB.

Then, $DE = \frac{1}{2} AB$ as in § 150.

152. THEOREM. *If a segment is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, then it bisects the other also and is equal to one half the sum of the bases.*



Given the trapezoid *ABCD* in which *AE* = *ED* and *EF* ∥ *AB*.

To prove that $BF = FC$ and $EF = \frac{1}{2}(AB + DC)$.

Proof: Draw the diagonal AC meeting EF in O.

In $\triangle ACD$, $AO = OC$ and $EO = \frac{1}{2} DC$. (Why?)

In $\triangle ABC$, $BF = FC$ and $OF = \frac{1}{2} AB$. (Why?)

Adding, $EO + OF = EF = \frac{1}{2}(AB + DC)$. (§ 61, V)

Prove this theorem also by drawing the diagonal *BD*.

State a similar theorem for a parallelogram and make a proof for that case which is simpler than the above.

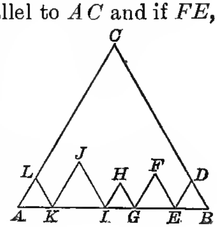
153. THEOREM. *If a segment connects the middle points of the two non-parallel sides of a trapezoid, it is parallel to the bases and equal to one half their sum.*

Proof: The argument is similar to that in § 151. Give it in full.

154. EXERCISES.

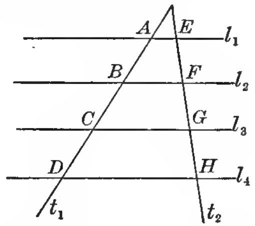
1. Does a theorem similar to that of § 153 hold for a parallelogram? If so, state it and give a simpler proof in this case.

2. If in the figure *DE*, *FG*, *HI*, etc., are parallel to *AC* and if *FE*, *HG*, *JI*, etc., are parallel to *BC*, find the sum of *BD*, *DE*, *EF*, *FG*, *GH*, etc. How, if at all, does the length of *AB* enter into the solution?



SUGGESTION. Produce *GF*, *IH*, *KJ*, to meet *BC*, forming ▱. Then show that $BD + EF + GH$, etc., $= BC$, and similarly $ED + GF + IH$, etc., $= AC$.

155. THEOREM. *If a series of parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given the parallel lines l_1, l_2, l_3, l_4, cutting the transversal t_1 so that $AB = BC = CD$.

To prove that on the transversal t_2 $EF = FG = GH$.

Proof: The figure $ACGE$ is a parallelogram or a trapezoid according as $t_1 \parallel t_2$ or not.

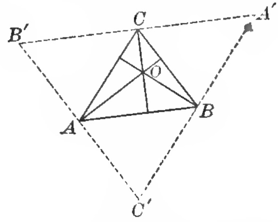
In either case BF, which bisects AC, also bisects EG (§ 152).

$$\therefore EF = FG.$$

Similarly, in the figure $DHFB$, $FG = GH$.

$$\therefore EF = FG = GH.$$

156. THEOREM. *The three altitudes of a triangle meet in a point.*



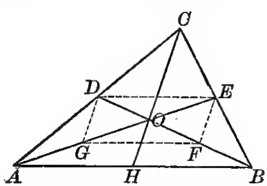
Outline of Proof: Through each vertex of the given triangle ABC draw a line parallel to the opposite side, forming a triangle $A'B'C'$.

Show that $ACA'B$, and $AB'CB$ are parallelograms, and hence that $B'C = AB = CA'$. That is, C is the middle point of $A'B'$.

In the same manner show that A and B are the middle points of $B'C'$ and $A'C'$ respectively. Also show that the altitudes of $\triangle ABC$ are the perpendicular bisectors of the sides of $\triangle A'B'C'$, and therefore meet in a point (§ 132).

157. Definition. A segment connecting a vertex of a triangle with the middle point of the opposite side is called **a median of the triangle.**

158. THEOREM. *The three medians of a triangle meet in a point which is two thirds the distance from each vertex to the middle point of its opposite side.*



Given $\triangle ABC$ with medians BD and AE meeting in O.

To prove that the median from C also passes through O, and that $AO = \frac{2}{3} AE$, $BO = \frac{2}{3} BD$, and $CO = \frac{2}{3} CH$.

Outline of Proof: Taking F and G, the middle points of OB and OA respectively, use §§ 151 and 144 to show that the figure $GFED$ is a parallelogram, and hence that

$$DO = OF = FB \text{ and } EO = OG = GA.$$

That is, O trisects AE and BD.

In the same way we find that AE and CH meet in a point O' which trisects each of them.

Hence O and O' are the same point. Therefore the three medians meet in a point which trisects each of them.

159. **EXERCISES.**

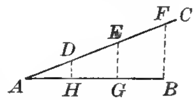
1. If one angle of a parallelogram is 120°, how many degrees in each of the other angles?

2. If the angles adjacent to one base of a trapezoid are equal, then those adjacent to the other base are equal.

3. If the angles adjacent to either base of a trapezoid are equal, then the non-parallel sides are equal and the trapezoid is isosceles.

4. In an isosceles trapezoid the angles adjacent to either base are equal.

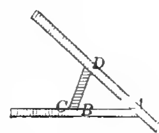
5. Divide a segment AB into three equal parts.



SUGGESTION. From A draw a segment AC, and on it lay off three equal segments, AD, DE, EF (§ 33).

Draw FB and construct EG and DH each parallel to FB. Prove $AH = HG = GB$.

6. To cut braces for a roof, as shown in the figure, a carpenter needs to know the angle DBC when the angle DAB is given, it being given that $AB = AD$. Show how to find this angle. (See § 123, Ex. 14.)



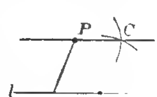
7. If each of the perpendicular bisectors of the sides of a triangle passes through the opposite vertex, what kind of a triangle is it? If it is given that two of the perpendicular bisectors of sides pass through the opposite vertices, what kind of a triangle is it? If only one?

8. Find the locus of the middle points of the segments joining a vertex of a triangle to all points on the opposite side.

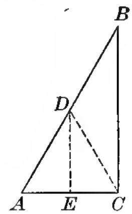
9. Given a line l and a point P not in the line. Find the locus of the middle points of all segments drawn from P to l.

10. The length of the sides of a triangle are 12, 14, 16. Four new triangles are formed by connecting middle points of the sides of this triangle. What is the sum of the sides of these four triangles?

11. Draw any segment PA meeting a line l in A. Lay off AB on l. With P and B as centers and with AB and AP as radii respectively, strike arcs meeting in C. Draw PC, and prove it parallel to l.



12. If the side of a triangle is bisected by the perpendicular upon it from the opposite vertex, the triangle is isosceles.



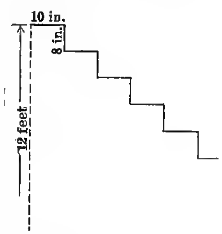
13. State and prove the converse of this theorem.

14. If in a right triangle the hypotenuse is twice as long as one side, then one acute angle is 60° and the other 30°.

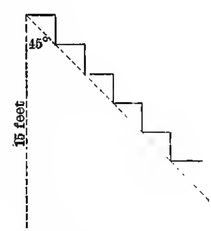
SUGGESTION. Let D be the middle point of AB. Use Ex. 12 and the hypothesis to show that $\triangle ACD$ is equilateral.

Prove the *converse* by drawing CD so as to make $\angle BCD = \angle B$.

15. A stairway leading from a floor to one 12 feet above it is constructed with steps 8 inches high and 10 inches wide. What is the length of the carpet required to cover the stairway, allowing 10 inches for the last step, which is on a level with the upper floor.



16. A stairway inclined 45° to the horizontal leads to a floor 15 feet above the first. What is the length of the carpet required to cover it if each step is 10 inches high? If each is 12 inches? If each is 9 inches?



Can this problem be solved without knowing the height of the steps? Is it necessary to know that the steps are of the same height?

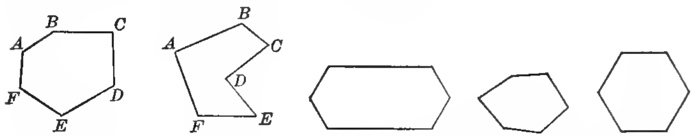
17. If the vertex angle of an isosceles triangle is 60°, show that it is equilateral.

18. By successively constructing angles of 60° divide the perigon about a point O into six equal angles. (This is possible because $360 \div 6 = 60$.) With O as a center construct a circle cutting the sides of these angles in points A, B, C, D, E, F. Draw the segments AB, BC, etc. Show by Ex. 17 that each of the six triangles thus formed is equilateral. Show also that $ABCDEF$ is equiangular and equilateral, that is, $AB = BC$, etc., and $\angle ABC = \angle BCD$, etc.

POLYGONS.

160. Definitions. A **polygon** is a figure formed by a series of segments, *AB*, *BC*, *CD*, etc., leading back to the starting point *A*.

The segments are the **sides** of the polygon and the points *A*, *B*, *C*, *D*, etc., are its **vertices**. The angles *A*, *B*, *C*, *D*, etc., are the angles of the polygon.



Convex. Concave. Equiangular. Equilateral. Regular.

A polygon is **convex** if no side when produced enters it. Otherwise it is **concave**.

Only convex polygons are here considered.

A polygon is **equiangular** if all its angles are equal and **equilateral** if all its sides are equal.

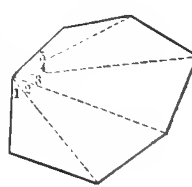
A polygon is **regular** if it is both equiangular and equilateral.

A segment connecting two non-adjacent vertices is a **diagonal** of the polygon.

The **perimeter** of a polygon is the sum of its sides.

161. THEOREM. *The sum of the angles of a polygon having n sides is* $(2n-4)$ *right angles.*

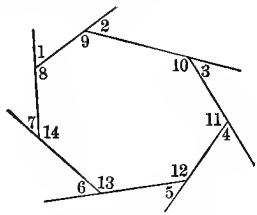
Proof: Connect one vertex with each of the other non-adjacent vertices, thus forming a set of triangles. Evidently the sum of the angles of these triangles equals the sum of the angles of the polygon.



Now show that if the polygon has n sides there are $(n-2)$ triangles. The sum of the angles of one triangle is 2 rt. ∠s. Hence, the sum of the angles of all the triangles, that is, the sum of the angles of the polygon, is

$$2(n-2) = (2n-4) \text{ rt. } \angle s.$$

162. THEOREM. *The sum of the exterior angles of a polygon, formed by producing the sides in succession, is four right angles.*



Outline of Proof: The sum of both exterior and interior angles is $2n$ rt. ∠s. (Why?)

The sum of the interior angles is $(2n-4)$ rt. ∠s. (Why?)

Hence, the sum of the exterior angles is 4 rt. ∠s. (Why?)

Write out the proof in detail, using the figure.

163. EXERCISES.

1. What is the sum of the angles of a polygon of 3 sides? of 4 sides? of 5 sides? of 6 sides? of 10 sides? of 18 sides?

2. Find each angle of a *regular* polygon of 3 sides, 4 sides, 5 sides, 6 sides, 8 sides, 14 sides, n sides.

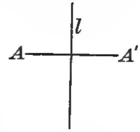
3. Construct a regular triangle, thus obtaining an angle of 60°.

4. Construct a regular quadrilateral. What is its common name?

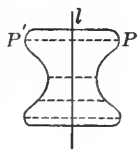
5. Prove that a regular hexagon $ABCDEF$ may be constructed as follows: Let A be any point on a circle with center O. With A as center and OA as radius describe arcs meeting the circle in B and in F. With B as center and the same radius describe an arc meeting the circle in C, and so for points to D and E. See § 159, Ex. 18.

SYMMETRY.

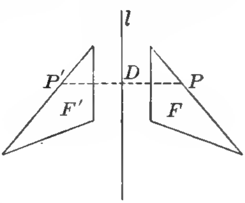
164. Two points A and A' are said to be **symmetric points** with respect to a line l if l is the perpendicular bisector of the segment AA'.



A figure is **symmetric** with respect to an axis l if for every point P in the figure there is also a point P' in the figure such that P and P' are symmetric points with respect to l. This is called **axial symmetry.** Two separate figures may have an axis of symmetry between them.



165. THEOREM. *Two figures which are symmetric with respect to a line are congruent.*

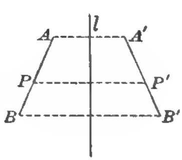


Given two figures F and F' symmetric with respect to a line l.

To Prove that $F \cong F'$.

Proof: This is evident, since, by folding figure F over on the line l as an axis, every point in F will fall upon a corresponding point in F' (Why?).

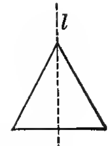
166. COROLLARY. *If points A and A' and also B and B' are symmetric with respect to a line l, then the segments AB and $A'B'$ are symmetric with respect to l.*



167. EXERCISES

1. How many axes of symmetry has a square? A rectangle? A rhombus? An isosceles trapezoid?

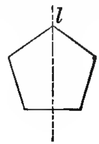
2. If a diagonal of a rectangle is an axis of symmetry, what kind of a rectangle is it?

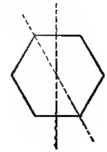


3. If a triangle has an axis of symmetry, what kind of a triangle is it? Assume that the axis passes through one vertex.

4. If a triangle has two axes of symmetry, what kind of a triangle is it?

5. How many axes of symmetry has an equilateral triangle?



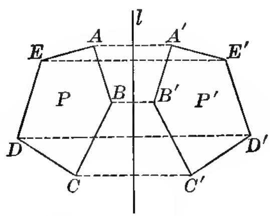


6. How many axes of symmetry has a regular pentagon (five-sided figure)?

7. How many axes of symmetry has a regular hexagon?

8. Show that one figure of § 40 has an axis of symmetry. State this as a theorem.

168. PROBLEM. *Given a polygon P and a line l not meeting it, to construct a polygon P' such that P and P' shall be symmetric with respect to l.*



SOLUTION. **Let A, B, C, D, E be the vertices of the given polygon P, and l the given line.**

Construct A', B', C', D', E' symmetric respectively to A, B, C, D, E with respect to the line l.

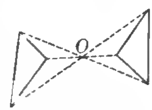
Then the polygon P' formed by joining the points A', B', C', D', E', A' in succession is symmetric to P.

Proof: Give the proof in full.

Figures having an axis of symmetry are very common in all kinds of decoration and architectural construction.

169. Two points A and A' are symmetric with respect to a point O if the segment AA' is bisected by O.

A figure is symmetric with respect to a point O if for every point P of the figure there is a point P' also in the figure such that P and P' are symmetric with respect to O.



Such a figure is said to have **central symmetry** with respect to the point. The point is called the **center of symmetry.** A circle has central symmetry.

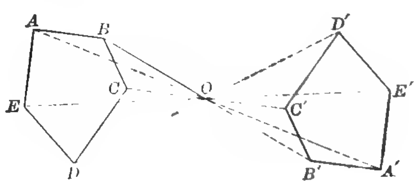
Two separate figures may have a center of symmetry between them.

170. EXERCISES.

1. Prove that if A and A' and also B and B' are symmetric with respect to a point O, then the segments AB and $A'B'$ are symmetric with respect to O.

2. Prove that if the triangles ABC and $A'B'C'$ are symmetric with respect to a point O, then they are congruent.

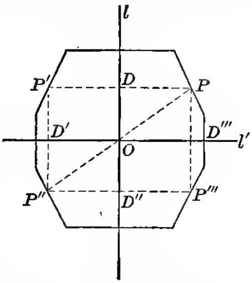
171. PROBLEM. *Given a polygon P and a point O outside of it, to construct a polygon P' symmetric to P with respect to O.*



SOLUTION. Construct points symmetric to the vertices of P. Connect these points, forming the polygon P', and prove this is the polygon sought.

172. THEOREM. *If a figure has two axes of symmetry at right angles to each other, their point of intersection is a center of symmetry of the figure.*

Outline of Proof: It is to be shown that for every point P in the figure, a point P'' also in the figure can be found such that $PO = P''O$ and POP'' is a straight line. Draw $PP' \perp l$, $P'P'' \perp l'$, $P''P''' \perp l$ and connect the points P''' and P.



Now use the hypothesis that $l \perp l'$, and each an axis of symmetry, to show that $PP'P''P'''$ is a rectangle of which DD'' and $D'D'''$ are the diameters. Hence O, the intersection of DD'' and $D'D'''$, bisects the diagonal PP'', making P'' symmetric to P with respect to O, § 147, Ex. 2. Give the proof in full.

173. EXERCISES.

1. Has a square a center of symmetry? has a rectangle?

2. If a parallelogram has a center of symmetry, does it follow that it is a rectangle?

3. Has a trapezoid a center of symmetry? has an isosceles trapezoid?

4. If two non-parallel straight lines are symmetric with respect to a line l, show that they meet this line in the same point and make equal angles with it. (Any point on the axis of symmetry is regarded as being symmetric to itself with respect to the axis.)

5. If segments AB and $A'B'$ are symmetric with respect to a point O, they are equal and parallel.

6. Has a regular pentagon a center of symmetry? See figure of Ex. 6, § 167.

7. Has a regular hexagon a center of symmetry? Ex. 7, § 167.

8. Has an equilateral triangle a center of symmetry?

METHODS OF ATTACK.

174. No general rule can be given for proving theorems or for solving problems.

In the case of theorems the following suggestions may be helpful.

(1) *Distinguish carefully the items of the hypothesis and of the conclusion.*

It is best to tabulate these as suggested in § 79.

(2) *Construct with care the figure described in the hypothesis.*

The figure should be as *general* as the terms of the hypothesis permit. Thus if a triangle is called for but no *special* triangle is mentioned, then a *scalene* triangle should be drawn. Otherwise some particular form or appearance of the figure may lead to unwarranted conclusions.

(3) *Study the hypothesis with care and determine whether any auxiliary lines may assist in deducing the properties required by the conclusion.*

Study the theorems previously proved in this respect. A careful review of these proofs will lead to some insight as to how they were evolved.

175. Direct Proof. The majority of theorems are proved by passing *directly* from the hypothesis to the conclusion by a series of logical steps. This is called **direct proof.**

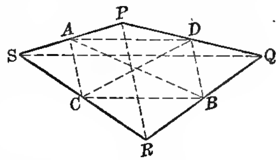
It is often helpful in *discovering* a direct proof to *trace it backward* from the conclusion.

Thus, we may observe that the conclusion C follows if statement B is true, and that B follows if A is true. If then we can *show that A is true*, it follows that B and C are true and the theorem is proved.

Having thus **discovered a proof**, we may then start from the beginning and follow it directly through.

As an example consider the following theorem:

The segments connecting the middle points of the opposite sides of any quadrilateral bisect each other.



Given segments AB and CD connecting the middle points of the quadrilateral $PQRS$.

To prove that AB and CD bisect each other.

Proof: Draw the diagonals PR and SQ and the segments AD, DB, BC, CA.

Now AB and CD bisect each other if $ADBC$ is a ▱, and $ADBC$ is a ▱ if $AD \parallel CB$ and $AC \parallel DB$.

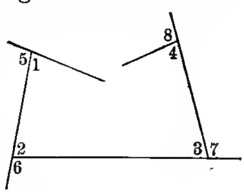
But $AD \parallel CB$ since each is $\parallel SQ$. See § 151.

And $AC \parallel DB$ since each is $\parallel PR$.

Hence $ADBC$ is a ▱ and AB and CD bisect each other.

Notice that the auxiliary lines PR and SQ divide the figure into triangles and this suggests the use of § 151.

176. Indirect Proof. In case a direct proof is not easily found, it is often possible to make a proof by assuming that the theorem is *not true* and showing that this leads to a conclusion *known to be false.*



As an example consider the following theorem:

A convex polygon cannot have more than three acute angles.

Proof: Assume that such a polygon may have four acute angles, as $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$.

Extend the sides forming the exterior angles 5, 6, 7, 8.

Since $\angle 1 + \angle 5 = 2$ rt. ∠s, $\angle 2 + \angle 6 = 2$ rt. ∠s, etc., and since $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$ are all acute by hypothesis, it follows that $\angle 5$, $\angle 6$, $\angle 7$, $\angle 8$ are all obtuse, and hence $\angle 5 + \angle 6 + \angle 7 + \angle 8 > 4$ rt. ∠s.

But this *cannot be true* since the sum of *all* the exterior angles is exactly 4 rt. ∠s by § 162.

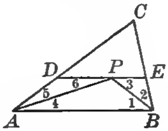
Hence the assumption that $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$ are all acute is false. That is, a convex polygon cannot have more than three acute angles.

The proof by the **method of exclusion** (§ 86) involves the *indirect* process in showing that all but one of the possible suppositions is false.

177. The **solution of a problem** often involves the same kind of analysis as that suggested for the discovery of a direct proof (§ 175).

For instance, consider the following problem:

To draw a line parallel to the base of a triangle such that the segment included between the sides shall equal the sum of the segments of the sides between the parallel and the base.



Given the $\triangle ABC$.

To find a point P through which to draw $DE \parallel AB$ so that $DP + PE = AD + EB$.

Construction. Draw the bisectors of $\angle s\ A$ and B meeting in point P.

Then P is the point required.

Proof: $\angle 1 = \angle 3$ since $DE \parallel AB$, and $\angle 1 = \angle 2$ since BP bisects $\angle B$.

$\therefore \angle 3 = \angle 2$ and $PE = BE$. (Why?)

Likewise in $\triangle ADP$, $AD = DP$.

Hence $DP + PE = AD + BE$. (Ax. V.)

Or $DE = AD + BE$.

This construction is discovered by observing that a point P must be found such that $PE = BE$ and $PD = AD$. This will be true if $\angle 3 = \angle 2$ and this follows if $\angle 1 = \angle 2$, while at the same time $\angle 6 = \angle 5$ and $\angle 4 = \angle 5$. Hence the bisectors of the base angles will determine the point P.

Having thus *discovered* the process, the construction and proof are made *directly*.

178. In general the most effective help is a ready knowledge of the *facts of geometry already discovered*, and skill in applying these will come with practice. It is important for this purpose that **summaries** like the following be made by the student and memorized:

179. (*a*) Two triangles are congruent if they have:

(1) *Two sides and the included angle of the one equal to the corresponding parts of the other.*

(2) *Two angles and the included side of the one equal to the corresponding parts of the other.*

(3) *Three sides of the one equal respectively to three sides of the other.*

In the case of right triangles:

(4) *The hypotenuse and one side of one equal to the corresponding parts of the other.*

(*b*) Two segments are proved equal if:

(1) *They are homologous sides of congruent triangles.*
(2) *They are legs of an isosceles triangle.*
(3) *They are opposite sides of a parallelogram.*
(4) *They are radii of the same circle.*

SUMMARY OF CHAPTER I.

1. Make a summary of ways in which two angles may be shown to be equal.

2. Make a summary of ways in which lines are proved parallel.

3. What conditions are sufficient to prove that a quadrilateral is a parallelogram?

4. Make a list of problems of construction thus far given.

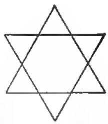
5. Make a list of definitions thus far given. Which of the figures defined are found on page 4?

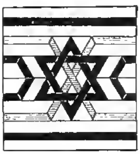
6. Tabulate all theorems on

(*a*) bisectors of angles and segments,
(*b*) perpendicular lines,
(*c*) polygons in general,
(*d*) symmetry.

7. What are some of the more important applications thus far given of the theorems in Chapter I?

PROBLEMS AND APPLICATIONS.





Border, Parquet Flooring.

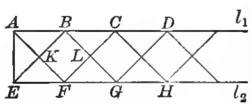
1. Divide each side of an equilateral triangle into three equal parts (§ 159, 5), and draw lines through the division points as shown in the figure.

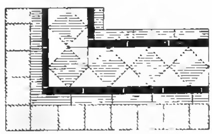
Prove (*a*) The six small triangles are equilateral and congruent to each other. (First prove them equiangular.)

(*b*) The two large triangles are congruent.

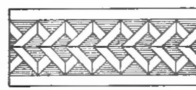
(*c*) The inner figure is a regular hexagon.

2. Let l_1 and l_2 be parallel lines, with AE perpendicular to both. Lay off segments AB, BC, CD, . . . and EF, FG, GH, . . . each equal to AE. Connect these points as shown in the figure.





Tile Floor Border.



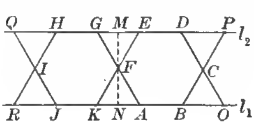
Parquet Floor Border.

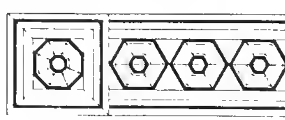
Prove (*a*) EKA, EFK, AKB, BLC, etc., are congruent right isosceles triangles.

(*b*) $FLBK$, etc., are squares.

Bisect AB, BC, ... EF, FG ... and join points by lines parallel to EB and AF respectively. Notice that the resulting figure forms the basis for the floor border to the right.

3. Let l_1 l_2 be parallel lines, with MN perpendicular to both. Through F, the middle point of MN, draw lines KE and AG, each making an angle of 30° with MN.





Parquet Floor Border.

Prove $FE = AF$ by showing $\triangle ANF \cong \triangle FEM$.

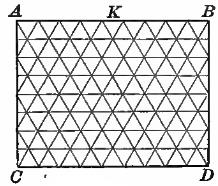
Also prove $KF = FG$ and that $\triangle$s FKA and FEG are equilateral.

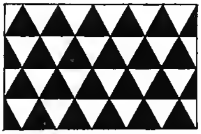
Lay off segments AB, BO, ED, DP, KJ, etc., each equal to AK. Connect points as shown in the figure.

Prove $\triangle BCO \cong \triangle CPD \cong \triangle AFK$, etc.

Prove that $ABCDEF$ and $JKFGHI$ are regular hexagons.

4. A network of congruent equilateral triangles is constructed in a rectangle $ABCD$ as shown in the figure.





Tile Flooring.

(*a*) How many of these triangles meet in a common vertex?

(*b*) Does this number of equilateral triangles exactly cover the whole plane about the vertices? Why?

(*c*) Do the triangles that meet in one point form a regular hexagon?

(*d*) At what angle to the horizontal lines are the oblique lines that form sides of the triangles? *e.g.* what is the angle DCK?

(*e*) Compare the lengths of CK and AK (see § 159, Ex. 14).

To construct this network when a side a of the triangles is given, proceed as follows: From the vertex A of a right angle lay off segments equal to a along one side AK. With one of these division points K as center and with a radius equal to twice AK strike an arc meeting the side AC at C. Draw CD parallel to AB and lay off segments on it equal to a. Connect these points as shown in the figure, and through the intersection points draw the horizontal lines, thus fixing the division points on AC and BD.

Prove that the resulting triangles are equilateral and congruent to each other.

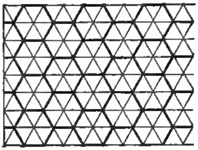
Suggestion. Use in order the converse of § 159, Ex. 14; § 144, and the fact that a diagonal divides a parallelogram into two congruent triangles.

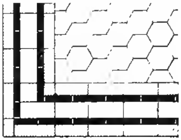
Notice how this construction is studied. The figure is *first* supposed constructed and its properties tabulated. Some of these (*c*) and (*d*) are then used in making the construction. This method is of very general application in problems of construction.

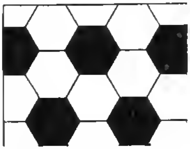
Note. This is the method by which a designer would construct a network of congruent regular triangles.

5. Construct a network of triangles as in Ex. 4, using pencil and then ink in parts of the lines, making a set of regular hexagons as shown in the figure.

How many such hexagons meet in a point? Will this number exactly cover the plane about the point? Why?

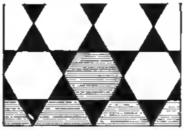




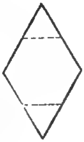


6. Point out how the figure constructed under Ex. 5 may be made the basis for the two tile floor patterns here given.

7. Given a rhombus whose acute angles are 60°. Show how to cut off two triangles so the resulting figure shall be a regular hexagon. What fraction of the whole rhombus is thus cut off? What fraction of the pattern is black?



Tile Flooring.



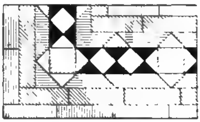
8. In a parallelogram, prove:

(*a*) The diameters divide it into four congruent parallelograms.

(*b*) The segments joining the midpoints of the sides in order form a parallelogram, and the four triangles thus cut off may be pieced together to form another parallelogram congruent to the one first formed.

(*c*) The diameters and diagonals all meet in a point.

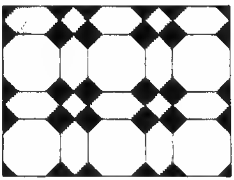
Suggestion. Prove that a diameter bisects a diagonal. Do these propositions hold for a square? What part of each of the three large squares in the adjoining figure is white?



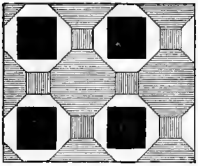
Tile Border.

9. The octagons (eight-sided polygons) in the figure are equiangular and the small quadrilaterals are squares.

Find the angles of the irregular hexagons. If the octagons were regular, would the hexagons be equiangular? equilateral? regular?



Tile Pattern.



Tile Pattern.

10. In the figure the octagons are equiangular.

(*a*) Will two such octagons and a square completely cover the plane about a point? Why?

(*b*) At what angles do the slanting sides of the octagons meet the horizontal line?

Find all the angles of the two small white trapezoids to the right.

(*c*) Could a pattern of this type be constructed by using regular octagons?

11. Equilateral triangles and regular hexagons have thus far been found to make a complete pavement. What other regular polygons can be used to make a complete pavement?

12. If three regular pentagons (five-sided polygons) meet in a common vertex, will they completely cover the plane about that point? If not, by how great an angle will they fail to do so?

13. If four regular pentagons be placed about a point, by how great an angle will they overlap?

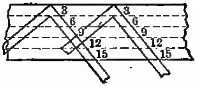
14. If three regular seven-sided polygons are placed about a point, by how great an angle will they overlap? If two, by how many degrees will they fail to cover the plane?

In Exs. 12, 13, 14 it is understood that, so far as possible, the polygons are so placed that no part of one of them lies within another; that is, they are not to overlap.

15. Answer questions like those in Ex. 14 for regular polygons of 8, 9, and 10 sides. Can any regular polygon of more than six sides be used to form a complete pavement?

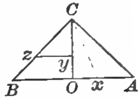
Note that the larger the number of sides of a regular polygon the larger is each angle.

16. A carpenter divides a board into strips of equal width as follows: Suppose five strips are desired. Place a steel square in two positions, as indicated in the figure, at such angles that the distance in inches diagonally across the board shall be some multiple of five (in the figure this distance is 15 inches). Mark the points and connect them as shown in the figure. Prove that these lines divide the board into equal strips.



(The black and white figures on this page are parquet floor patterns.)

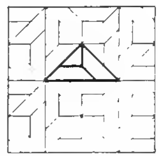
17. Given an isosceles right triangle ABC with the altitude CO upon the hypotenuse.



(*a*) Show how to draw $xy \parallel AC$ such that $xy = Cy$.

SUGGESTION. Bisect $\angle OCA$ and let the bisector meet AB in x.

Draw $xy \parallel AC$. Prove $\triangle xyC$ isosceles.

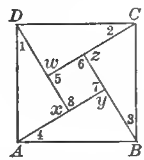


(*b*) Prove $xyCA$ an isosceles trapezoid.

(*c*) Draw $yz \parallel OB$ and prove $yz = yx$.

SUGGESTION. Prove $\triangle Cyz$ isosceles.

(*d*) Prove that $xyCA$ and $xyzB$ are congruent trapezoids.



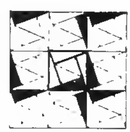
18. $ABCD$ is a square. Lines are drawn as shown in the figure so that $\angle 1 = \angle 2 = \angle 3 = \angle 4$.

(*a*) Prove $\triangle ABy \cong \triangle BCz \cong \triangle CDw \cong \triangle DAx$.

(*b*) Prove each of these triangles a right triangle.

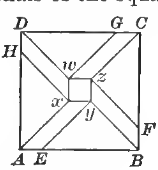
(*c*) Prove that $xyzw$ is a square.

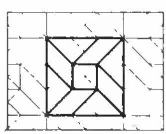
(*d*) If $\angle 1 = 45°$, what can be said of the figure $xyzw$?

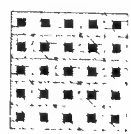


This and the following design are of Arabic origin.

19. On the sides of a square $ABCD$ the points E, F, G, H are laid off so that $AE = BF = CG = DH$. Ax and zC, By and wD are in the diagonals of the square and Ey, Fz, Gw, and Hx are parallel to these.







Prove that:

(*a*) AxH, EyB, etc., are congruent right isosceles triangles.

(*b*) $AEyx$, $BFzy$, etc., are congruent parallelograms.

(*c*) $xyzw$ is a square.

(*d*) If $AB = a$, find how long AE should be taken in order that xy shall equal EB; also equal to one half EB.

20. $ABCD$ is a square, $AE = BF = CG = DH$ and AG, CE, BH, and DF are drawn as shown in the figure.

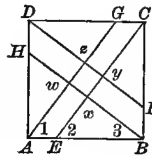
Prove that:

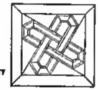
(*a*) $AG \parallel EC$ and $BH \parallel FD$.

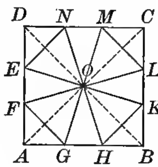
(*b*) Triangles HwA, ExB, etc., are congruent.

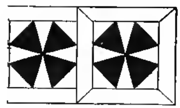
(*c*) The four trapezoids $AExw$, etc., are congruent.

(*d*) $xyzw$ is a square.









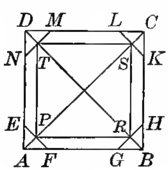
Parquet Flooring.

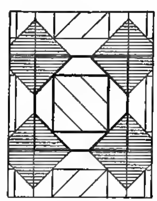
21. In the square $ABCD$, $AF = AG = HB = BK$, etc., and the figure is completed as shown.

(*a*) Pick out all isosceles triangles. Prove.

(*b*) Pick out all congruent triangles. Prove.

(*c*) Has this figure one or more axes of symmetry?





Linoleum Pattern.

22. On the sides of the square $ABCD$ points E, F, G, H, . . . are taken, so that $AE = AF = GB = BH =$ etc.

The middle points of EF, GH, KL and MN are connected, forming the quadrilateral $PRST$.

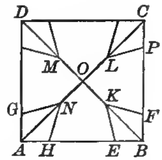
Prove that:

(*a*) AC is the perpendicular bisector of EF and KL.

(*b*) $FGRP$ is an isosceles trapezoid.

(*c*) PRT and RST are congruent right triangles.

(*d*) $PRST$ is a square.

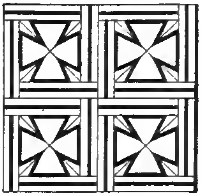


23. On the sides of a square $ABCD$ points G, H, E, F, P, etc., are taken, so that $GA = AH = EB = BF = PC$, etc. On the diagonals AC and BD points K, L, M, N, are laid off so that $OK = OL = OM = ON$.

(*a*) Prove that $GAHN \cong EBFK$, etc.

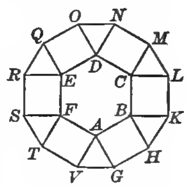
(*b*) Prove that $HEKON \cong FPLOK$.

SUGGESTION. Superpose one figure on the other.



Parquet Flooring.

24. The bisectors of the angles of a rhomboid form a rectangle; those of a rectangle form a square.



25. In the figure $ABCDEF$ is a regular hexagon. $ABHG$, $BCLK$, etc., are squares.

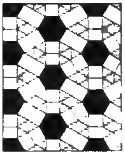
(*a*) What kind of triangles are BHK, CLM, etc.? Prove.

(*b*) Is the dodecagon (twelve-sided polygon) $HKLMN$... regular? Prove.

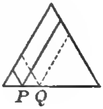
(*c*) How many axes of symmetry has this dodecagon?

(*d*) Has it a center of symmetry?

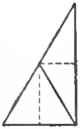
(*e*) Are the points S, F, B, K collinear? (That is, do they lie in the same straight line?)



Tile Pattern.



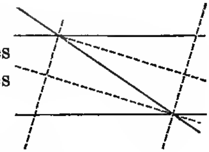
26. If from any two points P and Q in the base of an isosceles triangle parallels to the other sides are drawn, two parallelograms are formed whose perimeters are equal.



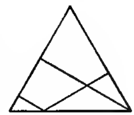
27. The middle point of the hypotenuse of a right triangle is equidistant from the three vertices. (This is a very important theorem.)

28. State and prove the converse of the theorem in Ex. 27.

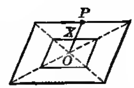
29. The bisectors of the four interior angles formed by a transversal cutting two parallel lines form a rectangle.



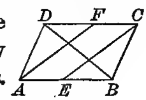
30. The sum of the perpendiculars from a point in the base of an isosceles triangle to the sides is equal to the altitude from the vertex of either base angle on the side opposite.



31. Find the locus of the middle points of all segments joining the center of a parallelogram to points on the sides. (See § 159, Ex. 8.)

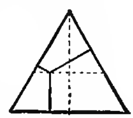


32. In the parallelogram $ABCD$ points E and F are the middle points of AB and CD respectively. Show that AF and CE divide BD into three equal segments.

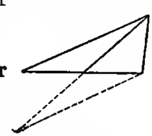


33. The sum of the perpendiculars to the sides of an equilateral triangle from a point P within is the same for all such points P (*i.e.*, the sum is a constant).

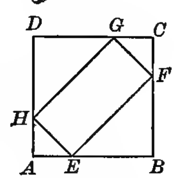
SUGGESTION. Prove the sum equal to an altitude of the triangle.



34. In any triangle the sum of two sides is greater than twice the median on the third side.



35. $ABCD$ is a square, and $EFGH$ a rectangle. Does it follow that $\triangle AEH \cong \triangle FCG$ and $\triangle EBF \cong \triangle HDG$? Prove.



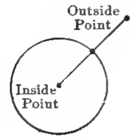
36. If a median of a triangle is equal to half the base, the vertex angle is a right angle.

37. State and prove the converse of the theorem in Ex. 36.

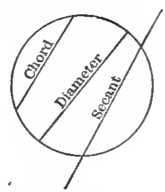
CHAPTER II.

STRAIGHT LINES AND CIRCLES.

180. A circle (§ 12) divides the plane into two parts such that any point which does not lie on the circle lies *within* it or *outside* it.

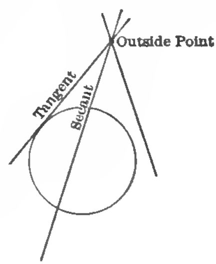


181. A line-segment joining any two points on a circle is called a **chord**. A chord which passes through the center is a **diameter**.



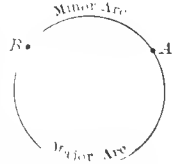
182. If a chord is extended in one or both directions, it **cuts** the circle and is called a **secant**.

183. A **tangent** is a straight line which touches a circle in one point but does not cut it. An indefinite straight line through a point outside a circle is a secant, a tangent, or does not meet the circle.

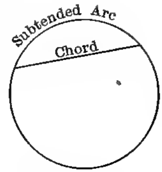


184. The portion of a circle included between any two of its points is called an **arc** (§ 12). An arc AB is denoted by the symbol $\widehat{AB}$.

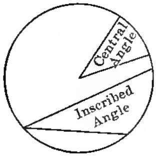
A circle is divided into two arcs by any two of its points. If these arcs are equal, each is a **semicircle**. Otherwise one is called the **major arc** and the other the **minor arc**.



Unless otherwise indicated $\widehat{AB}$ means the *minor* arc. In case of ambiguity a third letter may be used, as arc AmB.

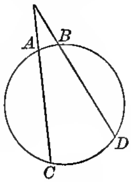


An arc is said to be **subtended** by the chord which joins its end-points. Evidently every chord of a circle subtends two arcs. Unless otherwise indicated the arc subtended by a chord means the *minor* arc.

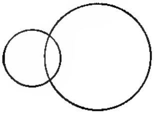


185. An angle formed by two radii is called a **central angle.** An angle formed by two chords drawn from the same point on the circle is called an **inscribed angle.**

If the sides of an angle meet a circle the arc or arcs which lie within the angle are called **intercepted arcs.**

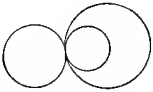


If the vertex of the angle is within or on the circle there is only one intercepted arc; if it is outside the circle there are two intercepted arcs, as $\widehat{AB}$ and $\widehat{CD}$ in the figure.



186. If a circle is partly inside and partly outside another circle, then they **cut each other.**

If two circles meet in one and only one point, they are said to be **tangent.**



Arcs of two circles are tangent to each other if the complete circles of which they form a part are tangent to each other.

187. Two circles which can be made to coincide are said to be **equal.**

The word *congruent* is unnecessary here, since all circles are *similar.*

188. EXERCISES.

1. Does the word *circle* as used in this book (§ 12) mean a curved line or the part of the plane inclosed by that line?

2. In how many points can a straight line cut a circle?

3. In how many points can two circles cut each other?

PRELIMINARY THEOREMS ON THE CIRCLE.

189. *Radii or diameters of the same circle or of equal circles are equal.*

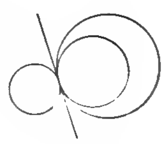
190. *If the radii or diameters of two circles are equal, the circles are equal.*

191. *A diameter of a circle is double the radius.*

192. *A point lies within, outside, or on a circle, according as its distance from the center is less than, greater than, or equal to the radius.*

193. *If an unlimited straight line contains a point within a circle, then it cuts the circle in two points.*

194. *If two circles intersect once, they intersect again. See figure,* § 186.



195. *If a straight line is tangent to each of two circles at the same point, then the circles do not intersect, but are tangent to each other at this point. See* § 186.

196. *If two arcs of the same circle or equal circles can be so placed that their end-points coincide and also their centers, then the arcs coincide throughout or else form a complete circle.*

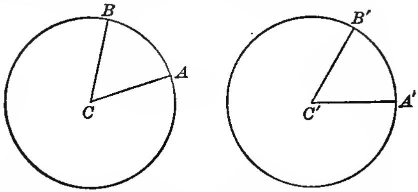
197. *If in two circles an arc of one can be made to coincide with an arc of the other, the circles are equal.*

198. A circle is conveniently referred to by indicating its center and radius.

Thus, ⊙ OA means *the circle whose center is O and radius OA.*

When no ambiguity arises, the letter at the center alone may be used to denote the circle. Thus, ⊙ C means *the circle whose center is C.*

199. THEOREM. *In the same circle or in equal circles equal central angles intercept equal arcs.*



Given the equal circles C and C' and $\angle C = \angle C'$.

To prove that $\widehat{AB} = \widehat{A'B'}$.

Proof: Place $\odot C$ on $\odot C'$ so that C falls on C', and $\angle C$ coincides with $\angle C'$.

Then A falls on A' and B on B'. (§ 189)

Hence $\widehat{AB} = \widehat{A'B'}$. (§ 196)

200. THEOREM. *In the same circle or in equal circles equal arcs are intercepted by equal central angles.*

Given $\odot C = \odot C'$, $\widehat{AB} = \widehat{A'B'}$. (See figure, § 199.)

To prove that $\angle C = \angle C'$.

Proof: Since $\widehat{AB} = \widehat{A'B'}$, the equal circles can be made to coincide in such manner that the arcs will also coincide.

That is, A will fall on A', B on B', and C on C'. Hence $\angle C$ coincides with $\angle C'$.

201. EXERCISES.

1. Show that in the same circle or in equal circles equal arcs subtend equal chords, and conversely.

2. Can two intersecting circles have the same center?

3. From a point on a circle construct two equal chords.

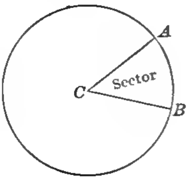
4. Show that the bisector of the angle formed by the chords in Ex. 3 passes through the center of the circle.

202. Measurement of Angles. If the perigon at the center of a circle be divided by radii into 360 equal angles, these radii will divide the circle into 360 equal arcs according to the theorem, § 199. Hence, we speak of an arc of 1°, 2°, 3°, etc., and similarly for minutes and seconds.

For this reason a central angle is said to be *measured by the arc which it intercepts*, meaning that a given central angle contains a *number of unit angles* equal to the *number of unit arcs* in the intercepted arc.

203. Definitions. A **quadrant** is an arc of 90°.

A semicircle is an arc of 180°. A right angle is, therefore, measured by a quadrant and a straight angle is measured by a semicircle.



A **sector** is a figure formed by two radii and their intercepted arc.

Thus, the sector BCA is formed by the radii CB, CA, and $\widehat{BA}$.

204. EXERCISES.

1. Show how to bisect an arc, using §§ 48, 199.

Divide an arc into four equal parts. Into eight equal parts.

2. Show that in the same circle or in equal circles two sectors having equal angles are congruent.

3. Show that if two sectors in the same circle or in equal circles have equal arcs, the sectors are congruent.

4. How many degrees in the arc which measures a right angle? a straight angle? half a right angle? three fourths of a straight angle? two thirds of a right angle? two thirds of a straight angle?

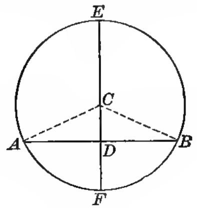
5. Show that the diameter is the longest chord of a circle.

6. Show that the two arcs into which the extremities of a diameter divide a circle are equal; that is, each is a semicircle.

SUGGESTION. Fold the figure over on the diameter.

7. Show that by bisecting the angles between two perpendicular diameters, a circle is divided into eight equal parts.

205. THEOREM. *A diameter perpendicular to a chord bisects the chord and also its subtended arc.*



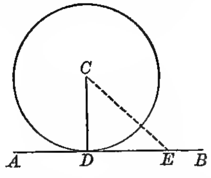
Given the diameter $EF \perp AB$ at D.

To prove that $AD = DB$ and $\overset{\frown}{AF} = \overset{\frown}{BF}$.

Proof: Draw the radii CA and CB.

If it can be shown that $\triangle ACD \cong \triangle BCD$, then (1) $AD = BD$ (Why?), and (2) $\angle ACD = \angle BCD$ (Why?); (3) $\overset{\frown}{AF} = \overset{\frown}{BF}$ (Why?).

206. THEOREM. *A line perpendicular to a radius at its extremity is tangent to the circle.*



Given $AB \perp CD$ at D.

To prove that AB is tangent to the circle; that is, does not meet it in any other point than D.

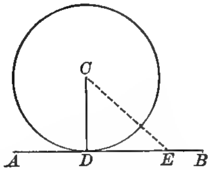
Proof: Let E be any point of AB other than D.

Draw segment CE. Then $CE > CD$ (Why?).

Hence E is outside the circle (§ 192).

That is, every point of AB except D is outside the circle, and hence AB is tangent to the circle (§ 183).

207. THEOREM. *If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.*



Given $\odot CD$ **with a line** AB **tangent to the circle at** D.

To prove that $AB \perp CD$ at D.

Proof: If CD is not $\perp AB$, then some other line, as CE, must be $\perp AB$ (§ 67), thus making $CE < CD$ (Why?).

The point E would then lie within the circle (§ 192), and the line AB would meet the circle in two points (§ 193).

But this contradicts the hypothesis that AB is *tangent* to the circle.

Hence no other line than CD can be perpendicular to AB from C, and as one such line exists, it must be CD.

208. EXERCISES.

1. What type of proof is used in the preceding paragraph?

2. How are the two theorems immediately preceding related to each other.

3. Show that there is only one tangent to a circle at a given point on it, and that the perpendicular from the center upon the tangent meets it at the point of contact.

4. A perpendicular to a tangent at the point of tangency passes through the center of the circle.

5. The perpendicular bisector of any chord passes through the center of the circle.

6. Two tangents at the extremities of a diameter are parallel.

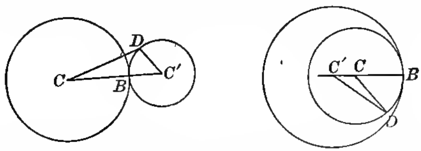
7. A diameter bisects all chords parallel to the tangents at its extremities, and also bisects the central angle subtended by each chord.

8. *A diameter bisecting a chord (or its subtended arc) is perpendicular to the chord.*

9. The mid-points of parallel chords all lie on a diameter.

10. A tangent to a circle at the mid-point of any arc is parallel to the chord of the arc.

209. THEOREM. *If two circles meet on the line joining their centers, they are tangent to each other at this point.*



Given ⓈS CB **and** $C'B$ **meeting in a point** B **on the line** $C'C$.

To prove that Ⓢ CB and $C'B$ are tangent to each other at B.

Proof: (1) *When each circle is outside the other.*

Let D be *any* point on ⊙ $C'B$ other than B.

Draw CD and $C'D$.

Then $CD + C'D > CB + C'B$. (Why?)

But $C'B = C'D$.

$\therefore CD > CB$. (Ax. IX, § 119)

$\therefore D$ is outside of ⊙ CB.

(2) *When* ⊙ CB *is inside of* ⊙ $C'B$.

Let D be *any* point on ⊙ CB other than B.

Draw CD and $C'D$.

Then $C'C + CD > C'D$. (Why?)

But $C'C + CD = C'C + CB = C'B$. (Why?)

$\therefore C'B > C'D$. (Ax. VII, § 82)

$\therefore D$ is within ⊙ $C'B$.

Therefore ⊙ $C'B$ and ⊙ CB have only one point in common and hence are tangent to each other (§ 186).

PROBLEMS AND APPLICATIONS.

1. If the distance between the centers of two circles is equal to the sum of their radii, how are the circles related? Construct and prove.

2. If the distance between the centers of two circles is equal to the difference of their radii, how are the circles related? Construct and prove?

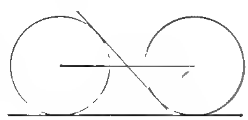
3. If the distance from the center of a circle to a straight line is equal to the radius, how is the line related to the circle? Construct and prove.

4. Given two circles having the same center, construct a circle tangent to each of them. Can more than one such circle be constructed? What is the locus of the centers of all such circles?

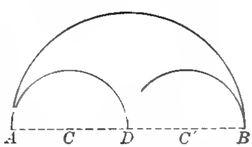
5. Prove the converse of the theorem in § 209.

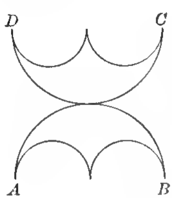
6. The straight line joining the centers of two intersecting circles bisects their common chord at right angles.

7. A line tangent to each of two equal circles is either parallel to the segment joining their centers or else it bisects this segment.



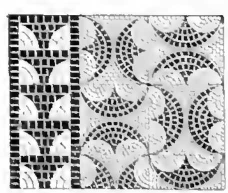
8. In the figure $AD = DB$. Semicircles are constructed on AD, DB, and AB as diameters. Which semicircles are tangent to each other?



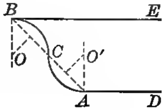


9. In the figure A, B, C, D are the vertices of a square. Show how to construct the entire figure. What semicircles are tangent to each other?

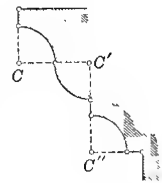
This construction occurs frequently in designs for tile flooring. See accompanying figure. This is from a Roman mosaic.



10. Given two parallel lines BE and AD, to construct arcs which shall be tangent to each other and one of which shall be tangent to BE at B and the other tangent to AD at A.



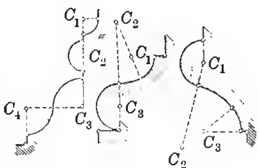
SOLUTION. Draw AB and bisect this segment at C; construct ⊥ bisectors of AC and BC. From A and B draw ⊥s to AD and BE respectively, thus locating the points O and O'.



Prove that O and O' are the centers of the required arcs.

SUGGESTION. Show that O, C, and O' lie in a straight line and use the theorem of § 209.

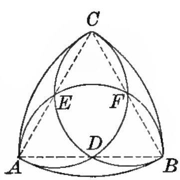
This construction occurs in architectural designs and in many other applications. In the accompanying designs pick out all the arcs that are tangent to each other and also the points of tangency.

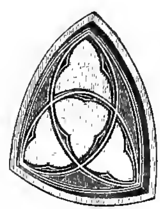


Scroll Work.

11. On the sides of the equilateral triangle ABC as diameters, semicircles are drawn, as $AEFB$. Also with A, B, C as centers and AB as radius arcs are drawn, as $\widehat{AB}$, $\widehat{BC}$.

(*a*) Prove that the arcs $AEFB$, $BDEC$, and $CFDA$ meet in pairs at the middle points D, E, F of the sides of the triangle.





Fourth Presbyterian Church, Chicago.

SUGGESTION. If the middle points of the sides of an equilateral triangle are joined, what kind of triangles are formed?

(*b*) What arcs in this figure are tangent to each other?

(*c*) Has the figure one or more axes of symmetry?

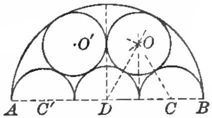
This figure and the two following occur frequently in church windows and other decorative designs.

12. Construct the design shown in the figure.

SUGGESTION. Divide the diameter AB into six equal parts and construct the three semicircles.

On DC and DC' as bases construct equilateral triangles with vertices O and O'.

With radius equal to CB and centers O and O' construct circles.



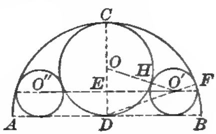
(*a*) Prove that ⊙ O is tangent to each of the three semicircles. Likewise ⊙ O'.

(*b*) Erect a ⊥ to AB at D and prove Ⓢ O and O' tangent to it.

(*c*) Prove circles with centers at O and O' tangent to each other.

(*d*) Has this figure one or more axes of symmetry?

13. In the figure AB, CD and OD are bisected, and $O'O'' \parallel AB$ through E. $DO' = DO'' = \frac{3}{4}\,DB$. Circles are constructed as shown in the figure.



(*a*) If AB is 4 feet, what is the radius of each circle?

(*b*) Prove that ⊙ O is tangent to ⊙ O' and also to ⊙ O''.

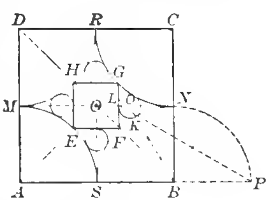
SUGGESTION. Show that OO' is the sum of the radii of the two circles.

(*c*) Is ⊙ O' tangent to the arc ACB and also to the line AB?

(*d*) Has this figure one or more axes of symmetry?

14. $ABCD$ is a square. Arcs are constructed with A, B, C, D as centers and with radii each equal to one half the side of the square The lines AC, BD, MN, and RS are drawn, and the points E, F, G, H are connected as shown in the figure.

The arc SN is extended to P, forming a semicircle. The line LP meets $\widehat{SN}$ in K, and BK meets MN in O'.



(*a*) Prove that $EFGH$ is a square.

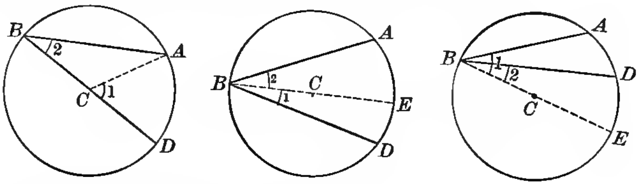
(*b*) Prove that ⚠ KLO' and KPB are mutually equiangular and each isosceles.

(*c*) Prove that ⊙ $O'K$ is tangent to FG and to $\widehat{SN}$.

(*d*) How many axes of symmetry has the figure inside the square?

(*e*) Show that ⊙ $O'K$ is tangent to $\widehat{RN}$ by drawing $O'C$ and folding the figure over on the axis of symmetry MN.

210. THEOREM. *An angle inscribed in a circle is measured by one half the intercepted arc.*



Given $\angle$ ***DBA*** **inscribed in** $\odot$ ***CB.***

To prove that $\angle DBA$ is measured by $\frac{1}{2}\ \widehat{AD}$.

Proof: (1) *If one side, as BD, is a diameter.*

Draw the radius CA. Show that $\angle 2 = \frac{1}{2} \angle 1$.

But $\angle 1$ is measured by $\widehat{AD}$ (§ 202).

Hence $\angle 2$ is measured by $\frac{1}{2}\ \widehat{AD}$.

(2) *If the center C lies within the angle.*

Draw the diameter BE.

Now $\angle DBA = \angle 1 + \angle 2$.

Complete the proof.

(3) *If the center C lies outside the angle.*

Draw BE and use the equation $\angle DBA = \angle 1 - \angle 2$.

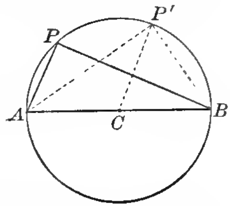
211. It follows from § 210 that if in equal circles two inscribed angles intercept equal arcs, they are equal; and conversely, that if equal angles are inscribed in equal circles, they intercept equal arcs.

212. EXERCISES.

1. If the sides of two angles BAD and $BA'D$ pass through the points B and D on a circle, and if the vertex A is on the *minor* arc BD and A' is on the *major* arc BD, find the sum of the two angles.

2. In Ex. 1 if the points B and D remain fixed while the vertex A of the angle is made to move along the *minor* arc of the circle, what can be said of the angle A? What if it moves along the *major* arc?

213. THEOREM. *The locus of the vertices of all right triangles on a given hypotenuse is a circle whose diameter is the given hypotenuse.*



Outline of Proof: Let AB be the given hypotenuse.

(1) If P is any point on the circle whose diameter is AB, $\angle APB =$ rt. $\angle$. (Why?)

(2) If $AP'B$ is any right triangle with AB as hypotenuse, then $AC = CB = CP'$. (See Ex. 27, p. 82.)

State the proof in full.

214. **PROBLEMS ON LOCI.**

Find the following loci:

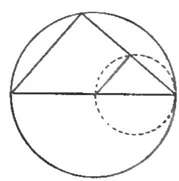
1. The centers of all circles of fixed radius tangent to a fixed line.

2. The centers of all circles tangent to two parallel lines.

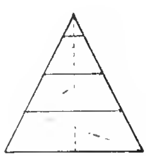
3. The centers of all circles tangent to both sides of an angle.

4. The centers of all circles tangent to a given line at a given point. Is the given point a part of this locus?

5. The vertices of all triangles which have a common base and equal altitudes.



6. The middle points of all chords through a fixed point on a circle. Use Ex. 8, § 208, and then § 213.

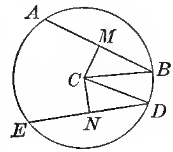


7. The points of intersection of the diagonals of trapezoids formed by the sides of an isosceles triangle and lines parallel to its base.

8. Two vertices of a triangle slide along two parallel lines. What is the locus of the third vertex if the triangle is fixed in size and shape?

9. $ABCD$ is a parallelogram all of whose sides are of fixed length. The side AB is fixed in position. Find the locus of the middle points of the remaining three sides.

10. *Prove that in the same circle or in equal circles equal chords are equally distant from the center.*



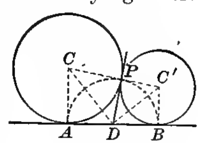
SUGGESTION. $MB = ND$. Why? Then prove $\triangle BMC \cong \triangle CND$.

11. State and prove the converse of the theorem in the preceding exercise. (What parts of $\triangle$s BMC and CND are *now* known?)

12. Find the locus of the middle points of all chords of equal length in the same circle.

13. Find the locus of the middle point of a segment AB of fixed length which moves so that its end-points slide along the sides of a right angle. (Use Ex. 27, p. 82.)

14. Find the locus of the points of contact of two varying circles tangent to each other, and each tangent to a given line at a given point.



SUGGESTION. A and B are the fixed points, and P one point of contact of the circles. Draw the common tangent PD. Prove $AD = DP$ and $DB = DP$.

Hence, D is the middle point of AB and DP is constant. That is, the locus is a circle of which AB is a diameter.

15. Find the locus of the centers of all circles tangent to a fixed circle at a fixed point P. Is the fixed point P a part of this locus? Is the center of the fixed circle a part of it?

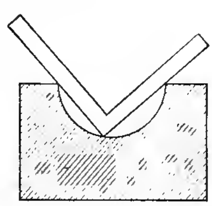
16. Find the locus of the centers of all circles of the same radius which are tangent to a fixed circle.

Under what conditions will this locus include the fixed circle itself? The center of this fixed circle?

Will the locus ever contain a circle within the fixed circle?

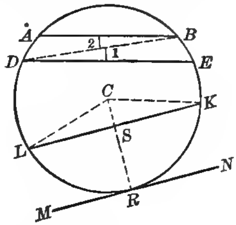
Under what conditions will the locus consist of two circles, each outside the fixed circle?

Under what condition does the locus consist of only one circle?



17. In making core-boxes, pattern makers use a square as indicated in the figure to test whether or not the core is a true semicircle. Is this method correct? Prove.

215. THEOREM. *The arcs intercepted by two parallel chords or by a tangent and a chord parallel to it are equal.*



Given $AB \parallel DE$ **and** $LK \parallel MN$.

To prove that $\widehat{AD} = \widehat{BE}$ and $\widehat{LR} = \widehat{KR}$.

Proof: (1) Draw chord DB.

Compare $\angle 1$ and $\angle 2$, and hence show that $\widehat{AD} = \widehat{BE}$. (Why?)

(2) Draw the radius CR to the point of tangency. Then $CR \perp MN$ and $CR \perp LK$. (Why?)

Prove $\triangle LCS \cong \triangle KCS$, and hence that $\widehat{LR} = \widehat{RK}$. (§ 199)

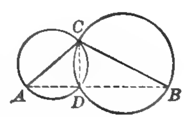
216. **EXERCISES.**

1. Prove that a tangent at the vertex of an inscribed angle forms equal angles with the two sides, if these are equal chords.

2. If the vertices of a quadrilateral lie on a circle, any two of its opposite angles are supplementary.

3. If two chords of a circle are perpendicular to each other, find the sum of each pair of opposite arcs into which they divide the circle.

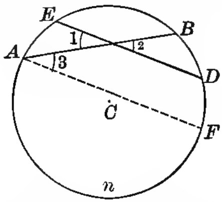
4. If the vertices of a trapezoid lie on a circle, its diagonals are equal.



5. Two circles intersect at C and D. Diameters CA and CB are drawn. Prove that A, D, B lie on a straight line.

SUGGESTION. Prove that $\angle ADC = \angle CDB =$ rt. $\angle$.

217. THEOREM. *An angle formed by two intersecting chords is measured by one half the sum of the arcs intercepted by the angle itself and its vertical angle.*



Given $\angle 1$ **formed by the chords** AB **and** DE.

To prove that $\angle 1$ is measured by $\frac{1}{2}(\overset{\frown}{AE} + \overset{\frown}{BD})$.

Proof: Through A draw the chord $AF \parallel ED$.

Compare $\angle 1$ and $\angle 3$.

Compare $\overset{\frown}{AE}$ and $\overset{\frown}{DF}$, also $\overset{\frown}{AE} + \overset{\frown}{BD}$ and $\overset{\frown}{BD} + \overset{\frown}{DF}$.

How is $\angle 3$ measured?

Hence, how is $\angle 1$ measured?

218. **EXERCISES.**

1. A chord AB is divided into three equal parts, AC, CD, and DB. OA, OC, OD, and OB are drawn. Compare the angles AOC, COD, and DOB.

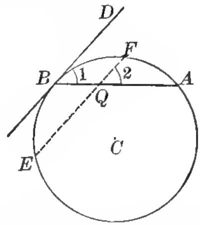
2. The accompanying table refers to the figure in § 217. Fill out blank spaces.

$\angle 1$	$\overset{\frown}{AE}$	$\overset{\frown}{BD}$	$\overset{\frown}{EB}$	$\overset{\frown}{AnD}$
35°	40°		80°	
48°	50°			216°
40°		50°	60°	
60°		54°		190°
		45°	90°	180°
	34°		108°	164°

3. In a circle C with a diameter AB a chord AD is drawn, and a radius $CE \parallel AD$. Prove that arcs DE and EB are equal.

4. The vertices of a square $ABCD$ all lie on a circle. E is any point on the arc AB. Prove that EC and ED divide the angle AEB into three equal parts.

219. THEOREM. *An angle formed by a tangent and a chord drawn from the point of tangency is measured by one half the intercepted arc.*



Given $\angle 1$ **formed by tangent** BD **and chord** BA.

To prove that $\angle 1$ is measured by $\frac{1}{2}\,\widehat{BA}$.

Proof: Draw a chord $EF \parallel BD$ intersecting BA in Q.

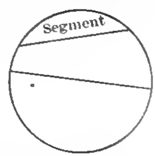
Compare $\angle 1$ and $\angle 2$, also $\widehat{EB}$ and $\widehat{BF}$.

How is $\angle 2$ measured?

Hence, how is $\angle 1$ measured?

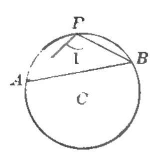
Give the proof in full.

220. Definitions. A **segment of a circle**, or a **circle-segment**, is a figure formed by a chord and the arc which it subtends. For each chord there are two circle-segments corresponding to the two arcs which it subtends.



If a chord is a *diameter* the two circle-segments are equal.

An angle is said to be **inscribed in an arc** if its vertex lies on the arc and its sides meet the arc in its end-points.



Such an angle is also said to be inscribed in the circle-segment formed by the arc and its chord.

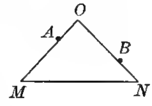
E.g. $\angle 1$ is inscribed in the arc APB or in the segment APB.

221. EXERCISES.

1. Show that an angle inscribed in a semicircle is a right angle.

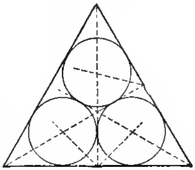
2. If the sides of a right angle pass through the extremities of a diameter, show that its vertex lies on the circle.

3. If a triangular ruler MNO, right-angled at O, is moved about in the plane so that two fixed points, A and B, lie always on the sides MO and NO respectively, what path does the point O trace?



4. Draw two concentric circles, having different radii, and show that all chords of the outer circle which are tangent to the inner circle are equal.

5. In an equilateral triangle construct three equal circles, each tangent to the two other circles and to two sides of the triangle.



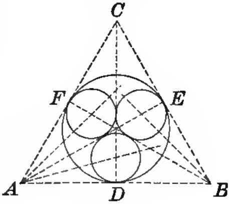
SUGGESTION. Construct the altitudes of the triangle and bisect angles as shown in the figure. Complete the construction and prove that the figure has the required properties.

(*a*) Has the figure consisting of the triangle and the three circles one or more axes of symmetry?

(*b*) Has it a center of symmetry?

6. Within a given circle construct three equal circles, each tangent to the other two and to the given circle.

SUGGESTION. Trisect the circle at D, E, and F by making angles at the center each equal to 120°. Draw tangents at D, E, and F, and prove that $\triangle ABC$ is equilateral.



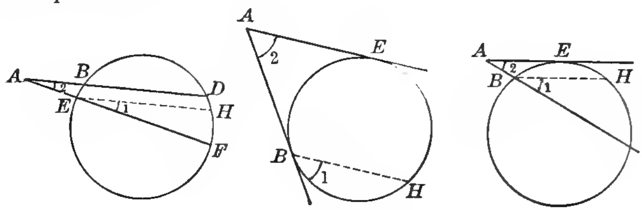
Construct the altitudes and prove that they meet the sides of the triangle at the points of tangency of the given circle with the sides of ABC, and also that they pass through the center of the given circle.

Bisect angles as shown in the figure and prove that the centers of the required circles are thus obtained.

(*a*) Has the figure consisting of the four circles one or more axes of symmetry?

(*b*) Has it a center of symmetry?

222. THEOREM. *The angle formed by two secants, two tangents, or a tangent and a secant, meeting outside a circle, is measured by one half the difference of the intercepted arcs.*



Outline of Proof: In each case the given angle is equal to $\angle 1$, and the arc which measures $\angle 1$ is the difference between two arcs, one of which is the larger of the two intercepted arcs and the other is equal to the smaller. For instance, in the first figure,

$$\overset{\frown}{FH} = \overset{\frown}{DF} - \overset{\frown}{DH} = \overset{\frown}{DF} - \overset{\frown}{BE}.$$

Give the proof in detail for each figure.

223. EXERCISES.

1. If (in left figure, § 222) $\angle A = 17°$ and $\overset{\frown}{EB} = 25°$, find $\overset{\frown}{DF}$.

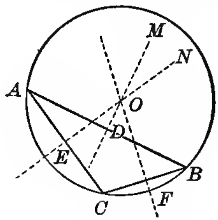
2. If $\angle A = 37°$ (in middle figure), find the arcs into which the points B and E divide the circle.

3. With a given radius construct a circle passing through a given point. How many such circles can be drawn? What is the locus of the centers of all such circles?

4. Draw a circle passing through two given fixed points. How many such circles are there? What is the locus of the centers of all such circles?

5. Construct a circle having a given radius and passing through two given points. How many such circles can be drawn? Is this construction ever impossible? Under what conditions is only one such circle possible?

224. PROBLEM. *To construct a circle through three fixed points not all in the same straight line.*



Given three points *A, B, C* not in the same straight line.

To construct a circle passing through them.

Construction. Let the student give the construction and proof in full. (See § 132.)

225. Definition. The circle *OA* in § 224 is said to be **circumscribed** about the triangle *ABC* and the triangle is said to be **inscribed** in the circle.

226. EXERCISES.

1. In the construction of § 224 why do *DM* and *EN* meet?

2. Why cannot a circle be drawn through three points all lying in the same straight line? Make a figure to illustrate this.

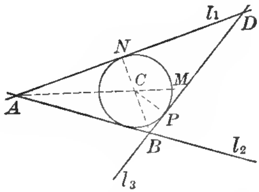
3. Show that an angle inscribed in an arc is greater than or less than a right angle according as the arc in which it is inscribed is less than or greater than a semicircle.

4. Prove that the bisectors of the angles of an equilateral triangle pass through the center of the circumscribed circle.

5. Draw a circle tangent to two fixed lines. How many such circles are there? What is the locus of their centers? Is the point of intersection part of this locus? Discuss fully.

6. Show that not more than one circle can be drawn through three given points, and hence that two circles which coincide in three points coincide throughout.

227. PROBLEM. *To construct a circle tangent to each of three lines, no two of which are parallel and not all of which pass through the same point.*



Given the lines l_1, l_2, l_3.

To construct a circle tangent to each of these lines.

Construction. Since no two of the lines are ∥, let l_1 and l_2 meet in A, l_2 and l_3 in B, and l_3 and l_1 in D, where A, B, and D are distinct points.

Draw the bisectors of $\angle A$ and $\angle B$ and let them meet in point C.

Then C is the center of the required circle. (See § 131.) Give the proof in full.

228. **Definitions.** The circle in the construction of § 227 is said to be **inscribed** in the triangle ABD.

Three or more lines which all pass through the same point are called **concurrent.** Hence the lines l_1, l_2, l_3 are not concurrent.

229. EXERCISES.

1. Why is the construction of § 227 impossible if l_1, l_2, and l_3 are concurrent?

2. If two of the lines are parallel to each other, show that the construction is possible. How many tangent circles can be constructed in this case? Draw a figure and give the construction and proof in full.

3. Is the construction possible when all three lines are parallel? Why?

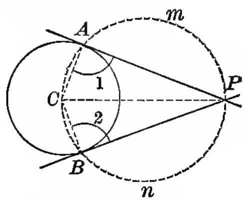
4. If two sides of the triangle are produced, as AB and AD in the figure of § 227, construct a circle tangent to the side BD and to the prolongations of the sides AB and AD.

This is called an **escribed** circle of the triangle.

5. How many circles can be constructed tangent to each of three straight lines if they are not concurrent and no two of them are parallel?

6. Draw a triangle and construct its inscribed and circumscribed circles and its three escribed circles.

230. PROBLEM. *From a given point outside a circle to draw a tangent to the circle.*



Given ⊙ CA **and an outside point** P.

To construct a tangent from P to the circle.

Construction. Draw CP. On CP as a diameter construct a circle, cutting the given circle in the points A and B.

Draw the lines PA and PB.

Then PA and PB are both tangents.

Give the proof.

231. EXERCISES.

1. If in the figure of § 230 the point P is made to move towards the circle along the line PC until it finally reaches the circle, while PA and PB remain tangent to the circle, describe the motion of the points A and B and also of the lines PA and PB. How does this agree with the fact that through a point on the circle there is only *one* tangent to the circle?

2. Can a tangent be drawn to a circle from a point inside the circle? Why?

3. Show that the line connecting a point outside a circle with the center bisects the angle formed by the tangents from that point.

4. Why are not more than two tangents possible from a given point to a circle?

5. The two tangents which can be drawn to a circle from an exterior point are equal.

6. In a right triangle the hypotenuse plus the diameter of the inscribed circle is equal to the sum of the two legs of the triangle.

7. If an isosceles triangle inscribed in a circle has each of its base angles double the vertex angle, and if tangents to the circle are drawn through the vertices, find the angles of the resulting triangle.

8. If the angles of a triangle ABC inscribed in a circle are 64°, 72°, and 44°, find the angles of the triangle formed by the tangents to the circle at the points A, B, and C.

SUMMARY OF CHAPTER II.

1. Make a list of all the definitions involving the circle.

2. State the theorems on the measurement of angles by intercepted arcs.

3. State the theorems involving equality of chords, central angles, and intercepted arcs.

4. State the theorems on the tangency of straight lines and circles.

5. State the theorems involving the tangency of two circles.

6. Make a list, to supplement that in the summary of Chapter I, of ways in which two angles or two line-segments may be proved equal.

7. State the ways in which two arcs of the same or equal circles may be proved equal.

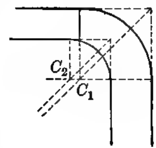
8. State the problems of construction given in Chapter II.

9. Explain what is meant by saying that a central angle is measured by its intercepted arc.

10. State some of the important applications of Chapter II. (Return to this question after studying those which follow.)

PROBLEMS AND APPLICATIONS.

1. Given two roads of different width at right angles to each other, to connect them by a road whose sides are arcs of circles tangent to the sides of the roads.



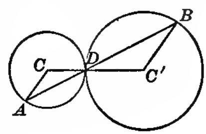
(*a*) Make the construction shown in the figure and prove that it has the required properties.

(*b*) Is this construction possible when the given roads are not at right angles to each other? Illustrate.

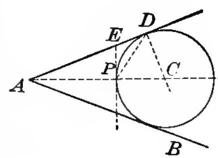
(*c*) Can the curve be made long or short at will?

(*d*) Make the construction if the given roads have the same width.

2. Two circles C and C' are tangent at the point D. AB is a segment through D terminating in the circles. Prove that the radii CA and $C'B$ are parallel.



3. Through a point on the bisector of an angle to construct a circle tangent to both sides of the angle.

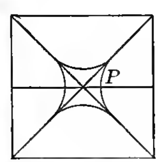


CONSTRUCTION. Through the given point P draw $EP \perp$ to AP. Lay off $ED = EP$ and at D construct $DC \perp AD$ meeting the line AP in C. Then C is the center of the required circle and CD is its radius.

PROOF: Draw PD and prove that $\triangle DEP$ is isosceles and hence also $\triangle PDC$.

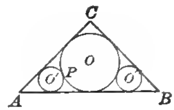
Is it possible to construct another circle having the properties required? If so, construct it.

This construction is used in the accompanying design in which the shape is determined by fixing the point P in advance.



Such designs are of frequent occurrence in decorative work such as the steel ceiling panel given here.

4. In an isosceles triangle construct three circles as shown in the figure.



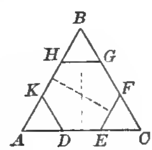
Suggestion. First construct the inscribed circle with center O. Let the bisector of $\angle A$ meet this circle in a point P. Then use Ex. 3.

5. The angles formed by a chord and a tangent are equal respectively to the angles inscribed in the arcs into which the end-points of the chord divide the circle.

6. If a triangle whose angles are 48°, 56°, and 76° is circumscribed about a circle, find the number of degrees in the arcs into which the points of tangency divide the circle.

7. Divide each side of an equilateral triangle into three equal parts (Ex. 5, § 159) and connect points as shown in the figure.

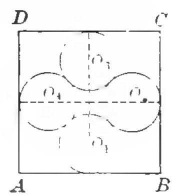
Prove that $DEFGHK$ is a regular hexagon.



8. If a circle is inscribed in the triangle of Ex. 7, prove that all sides of the hexagon are tangent to the circle.

Suggestion. Show that the perpendicular bisectors of the segments HK, KD, DE meet in a point equidistant from these segments.

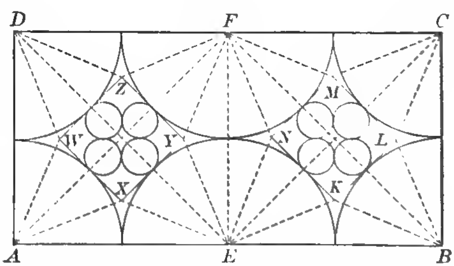
9. Within a given square construct four equal circles so that each circle is tangent to one side of the square and to two of the circles.



Suggestion. First construct the diagonals of the square.

10. In the figure, $ABCD$ is a rectangle with $AD = \frac{1}{2} AB$. E and F are the middle points of AB and CD respectively.

Semicircles are constructed with E and F as centers and $\frac{1}{2} AE$ as a radius, etc.



Fan vaulting from Gloucester Cathedral, England.

(*a*) Prove that these quadrant arcs are tangent to each other in pairs and also to the semicircles.

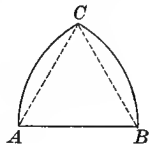
(*b*) Lines are drawn tangent to the arcs at the points where these are met by the diagonals of the squares $AEFD$ and $BCFE$. Prove that these lines form squares $KLMN$ and $XYZW$.

(*c*) Construct the small circles within each of these squares.

The above design occurs in fan-vaulted ceilings.

The gothic or pointed arch plays a conspicuous part in modern architecture, and examples of it may be found in almost any city. Its most common use is in church windows.

The figure represents a so-called equilateral gothic arch. The arcs AC and BC are drawn from B and A as centers respectively, and with AB as a radius.

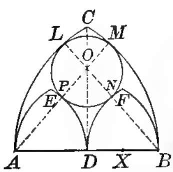


The segment AB is called the *span of the arch*, and the point C its *apex*.

11. In the figure $AD = DB$. ABC, ADE, and DBF are equilateral gothic arches.

(*a*) Construct the circle with center O tangent to the four arcs as shown.

SUGGESTION. Take X so that $DX = XB$. With centers A and B and radius AX draw arcs meeting at O. Complete the construction and prove that the figure has the required properties.



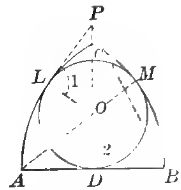
Door, Union Park Church, Chicago.

(*b*) Prove that $\overset{\frown}{DE}$ and $\overset{\frown}{DF}$ are tangent to each other. Also $\overset{\frown}{BF}$ and $\overset{\frown}{BC}$, and $\overset{\frown}{AE}$ and $\overset{\frown}{AC}$.

(*c*) What axis of symmetry has this figure?

12. A triangle ABC whose angles are 45°, 80°, and 55° is inscribed in a circle. Find the angles of the triangle formed by the tangents at A, B, and C.

13. Inscribe a circle in an equilateral gothic arch ABC.



SUGGESTIONS. Construct $CD \perp$ to AB and extend it to P, making $DP = AB$. From P construct a tangent to $\widehat{AC}$ at L.

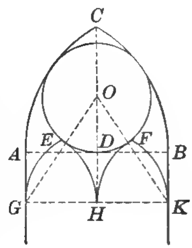
(*a*) Prove that $\triangle BDP \cong \triangle BLP$ and hence $PL = BD$.

(*b*) $\triangle OLP \cong \triangle BDO$ and hence $OD = OL$.

Then $\odot OD$ is the required circle. See § 209.

Notice that this figure is symmetrical with respect to the line PD, and hence if the circle is proved tangent to $\widehat{AC}$, we know at once that it is tangent to $\widehat{BC}$.

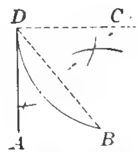
14. In the figure ABC is an equilateral gothic arch with a circle inscribed, as in Ex. 13.



(*a*) Construct the two equilateral arches GHE and HKF, as shown in the figure.

CONSTRUCTION. Draw BK and $AG \perp AB$. With a radius equal to $OD + DB$, and with O as center draw arcs meeting BK and AG in K and G respectively. Draw GK, construct the arches and show that each is tangent to the circle.

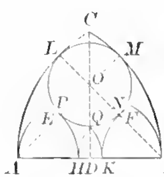
(*b*) Do the points E and F lie on the circle?

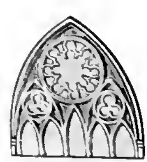


SUGGESTION. Suppose KF to be drawn, and compare $\angle HKF$ with $\angle HKO$ by comparing the sides HK and KF and also GK and KO.

15. Construct an arc passing through a given point B, and tangent to a given line AD at a given point D.

16. In the figure ABC is an equilateral arch. BK is $\frac{5}{6}$ of BD. KBF and AHE are equal equilateral arches. Arcs KQ and HQ are tangent to arcs KF and HE respectively.



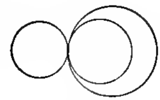


From Lincoln Cathedral, England.

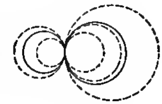
(*a*) Find by construction the center O of the circle tangent to $\widehat{AC}$, $\widehat{BC}$, $\widehat{KF}$, and $\widehat{HE}$, and give proof.

(*b*) Find by construction the centers of the arcs KQ and HQ. How is this problem related to Ex. 15?

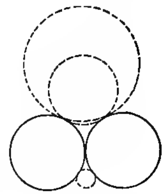
17. Two circles are tangent to each other *internally*. Find the locus of the centers of all circles tangent to both *externally*.



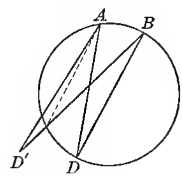
18. Two circles are tangent to each other *externally*. Find the locus of the centers of all circles tangent to both, but *external* to one and *internal* to the other.



19. Two equal circles are tangent to each other externally. Find the locus of the centers of all circles tangent to both.

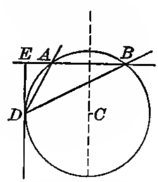


20. $AD'B$ is an angle whose vertex is outside the circle and whose sides meet the circle in the points A and B, while $\angle ADB$ is an inscribed angle intercepting the arc AB. Prove that $\angle ADB > \angle AD'B$, provided each of the segments $D'A$ and $D'B$ cuts the circle at a second point.

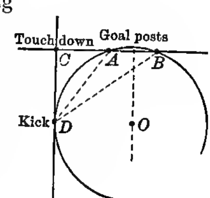


21. Through two given points A and B construct a circle tangent to a given line which is perpendicular to the line AB.

Is this construction possible if the given line passes through either of the points A or B? If it meets AB between these points?



22. In kicking a goal after a touchdown in the game of football, the ball is brought back into the field at right angles to the line marking the end of the field. The distance between the goal posts being given, and also the point at which the touchdown is made, find by a geometrical construction how far back into the field the ball must be brought in order that the goal posts may subtend the greatest possible angle.



CHAPTER III.

THE MEASUREMENT OF STRAIGHT LINE-SEGMENTS.

232. A straight line-segment is said to be **exactly measured** when we find how many times it contains a certain other segment which is taken as a unit. The number thus found is called the **numerical measure**, or the **length** of the segment.

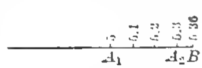
E.g. a line-segment is 9 in. long if a segment 1 in. long can be laid off on it 9 times in succession.

Thus, 9 is the numerical measure, or the **length** of the segment, when 1 in. is taken as a unit.

233. In selecting a unit of measure it may happen that it is not contained an *integral* number of times in the segment to be measured.

Thus, in measuring a line-segment the *meter* is often a convenient unit. Suppose it has been applied five times to the segment AB and that the last time the end falls on A_1, A_1B being less than one meter.

Then, taking a decimeter (one tenth of a meter) as a new unit, suppose this is contained three times in A_1B with a remainder A_2B less than a decimeter.



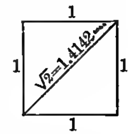
Finally, using as a unit a *centimeter* (one tenth of a decimeter), suppose this is contained exactly six times in A_2B.

Then, the length of AB is 5 meters, 3 decimeters, and 6 centimeters, or 5.36 meters.

The process of measuring considered here is ideal. In *practice* we cannot say that a given segment is contained *exactly* an integral number of times in another segment. See § 235.

234. It may also happen that, in continuing this ideal process of measuring as just described, no subdivided unit can be found which exactly measures the last interval, that is, such that the final division point falls *exactly* on *B*.

E.g. it is known that in a square whose sides are each one unit the diagonal is $\sqrt{2}$, and that this cannot be *exactly* expressed as an integer or a fraction whose numerator and denominator are both integers.



By the ordinary process of extracting square root we find $\sqrt{2} = 1.4142 \cdots$, each added decimal making a nearer approximation. But this process never terminates.

Hence, in attempting to measure the diagonal of a square whose side is one meter, we find 1 meter, 4 decimeters, 1 centimeter, 4 millimeters, etc., or 1.414 meters approximately.

It should be noticed, however, that 1.415 is greater than the diagonal and hence the approximation given is correct within one millimeter.

235. Evidently any line-segment can be measured either exactly or to a degree of approximation, depending upon the fineness of the instruments and the skill of the operator. The word *measure* is commonly used to include both *exact* and *approximate* measurement.

For **practical purposes**, a line-segment is measured as soon as the last remainder is smaller than the smallest unit available. It should be noticed that *all practical* measurements are in reality only approximations, since it is quite impossible to say that a given distance is, for instance, exactly 25 ft. It may be a fraction of an inch more or less.

E.g. in the above example 1.414 meters gives the length of the diagonal for *practical purposes* if the millimeter is the smallest unit available. The error in this case is less than one millimeter.

236. Definition. Two straight line-segments are **commensurable** if they have a common unit of measure. Otherwise they are **incommensurable.**

E.g. two line-segments whose lengths are *exactly* 5.27 and 3.42 meters respectively have one centimeter as a common unit of measure, it being contained 527 times in the first segment and 342 times in the second.

But the side and the diagonal of a square have *no common unit of measure.*

In the example of § 234, the millimeter is contained 1000 times in the side and 1414 times in the diagonal, *plus a remainder* less than one millimeter. A similar statement holds for *any unit* of measure, however small.

237. For the purposes of *practical measurement* any two line-segments *may be considered as commensurable*, but for theoretical purposes it is necessary to take account of incommensurable segments also.

The theorems in this chapter are here proved for commensurable segments only. They are proved for incommensurable segments also in Chapter VII.

RATIOS OF LINE-SEGMENTS.

238. The **ratio of two commensurable line-segments** is the quotient of their numerical measures taken with respect to the same unit.

E.g. if two segments are respectively 3 ft. and 4 ft. in length, the ratio of the first segment to the second is $\frac{3}{4}$ and the ratio of the second to the first is $\frac{4}{3}$.

239. The ratio of two commensurable segments is the *same*, no matter what common unit of measure is used.

E.g. two segments whose numerical measures are 3 and 4 if one *foot* is the common unit, have 36 and 48 as their numerical measures if one *inch* is the common unit. But the ratio is the same in both cases, namely: $\frac{36}{48} = \frac{3}{4}$.

240. The **approximate ratio of two incommensurable line-segments** is the quotient of their approximate numerical measures. It will be seen that this approximate ratio depends upon the length of the smallest measuring unit available, and that the approximation can be made as close as we please by taking the measuring unit small enough.

E.g. an approximate ratio of the side of a square to its diagonal is $\frac{1}{1.41} = \frac{100}{141}$. Another and closer approximation is $\frac{1}{1.414} = \frac{1000}{1414}$. In this case the numerical measure of one of the segments is *exact*.

An approximate ratio of $\sqrt{2}$ to $\sqrt{3}$, in which neither has an exact measure, is $\frac{1.41}{1.73} = \frac{141}{173}$. Another is $\frac{1.414}{1.732} = \frac{1414}{1732}$.

241. It should be clearly understood that the numerical measure of a line-segment is a **number**, as is also the *ratio* of two such segments. Hence they are subject to the same laws of operation as other arithmetic numbers.

For example, the following are axioms pertaining to such numbers:

(1) *Numbers which are equal to the same number are equal to each other.*

(2) *If equal numbers are added to or subtracted from equal numbers, the results are equal numbers.*

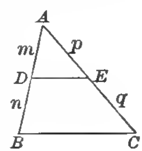
(3) *If equal numbers are multiplied by or divided by equal numbers, the results are equal numbers.*

It is understood, however, that all the numbers here considered are positive. For a more complete consideration of axioms pertaining to numbers, see Chapter I of the Advanced Course of the authors' High School Algebra.

242. A **proportion** is an equality, each member of which is a ratio. Four numbers, a, b, c, and d, are said to be in **proportion**, in the order given, if the ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equal. In this case a and c are called the **antecedents** and b and d the **consequents.** Also a and d are called the **extremes** and b and c the **means.**

The proportion $\frac{a}{b}=\frac{c}{d}$ is sometimes written $a : b = c : d$, and in either case may be read *a is to b as c is to d.*

If D and E are points on the sides of the triangle ABC, and if m, n, p, and q, the numerical measures respectively of AD, DB, AE, and EC, are such that $\frac{m}{n}=\frac{p}{q}$, then the points D and E are said to divide the sides AB and AC **proportionally**, that is, **in the same ratio.** For convenience it is common to let AD, DB, AE, and EC stand for the *numerical measures* of these segments, and thus to write the above proportion, $\frac{AD}{DB}=\frac{AE}{EC}$ or $AD : DB = AE : EC$.

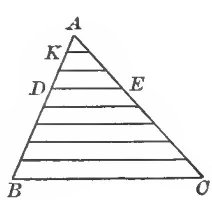


THEOREMS ON PROPORTIONAL SEGMENTS.

243. THEOREM. *If a line is parallel to one side of a triangle and cuts the other two sides, then it divides these sides in the same ratio.*

Given $\triangle ABC$ **in which** $DE \parallel BC$.

To prove that $\frac{AD}{DB}=\frac{AE}{EC}$.



Proof: Choose some common measure of AD and DB, as AK. Suppose it is contained 3 times in AD and 5 times in DB.

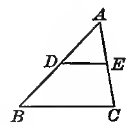
Then $$\frac{AD}{DB} = \frac{3}{5}. \qquad (1)$$

Through the points of division on *AD* and *DB* draw lines parallel to *BC*, cutting *AE* and *EC*. By § 155 these parallels divide *AE* into three equal parts and *EC* into five equal parts. Hence, $$\frac{AE}{EC} = \frac{3}{5}. \qquad (2)$$

$$\therefore \text{ from (1) and (2) } \frac{AD}{DB} = \frac{AE}{EC}. \qquad \text{§ 241}$$

For a proof in case *AD* and *DB* are incommensurable, see § 410.

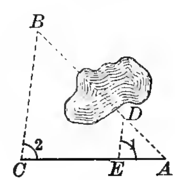
244. EXERCISES.



1. If $DE \parallel BC$ in $\triangle ABC$, compute the segments left blank from those given in the following table:

AD	*DB*	*AE*	*EC*	*AB*	*AC*
20	24	15			
4	56		42		
	102	12	408		
25		18	342		

2. *B* is a point visible from *A* but inaccessible. Required to compute the distance from *A* to *B*.



Suggestion. Select some accessible point *C* from which *A* and *B* are both visible. Through *E*, a point near *A* and on the line of sight from *A* to *C*, draw $ED \parallel CB$ and meeting the line of sight from *A* to *B* at *D*.

Now *AE*, *EC*, and *AD* can be *measured*. Then *DB* and hence *AB* can be *computed* by § 243.

Note. The advantage of this method over that in § 34, Ex. 5, is that here a *small* triangle *AED* is made to do the service, which was there performed by another triangle the same size as *ABC*.

245. Theorem. *If four numbers m, n, p, q are such that* $\frac{m}{n}=\frac{p}{q}$, *then it follows that:*

(1) $\frac{n}{m}=\frac{q}{p}$. (2) $\frac{m}{p}=\frac{n}{q}$.

(3) $\frac{m+n}{n}=\frac{p+q}{q}$. (4) $\frac{m-n}{n}=\frac{p-q}{q}$.

(5) $\frac{m+n}{m-n}=\frac{p+q}{p-q}$.

Given $\frac{m}{n}=\frac{p}{q}$. (*a*)

To prove (1) divide the members of $1=1$ by those of (*a*).

To prove (2) multiply each member of (*a*) by $\frac{n}{p}$.

To prove (3) add 1 to each member of (*a*) and reduce each side to a common denominator.

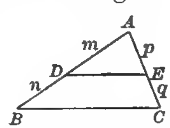
To prove (4) subtract 1 from each member of (*a*) and reduce each side to a common denominator.

To prove (5) divide the members of (3) by the members of (4).

Write out these proofs in full, giving the reason for each step, and read off the results as applied to the figure.

For example, show that (3) gives, when applied to the figure,

$$\frac{AB}{DB}=\frac{AC}{EC}.$$



246. The results in the above theorem are sometimes named as follows:

The proportion (*a*) is said to be taken by **inversion** in (1), by **alternation** in (2), by **composition** in (3), by **division** in (4), by **composition** and **division** in (5).

247. **EXERCISES.**

1. If $\frac{m}{n}=\frac{p}{q}$, prove that $\frac{m}{m+n}=\frac{p}{p+q}$, and hence show in the above figure that

$$\frac{AD}{AB}=\frac{AE}{AC}.$$

2. If in the figure on page 118 $DE \parallel BC$, compute the segments indicated by blanks in the accompanying table.

AD	AB	DB	AE	AC	EC
8	12		6		
6	10			16	
6	9				7
12		10	10		
10		8		18	
10		7			14
240			200	380	
160			140		20
120				100	50
	35	21	14		
	40	15		30	
	500	200			400
	90		40	70	
	20		30		8
	800			360	300
		27	30	48	
		30	20		50
		27		560	48

3. If $\frac{m}{n}=\frac{p}{q}$, show that $\frac{m+p}{p}=\frac{n+q}{q}$.

4. If $\frac{m}{n}=\frac{p}{q}$, show that $\frac{m+p}{n+q}=\frac{m}{n}$.

5. If $\frac{m}{x}=\frac{p}{q}$ and $\frac{m}{y}=\frac{p}{q}$, show that $x=y$.

6. If $\frac{m}{x}=\frac{m}{y}$, show that $x=y$.

7. State Exs. 4, 5, and 6 in words.

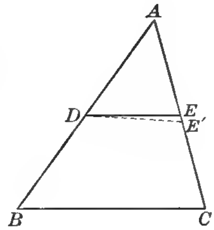
8. A triangle is formed by a chord and the tangents to the circle at its extremities. Prove that the triangle is isosceles.

9. A triangle with angles A, B, C is circumscribed about a circle. Find the angles of the triangle formed by the chords joining the points of tangency.

248. THEOREM. *If a line divides two sides of a triangle in the same ratio, it is parallel to the third side.*

Given the points D and E on the sides of the $\triangle ABC$ such that $\frac{AD}{DB} = \frac{AE}{EC}$.

To prove that $DE \parallel BC$.



Proof: Suppose DE' is drawn parallel to BC. It is proposed to prove that the point E' coincides with E.

Since $DE' \parallel BC$, we have $\frac{AB}{AD} = \frac{AC}{E'C}$.

But by (3), § 245, $\frac{AB}{DB} = \frac{AC}{EC}$.

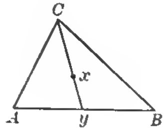
Now use the proof of Ex. 5, § 247, to show that $E'C = EC$, and hence that E' and E coincide, so that $DE \parallel BC$. Give the proof in full detail.

249. EXERCISES.

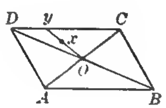
1. Show by § 248 that the line joining the middle points of two sides of a triangle is parallel to the third side. Compare § 151.

2. Show by § 243 that the line which bisects one side of a triangle and is parallel to a second side bisects the third side.

3. If in the $\triangle ABC$ a segment Cy connects the vertex to any point y of the base, find the locus of the point x on this segment such that $Cx : Cy$ is the same for all points y.



4. $ABCD$ is a ▱ whose diagonals meet in O. If y is a point on any side of the ▱, find the locus of a point x on the segment Oy such that $Oy : xy$ is the *same* for every such point y.

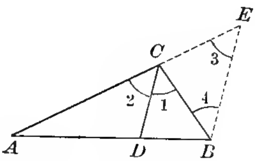


5. Find the locus of the points of intersection of the medians of all triangles having the same base and equal altitudes. (Use §§ 158, 248.)

250. THEOREM. *The bisector of an angle of a triangle divides the opposite side into segments whose ratio is the same as that of the adjacent sides.*

Given *CD* bisecting $\angle C$ in $\triangle ABC$.

To prove that $\frac{AD}{DB}=\frac{AC}{BC}$.



Proof: Through B draw $BE \parallel DC$. Prolong AC to meet BE at E.

In $\triangle ABE$ $\qquad \frac{AD}{DB}=\frac{AC}{CE}$. (Why?)

Now show that $\triangle BCE$ is isosceles, and hence that BC may be substituted for CE in the above proportion. Complete the proof.

251. EXERCISES.

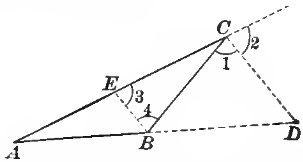
1. Fill in the blank spaces in the table, if in the figure of § 250 *CD* is the bisector of $\angle ACB$.

AC	*CB*	*AD*	*DB*	*AB*
8	10	6		
20	16		12	
35	17			40
3		2	1	
7		9		12
$12\frac{1}{2}$			8	16
	364	200	480	
	54	65		105
	24.5		18.3	32.6

252. Definition. A segment is said to be **divided externally** by any point which lies on the *line of the segment* but not on the segment itself.

E.g. point C divides the segment AB *externally*, the parts being AC and CB, while point B divides the segment AC *internally*, the parts being AB and BC.

253. Theorem. *A line which bisects an exterior angle of a triangle divides the opposite side externally into two segments whose ratio is the same as that of the adjacent sides of the triangle.*



Given CD, the bisector of the exterior angle at C of the triangle ABC.

To prove that $$\frac{AD}{BD} = \frac{AC}{BC}.$$

Proof: Through B draw $BE \parallel DC$.

In $\triangle ACD$ $$\frac{AD}{BD} = \frac{AC}{EC}. \qquad \text{(Why?)}$$

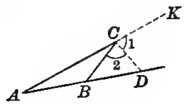
Now show that $\triangle EBC$ is isosceles, and hence that $$EC = BC.$$

Complete the proof.

254. EXERCISES.

1. Draw a triangle with an *acute* exterior angle bisected. Using different lettering from that in § 253, prove the theorem again.

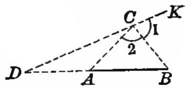
2. Compare the proofs in §§ 250 and 253. Give the proof in § 253 for a figure in which $AC < BC$.



3. Fill in the blank spaces in the table below if *CD* is the bisector of the exterior angle *BCK*.

AC	*CB*	*AD*	*DB*	*AB*
7	4	9		
14.3	9.6		18	
164	48			144
13.7		84	60	
32		60		25
56			80	40
	4.8	12.5	6	
	550	600		400
	350		200	300

4. To measure indirectly the distance from an accessible point *A* to an inaccessible point *B* by means of § 253.



SUGGESTION. Through *C*, a point where *A* and *B* are both visible, draw *CK* making $\angle 1 = \angle 2$. Produce *KC* to a point *D* on the line *BA* extended. What lines must now be measured in order to compute *AB*?

5. What methods have been used so far for the indirect measurement of the distance from an accessible to an inaccessible point? Compare these

(*a*) As to the simplicity of the theory involved.

(*b*) As to the simplicity and ease of the direct measurements required.

6. Divide a given line-segment in a given ratio without constructing a line parallel to another.

7. Similarly divide a given line-segment externally in a given ratio.

8. Solve Ex. 3, § 249, if x is on Cy extended.

9. Solve Ex. 4, § 249, if x is on Oy extended.

SIMILAR POLYGONS.

255. Two polygons, in which the angles of the one are equal respectively to the angles of the other, taken in order, are said to be **mutually equiangular**.

The angles of the two polygons are thus arranged in pairs of equal angles, which are called **corresponding angles**.

Two sides, one of each polygon, included between corresponding angles, are called **corresponding sides**.

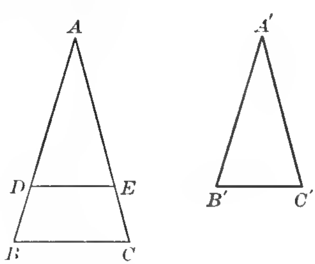
256. Two polygons are **similar** if (1) they are mutually **equiangular** and if (2) their pairs of corresponding sides are **proportional**.

Two polygons may have property (1) but not (2). For example, a rectangle and a square. Or they may have property (2) and not (1). For example, a square and a rhombus.

Hence any proof that two polygons are similar must show that both (1) and (2) hold concerning them.

In the case of triangles it will be proved that either property specified in the definition of similar polygons is sufficient to make them similar.

257. THEOREM. *If two triangles are mutually equiangular, they are similar.*



Given $\triangle ABC$ **and** $A'B'C'$, **in which** $\angle A = \angle A'$, $\angle B = \angle B'$, **and** $\angle C = \angle C'$.

To prove that the other property of similarity holds, namely that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$.

Proof: Place $\triangle A'B'C'$ on $\triangle ABC$ with $\angle A'$ upon its equal $\angle A$, and $B'C'$ taking the position DE.

Now show that $DE \parallel BC$ and hence $\frac{AB}{AD} = \frac{AC}{AE}$,

that is, $$\frac{AB}{A'B'} = \frac{AC}{A'C'}.$$

In like manner, placing $\angle B'$ upon $\angle B$,

show that $$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

Hence, $$\frac{AB}{A'B} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$ (Why?)

Give the full details of this proof.

258. **EXERCISES.**

1. To measure indirectly the distance from an accessible point A to an inaccessible point B.

SUGGESTION. Construct $AD \perp$ the line of sight from A to B, and $ED \perp AD$. Let C be the point on AD which lies in line with E and B.

Now show that $\triangle EDC$ and BAC are mutually equiangular and hence similar. What segments need to be measured in order to compute AB? Give full details of proof.

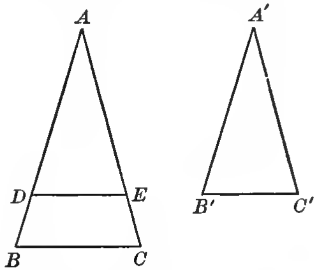
2. Prove that two right triangles are similar if they have an acute angle of one equal to an acute angle of the other.

3. Two isosceles triangles are similar if they have the vertical angle of one equal to the vertical angle of the other.

4. Two triangles which have the sides of one respectively parallel or perpendicular to the sides of the other are similar.

5. Show by similar triangles that the segment joining the mid-points of two sides of a triangle is equal to one half the third side.

259. THEOREM. *If two triangles have an angle of one equal to an angle of the other and the pairs of adjacent sides in the same ratio, the triangles are similar.*



Given $\triangle$ ABC **and** $A'B'C'$ **in which** $\angle A = \angle A'$ **and** $\frac{AB}{A'B'} = \frac{AC}{A'C'}$.

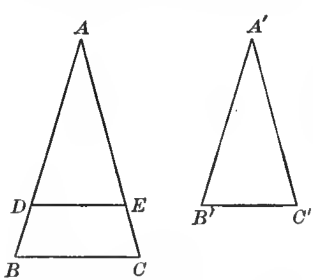
To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof: Place $\triangle A'B'C'$ upon $\triangle ABC$ with $\angle A'$ on $\angle A$, $B'C'$ taking the position DE.

Then, $\frac{AB}{AD} = \frac{AC}{AE}$ (Why?)

and hence, $DE \parallel BC$. (Why?)

Now show that $\triangle$ ADE and ABC are mutually equiangular and hence similar.



260. THEOREM. *If two triangles have their pairs of corresponding sides in the same ratio, they are similar.*

Given $\triangle$ ABC **and** $A'B'C'$ **in which** $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$.

To prove that $\triangle ABC \sim \triangle A'B'C'$, that is, to prove $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

Proof: Lay off on AB and AC respectively $AD = A'B'$ and $AE = A'C'$, and draw DE.

Now prove, as in § 259, that $\triangle ADE \sim \triangle ABC$,

and hence, that $$\frac{AD}{DE} = \frac{AB}{BC}. \qquad (1)$$

But, $$\frac{A'B'}{B'C'} = \frac{AB}{BC}. \quad \text{(Why?)} \qquad (2)$$

Hence, since $AD = A'B'$, it follows from (1) and (2) that $DE = B'C'$, as in Ex. 5, § 247.

Now show that $\triangle A'B'C' \cong \triangle ADE$,

and hence, that $\triangle A'B'C' \sim \triangle ABC$.

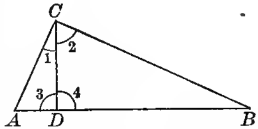
Make an outline of the steps in this proof and show how each is needed for the one that follows.

261. **EXERCISES.**

1. Given a triangle whose sides are 2, 3, 4. Construct a triangle having its angles equal respectively to those of the given triangle and having a side 10 corresponding to the given side 2.

2. If each of two triangles is similar to a third triangle, they are similar to each other.

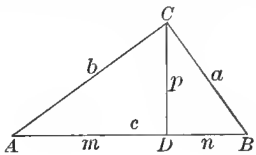
3. If in a right triangle a perpendicular be drawn from the vertex of the right angle to the hypotenuse, show that each of the triangles thus formed is similar to the given triangle, and hence that they are similar to each other.



4. In the figure of Ex. 3, make a table showing which angles are equal and which pairs of sides are corresponding in the following pairs of triangles: *ACD* and *ACB*, *CDB* and *ACB*, *ACD* and *CDB*.

5. On a given segment as a side show how to construct a triangle similar to a given equilateral triangle.

262. THEOREM. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.*



Given $\triangle ABC$ **with a right angle at** C. **Call the lengths of the sides opposite** $\angle s\ A, B, C$, **respectively,** a, b, c.

To prove that $c^2 = a^2 + b^2$.

Proof: Let the perpendicular p divide the hypotenuse into the two parts m and n so that $c = m + n$.

From $\triangle s\ ACD$ and ACB show that $\frac{m}{b} = \frac{b}{c}$. (1)

From $\triangle s\ CDB$ and ACB show that $\frac{n}{a} = \frac{a}{c}$. (2)

From (1) $mc = b^2$ (3)

and from (2) $nc = a^2$ (Why?). (4)

From (3) and (4) $(m + n)c = a^2 + b^2$ (Why?).

That is, $c \cdot c = c^2 = a^2 + b^2$.

For another proof of this theorem see § 319.

HISTORICAL NOTE. The proof given above is supposed to be that given by Pythagoras, who first discovered the theorem.

263. EXERCISES.

1. The radius of a circle is 8. What is the distance from the center to a chord whose length is 6?

2. In the same circle, what is the length of a chord whose distance from the center is 5?

3. Find the diagonal of a square whose side is 5; whose side is a.

4. What is the side of a square whose diagonal is 8? whose diagonal is d?

5. The hypotenuse of a right isosceles triangle is 12 inches. Find the lengths of its sides.

6. The diagonals of a rhombus are 14 and 10 inches respectively. Find the length of its sides. (See § 147, Ex. 4.)

7. The square on the hypotenuse of a right triangle is equal to four times the square on the median to the hypotenuse.

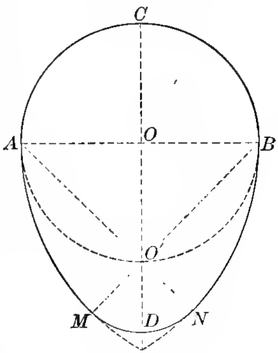
8. What is the radius of a circle if a chord 12 inches long is 9 inches from the centre?

9. Find the altitude of an equilateral triangle whose side is 8; whose side is a.

10. If the altitude of an equilateral triangle is h, find its side. (Use the formula obtained under Ex. 9.)

11. Find the altitude of a triangle whose sides are 6, 8, and 10.

12. The oval in the figure is a design used in the construction of sewers. It is constructed as follows: In the $\odot OA$ let CD, the perpendicular bisector of AB, meet the arc $AO'B$ at O'. Arcs AM and BN are drawn with the same radius AB and with centers B and A respectively.



The lines BO' and AO' meet these arcs in M and N respectively.

The arc MDN has the center O' and radius $O'M$.

(*a*) Is arc ACB tangent to $\widehat{AM}$ and $\widehat{BN}$ at A and B respectively? Why?

(*b*) Is arc MDN tangent to $\widehat{AM}$ and $\widehat{BN}$ at M and N respectively? Why?

(*c*) If $AB = 8$ feet, find BO', and hence, $O'M$, and finally CD. That is, if the sewer is 8 feet wide, what is its depth?

(*d*) If the width of the sewer is a feet, show that its depth is $\frac{a}{2}(4 - \sqrt{2})$.

(*e*) If the depth of the sewer is d feet, show that its width is $\frac{d(4 + \sqrt{2})}{7}$.

(*f*) Compute to two places of decimals the width of a sewer whose depth is 12 feet.

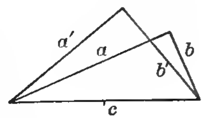
264. THEOREM. *If in two right triangles the hypotenuse of the one equals the hypotenuse of the other, and if the sides a, b and a', b' are such that $a > a'$, then $b < b'$.*

Proof: Let c be the length of the hypotenuse in each.

Then $a^2 + b^2 = c^2$ and $a'^2 + b'^2 = c^2$. (Why?)

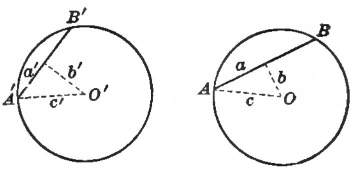
$\therefore a^2 + b^2 = a'^2 + b'^2$. (Why?)

and $a^2 - a'^2 = b'^2 - b^2$. (Why?)



Since $a > a'$, the left member of the last equation is *positive*, and hence the right member is also positive, that is, $b < b'$.

265. THEOREM. *In the same circle or in equal circles, of two unequal chords, the greater is nearer the center.*



Given ⓢ OA and $O'A'$ in which $OA = O'A'$ and $AB > A'B'$.

To prove that AB is nearer the center than $A'B'$.

Proof: Draw the ⊥s b and b' and the radii c and c'.

Then a and a' are halves of AB and $A'B'$ respectively.

Now complete the proof, using § 264.

266. THEOREM. *State and prove the converse of the theorem in § 265, using the same figure.*

267. Definitions. A **continued proportion** is a series of equal ratios connected by signs of equality.

$$E.g.\quad \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$$

The **perimeter** of a polygon is the sum of its sides.

268. THEOREM. *In a continued proportion, the sum of the antecedents and the sum of the consequents form a ratio equal to any one of the given ratios.*

Given the continued proportion $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$

To prove that $\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}.$

Proof: Let $\frac{a}{b}=r.$

Then, $\frac{c}{d}=r,\ \frac{e}{f}=r,\ \frac{g}{h}=r.$

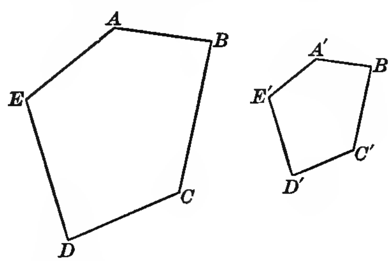
Hence, $a=br,\ c=dr,\ e=fr,\ g=hr,$

and $a+c+e+g=br+dr+fr+hr=(b+d+f+h)r,$

or $$\frac{a+c+e+g}{b+d+f+h}=r=\frac{a}{b}.$$

Give all reasons in full.

269. THEOREM. *The perimeters of two similar polygons are in the same ratio as any two corresponding sides.*

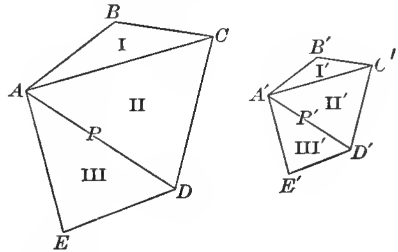


Proof: By definition of similar polygons

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\frac{DE}{D'E'}=\frac{EA}{E'A'}.$$

Complete the proof.

270. THEOREM. *If the diagonals drawn from one vertex in each of two polygons divide them into the same number of triangles, similar each to each and similarly placed, then the two polygons are similar.*



Given the diagonals drawn from the vertices A and A' in the polygons P and P', forming the same number of triangles in each, such that $\triangle I \sim \triangle I'$, $\triangle II \sim \triangle II'$, $\triangle III \sim \triangle III'$.

To prove that $P \sim P'$.

Outline of Proof : (1) Use the hypothesis to show that

$$\angle A = \angle A',\quad \angle B = \angle B',\quad \angle C = \angle C', \text{ etc.}$$

(2) Show that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.

Notice that the proportion $\frac{BC}{B'C'} = \frac{CD}{C'D'}$ follows from

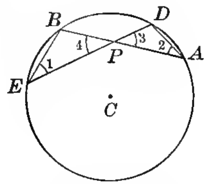
$$\frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'}. \qquad \text{(Why ?)}$$

Give the proof in detail.

271. THEOREM. *State the converse of the preceding theorem, and give the proof in full detail.*

Make an *outline of all the steps* in the proofs of these two theorems.

272. THEOREM. *If through a fixed point within a circle any number of chords are drawn, the product of the segments of one chord is equal to the product of the segments of any other.*



Given $\odot C$ with any two chords AB and DE intersecting in P.

To prove that $AP \cdot PB = EP \cdot PD.$

Proof: Draw EB and DA.

Then, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4.$ (Why?)

Hence, $\triangle EPB \sim \triangle PDA.$ (Why?)

Which are corresponding angles, and which are corresponding sides?

Show that $\frac{BP}{EP} = \frac{PD}{PA}.$

Complete the proof.

It follows from this theorem that if a chord AB is made to swing around the fixed point P, the product $AP \cdot PB$ does not change, that is, it is *constant.*

273. EXERCISES.

1. Which chord through a point is bisected by the diameter through that point? Why?

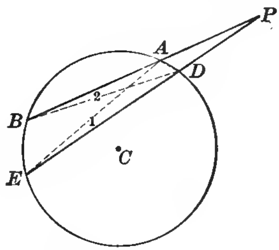
2. Through a given point within a circle which chord is the shortest? Why?

3. The product $AP \cdot PB$ is the area of the rectangle whose base and altitude are the segments of AB. (See § 307.)

Note that this area is *constant* as the chord swings about the point P as a pivot.

274. Definition. If a secant of a circle is drawn from a point P without it, meeting the circle in the points A and B, then PB is called the **whole secant** and PA the **external segment**, provided A lies between B and P.

275. THEOREM. *If from a fixed point outside a circle any number of secants are drawn, the product of one whole secant and its external segment is the same as that of any whole secant and its external segment.*



Given secants PB and PE drawn from a point P.

To prove that $PA \cdot PB = PD \cdot PE$.

Proof: In the figure show that $\triangle PDB \sim \triangle PAE$. Complete the proof.

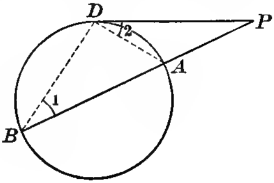
276. EXERCISES.

1. A point P is 8 inches from the center of a circle whose radius is 4. Any secant is drawn from P, cutting the circle. Find the product of the whole secant and its external segment.

2. From the same point without a circle two secants are drawn. If one whole secant and its external segment are 14 and 5 respectively and the other external segment is 7, find the other whole secant.

3. Two chords intersect within a circle. The segments of one are m and n and one segment of the other is p. Find the remaining segment.

277. THEOREM. *If a tangent and a secant meet outside a circle, the square on the tangent is equal to the product of the whole secant and its external segment.*



Proof: Show that $\triangle APD \sim \triangle BPD$ and hence that $PB : PD = PD : PA$.

Complete the proof.

278. EXERCISES.

1. If a square is constructed on PD as a side, and a rectangle with PB as base and PA as altitude, compare their areas as the secant revolves about P as a pivot.

2. Show that the theorem in § 277 may be obtained as a direct consequence of that in § 275 by supposing one secant to swing about P as a pivot till it becomes a tangent.

3. A point P is 10 inches from the center of a circle whose radius is 6 inches. Find the length of the tangent from P to the circle.

4. The length of a tangent from P to a circle is 7 inches, and the external segment of a secant is 4 inches. Find the length of the whole secant.

5. What theorems are included in the following statement: "*From a point P in a plane a line is drawn cutting a circle in A and B. Then the product $PA \cdot PB$ is the same for all such lines*"?

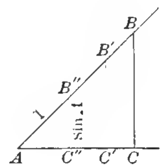
6. In a circle of radius 10 a point P divides a chord into two segments 4 and 6. How far from the center is P?

SUGGESTION. Use Ex. 5.

7. In two similar polygons two corresponding sides are 3 and 7. If the perimeter of the first polygon is 45, what is the perimeter of the second?

8. The perimeters of two similar polygons are 32 and 84. A side of the first is 11. What is the corresponding side of the second polygon?

279. Definitions. In a right triangle ABC, right-angled at C, the ratio $\frac{CB}{AB}$ is called the **sine of $\angle A$** and is written $\sin A$.



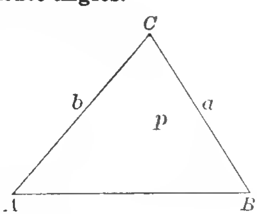
If any other point B' be taken on the hypotenuse or the hypotenuse extended, and a perpendicular $B'C'$ be let fall to AC, then $\frac{C'B'}{AB'} = \frac{CB}{AB}$. (Why?)

Likewise in $\triangle AB''C''$, in which $AB'' = 1$ unit, we have $\frac{C''B''}{AB''} = \frac{C''B''}{1} = C''B'' = \sin A$.

Hence, in a right triangle whose hypotenuse is unity, *the length of the side opposite an acute angle is the sine of that angle.*

280. THEOREM. *The ratio of the sides opposite two acute angles of a triangle is equal to the ratio of the sines of these angles.*

Given $\triangle ABC$ **with** $\angle A$ **and** $\angle B$ **both acute angles.**



To prove that $\frac{\sin A}{\sin B} = \frac{a}{b}$.

Proof: Draw the perpendicular p.

Then $\sin A = \frac{p}{b}$ and $\sin B = \frac{p}{a}$.

Hence, $\frac{\sin A}{\sin B} = \frac{p}{b} \div \frac{p}{a} = \frac{a}{b}$. See § 241, (3).

NOTE. The definitions of § 279 and the theorem of § 280 are given here for acute angles only. In trigonometry, where the subject is studied in full detail, they are extended to apply to any angles whatever. Other ratios called cosines, tangents, etc., are also introduced.

281. The theorem of § 280 is of great importance in finding certain parts of a triangle when other parts are known. By careful measurement (and in other ways) tables may be constructed giving the sine of any angle.

282. EXERCISES.

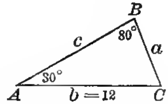
1. By means of a protractor construct angles of 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, and *measure* the sine of each angle, and so construct a table of these sines.

If one decimeter is used as a unit for the hypotenuse, then the length of the side opposite $\angle A$, expressed in terms of decimeters, is the sine of the angle A.

Notice that the values of the sines are the same no matter what unit is used, but in general the larger the unit the more accurately is the sine determined.

By means of the table just constructed solve the following problems, using the notation of the figure:

2. Given $\angle A = 30°$, $B = 80°$, $b = 12$, find a, c.

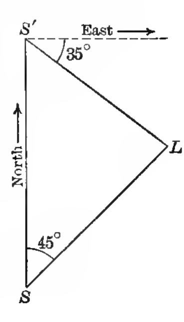


SOLUTION. By the theorem, § 280, $\frac{a}{b} = \frac{\sin A}{\sin B}$.

Substituting the values $b = 12$, and $\sin A = \sin 30°$, $\sin B = \sin 80°$ from the table, we find a. In the same manner find c.

3. Given $a = 16$, $\angle A = 60°$, $\angle C = 70°$, find $\angle B$, b, c.

4. A lighthouse L is observed from a ship S to be due northeast. After sailing north 9 miles to S', the lighthouse is observed to be 35° south of east. Find the distance from the ship to the lighthouse at each point of observation. Use § 280.

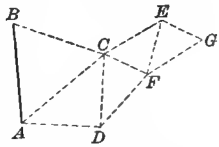


5. A ladder 25 feet long rests with one end on the ground at a point 12 feet from a wall. At what angle does the ladder meet the ground.

6. If two sides and the included angle of a triangle are known, can the remaining parts be found by means of § 280?

283. In land surveying on an extensive scale, processes similar to that used on the preceding page are constantly employed in finding the sides of triangles.

To begin with, a level piece of ground is selected and a line AB measured with great care. Then a point C is selected, and $\angle ABC$ and $\angle BAC$ measured very accurately with an instrument. Sides AC and BC may now be computed by means of the theorem, § 280, and a table of sines (see page opposite). By measuring $\angle DAC$ and $\angle ACD$, CD may be computed. By this process, called triangulating, it is possible to survey over a large territory without *directly measuring* any line except the first.



284. The saving of labor afforded by this *indirect method* of measuring is very great, and especially so in a rough and mountainous country, since measuring the straight line distance from one mountain peak to another by means of a measuring chain is *impossible.*

In practice, tables of logarithms are used and the sines are carried out to a larger number of decimal places, but the general process is that used on the preceding page.

285. EXERCISES.

Using the table on the next page, solve the following examples:

1. Given $A = 53°$, $B = 65°$, $a = 11.5$. Find b, c, and $\angle C$.

2. Given $B = 49°$, $C = 71°$, $a = 19.3$. Find b, c, and $\angle A$.

3. Given $A = 65°$, $a = 14$, $b = 12$. Find $\angle B$, $\angle C$, and c.

SOLUTION. $\frac{\sin B}{\sin A} = \frac{b}{a}$ or $\sin B = \sin A \times \frac{b}{a} = .91 \times \frac{12}{14} = .78.$

From the table we find that $\sin 51° = .78$. Hence $B = 51°$. $\angle C$ and c may now be found as before.

4. Given $A = 71°$, $a = 19.5$, $b = 17$. Find $\angle B$, $\angle C$, and c. As before, $\sin B = \sin A \times \frac{b}{a} = \sin 71° \times \frac{17}{19.5} = .95 \times \frac{17}{19.5} = .828$. From the table we find $\sin 55° = .82$ and $\sin 56° = .83$. But $\sin B$ is nearer .83 than .82, and hence $B = 56°$ is the nearest approximation using a degree as the smallest unit.

Angle	Sin	Angle	Sin	Angle	Sin	Angle	Sin	Angle	Sin	Angle	Sin
0°	0	15°	.26	30°	.50	45°	.71	60°	.87	75°	.97
1°	.02	16°	.28	31°	.52	46°	.72	61°	.87	76°	.97
2°	.03	17°	.29	32°	.53	47°	.73	62°	.88	77°	.97
3°	.05	18°	.31	33°	.54	48°	.74	63°	.89	78°	.98
4°	.07	19°	.33	34°	.56	49°	.75	64°	.90	79°	.98
5°	.09	20°	.34	35°	.57	50°	.77	65°	.91	80°	.98
6°	.10	21°	.36	36°	.59	51°	.78	66°	.91	81°	.99
7°	.12	22°	.37	37°	.60	52°	.79	67°	.92	82°	.99
8°	.14	23°	.39	38°	.62	53°	.80	68°	.93	83°	.99
9°	.16	24°	.41	39°	.63	54°	.81	69°	.93	84°	.99
10°	.17	25°	.42	40°	.64	55°	.82	70°	.94	85°	1.
11°	.19	26°	.44	41°	.66	56°	.83	71°	.95	86°	1.
12°	.21	27°	.45	42°	.67	57°	.84	72°	.95	87°	1.
13°	.22	28°	.47	43°	.68	58°	.85	73°	.96	88°	1.
14°	.24	29°	.48	44°	.69	59°	.86	74°	.96	89°	1.

286. Definition. An object *AB* is said to **subtend an angle** ***APB*** from a point *P* if the lines *PA* and *PB* are supposed to be drawn from *P* to the extremities of the object.

287. EXERCISES.

1. A building known to be 150 feet high is seen to subtend an angle of 20°. How far is the observer from the building if he is standing on a level with its base?

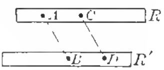
2. Show how to find the distance from an accessible to an inaccessible point by means of § 280 and the table of sines.

3. A flagstaff is 125 feet tall. How far from it must one be in order that the flagstaff shall subtend an angle of 25°?

PROBLEMS OF CONSTRUCTION.

288. Instruments. In addition to the *ruler*, *compasses*, and *protractor* described in § 44, the **parallel ruler** is convenient for drawing lines through given points parallel to given lines without each time making the construction of § 101.

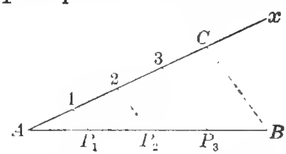
DESCRIPTION. R and R' are two rulers of equal length and width. AB and CD are arms of equal length pivoted at the points A, B, C, and D, making $AC = BD$.



Why do the rulers, when thus constructed, remain parallel as they are spread?

289. Segments of equal length can be laid off with great accuracy on a line-segment by means of the compasses. The following construction makes use of the compasses and parallel ruler.

290. PROBLEM. *To divide a given line-segment into any number of equal parts.*



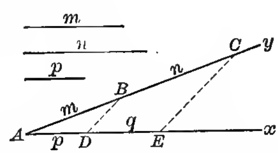
Construction. Proceed as in § 159, Ex. 5, using the parallel ruler to draw the parallel lines.

Give the construction and proof in full.

HISTORICAL NOTE. The idea of similarity of geometric figures, or "sameness of shape," is one of early origin, as is also the simple theory of proportion. It was probably used by Pythagoras to prove the famous theorem known by his name. (See § 262.) But the discovery by him of the incommensurable case (§ 236) showed that this theory was inadequate for the rigorous proof of all theorems on similar figures. It remained for Eudoxus, the teacher of Plato, to perfect a rigorous theory of ratio and proportion.

Euclid, following his predecessors, deals with ratios of *magnitudes* in general as well as of *numbers*. Later writers have frequently insisted that ratios in general are not numbers. But nothing is gained by this procedure, since they possess all the properties of numbers. In this book a ratio is treated simply as the quotient of two numbers. See §§ 238–242.

291. PROBLEM. *Given three line-segments m, n, p, to construct a fourth segment q such that $\frac{m}{n}=\frac{p}{q}$; that is, to find a fourth proportional to m, n, and p.*



Construction. Draw two indefinite straight lines, Ax and Ay making a convenient angle.

On Ay lay off $AB = m$, $BC = n$. On Ax lay off $AD = p$. Draw BD. Through C draw $CE \parallel BD$. Then $DE = q$ is the required segment.

Give the proof in detail.

292. EXERCISES.

1. Apply the method of § 290 to bisect a given line-segment, and compare this with the method of § 51.

2. Divide a line $7\frac{3}{4}$ inches long into 11 equal parts by the method of § 290, and compare with the process of *measuring* by means of an ordinary ruler giving inches and sixteenths.

3. Using the same segments as in § 291, construct a segment q such that $\frac{m}{p}=\frac{n}{q}$; also such that $\frac{n}{p}=\frac{m}{q}$.

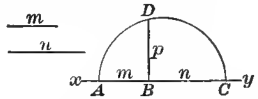
4. Using the same segments as in the preceding, construct a segment q such that $\frac{p}{m}=\frac{n}{q}$; also such that $\frac{p}{n}=\frac{m}{q}$.

What is q called in each case?

5. If the given segments m, n, p are respectively $3\frac{3}{8}$ inches, $5\frac{5}{16}$ inches, $4\frac{5}{8}$ inches, compute q such that $\frac{m}{n}=\frac{p}{q}$. Also construct q as in § 291 and compare results.

293. Definitions. If three numbers a, m, and b are such that $a : m = m : b$, then m is the **mean proportional** between a and b, and b is the **third proportional** to a and m.

294. PROBLEM. *To construct a mean proportional between two given segments m and n.*



Construction. On an indefinite line xy lay off $AB = m$ and $BC = n$. On AC as a diameter construct a semicircle.

At B erect a perpendicular to AC meeting the semicircle in D. Then $BD = p$ is the required segment.

Proof. Draw AD and CD and prove

$$\triangle ABD \sim \triangle CBD.$$

Complete the proof.

295. EXERCISES.

1. Show that in § 277 DP is a mean proportional between PB and PA.

2. If in the above problem $m = 3$, and $n = 5$ show that the construction gives $\sqrt{15}$.

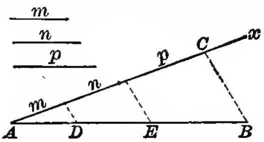
3. Show how to construct a segment $p = \sqrt{5}$, also $p = \sqrt{3}$, $p = \sqrt{2}$, by means of the above process.

4. Show how to construct a square equal in area to a given rectangle.

5. Show how to construct on a given base a rectangle equal in area to a given square.

Suppose $AB = m$ is the given base and $BD = p$ a side of the given square, the two segments being placed at right angles as in the figure above. The problem is then to find on the line AB the center of a circle which passes through the points A and D. To do this connect A and D and construct a perpendicular bisector of this segment meeting AC in a point which is the center of the required circle. $BC = n$ is then the required side of the rectangle since $m \cdot n = p^2$.

296. PROBLEM. *To divide a line-segment into three parts proportional to three given segments m, n, p.*



Construction. Let AB be the given segment. Construct Ax and on it lay off m, n, p as shown in the figure. Complete the figure and give proof in full.

297. EXERCISES.

1. Divide a line-segment 5 inches long into three parts proportional to 2, 3, and 4.

2. Divide a segment 11 inches long into parts proportional to 3, 5, 7, and 9.

First *compute* the lengths of the required segments, then construct them and *measure* the segments obtained. Compare the results. Which method is more convenient? Which is more accurate?

3. Divide a segment 9 inches long into two parts proportional to 1 and $\sqrt{2}$.

Also *compute* the required segments. Which is more convenient? More accurate?

4. Divide a segment whose length is $\sqrt{11}$ into two parts proportional to $\sqrt{2}$ and $\sqrt{5}$.

First construct the segments whose lengths are $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{11}$. Also *compute* the required segments.

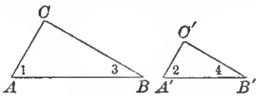
5. Divide a given line-segment into parts proportional to two given segments m and n, (*a*) if the division point falls on the segment; (*b*) if the division point falls on the segment produced. See § 252.

6. A triangle is inscribed in another by joining the middle points of the sides.

(*a*) What is the ratio of the perimeters of the original and the inscribed triangles?

(*b*) Is the inscribed triangle similar to the original?

298. PROBLEM. *On a given line-segment as a side construct a triangle similar to a given triangle.*



Construction. Let ABC be the given triangle and $A'B'$ the given segment, and let it correspond to AB of the given triangle.

Construct $\angle 2 = \angle 1$ and $\angle 4 = \angle 3$ and produce the sides till they meet in C'. Then $A'B'C'$ is the required triangle (Why?).

299. EXERCISES.

1. In the problem of § 298 construct on $A'B'$ a triangle similar to ABC such that $A'B'$ and BC are corresponding sides.

2. Show that on a given segment three different triangles may be constructed similar to a given triangle.

3. Solve the problem of § 298 by making $\angle 2 = \angle 1$ and constructing $A'C'$ so that $\frac{AB}{A'B'} = \frac{AC}{A'C'}$. Give the solution and proof in full.

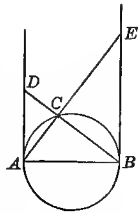
4. On a given segment as a side construct a polygon similar to a given polygon, by first dividing the given polygon into triangles and then constructing triangles in order similar to these. Apply § 270.

5. On a given segment construct a polygon similar to a given polygon in a manner analogous to the method used in Ex. 3 for constructing a triangle similar to a given triangle.

6. ABC is a right triangle with the hypotenuse AB, and $CD \perp AB$. Prove AC a mean proportional between AB and AD and likewise CB a mean proportional between AB and DB.

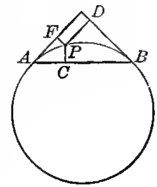
7. Use the theorem of § 277 to construct a square equal in area to that of a given rectangle.

8. The tangents to two intersecting circles from any point in their common chord produced are equal. Use § 277.



9. In the figure AD and BE are tangents at the extremities of a diameter. If BD and AE meet in a point C on the circle prove that AB is a mean proportional between AD and BE.

10. The greatest distance to a chord 8 inches long from a point on its intercepted arc (minor arc) is 2 inches. Find the diameter of the circle. Use § 272.



11. At the extremities of a chord AB tangents are drawn. From a point in the arc AB perpendiculars PC, PD, PF are drawn to the chord and the tangents. Prove that PC is a mean proportional between PD and PF.

SUGGESTION. Draw segments AP and BP and prove $\triangle APF \sim \triangle BCP$ and $\triangle APC \sim BPD$. Then

$$\frac{AP}{PB}=\frac{FP}{PC} \text{ and } \frac{AP}{PB}=\frac{PC}{PD}.$$

SUMMARY OF CHAPTER III.

1. Make a list of the definitions in Chapter III.

2. Make a list of the theorems on proportional segments involving triangles.

3. State the various conditions which make triangles similar.

4. State the conclusions which can be drawn when it is known that two triangles are similar.

5. State the theorems on proportional segments involving polygons.

6. State the theorems on proportional segments involving straight lines and circles.

7. State in the form of theorems the various ways in which a proportion may be taken so as to leave the four terms still in proportion.

8. Under what conditions is a segment a mean proportional between two given segments. For instance, see § 277.

9. Make a list of the problems of construction in Chapter III.

10. State some important applications of the theorems on proportion. (Return to this question after studying those which follow.)

PROBLEMS AND APPLICATIONS.

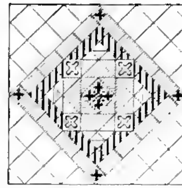
1. The middle points of adjacent sides of a square are joined.

(*a*) Prove that the inscribed figure is a square.

(*b*) What is the ratio of the inscribed and the original squares.

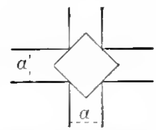
(*c*) If a side of the original square is *a*, find a side of the inscribed square.

(*d*) If a side of the inscribed square is *b*, find a side of the original square.



Tile Pattern.

2. Two strips intersect at right angles.

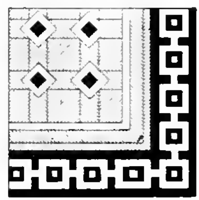


(*a*) If the width of each strip is 3 inches, what is the largest square which can be placed on them so that its sides will pass through the corner points as shown in the figure? (The corners will bisect the sides of the square.)

(*b*) What is the side of this square if the width of each strip is *a*?

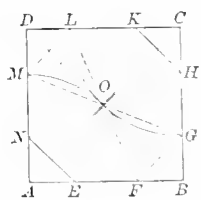
(*c*) Find the side of the square if the width of one strip is 3 and that of the other 5.

(*d*) Find the side of the square if the width of one strip is *a* and that of the other *b*.



Tile Pattern.

3. In the figure *ABCD* is a square and *EFGH* ... is a regular octagon.



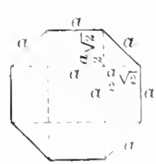
(*a*) Show that *AE*, *EF*, *FB* are proportional to 1, $\sqrt{2}$, 1. (Assume $AE = 1$ and find *NE*.)

(*b*) If $AB = a$, show that $\frac{AF}{a} = \frac{1+\sqrt{2}}{2+\sqrt{2}} = \frac{\sqrt{2}}{2}$.

(*c*) Find $\angle FEN$. See § 163, Ex. 2.

(*d*) Show that $AO = AF$, and hence that the regular octagon may be constructed as shown in the figure.

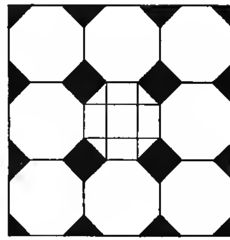
(*e*) Show that $\angle AOE = 22\frac{1}{2}°$ by § 219, and use this to make another proof that *EFGH* ... is a regular octagon.



4. Find the area of a square whose diagonal is *d*.

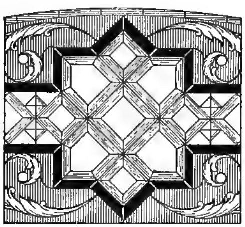
5. If a side of a regular octagon is a, find its area.

6. A floor is tiled with regular white octagons and black squares as shown in the design. What per cent (approximately) of the floor is black?

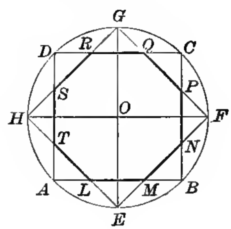


7. Divide a circle into eight equal parts and join alternate division points.

(a) Prove that $LMNPQ \cdots$ is a regular octagon.



Cut Glass Design.

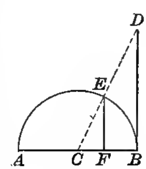


(b) If the diameter of the circle is a, show that $LE = \frac{a}{2}(\sqrt{2} - 1)$.

SUGGESTION. Find HE and then use Ex. 3, (d).

8. PROBLEM. Inscribe a square in a given semicircle.

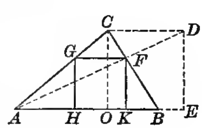
CONSTRUCTION. Let AB be the diameter of the given semicircle. At B construct a perpendicular to AB, making $BD = AB$. Connect D with the center C, meeting the circle at E. Let fall EF perpendicular to AB. Then EF is a side of the required square. Complete the figure and make the proof by showing that $EF = 2\,CF$.



9. PROBLEM. Inscribe a square in a given triangle.

CONSTRUCTION. Let ABC be the given triangle. Draw CD parallel to AB, making $CD = CO$. Complete the square $CDEO$.

Draw DA meeting CB in F. Draw $FG \parallel AB$, $GH \parallel CO$, and $FK \parallel CO$. Then $FGHK \sim CDEO$ and hence is the required square.

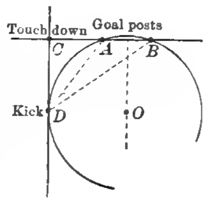


10. The sides of a triangle are 13, 17, 19. Find the lengths of the segments into which the angle bisectors divide the opposite sides.

11. The angles of a triangle are 30°, 60°, 90°. Find the lengths of the segments into which the angle bisectors divide the opposite sides if the hypotenuse is 10.

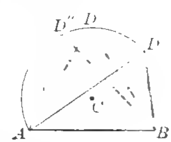
12. Prove that the perpendicular bisectors of all the sides of a polygon inscribed in a circle meet in a point.

13. An equilateral polygon inscribed in a circle is equiangular.



14. The goal posts on a football field are $18\frac{1}{2}$ feet apart. If in making a touchdown the ball crosses the goal line 25 ft. from the nearest goal post, how far back should it be carried so that the goal posts shall subtend the greatest possible angle from the place where the ball is placed? See Ex. 22, page 111, and § 277.

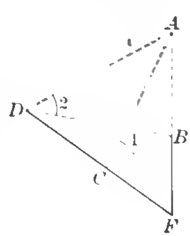
15. Solve the preceding problem if the touchdown is made *a* feet from the nearest goal post, and thus obtain a formula by means of which the distance may be computed in any case.



16. A triangle has a fixed base and a constant vertex angle. Show that the locus of the vertex consists of two arcs whose end-points are the extremities of the base. See Ex. 20, page 111.

Note that the locus also includes an arc on the opposite side of AB from the one shown in the figure.

It follows that if two points A and B subtend the same angle from P and from Q, then a circle may be passed through A, B, P, Q.



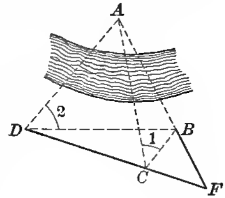
17. In the figure, A, B, F lie in the same straight line, as do also D, C, F, and $\angle 1 = \angle 2$.

(a) If $BF = 8$, $CF = 9$, and $DC = 12$, find AB.

(b) If $AB = 100$, $AF = 250$, $CF = 60$, find DC.

Note. By holding a small object, say a pencil, at arm's length, and sighting across the ends of it, we may determine approximately whether two given objects subtend the same angle.

18. Not having any instruments, an engineer proceeds as follows to obtain approximately the distance from an accessible point B to an inaccessible point A. Walking from B along the line AB he takes 50 steps to F. Then he walks in a convenient direction 50 steps to C, and notes that A and B subtend a certain angle. He then proceeds along the same straight line until he reaches a point D at which A and B again subtend the same angle as at C. He then concludes that $DC = AB$. Is this conclusion correct? Give proof.



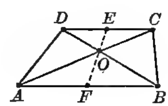
19. If the height of a building is known, show how the method of Ex. 18 can be used to determine the height of a flagstaff on it.

20. A building is 130 feet high, and a flagstaff on the top of it is 60 ft. high; 130 feet from the base of the building in a horizontal plane, the flagstaff subtends a certain angle. How far from the building along the same line is there another point at which the staff subtends the same angle? At what distance does it subtend the same angle as it does at 300 feet?

21. If the diagonals of an isosceles trapezoid are drawn, what similar triangles are produced?

22. Find the locus of points at a fixed distance from a given triangle, always measuring to the nearest point on it.

23. The line bisecting the bases of any trapezoid passes through the point of intersection of its diagonals.

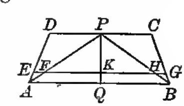


SUGGESTION. Let E bisect DC, and draw EO meeting AB in F. Prove $AF = FB$.

24. If two triangles have equal bases on one of two parallel lines, and their vertices on the other, then the sides of these triangles intercept equal segments on a line parallel to these and lying between them.

25. A segment bisecting the two bases of a trapezoid bisects every segment joining its other two sides and parallel to the bases.

(Prove $EF = HG$ and $FK = KH$, using Ex. 24.)



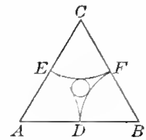
26. In a triangle lines are drawn parallel to one side, forming trapezoids. Find the locus of the intersection points of their diagonals.

27. In the figure D, E, F are the middle points of the sides of an equilateral $\triangle ABC$. Arcs are constructed with centers A, B, C as shown in the figure.

(*a*) Prove that these arcs are tangent in pairs at the points D, E, F.

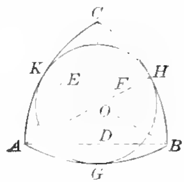
(*b*) Construct a circle tangent to the three arcs.

(*c*) What axes of symmetry has this figure?



28. In the figure ABC is an equilateral triangle. Arcs AB, BC, and CA are constructed with C, A, and B as centers. AF, BE, CD are altitudes of the triangle.

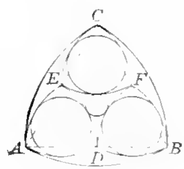
Prove $\odot OG$ tangent to $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CA}$.



29. In the equilateral triangle ABC, $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CA}$ are constructed as in the preceding example. $\widehat{DE}$, $\widehat{EF}$, $\widehat{FD}$ are constructed as in Ex. 27. The figure is completed making ADE, DFB, and EFC similar to ABC.

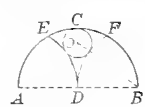
(*a*) Construct a circle tangent to $\widehat{AD}$, $\widehat{DE}$, and $\widehat{EA}$ as shown in the figure.

(*b*) Construct a circle tangent to $\widehat{ED}$, $\widehat{DF}$, and $\widehat{FE}$.

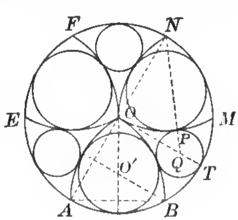


30. In the figure ACB is a semicircle. Arcs DE and DF are constructed with B and A as centers. If $AD = 4$ feet, find the radius of $\odot OC$.

Solution. Show that the value of r is derived from the equation $(4 + r)^2 = (4 - r)^2 + 4^2$.



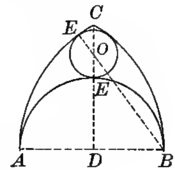
31. Construct the accompanying design. Notice that the points A, B, M, N, F, E divide the circle into six equal arcs. See Ex. 5,



From Boynton Cathedral, England.

§ 163. The arcs AO, OB, etc., and the circle with center O', are constructed as in Ex. 28. The small circle with center at Q is constructed as in the preceding example.

32. ABC is an equilateral arch and AEB is a semicircle.



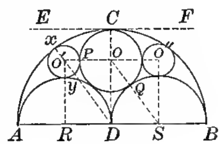
(*a*) If AB (the span of the arch) is 10 feet, find the radius of the small circle tangent to the semicircle and the arcs of the arch.

(*b*) Find the radius OE of the circle if $AB = s$.

(*c*) If $OE = 2$, find AB.

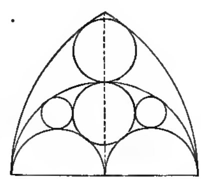
33. The accompanying design consists of three semicircles and three circles related as shown in the figure.

(*a*) If $AB = 12$, find OQ by using the triangle OSD. The method is similar to that used in Ex. 30.

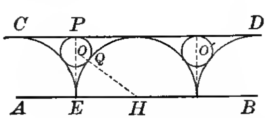


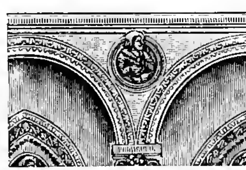
(*b*) Using the right triangle DRO', find $O'x$. Notice that in order to have the circles with the centers O and O' tangent to each other the sum of their radii must be equal to RD.

(*c*) If $AB = s$ show that $OQ = \frac{s}{6}$ and $O'x = \frac{s}{12}$, and construct the figure.



34. Upon a given segment AB construct the design shown in the figure. Notice that it consists of the two preceding figures put together. Compare the radii of the circles in this design.



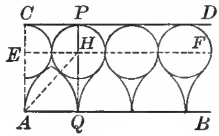


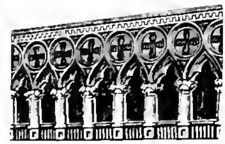
Ospedale Maggiore, Milan.

35. Between two parallel lines construct semicircles and circles as shown in the figure.

Solution. Suppose the construction made. Let $HE = r$, and $OQ = r'$. Using the triangle EHO, show that $r' = \frac{r}{4}$ and then construct the figure.

36. Between two parallel lines construct circles and equilateral arches as shown in the figure.





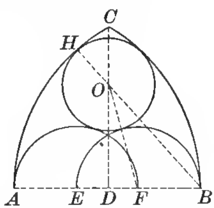
The Doge's Palace, Venice.

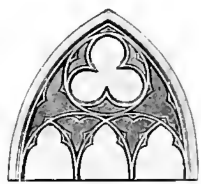
HINT. $AQ = 2\,PH$ and $AH = 3\,PH$.

Hence $\overline{HQ}^2 = \overline{3\,PH}^2 - \overline{2\,PH}^2 = 5\,\overline{PH}^2$.

That is, $PH : HQ = 1 : \sqrt{5}$. (Why ?)

Now divide AC into two segments proportional to 1 and $\sqrt{5}$, and so construct the figure.





37. ABC is an equilateral arch. $AD = DB$, $AF = EB = \frac{2}{3}AB$. Semicircles are constructed with E and F as centers, and radius FB.

(*a*) If $AB = s$, find OH and construct the figure.

SOLUTION. Let $OH = r$. Then $BO = s - r$, $FB = \frac{s}{3}$, $DF = \frac{s}{6}$, $FO = \frac{s}{3} + r$.

From $\triangle ODB$, $\overline{OD}^2 = \overline{OB}^2 - \overline{DB}^2 = (s - r)^2 - \left(\frac{s}{2}\right)^2$. (1)

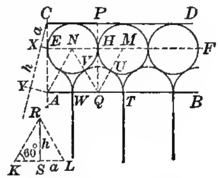
From $\triangle ODF$, $\overline{OD}^2 = \overline{OF}^2 - \overline{DF}^2 = \left(\frac{s}{3} + r\right)^2 - \left(\frac{s}{6}\right)^2$. (2)

From (1) and (2) $(s - r)^2 - \left(\frac{s}{2}\right)^2 = \left(\frac{s}{3} + r\right)^2 - \left(\frac{s}{6}\right)^2$.

Hence, show that $r = \frac{1}{4}s$.

Construct the figure.

(*b*) If in the preceding $OH = 4$ feet, find AB and AF.



38. Between two parallel lines construct circles and semicircles having equal radii as shown in the figure.

Prove that the ratio $\frac{PH}{HQ} = \frac{SL}{RS}$, RLK being an equilateral triangle and $KS = SL$. Divide CA in the ratio of $SL : RS$.

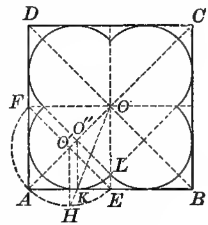
39. Through a fixed point on a circle chords are drawn and each extended to twice its length. Find the locus of the end-points of these segments. Compare Ex. 6, § 214.

40. If a quadrilateral is circumscribed about a circle, show that the sums of its pairs of opposite sides are equal.

41. On a diameter produced of a given circle, find a point from which the tangents to the circle are of a given length. Solve this problem by construction, and also algebraically.

42. Compare the perimeters of equilateral triangles circumscribed about and inscribed in the same circle.

43. In a given square construct semicircles each tangent to two sides of the square and terminating on the diameters of the square.



CONSTRUCTION. Connect E and F two extremities of diameters and on EF as a diameter construct a semicircle with center at O'. Draw $O'H$ perpendicular to AB meeting the arc in H. Draw OH meeting AB in K. Draw KO'' perpendicular to AB meeting the diagonal AC in O''. Then O'' is the center of the required circle.

PROOF. Draw $O''L \parallel FE$. We need to prove that $O''K = O''L$.

$$\triangle OO'E \sim \triangle OO''L, \text{ and hence } \frac{OO''}{OO'} = \frac{O''L}{O'E}. \quad \text{(Why?)} \quad (1)$$

Also $\triangle OO'H \sim \triangle OO''K$, and hence $\frac{OO''}{OO'} = \frac{O''K}{O'H}$. (2)

From (1) and (2) $\frac{O''L}{O'E} = \frac{O''K}{O'H}$.

But $O'E = O'H$, and hence $O''L = O''K$. (Why?)

This figure occurs in designs for steel ceilings.

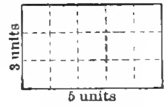
CHAPTER IV.

AREAS OF POLYGONS.

AREAS OF RECTANGLES.

300. Heretofore certain properties of plane figures have been studied, such as *congruence* and *similarity*, but no attempt has been made to measure the extent of surface inclosed by such figures. For this purpose we first consider the rectangle.

301. The surface inclosed by a rectangle is said to be **exactly measured** when we find how many times some unit square is contained in it.



E.g. if the base of a rectangle is five units long and its altitude three units, its surface contains a square one unit on a side fifteen times.

302. The number of times which a unit square is contained in the surface of a rectangle is called the **numerical measure** of the surface, or its **area.**

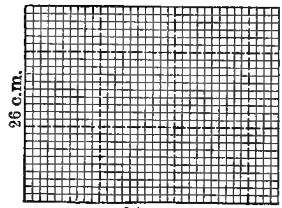
We distinguish three cases.

303. Case 1. If the sides of the given rectangle are *integral multiples* of the sides of the unit square, then the area of the rectangle is determined by finding into how many unit squares it can be divided.

Thus if the sides of a rectangle are m and n units respectively, then it can be divided into m rows of unit squares, each row containing n squares. Hence the area of such a rectangle is $m \times n$ unit squares, that is, in this case,

$$\textbf{area} = \textbf{base} \times \textbf{altitude}. \qquad (1)$$

304. Case 2. If the sides of the rectangle are *not integral multiples* of the side of the chosen unit square, but if the side of this square can be divided into equal parts such that the sides of the rectangle are integral multiples of one of these parts, then the area of the rectangle may be expressed *integrally* in terms of this smaller unit square, and *fractionally* in terms of the original unit.



For example, if the base is 3.4 decimeters and the altitude 2.6 decimeters, then the rectangle cannot be exactly divided into square decimeters, but it can be exactly divided into square centimeters. Each row contains 34 centimeters and there are 26 such rows.

Hence, the area is $34 \times 26 = 884$ small squares or 8.84 square decimeters.

But $3.4 \times 2.6 = 8.84$. Hence, in this case also,

$$\text{area} = \text{base} \times \text{altitude}. \tag{2}$$

305. Case 3. If a rectangle is such that there exists **no common measure whatever** of its base and altitude, then there is no surface unit in terms of which its area can be *exactly* expressed. But by choosing a unit sufficiently small we may determine the area of a rectangle which differs as little as we please from the given rectangle.

E.g. if the base is 5 inches and the altitude is $\sqrt{5}$ inches, then the rectangle cannot be exactly divided into equal squares, however small.

But since $\sqrt{5} = 2.2361\cdots$, if we take as a unit of area a square whose side is one one-thousandth of an inch, then the rectangle whose base is 5 inches and whose altitude is 2.236 inches can be exactly measured as in cases 1 and 2, and its area is $5 \times 2.236 = 11.18$ square inches.

The small strip by which this rectangle differs from the given rectangle is less than .0002 of an inch in width, and its area is less than $5 \times .0002 = .001$ of a square inch.

By expressing $\sqrt{5}$ to further places of decimals and thus using smaller units of area, successive rectangles may be found which differ less and less from the given rectangle.

306. An area thus obtained is called an **approximate area of the rectangle.**

For *practical purposes* the surface of a rectangle is measured as soon as the width of the remaining strip is less than the width of the smallest unit square available.

From the foregoing considerations we are led to the following **preliminary theorem**:

307. THEOREM. *The area of a rectangle is equal to the product of its base and altitude.*

308. The argument used above shows that the theorem (§ 307) holds for all rectangles used in the process of approximation, and hence it applies to all *practical measurements* of the areas of rectangles.

AREAS OF POLYGONS.

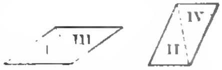
309. From the formula for rectangles,

area = base × altitude,

we deduce the areas of other rectilinear figures by means of the principle:

Two rectilinear figures are equivalent (that is, have the same area) if they are congruent, or if they can be divided into parts which are congruent in pairs.

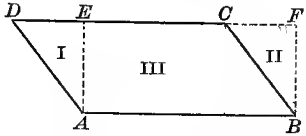
E.g. the two figures here shown are equivalent since ▵ I and III are congruent respectively to II and IV.



It can be shown that for any two given rectilinear figures, either it is possible to divide them into parts which are congruent in pairs, or one of the figures incloses a greater area than the other. Hence the test specified is sufficient for all rectilinear figures.

The symbol = joining two polygons means *equivalent* or *equal in area.*

310. THEOREM. *The area of a parallelogram is equal to the product of its base and altitude.*



Given the parallelogram $ABCD$ whose base is AB and whose altitude is AE.

To prove that area $ABCD = AB \times AE$.

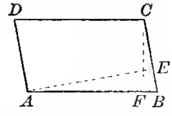
Proof: Draw $BF \perp$ to DC produced, forming the rectangle $ABFE$, whose base is AB and altitude AE.

Then area $ABFE = AB \times AE$ (Why?).

If now we prove $\triangle \text{I} \cong \triangle \text{II}$, then the parallelogram is composed of parts I and III which are congruent respectively to parts II and III of the rectangle, and hence the parallelogram and rectangle are equal in area. Give this proof in full.

311. EXERCISES.

1. In the figure CF is perpendicular to AB and AE to CB. Prove that $AB \times CF = BC \times AE$.



SUGGESTION. (Show that $\triangle ABE \sim \triangle CBF$.)

It follows that the same result is obtained if either of two adjacent sides of a parallelogram is taken as the base.

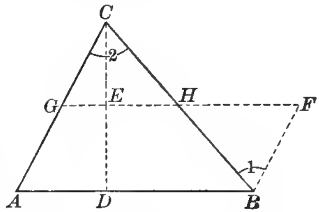
2. Construct the figure described in Ex. 1 in such manner that the point E falls on BC extended and prove the theorem in that case.

In the following l_1 and l_2 are parallel lines.

3. Two parallelograms have equal bases lying on l_1 and l_2. Show that they are equivalent.

4. One base of a parallelogram is fixed on l_1 and the other moves along l_2. Does the area change?

312. Theorem. *The area of a triangle is equal to one half the product of its base and altitude.*



Given the $\triangle ABC$ whose altitude upon the side AB is CD.

To prove that the area of $\triangle ABC = \frac{1}{2}(AB \times CD)$.

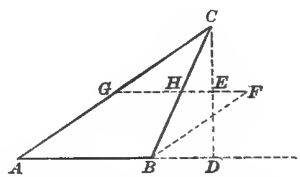
Proof: Let G be the middle point of AC. Complete the parallelogram $ABFG$.

Prove that $\triangle CGH \cong \triangle BFH$, and hence show that $\triangle ABC$ and $\square ABFG$ have equal areas.

Hence the area of $\triangle ABC = ED \times AB$. But $ED = \frac{1}{2} CD$.

Hence the required area is $\frac{1}{2}(AB \times CD)$.

313. EXERCISES.



1. Prove the theorem of § 312 using the accompanying figure.

2. Prove this theorem also by drawing a figure in which the given triangle is one half of a parallelogram.

3. If the middle points of two adjacent sides of a parallelogram are joined, a triangle is formed whose area is equal to one eighth of the area of the parallelogram.

In the following exercises l_1 and l_2 are parallel lines.

4. Show that two triangles are equivalent if their vertices lie in l_2 and their equal bases in l_1.

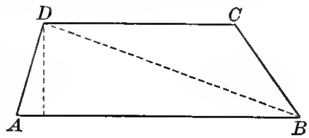
5. A triangle has a fixed base in l_2. If its vertex moves along l_1, what can you say of its area?

6. If the base of a triangle is fixed and if its vertex moves so as to preserve the area constant, what is the locus of the vertex?

7. A line is drawn from a vertex of a triangle to the middle point of the opposite side. Compare the areas of the triangles thus formed.

If a segment is drawn from the vertex to the point P in the base AB, show that the areas of the triangles are in the ratio $AP : PB$.

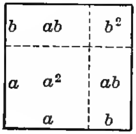
314. THEOREM. *The area of a trapezoid is equal to one half the sum of its bases multiplied by its altitude.*



Proof: Draw a diagonal, thus forming two triangles having a common altitude. Form the expression for the sum of the areas of these triangles.

Give the proof in detail.

315. THEOREM. *The square erected on the sum of two line-segments as a side is equal to the sum of the squares erected on the two segments separately, plus twice the rectangle whose base is one segment and whose altitude is the other segment.*



Proof: Let a and b represent the numerical measures of the two line-segments. Then, the square erected upon the segment $a+b$ may be subdivided as shown in the figure, giving

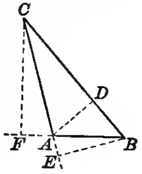
$$(a+b)^2 = a^2 + 2\,ab + b^2.$$

Give the construction and proof in full.

316. EXERCISES.

1. Show by a figure that
$(a+b+c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$

2. If ABC is any triangle and AD, BE, and CF are its altitudes, show that $\frac{AD}{BE} = \frac{AC}{BC}$, and $\frac{BE}{CF} = \frac{AB}{AC}$.

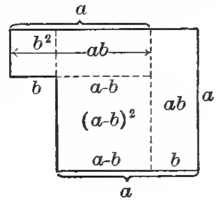


Hence show that

$$AD \times CB = BE \times CA = CF \times AB.$$

3. The side AB of ▱ $ABCD$ is fixed. What is the locus of the points C and D if CD moves so as to leave the area of the parallelogram fixed?

317. THEOREM. *The square erected on the difference of two line-segments as a side is equal to the sum of the squares erected on the two segments separately, minus twice the rectangle whose base is one of the segments and whose altitude is the other.*



Proof: Use the figure to show that

$$(a-b)^2 = a^2 + b^2 - 2\,ab.$$

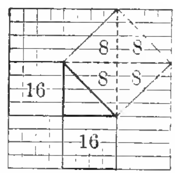
Give the construction and proof in detail.

318. **EXERCISES.**

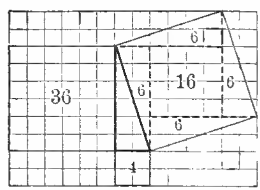
1. Show by a figure that

$$(a-b)(a-c) = a^2 + bc - ab - ac.$$

2. Likewise, show that $(a+b)(a-b) = a^2 - b^2$.

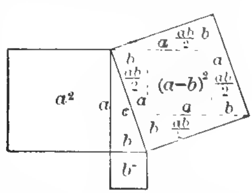


3. Show by counting squares in the accompanying figure that in case of an isosceles right triangle the square constructed on the hypotenuse is equal in area to the sum of the squares on the two legs.



4. In case the triangle is scalene, show by counting squares and congruent parts of squares that the square constructed on the hypotenuse is equal in area to the sum of the squares on the other two sides.

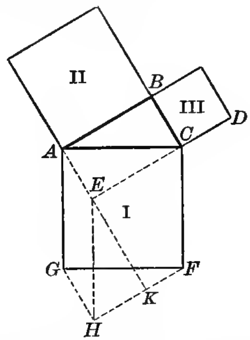
5. Using the third figure, show that the same theorem holds for any right triangle.



HINT. Call the legs of the given triangle a and b. Describe the construction of the auxiliary lines and give the proof in full.

This theorem is proved again in the next paragraph and was also proved in § 262. The proof just given is due to Bhâskara (born 1114 A.D.), who constructed the figure and simply said "Behold!"

319. THEOREM. *The area of the square described on the hypotenuse of a right triangle is equal to the sum of the areas of the squares described on the other two sides.*



Given the right triangle *ABC*, and the squares I, II, and III constructed as shown in the figure.

To prove that □ I = □ II + □ III.

Proof: Complete the rectangle *ABCE*.

Construct △ *GFH* ≅ △ *ACE*.

Draw *EH* and produce *AE* to meet the line *FH* at *K*.

Prove that (*a*) △ *AEC* ≅ △ *HEK*.

(*b*) *AEHG* is a parallelogram whose base *AE* and altitude *HK* are each equal to a side of □ III.

(*c*) *ECFH* a parallelogram whose base *EC* and altitude *EK* are each equal to a side of □ II.

But these two parallelograms together are equivalent to □ I (Why?).

The student should give the proof in detail, making an outline of the steps and showing how each step is needed for the next. For example, why is it necessary to prove △ *HEK* ≅ △ *AEC*.

Compare the various proofs of this theorem that are given in this book. The method of counting squares shown on the opposite page is applicable to all cases where the two sides are commensurable.

Historical Note. The theorem of § 319 is one of the most important in all mathematics. It is now fairly certain that the general theorem was first stated and proved by Pythagoras, though the story that he sacrificed 100 oxen to the gods on the occasion may be questioned. Special cases of the theorem were known to the Egyptians as early as 2000 B.C., *e.g.* that a triangle whose sides are 3, 4, 5 is right-angled.

In this connection the Pythagoreans also discovered the irrational number, that is, that there are numbers such as $\sqrt{2}$ which cannot be expressed exactly as integers or as ordinary fractions.

320. EXERCISES.

1. The bases of a trapezoid are 8 and 12 inches, and its altitude is 8 inches. Find its area.

2. The bases of a trapezoid are 16 and 20 inches, respectively, and the area of the smaller of its component triangles is 80 square inches. Find the area of the trapezoid. See the figure in § 152.

3. State in words the geometric theorem indicated in each of the following, and draw a figure to illustrate each case:

(*a*) $h(a+b) = ah + bh.$

(*b*) $(a+b)^2 + (a-b)^2 = 2(a^2+b^2).$

(*c*) $(a+b)^2 - (a-b)^2 = 4ab.$

4. Show that the rectangle whose base is $a+b$ and whose altitude is $a-b$ has the same perimeter as the square whose side is a. By means of Ex. 2, § 318, compare their areas.

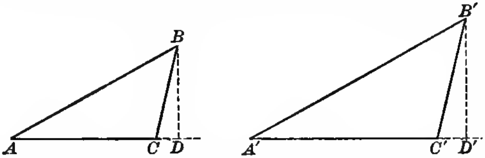
5. If two triangles have the same base but their vertices are on opposite sides of it, and if the segment joining their vertices is bisected by the common base, extended if necessary, then the two triangles are equivalent.

6. State and prove the converse of the theorem in the preceding exercise.

7. The bases of a parallelogram lie on the parallel lines l_1 and l_2. A triangle whose base is equal to that of the parallelogram has its vertex in l_1 and its base in l_2. Compare their areas.

8. Prove that all triangles having the same vertex and equal bases lying in the same straight line are equal in area.

321. THEOREM. *The areas of similar triangles are in the same ratio as the squares of any two corresponding sides, or as the squares of any two corresponding altitudes.*



Given $\triangle s$ ABC **and** $A'B'C'$, **with altitudes** BD **and** $B'D'$.

To prove that

$$\frac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{CA}^2}{\overline{C'A'}^2} = \frac{\overline{BD}^2}{\overline{B'D'}^2}.$$

Proof:
$$\frac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \frac{\frac{1}{2}(AC \times BD)}{\frac{1}{2}(A'C' \times B'D')} = \frac{AC}{A'C'} \times \frac{BD}{B'D'}.$$

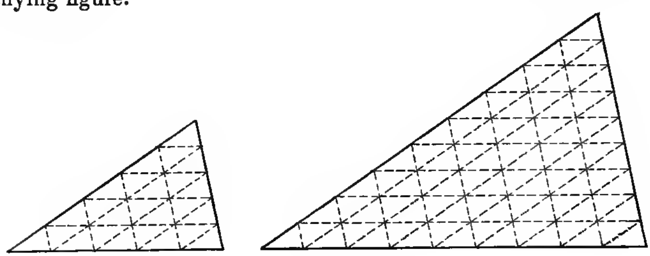
But
$$\frac{AC}{A'C'} = \frac{BD}{B'D'}.$$
Why?

Hence
$$\frac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{BD}^2}{\overline{B'D'}^2}.$$
Why?

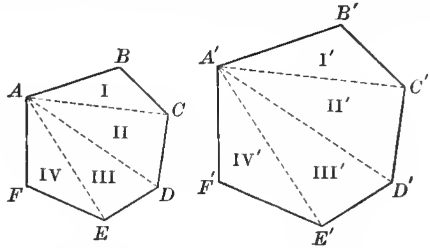
Give the proof for each of the other pairs of corresponding sides.

322. EXERCISE.

Verify the preceding theorem by counting triangles in the accompanying figure.



323. THEOREM. *The areas of two similar polygons are in the same ratio as the squares of any two corresponding sides or any two corresponding diagonals.*



Outline of Proof: Let I, II, ⋯, I′, II′, ⋯ stand for the areas of pairs of similar triangles. (§ 271.)

Show (1) $\frac{\text{I}}{\text{I}'}=\frac{\text{II}}{\text{II}'}=\frac{\text{III}}{\text{III}'}=\frac{\text{IV}}{\text{IV}'}$. (By § 321.)

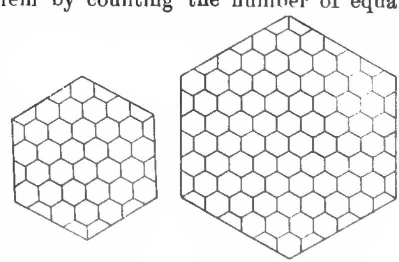
(2) $\frac{\text{I}+\text{II}+\text{III}+\text{IV}}{\text{I}'+\text{II}'+\text{III}'+\text{IV}'}=\frac{\text{I}}{\text{I}'}$. (By § 268.)

(3) $\frac{ABCDEF}{A'B'C'D'E'F'}=\frac{\text{I}}{\text{I}'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}=\frac{\overline{BC}^2}{\overline{B'C'}^2}=\cdots$

Give the proof in detail.

324. EXERCISES.

1. Verify the above theorem by counting the number of equal hexagons into which these two similar hexagons are divided, and also taking the square of the length of one side in each, using a side of a small hexagon as a unit.



2. A certain triangular field containing 2 acres is 10 rods long on one side. Find the area of a similar triangular field whose corresponding side is 50 rods.

3. The areas of two similar triangular flower beds are 24 square feet and 36 square feet respectively. If a side of one bed is 8 feet, find the corresponding side of the other.

4. If similar polygons are constructed on the three sides of a right triangle, show that the one described on the hypotenuse is equivalent to the sum of the other two.

SUGGESTIONS. Let a, b, c represent the two legs and hypotenuse of the triangle and A, B, C the areas of the corresponding polygons. Then $\frac{A}{C}=\frac{a^2}{c^2}$ and $\frac{B}{C}=\frac{b^2}{c^2}$.

Hence $\frac{A+B}{C}=\frac{a^2+b^2}{c^2}=\frac{c^2}{c^2}=1$ or $A+B=C$. Give all reasons in full.

5. Find a line-segment such that the equilateral triangle described upon it has four times the area of the equilateral triangle whose side is 3 inches long.

6. Show that the square on the altitude of an equilateral triangle is three fourths the square on a side.

7. If in a right triangle a perpendicular is let fall from the vertex of the right angle to the hypotenuse, show that the areas of the two triangles thus formed are in the same ratio as the adjacent segments of the hypotenuse, and also as the squares of the adjacent sides of the triangle.

8. Draw a line from a vertex of a triangle to a point in the opposite side which shall divide the triangle into two triangles whose ratio is 2 : 5. Also 2 : 1.

9. Divide a parallelogram into three equivalent parts by lines drawn from one vertex. Use the last construction in Ex. 8.

10. The sides of two equilateral triangles are 8 and 6 respectively. Find the side of an equilateral triangle whose area shall be equivalent to their sum. Use the result in Ex. 4.

11. State and solve a problem like the preceding for the difference of the areas.

12. Corresponding sides of two similar triangles are a and b. Find the side of a third triangle similar to these whose area is equal to the sum of their areas.

13. Likewise for any two similar polygons.

CONSTRUCTIONS.

The theorems of this chapter lead to numerous constructions of practical importance.

325. PROBLEM. *To construct a square equivalent to the sum of two or more given squares.*

Construction. In the case of two given squares, construct a right triangle whose legs are sides of the two given squares. How is the desired square obtained?

In the case of three given squares, use the square resulting from the first construction together with the third square, and so on.

Give the construction and proof in full.

326. PROBLEM. *To construct a square equivalent to a given rectangle.*

Construction. If the base and altitude of the rectangle are b and a respectively, we seek the side s of a square such that $s^2 = ab$; that is, we seek a mean proportional between a and b. See Ex. 4, § 295.

Give the construction and proof in full here.

327. PROBLEM. *To construct a square equivalent to a given triangle.*

Construction. Show how to modify the preceding construction to suit this case.

328. EXERCISES.

Show how to construct each of the following:

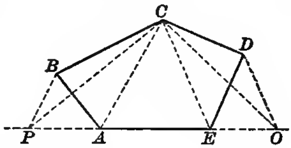
1. A square equivalent to the difference of two given squares.

2. A square equivalent to the sum of two given rectangles.

3. A square equivalent to the sum of two given triangles.

4. A square equivalent to the difference of two given rectangles, also to the difference of two given triangles.

329. Problem. *To construct a triangle equivalent to a given polygon.*



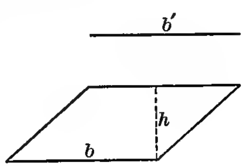
Given the polygon $ABCDE$.

To construct the triangle PCO equivalent to $ABCDE$.

Construction. Cut off $\triangle ABC$ by the segment AC.
Through B draw $BP \parallel AC$ to meet EA extended at P.
Draw CP.
In a similar manner draw CO.
Then PCO is the required triangle.

Proof: $PCDE = ABCDE$ since $\triangle APC = \triangle ABC$ (Why?).
Further, $PCDE$ has one less vertex than $ABCDE$.
But $\triangle PCO = PCDE$ since $\triangle ECD = \triangle ECO$ (Why?).
Hence, $\triangle PCO$ is equivalent to the given polygon.

330. Problem. *To construct a rectangle on a given base and equivalent to a given parallelogram.*



Construction. Let b and h be the base and altitude of the given parallelogram, b' the base of the required rectangle, and x the unknown altitude.

Then we are to determine x so that $b'x = bh$, that is, $b' : b = h : x$.

Hence, x is the fourth proportional to b', b, and h. (See § 291.) Construct this fourth proportional, showing the complete solution.

This construction is attributed to Pythagoras. It represents a much higher achievement than the discovery of the Pythagorean proposition itself.

331. **EXERCISES.**

1. Show how to modify the last construction in case the given figure is a triangle. Give the construction.

2. Construct a rectangle on a given base equivalent to a given irregular quadrilateral.

3. Construct a rectangle on a given base equivalent to an irregular hexagon.

4. On a side of a regular hexagon as a base construct a rectangle equivalent to the hexagon.

5. Construct a parallelogram on a given base equivalent to a given triangle. Is there more than one solution?

6. Construct a square whose area shall be three times that of a given square; five times; one half the area; one fifth.

7. Construct an isosceles triangle, with a given altitude h, equivalent to a given triangle.

8. Draw a line parallel to the base of a triangle and cutting two of its sides. How will the resulting triangle and trapezoid compare in area,

(*a*) If each of the two sides of the triangle is bisected?

(*b*) If each of the two sides of the triangle is three times the length of the corresponding side of the trapezoid?

9. Construct a triangle whose base and altitude are equal and whose area is equal to that of a given triangle.

10. In a parallelogram $ABCD$, any point E on the diagonal BD is joined to A and C. Prove that $\triangle BEA$ and BEC are equivalent, and also that $\triangle DEA$ and DEC are equivalent.

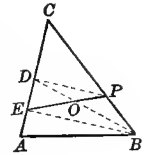
11. The sides of a triangle are 6, 8, 9. A line parallel to the longest side divides the triangle into a trapezoid and a triangle of equal areas. Find the ratio in which the line divides the two sides.

12. Draw a line parallel to the base of any triangle, and cutting two of its sides. How do the altitudes of the resulting triangle and trapezoid compare,

(*a*) If they are equal in area?

(*b*) If the area of the triangle is three times that of the trapezoid?

13. Through a point on a side of a triangle draw a line dividing it into two equivalent parts.

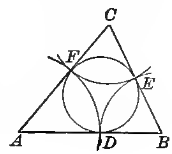


SOLUTION. Let P be the given point. Draw the median BD. Draw $BE \parallel PD$ and draw PE. This is the required line. Prove.

SUGGESTION. Notice that $\triangle DPE = \triangle DPB$ by Ex. 4, § 313, and hence that $\triangle EOD = \triangle BOP$.

14. Through a given point on a triangle draw a line which divides it into two figures whose areas are in the ratio $\frac{1}{8}$.

15. Inscribe a circle in a triangle, touching its sides in the points D, E, F. With the vertices as centers, construct circles passing through these points in pairs. Show that each of these latter circles is tangent to the other two.



SUMMARY OF CHAPTER IV.

1. State what is meant by the area of a rectangle. Give the formula.

2. Give formulas for areas of parallelograms and triangles.

3. How is the formula for the area of a trapezoid obtained?

4. What theorems of this chapter can be stated algebraically, as $(a+b)^2 = a^2 + 2ab + b^2$.

5. State the theorem on the ratio of the areas of two similar triangles; two similar polygons. Give examples.

6. Tabulate the problems of construction given in this chapter.

7. If two rectangles have the same base, how does the ratio of their areas compare with the ratio of their altitudes?

8. If two triangles have equal altitudes, how does the ratio of their areas compare with the ratio of their bases?

9. State all theorems of this chapter proved by means of the Pythagorean proposition.

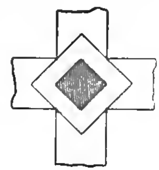
10. State some of the more important applications of the theorems in this chapter. Return to this question after studying the succeeding list of problems.

PROBLEMS AND APPLICATIONS.

1. Find the area of a square whose diagonal is 6 inches.

2. Find the area of a square whose diagonal is d inches.

3. $ABCD$ is a square placed at the crossing of two strips of equal width, as shown in the figure. The small black square has two vertices on the sides of the horizontal strip and two on the sides of the vertical strip.



(*a*) Find the area of each square when the width of the strips is 4 inches.

(*b*) Compare the area of the black square and the white border surrounding it.

(*c*) Can squares be placed as in the figure in case the strips are of unequal width? In the two following questions let the small square be drawn with two vertices on the sides of the horizontal strip and one diagonal parallel to these sides.

(*d*) If the horizontal strip is 4 inches wide, what must be the width of the vertical strip in order that the large square may have twice the area of the small one?

HINT. The diagonal of the small square is 4 inches.

(*e*) If the horizontal strip is a inches wide, what must be the width of the vertical strip in order that the area of the black square shall be $\frac{1}{n}$ the area of the larger square?

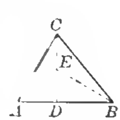
4. Prove that the area of a rhombus is one half the product of its diagonals.

5. Prove that the area of an isosceles right triangle is equal to the square on the altitude let fall upon the hypotenuse.

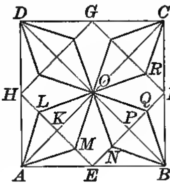
6. If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other two sides.

7. Inscribe a square in a semicircle and in a quadrant of the same circle. Compare their areas. See Ex. 8, page 147.

8. In the triangle ABC, CD is an altitude. E is any point on CD. If DE is one half CD, compare the area of the triangle AEB and the sum of the areas of the triangles AEC and BEC. Also compare these areas if DE is one nth of DC.



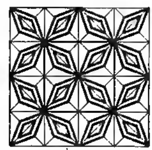
9. $ABCD$ is a square, and E, F, G, H are the middle points of its sides. On EF, points N and Q are taken so that $PN = PQ$. Similarly $KL = KM$, also $KL = NP$, and so on.



(*a*) Prove that $AMOL$ is a rhombus.

(*b*) If $AB = 6$ inches and if $KM = ME$, find the sum of the areas of the four rhombuses $AMOL$, $BQON$, etc.

(*c*) If $HE = 8$ inches, what is the area of the whole square?



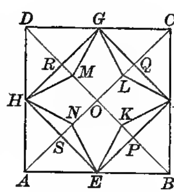
Parquet Flooring.

(*d*) What part of KE must KM be in order that the sum of the rhombuses shall be $\frac{1}{3}$ the area of the square?

(*e*) If $AB = a$, find KM so that the sum of the rhombuses shall be $\frac{1}{n}$ the area of the square.

(*f*) Prove that L, O, R lie in the same straight line.

10. $ABCD$ is a square and E, F, G, H are the middle points of its sides. $SN = PK = QL = RM$.



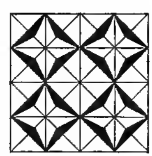
(*a*) If $AB = 6$ inches and if $PK = 1$ inch, find the sum of the areas of the triangles EHN, EFK, FGL, and GHM.

(*b*) If $AB = 6$, find PK so that the sum of the four triangles EFK, etc., shall be one fifth of the whole square.

(*c*) If $AB = 6$ and if the points E, K, Q lie in a straight line, find the sum of the areas of these triangles.

(*d*) If $AB = a$ and if $PK =$ one nth of PO, find the sum of the areas of the triangles.

(*e*) If $AB = a$, find PK so that the sum of the triangles shall be one mth of the whole square. What will be the length of PK in case the triangles occupy one half of the whole square?

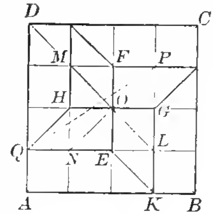


Parquet Flooring.

11. The sides of two equilateral triangles are a and b respectively. Find the side of an equilateral triangle whose area is equal to the sum of their areas.

12. Construct a triangle similar to a given triangle and having 16 times the area.

13. The middle points of the sides of a quadrilateral are connected. Show that the area of the parallelogram so formed is half the area of the quadrilateral.



14. $ABCD$ is a square. Each side is divided into four equal parts and the construction completed as shown in the figure.

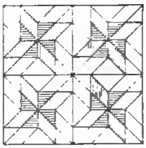
(*a*) Prove that $QNOH$, $KLOE$, etc., are parallelograms.

(*b*) What part of the area of the square is occupied by these four parallelograms?

(*c*) What part of the area of the square is occupied by the four triangles NEO, LGO, PFO, and MHO?

(*d*) If $AB = 6$, find the lengths of KO and QP.

(*e*) Find the ratio of the segments KO and QP. Does this ratio depend upon the length of AB?



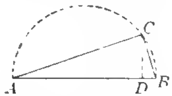
Parquet Flooring.

(*f*) In the parquet floor design what fraction is made of the dark wood? Does this depend upon the size of the original square?

15. On a given line-segment AB as a hypotenuse construct a right triangle such that the altitude upon the hypotenuse shall meet it at a given point D.

16. If ABC is a right triangle and $CD \perp AB$, prove that

$$\frac{\overline{AC}^2}{\overline{CB}^2} = \frac{AD}{DB}.$$



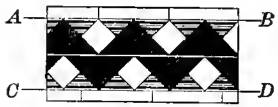
17. By means of Exs. 15 and 16 construct two segments HK and LM such that the ratio of the squares on these segments shall equal a given ratio.

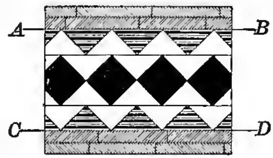
18. Divide a given segment into two segments such that the areas of the squares constructed upon them shall be in a given ratio.

19. On a given segment AB find a point D such that $\dfrac{\overline{AB}^2}{\overline{AD}^2} = 2.$

20. Construct a line parallel to the base of a triangle such that the resulting triangle and trapezoid shall be equivalent.

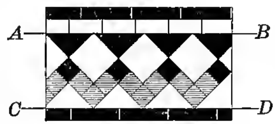
21. Construct two lines parallel to the base of a triangle so that the resulting two trapezoids and the triangle shall have equal areas.



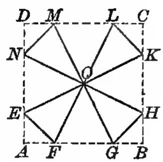


22. In each of the accompanying designs for tile flooring find what fraction of the space between the lines AB and CD is occupied by tiles of each color.

Study each design with care to see that the character of the figure determines the relative sizes of the various pieces of tile.



23. $ABCD$ is a square. Points E, F, G, ... are so taken that $AE = AF = GB = BH = \cdots$.



(*a*) If $AB = 6$ and $AF = 1$, find the sum of the areas of the four triangles EFO, GHO, KLO, MNO.

(*b*) Find the sum of these areas if $AB = a$ and $AF = h$.

(*c*) If $AB = 6$ and if the sum of the areas of the triangles is 9 square inches, find AF.

(*d*) If $AB = a$ and if the sum of the areas of the triangles is $\frac{a^2}{n}$, find AF. Interpret the two results.

24. Show how to construct a square whose area is n times the area of a given square.

25. Construct a triangle similar to a given triangle and equivalent to n times its area.

26. Construct a hexagon similar to a given hexagon and equivalent to n times its area.

27. Show how to construct a polygon similar to a given polygon and equivalent to n times its area.

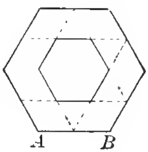
28. The alternate middle points of the sides of a regular hexagon are joined as shown in the figure.

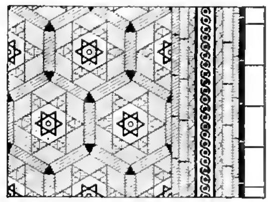
(*a*) Are the triangles thus formed equilateral? Prove.

(*b*) Is the star regular (*i.e.* are its six acute angles equal and its sides equal)?

(*c*) Compare the three segments into which each triangle divides the sides of the other.

(*d*) Is the inner hexagon regular? Prove. See Ex. 1, p. 76.





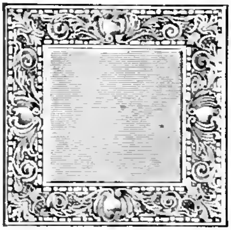
Tile Pattern.

(*e*) If $AB = 6$ in., find the area of the large hexagon, the star, and the small hexagon.

29. A border is to be constructed about a given square with an area equal to one half that of the square.

(*a*) By geometrical construction find the outer side of the border if the side of the square is given.

(*b*) If an outer side of the border is 24 in., find a side of the square, its area being two thirds that of the border.



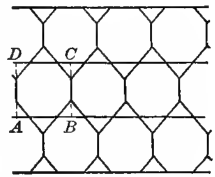
Ceiling Pattern.

30. A border is constructed about a given regular octagon, such that its area is equal to that of the octagon.

(*a*) If a side of the given octagon is a given segment AB, find by geometrical construction a segment equal to an outer side of the border.

(*b*) If a side of the given octagon is 16 in., find an outer side of the border.

Ceiling Pattern.

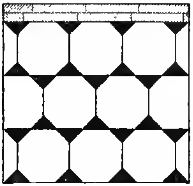


31. The accompanying design is based on a set of squares such as $ABCD$. The small triangles are equal isosceles right triangles constructed as shown.

(*a*) Are the vertical and the horizontal sides of the octagons equal?

(*b*) Are the octagons regular?

(*c*) If a side of one of the squares is 6, find the area of one of the octagons.



(*d*) What fraction of the whole tile design is occupied by the light-colored tiles? Does this depend upon the size of the original squares?

32. Given two lines at right angles to each other. Find the locus of all points such that the sum of the squares of the distances from the lines is 25.

33. Given two concentric circles whose radii are r and r'. Find the length of a chord of the greater which is tangent to the smaller.

34. If two equal circles of radius r intersect so that each passes through the center of the other, find the length of the common chord.

35. The square on the hypotenuse of a right triangle is four times the square on the altitude upon the hypotenuse. Prove it isosceles.

36. In a right triangle the hypotenuse is 10 feet and the difference between the other sides is 2 feet. Find the sides.

37. Two equal circles are tangent to each other and each circle is tangent to one of two lines perpendicular to each other. Find the locus of the points of tangency of the two circles.

SUGGESTION. Note that the point of tangency bisects their line of centers and that the centers move along lines at right angles to each other.

38. The square on a diagonal of a rectangle is equal to half the sum of the squares on the diagonals of the squares constructed on two adjacent sides of the rectangle.

39. Show that the diagonals of a trapezoid form with the non-parallel sides two triangles having equal areas.

CHAPTER V.

REGULAR POLYGONS AND CIRCLES.

REGULAR POLYGONS.

332. A **regular polygon** is one which is both equilateral and equiangular.

According to this definition, determine whether each of the following polygons is *regular* or not and state why:

An equilateral triangle, an equiangular triangle, a rectangle, a square, a rhombus. Draw a figure to illustrate each. Make a triangle which fulfills neither condition of the definition, also a quadrilateral.

333. The general problem of constructing a regular polygon depends upon the division of a circle into as many equal parts as the polygon has sides.

The problem of dividing the circle into equal parts can be solved in some cases by the methods of elementary geometry, and some of these methods will be considered in this chapter. In most cases this problem cannot be solved by elementary methods.

E.g. the circle may be divided into 2, 3, 4, 5, 6, 8, 10, 12, 15, equal parts, but not into 7, 9, 11, 13, 14, equal parts.

If a circle has already been divided into a certain number of equal parts, it may then be divided into twice, four times, eight times, etc., that number of parts by repeated bisection of the arcs. (See § 204, Ex. 1.)

The division of the circle into equal parts depends upon the theorem (§ 199) that equal central angles intercept equal arcs on the circle, and hence it involves the subdivision of angles into equal parts.

The following exercises depend upon principles already familiar.

With each solution give the reasons in full for each step.

All constructions are to be made with ruler and compasses only.

334. EXERCISES.

1. Divide a given angle or arc into four equal parts.

2. Divide a given angle or arc into eight equal parts.

3. If an angle or arc is already divided into a certain number of equal parts, show how to divide it into twice that number of equal parts.

4. Divide a circle into four equal parts.

5. Divide a circle into eight equal parts.

6. If a circle is already divided into a certain number of equal parts, show how to divide it into twice that number of equal parts.

7. Divide a circle into six equal parts.

SUGGESTION. Construct at the center an angle of 60°.

8. Draw the chords connecting in order the four division points in Ex. 4, and show that the figure is a regular quadrilateral.

9. Draw the tangents at the four division points in Ex. 4, and show that a regular quadrilateral is formed.

10. Draw the chords connecting in order the division points in Ex. 7, and show that a regular hexagon is formed. Prove that the side of the hexagon is equal to the radius of the circle.

11. Draw chords connecting alternate division points in Ex. 7, and show that a regular triangle is formed.

12. Construct tangents to the circle at alternate division points in Ex. 7, and show that a regular triangle is formed.

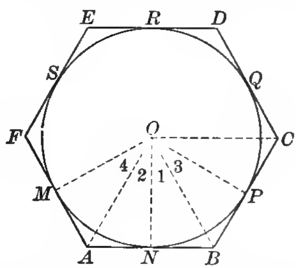
13. Draw tangents to the circle at the division points in Ex. 7, and show that a regular hexagon is formed.

NOTE. See also the construction of regular polygons of 3, 4, and 6 sides, § 163, Exs. 3, 4, 5.

335. THEOREM. *If a circle is divided into any number of equal arcs, the chords joining the division points, taken in order, form a regular polygon.*

Proof: Show (1) that the chords are equal; (2) that the angles are inscribed in equal arcs and hence are equal.

336. THEOREM. *If a circle is divided into any number of equal parts, the tangents at the points of division, taken in order, form a regular polygon.*



Given the ⊙ *OA* divided into equal arcs by the points *M*, *N*, *P*, etc., with tangents drawn at these points forming the polygon *ABCDEF*.

To prove that *ABCDEF* is a regular polygon.

Analysis of Proof: (1) To prove that $AB = BC = CD$, etc.

We know that $BP = BN$ (Why?).

Hence, if we can show that $AN = NB$ and $BP = PC$, then it will follow that $AB = BC$.

To prove $AN = NB$, it must be shown that $\angle 1 = \angle 2$, and this is done by showing that $\angle 1$ and $\angle 2$ are halves of the equal angles *NOM* and *NOP*.

This necessitates proving $\triangle AON \cong \triangle AOM$.

Now state the proof in full.

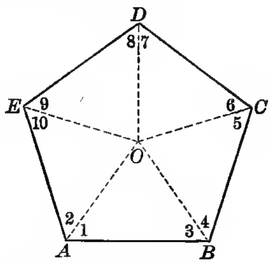
(2) To prove $\angle ABC = \angle BCD = \angle CDE$, etc.

From the triangles proved congruent under (1) show that $\angle MAO = \angle OAN = \angle NBO = \angle OBP$; and hence, $\angle MAO + \angle OAN = \angle NBO + \angle OBP$, or $\angle FAB = \angle ABC$.

In like manner it is proved that $\angle ABC = \angle BCD$, etc.

Hence the polygon is equilateral and also equiangular, and hence regular.

337. THEOREM. *If a polygon is regular, a circle may be circumscribed about it.*



Given a regular polygon *ABCDE*.

To prove that a point O can be found such that

$$OA = OB = OC = OD = OE.$$

Outline of Proof: Bisect $\angle$s A and B and let the bisectors meet in some point O. Then O is the center sought.

For we have (1) $OA = OB$, Why?

(2) $\triangle AOB \cong BOC$. $\therefore OA = OC$.

(3) $\triangle BOC \cong \triangle COD$. $\therefore OB = OD$.

(4) Now prove $OC = OE$.

$\therefore OA = OB = OC = OD = OE$.

Prove each step, showing how it depends upon the preceding.

338. EXERCISES.

1. Find the locus of the vertices of all regular polygons of the same number of sides which can be circumscribed about the same circle.

2. Find the locus of the middle points of the sides of all regular polygons of the same number of sides which are inscribed in the same circle.

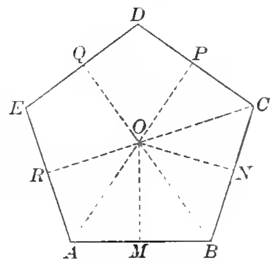
3. Is an equilateral circumscribed polygon regular? Prove.

4. Is an equiangular circumscribed polygon regular? Prove.

339. THEOREM. *If a polygon is regular, a circle may be inscribed in it.*

Given the regular polygon *ABCDE*.

To prove that a point O may be found such that the perpendiculars OM, ON, OP, OQ, OR, are equal.



Outline of Proof: Determine a point O as in the proof of the preceding theorem. Then $\triangle$ AOB, BOC, etc., are equal isosceles triangles, and their altitudes are equal. Hence O is the required point.

Give proof in full.

340. THEOREM. *An equilateral polygon inscribed in a circle is regular.*

Suggestion for Proof: Use the fact that equal chords subtend equal arcs, and apply § 335.

341. EXERCISE.

Show that the inscribed and circumscribed circles of a regular polygon have the same center.

342. Definitions. The **center of a regular polygon** is the common center of its inscribed and circumscribed circles.

The **radius of a regular polygon** is the distance from the center to one of its vertices.

The **apothem of a regular polygon** is the perpendicular distance from its center to a side.

343. THEOREM. *The area of a regular polygon is equal to half the product of its apothem and perimeter.*

Suggestion for Proof: A regular polygon is divided by its radii into a series of congruent triangles, the area of each of which is one half the product of the apothem and a side of the polygon.

Complete the proof.

344. THEOREM. *The area of any polygon circumscribed about a circle is equal to one half the product of the perimeter of the polygon and the radius of the circle.*

The proof is left to the student.

345. EXERCISES.

1. Is every equiangular polygon inscribed in a circle regular? Prove.

2. Show that the radius of a regular polygon bisects the angle at the vertex to which it is drawn.

3. Show that the perimeter of a regular polygon of a given number of sides is less than that of one having twice the number of sides, both being inscribed in the same circle.

4. Show that the perimeter of a regular polygon is greater than that of one having twice the number of sides, both being circumscribed about the same circle.

5. Compare the areas of the two polygons in Ex. 3.

6. Compare the areas of the two polygons in Ex. 4.

7. Show that the area of a square inscribed in a circle of radius r is $2r^2$. How does this compare with the area of the circumscribed square.

8. Compute the apothem and area of a regular inscribed hexagon if the radius of the circle is r; also of the regular circumscribed hexagon.

9. A regular triangle is inscribed in a circle of radius 10. Find the apothem and a side of the triangle.

346. THEOREM. *Two regular polygons of the same number of sides are similar.*

Outline of Proof: (1) Show that all pairs of corresponding angles are equal.

(2) Show that the ratios of pairs of corresponding sides are equal. Hence the polygons are similar (Why?).

347. THEOREM. *The perimeters of two regular polygons having the same number of sides are in the same ratio as their radii or their apothems.*

Outline of Proof: Show (1) that each triangle formed by a side and two radii in one polygon is similar to the corresponding triangle in the other polygon.

(2) That $\frac{AB}{A'B'} = \frac{r}{r'} = \frac{a}{a'}$, where AB and $A'B'$ are the two sides, r and r' the corresponding radii and a and a' the corresponding apothems. And so for the remaining pairs of triangles.

(3) That $\frac{AB + BC + CD + \cdots}{A'B' + B'C' + C'D' + \cdots} = \frac{AB}{A'B'} = \frac{r}{r'} = \frac{a}{a'}$.

Draw the figure and give the proof in full.

348. THEOREM. *The areas of two regular polygons of the same number of sides are in the same ratio as the squares of the corresponding radii or apothems.*

Outline of Proof: Divide the polygons into pairs of corresponding triangles as in the preceding proof.

(1) Show that $\frac{\triangle \text{I}}{\triangle \text{I}'} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2}$, and so for each pair of triangles.

(2) Hence $\frac{\triangle \text{I} + \triangle \text{II} + \triangle \text{III} + \cdots}{\triangle \text{I}' + \triangle \text{II}' + \triangle \text{III}' + \cdots} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2}$.

Draw figure and give the proof in full.

PROBLEMS AND APPLICATIONS.

1. Find the ratio of the perimeters of squares inscribed in and circumscribed about the same circle.

2. Find the ratio of the perimeters of regular hexagons inscribed in and circumscribed about the same circle.

3. Find the ratio between the perimeters of regular triangles inscribed in and circumscribed about the same circle.

4. Find the ratio of the areas of regular triangles inscribed in and circumscribed about the same circle. Also find the ratio of the areas of such squares and of such hexagons.

5. The perimeter of a regular hexagon inscribed in a circle is 24 inches. Find the perimeter of a regular hexagon circumscribed about a circle of twice the diameter.

6. The area of a regular triangle circumscribed about a circle is 64 square inches. What is the area of a regular triangle inscribed in a circle of one third the radius?

7. The area of a regular hexagon inscribed in a circle is 48 square inches. What is the area of a regular hexagon circumscribed about a circle whose diameter is $1\frac{3}{4}$ times that of the first?

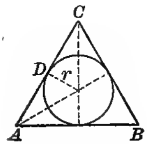
8. A chord AB bisects the radius perpendicular to it. Find the central angle subtended by the chord. State the result as a theorem.

9. State and prove the converse of the theorem in Ex. 8.

10. Find the area of a regular triangle inscribed in a circle whose radius is 6 inches.

11. Find the area of a regular triangle inscribed in a circle whose radius is r inches.

12. One of the acute angles of a right triangle is 60° and the side adjacent to this angle is r inches long. Find the remaining sides of the triangle.



13. A regular triangle is circumscribed about a circle of radius r. Find its area.

SUGGESTION. First find DC in the figure.

14. A regular triangle of area 36 square inches is inscribed in a circle. Find the radius of the circle.

15. Find the radius of a circle if the area of its regular inscribed triangle is a.

16. Find the radius of the circle if the area of the regular circumscribed triangle is a.

17. Find the radius of a circle if the difference between the perimeters of the regular inscribed and circumscribed triangles is 12 inches.

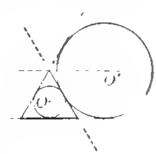
18. Find the radius of a circle if the difference of the perimeters of the regular inscribed and circumscribed hexagons is 10 inches.

19. If the area of a circumscribed square is 25 square inches greater than that of an inscribed square, what is the diameter of the circle?

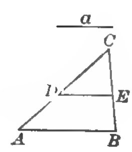
20. Find the radius of a circle if the difference between the areas of the inscribed and circumscribed regular triangles is 25 square inches.

21. Find the radius of a circle if the difference between the areas of the regular inscribed and circumscribed hexagons is 25 square inches.

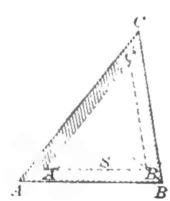
22. The difference between the areas of the squares circumscribed about two circles is 50 square inches and the difference of their diameters is 4 inches. Find each diameter.



23. If the inscribed and escribed circles O and O' of an equilateral triangle are constructed as shown in the figure, find the ratio of their radii. Does this ratio depend upon the size of the triangle?



24. Given a triangle ABC and a segment a, show how to construct a segment $DE \parallel AB$ and equal to the segment a, such that the points D and E shall lie on the sides CA and CB respectively or on these sides extended.



25. A triangular plot of ground ABC is to be laid out as a triangular flower bed with a walk of uniform width extending around it.

(a) Prove that the flower bed is similar to the original triangle.

(b) Show that the corners of the flower bed lie on the bisectors of the angles of the original triangle.

(*c*) Find a segment S so that $\frac{\overline{AB}^2}{S^2} = 2$ and construct $A'B'$ equal to S and parallel to AB, the points A' and B' lying on the bisectors of the angles A and B respectively. See Exs. 16–19, page 172.

(*d*) Draw $A'C' \parallel AC$ and $B'C' \parallel BC$.

Prove that the area of the flower bed $A'B'C'$, as thus constructed, is equal to the area of the walk.

(*e*) Construct the figure for the flower bed so that its area is five times that of the walk.

26. Given a rectangular plot of ground. Is it always possible to lay off on it a walk of uniform width running around it so that the plot inside the walk shall be similar to the original figure? Prove.

27. Is the construction proposed in Ex. 26 always possible in the case of a square? of a rhombus? Prove.

28. If a side of a regular hexagon circumscribed about a circle is a, find the radius of the circle.

29. On a regular hexagonal plot of ground whose side is 12 feet a walk of uniform width is to be laid off around it. Find by algebraic computation the width of the walk if it is to occupy one half the whole plot.

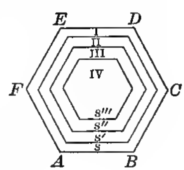
30. Find by geometric construction the width of the walk in Ex. 29.

31. Show that the figure inside the walk in Ex. 29 is a regular hexagon.

32. Given a segment AB, find three points C, D, E on it so that,

$$\frac{\overline{AC}^2}{\overline{AB}^2} = \frac{1}{4}, \quad \frac{\overline{AD}^2}{\overline{AB}^2} = \frac{1}{2}, \quad \text{and} \quad \frac{\overline{AE}^2}{\overline{AB}^2} = \frac{3}{4}.$$

33. A regular hexagon $ABCDEF$ is to be divided into four pieces of equal area by segments drawn parallel to its sides forming hexagons as shown in the figure. If $AB = 24$ feet, find a side of each of the other hexagons and also the apothem of each.



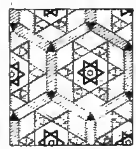
34. Without computing *algebraically* the apothem or sides of the inner hexagons in Ex. 33, show how to construct the figure *geometrically*. See Ex. 6, § 331. Also Ex. 18, page 172. Use § 348.

35. In a given hexagonal polygon whose side is a find in terms of a and n the width of a walk around it which will occupy one nth of the area of the whole polygon.

36. By means of hexagons similar to those in Ex. 33, divide a given regular hexagon into three parts such that the outside part is $\frac{1}{4}$ the whole area and the next $\frac{1}{2}$ of the whole area.

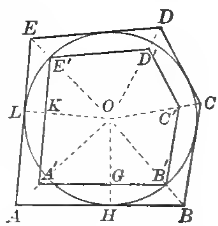
37. Compute the sides and the apothem of the two hexagons constructed in Ex. 36 if the side of the given hexagon is 12 inches.

38. In the adjoining pattern find two regular hexagons whose areas are in the ratio 1 : 4 and show that this agrees with theorem, § 348.



39. If a polygon is circumscribed about a circle, show that the bisectors of all its angles meet in a point.

40. Given any polygon circumscribed about a circle. Within it draw segments parallel to each of its sides and at the same distance from each side. Show that these segments form a polygon similar to the first.

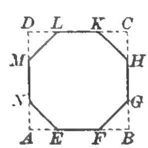


41. If the bisectors of all the angles of a polygon meet in a point prove that a circle may be inscribed in it (tangent to all its sides).

What is the relation of the theorems in Exs. 39 and 41?

42. Given a polygonal plot of ground (boundary is a polygon) such that a path of equal width on it around the border leaves a polygonal plot similar to the first. Prove that a circle may be inscribed in the polygon which forms the boundary of either plot.

43. In the figure, $ABCD$ is a square and $EFGHKLMN$ is a regular octagon.



(*a*) If $EF = 4$ inches, find AB.
(*b*) If $AB = 12$ inches, find EF.
(*c*) If $EF = a$, find AB.
(*d*) If $AB = s$, find EF.
(*e*) If $AB = s$, find the apothem.
(*f*) If $AB = s$, find the radius of the octagon.

44. Find the ratio between the areas of regular octagons inscribed in and circumscribed about the same circle.

45. (*a*) The apothem of a regular octagon is 10 feet. Find the width of a uniform strip laid off around it which occupies $\frac{1}{4}$ its area.

(*b*) Show how to construct this strip geometrically without first computing its width. Prove that the inside figure is a regular octagon.

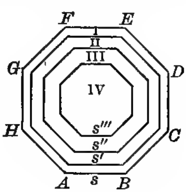
46. A regular octagon *ABCDEFGH* is to be divided into four parts of equal area by means of octagons as shown in the figure.

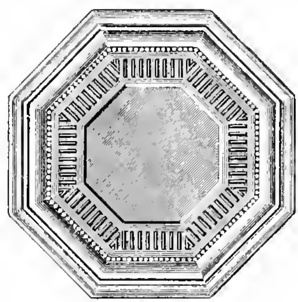
(*a*) If $AB = 12$ inches find a side of each octagon.

(*b*) Show how to construct the figure without first computing the sides.

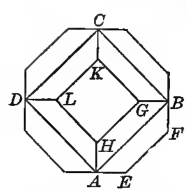
(*c*) Show how to construct such a figure if the four parts I, II, III, IV, are to be in any required ratios.

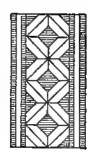
(*d*) Measure the sides of the two inner parts of the ceiling pattern and hence find the ratio of their areas.





47. The middle points, A, B, C, D, of alternate sides of a regular octagon are joined as shown in the figure. AH is perpendicular to AE and equal to it. BG, CK, and DL are constructed in the same manner.





(*a*) Prove that *ABCD* and *HGKL* are squares.

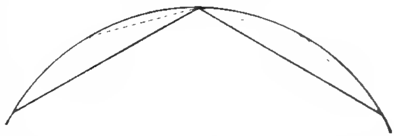
(*b*) Find the areas of *ABCD* and *HGKL*, if $EF = 10$ inches.

(*c*) What fraction of the whole octagon is occupied by the square *HGKL*.

See the accompanying tile border.

MEASUREMENT OF THE CIRCLE.

349. If in a circle a regular polygon is inscribed, its perimeter may be measured.



For example, the perimeter of a regular inscribed hexagon is $6r$ if r is the radius of the circle. See § 334, Ex. 10.

If the number of sides of the inscribed polygon be doubled, the resulting perimeter may be *measured* or *computed* in terms of r. See § 357.

If again the number of sides be doubled, the resulting perimeter may be computed in terms of r, and so on.

In a similar manner a regular polygon, say a hexagon, may be circumscribed about a circle and its perimeter expressed in terms of r.

If the number of sides of the circumscribed polygon be doubled, its perimeter may again be computed, and so on as often as desired.

350. By continuing either of these processes it is evident that the inscribed or the circumscribed polygon may be made to lie as close to the circle as we please.

351. The word *length* has thus far been used in connection with *straight* line-segments only. Thus, the perimeter of any polygon is the *sum of the lengths of its sides.*

352. We now *assume* that a **circle has a definite length** and that this can be **approximated** as nearly as we please by taking the perimeters of the successive inscribed or circumscribed polygons.

The length of a circle is often called its **perimeter** or **circumference**.

It is evident that *approximate measurement* is the only kind possible in the case of the circle, since no straight line unit of measure, however small, can be made to coincide with an arc of a circle.

353. Comparison of the Lengths of Two Circles. In each of two circles, O and O', let regular polygons of 6, 12, 24, 48, 96, 192, etc., sides be inscribed. Call the perimeters of the polygons in $\odot O$, P_6, P_{12}, P_{24}, etc., and those in $\odot O'$, P'_6, P'_{12}, P'_{24}, etc. Let the radii of the circles be R and R' respectively.

Then by the theorem of § 347, we have

$$\frac{R}{R'} = \frac{P_6}{P'_6} = \frac{P_{12}}{P'_{12}} = \frac{P_{24}}{P'_{24}} = \text{etc.},$$

however great the number of sides of the inscribed polygons. Show that the same relations hold if polygons are circumscribed about the circle.

From these considerations we are led to the following:

354. PRELIMINARY THEOREM. *The lengths of two circles are in the same ratio as their radii.*

355. Hence, if C and C' are the circumferences of two circles, R and R' their radii, and D and D' their diameters,

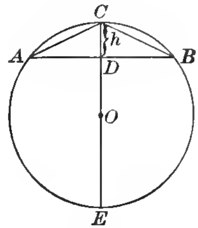
we have $\frac{C}{C'} = \frac{R}{R'} = \frac{D}{D'}$; and, also $\frac{C}{D} = \frac{C'}{D'}$. (See § 245)

Hence, the ratio of the circumference to the diameter in one circle is the same as this ratio in any other circle.

356. This constant ratio is denoted by the Greek letter π, pronounced *pi*.

The argument used above shows that a theorem like that of § 354 holds for every pair of polygons used in the approximation process, and *hence it is established for all purposes of practical measurement or computation.*

357. PROBLEM. *To compute the approximate value of π.*



SOLUTION. Suppose a regular polygon of n sides is inscribed in a circle whose radius is r and let one of the sides AB be called s_n.

We first obtain in terms of s_n the length AC, or s_{2n}, of a side of a regular polygon of $2n$ sides inscribed in the same circle.

Let AB be a side of the first polygon. Bisect $\widehat{AB}$ at C and draw AC and BC. Then AC is a side of a regular inscribed polygon of $2n$ sides (Why?).

Then in the figure

$$(s_{2n})^2 = \overline{AC}^2 = (\tfrac{1}{2}AB)^2 + h^2 = (\tfrac{1}{2}s_n)^2 + h^2. \quad \text{(Why?)} \quad (1)$$

But $\quad h \times DE = h(2r - h) = AD \times DB = \overline{AD}^2$, (Why?)

or $$\overline{AD}^2 = (\tfrac{1}{2}s_n)^2 = h(2r - h). \qquad (2)$$

Solving equation (2) for h, we have

$$h = \frac{2r \pm \sqrt{4r^2 - s_n^2}}{2}.$$

Taking only the negative sign, since $h < r$, and squaring,

we have, $\quad h^2 = 2r^2 - r\sqrt{4r^2 - s_n^2} - \tfrac{1}{4}s_n^2$,

or, $\quad h^2 + (\tfrac{1}{2}s_n)^2 = 2r^2 - r\sqrt{4r^2 - s_n^2}$.

Hence, from (1), $s_{2n}^2 = 2r^2 - r\sqrt{4r^2 - s_n^2}$.

and $$s_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - s_n^2}}.$$

Since, by § 355, the value of π is the same for all circles, we take a circle whose radius is 1.

In this case $$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}. \qquad (3)$$

If the first polygon is a regular hexagon, then $s_6 = 1$.

Hence, $$s_{12} = \sqrt{2 - \sqrt{4 - 1}} = 0.51763809.$$

Denoting the perimeter of a regular inscribed polygon of n sides by P_n, we have,

$$P_{12} = 12(0.51763809) = 6.21165708.$$

In the formula (3) let $n = 12$.

Then $$s_{24} = \sqrt{2 - \sqrt{4 - (0.51763809)^2}} = 0.26105238.$$

Hence, $$P_{24} = 24(0.26108238) = 6.26525722.$$

Computing s_{48}, P_{48}, etc., in a similar manner, we have,

$$S_{12} = \sqrt{2 - \sqrt{4 - 1}} = .51763809. \quad \therefore P_{12} = 6.21165708.$$

$$S_{24} = \sqrt{2 - \sqrt{4 - (.51763809)^2}} = .26105238. \quad \therefore P_{24} = 6.26525722.$$

$$S_{48} = \sqrt{2 - \sqrt{4 - (.26105238)^2}} = .13080626. \quad \therefore P_{48} = 6.27870041.$$

$$S_{96} = \sqrt{2 - \sqrt{4 - (.13080626)^2}} = .06543817. \quad \therefore P_{96} = 6.28206396.$$

$$S_{192} = \sqrt{2 - \sqrt{4 - (.06543817)^2}} = .03272346. \quad \therefore P_{192} = 6.28290510.$$

$$S_{384} = \sqrt{2 - \sqrt{4 - (.03272346)^2}} = .01636228. \quad \therefore P_{384} = 6.28311544.$$

$$S_{768} = \sqrt{2 - \sqrt{4 - (.01636228)^2}} = .00818126. \quad \therefore P_{768} = 6.28316941.$$

358. The Length of the Circle. By continuing this process it is found that the first five figures in the decimal remain unchanged. Hence 6.28317 is an approximation to the circumference of a circle whose radius is 1.

359. By a process similar to the preceding, if circumscribed polygons of 4, 8, 16, etc., sides are used, the following results are obtained:

NUMBER OF SIDES	LENGTH OF EACH SIDE	PERIMETER
4	2.000000	8.000000
8	0.828428	6.627418
16	0.397824	6.365196
32	0.196984	6.303450
64	0.098254	6.288236
128	0.049078	6.284448
256	0.024544	6.283500
512	0.012272	6.283264
1024	0.006136	6.283205
2048	0.003068	6.283190

360. Since the diameter is 2, the ratio of the circumference to the diameter is

$$\frac{6.28317}{2} = 3.14159, \text{ or } \frac{6.28319}{2} = 3.14159$$

according as the inscribed or circumscribed polygons are used.

That is, these approximations of π agree to five decimal places.

If so great accuracy is not required, we may use a smaller number of decimal places; such as, 3.1416, 3.14, or $3\frac{1}{7}$.

Since $\frac{C}{D} = \pi$, it follows that $C = \pi D = 2\pi R$.

That is, *the circumference is* 2π *times the radius.*

361. The area of the circle. In § 359 the area of each circumscribed polygon is half the product of the perimeter and the apothem, which in this case is the radius of the circle. The area inclosed by the circle is called the area of the circle. That is, for every such polygon, $A = \frac{1}{2} P \cdot R$, or **area equals one half perimeter times radius.**

We now assume that a circle has a definite area which can be approximated as closely as we please by taking the areas of the successive circumscribed polygons.

362. Since the perimeters of the circumscribed polygons can be made to approximate the length of the circle as nearly as we please, and since $C = 2\pi r$ we have,

$$\textbf{area of circle} = \tfrac{1}{2} \cdot 2\pi r \cdot r = \pi r^2.$$

The degree of accuracy to which this formula leads depends entirely upon the accuracy with which π is determined.

The old problem of *squaring the circle,* that is, finding the side of a square whose area equals that of a given circle, involves therefore determining the value of π. Much time and labor have been expended upon this in the hope that this value could be exactly constructed by means of the ruler and compasses, but it is now known that this is impossible.

363. Since the area of a circle is πr^2, if we have two given circles whose radii are r and r' and whose diameters are d and d', then the ratio of their areas A and A' is

$$\frac{A}{A'} = \frac{\pi r^2}{\pi r'^2} = \frac{r^2}{r'^2} = \frac{d^2}{d'^2}; \text{ that is,}$$

THEOREM. *The areas of two circles are in the same ratio as the squares of their radii, or of their diameters.*

364. The area of a sector bears the same ratio to the area of the circle as the angle of the sector does to the perigon.

E.g. the area of the sector whose arc is a quadrant is one fourth of the area of the circle, that is, $\frac{\pi R^2}{4}$. The area of a semicircle is $\frac{\pi R^2}{2}$.

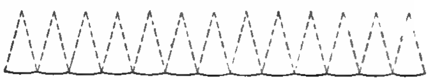
Find the area of a sector whose angle is 60°; 30°; 45°; 72°.

365. The area of a segment of a circle is known if that of the sector having the same arc, and of the central triangle on the same chord, can be determined.

Show that the areas of segments of a circle whose arcs are respectively 90° and 60° are

$$\frac{\pi R^2}{4} - \frac{R^2}{2} = \frac{R^2}{4}(\pi - 2) \text{ and } \frac{\pi R^2}{6} - \frac{R^2}{4}\sqrt{3} = \frac{R^2}{12}(2\pi - 3\sqrt{3}).$$

The accompanying figure shows a circle cut into sectors by a series of radii. Each sector approximates the shape of a triangle, whose altitude is the radius of the circle, and whose base is an arc of the circle. Since the sum of the areas of these triangles is the product of the altitude and the sum of the bases, we obtain a verification of the theorem that the area of a circle is one half the product of the circumference and radius.



SUMMARY OF CHAPTER V.

1. Make a list of the definitions pertaining to regular polygons.

2. State the theorems concerning regular polygons inscribed in or circumscribed about a circle.

3. State the theorems involving similar regular polygons.

4. Give an outline of the discussion concerning the value of π and its use in approximating the length and area of the circle.

5. What are some of the more important applications of the theorems in Chapter V? (Return to this question after studying the applications which follow.)

PROBLEMS AND APPLICATIONS ON CHAPTER V.

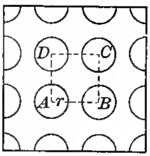
NOTE. In the following problems, unless otherwise specified, use the value $\pi = 3\frac{1}{7}$, which differs from 3.1416 by about $\frac{1}{10}$ of one per cent.

1. The diameter of a circle is 8 inches. Find the circumference and area.

2. The circumference of a circle is 10 feet. Find its radius and area.

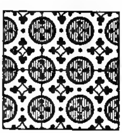
3. The area of a circle is 24 square inches. Find its radius and circumference.

4. Centers of circles are arranged at equal distances on a network of lines at right angles to each other as shown in the figure. If $r = \frac{1}{3}AB$, what part of the whole area is inclosed by the circles?



SUGGESTION. Consider what part of the square $ABCD$ lies within the circles.

5. If in Ex. 4 the radius of each circle is 3 inches, how far apart must the centers be in order that one half the area shall lie within the circles?



6. If the radius of each circle is r, how far apart must the centers be located in order that $\frac{1}{n}$ of the whole area may lie within the circles?

7. If the circles occupy $\frac{1}{n}$ of the area, and if the centers are d inches apart, find the radii.

8. A square is inscribed in a circle. Find the ratio between the areas of the square and the circle.

9. A square is circumscribed about a circle. Find the ratio between their areas.

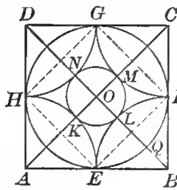
Do the ratios required in Exs. 8 and 9 depend upon the radius of the circle?

10. Find the ratio between the area of a circle and its regular inscribed hexagon.

11. Find the ratio between the area of a circle and its regular circumscribed hexagon.

Do the ratios required in Exs. 10 and 11 depend upon the radius of the circle?

12. Find the ratio between the area of a square inscribed in a circle and another circumscribed about a circle having 3 times the radius.



13. Find the ratio of the area of a regular hexagon inscribed in a circle to that of another circumscribed about a circle having a radius $\frac{1}{4}$ as great.

14. *ABCD* is a square whose side is a and the points *A*, *B*, *C*, *D*, are the centers of the arcs *HE*, *EF*, etc.

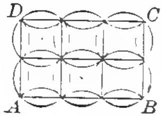
(*a*) Find the area of the figure formed by the arcs *FLE*, *EKH*, *HNG*, *GMF*.

(*b*) Find the area of the figure formed by the arcs *ELF* and *EQF*.

SUGGESTION. Find the difference between the areas of a circle and its inscribed square.

(*c*) Find the area of the figure formed by the arcs *EK*, *KL*, and *LE*.

15. If the sides of the rectangle *ABCD* are 8 and 12 inches, respectively, find the radii of the circles..



(*a*) What fraction of the area of the rectangle lies within only one circle?

(*b*) Prove that at each vertex of a square two circles are tangent to a diagonal of the square.

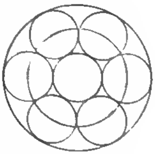
16. Two concentric circles are such that one divides the area of the other into two equal parts.

(*a*) Find the ratio of the radii of the circles.

(*b*) Given the outer circle, construct the inner one.

17. Construct three circles concentric with a circle of radius r, which shall divide its area into four equal parts.

18. Prove that six circles of equal radii can be constructed each tangent to two of the others and to a given circle.

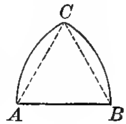


(*a*) Show that a circle can be constructed around the six circles tangent to each of them.

(*b*) What fraction of the area of the last circle is occupied by the seven circles within it?

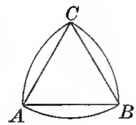
19. *AOB* is a central angle of 60°. Find the area bounded by the chord *AB* and $\widehat{AB}$ if the radius of the circle is 3.

20. ABC is an equilateral gothic arch. (See page 109.) Find the area inclosed by the segment AB and the arcs AC and BC, if $AB = 3$ feet.



Suggestion. Find the area of the *sector* with center A, arc BC, and radii AB and AC, and add to this the area of the circle-segment whose chord is AC.

21. ABC is an equilateral triangle and A, B, and C are the centers of the arcs. Show that the area of the figure formed by the arcs is three times the area of one of the *sectors* minus twice the area of the $\triangle ABC$.

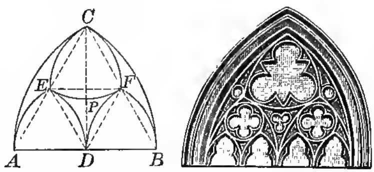


22. In the figure ABC is an equilateral triangle and D, E, and F are the middle points of its sides. Arcs are constructed as shown.

(*a*) If $AB = 6$ feet, find the area inclosed by $\widehat{CE}$, $\widehat{EF}$, $\widehat{FC}$.

(*b*) Find the area inclosed by $\widehat{AE}$, $\widehat{EC}$, and $\widehat{CA}$.

(*c*) Find the area inclosed by $\widehat{DF}$, $\widehat{FE}$, and $\widehat{ED}$.



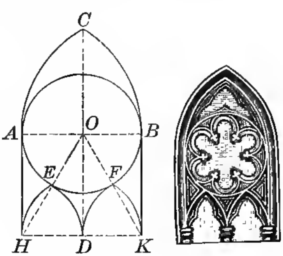
From the Union Park Church, Chicago.

23. In the figure ABC is an equilateral arch and $\odot O$ is constructed on AB as a diameter. AH and BK are perpendicular to AB.

(*a*) Construct the equilateral arches HED and DFK tangent to the circle as shown in the figure.

Suggestion. $OH = OA + DH$.

(*b*) Prove that the vertices E and F lie on the circle.



From the First Presbyterian Church, Chicago.

Suggestion. What kind of a triangle is HOK?

In the following let $AB = 8$ feet.

(*c*) Find the area bounded by the arcs DF, FE, and ED.

(*d*) Find the area of the rectangle $ABKH$.

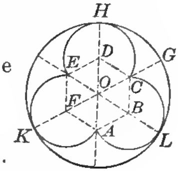
(*e*) Find the area bounded by AH and the arcs HE and EA.

(*f*) Find the area bounded by the upper semicircle AB and the arcs AC and BC.

(*g*) Find all the areas required in (*c*) ⋯ (*f*) if $AB = a$.

24. In the figure, $ABCDEF$ is a regular hexagon. B is the center of the arc ALC, D the center of CHE, and F the center of EKA.

If $AB = 16$ inches, find



(*a*) The circumference and area of the circle,

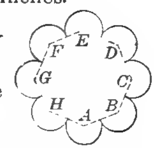
(*b*) The area bounded by the arc ALC and the segments AB and BC,

(*c*) The area bounded by $\overset{\frown}{KA}$, $\overset{\frown}{AL}$ and $\overset{\frown}{LK}$,

(*d*) The area bounded by $\overset{\frown}{ALC}$, $\overset{\frown}{CHE}$ and $\overset{\frown}{EKA}$.

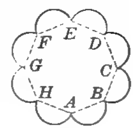
(*e*) Find the areas required in (*a*)–(*d*) if $AB = a$ inches.

25. $ABC \ldots\ldots$ is a regular octagon. Arcs are constructed with the vertices as centers as in the figure.



If $AB = 10$ inches, find the area inclosed by the whole figure. Also if $AB = a$.

26. $ABCDEFGH$ is a regular octagon. Semicircles are constructed with the sides as diameters.



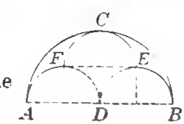
(*a*) If $AB = 10$ inches, find the area of the whole figure. Also if $AB = a$.

(*b*) Complete the drawing in the outline figure to make the steel ceiling pattern here shown.

27. In the figure ACB, AFD, DEB, and ECF are semicircles. EF is tangent to two semicircles.

(*a*) Prove that the semicircles AFD, FCE, and DEB are equal, D being given the middle point of AB.

If $AB = 48$ inches, find:



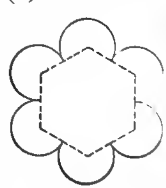
(*b*) The area bounded by $\overset{\frown}{AC}$, $\overset{\frown}{FC}$ and $\overset{\frown}{FA}$,

(*c*) The area bounded by $\overset{\frown}{FD}$ and $\overset{\frown}{DE}$, and the line-segment FE,

(*d*) The area bounded by $\overset{\frown}{AF}$, $\overset{\frown}{FCE}$, $\overset{\frown}{EB}$, and the line-segment AB.

(*e*) Find the areas required in (*b*)–(*d*) if $AB = a$.

28. (*a*) If a side of the regular hexagon in the figure is a, find the area inclosed by the arcs (including the area of the hexagon).



(*b*) Show that a circle may be circumscribed about the whole figure.

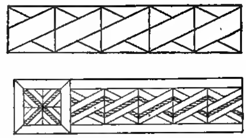
(*c*) Find the area inside this circle and outside the figure in (*a*).

MISCELLANEOUS PROBLEMS AND APPLICATIONS.

1. Given a rhombus, two of whose angles are 60°, to divide it into a regular hexagon and two equilateral triangles. See Ex. 7, page 78.

2. What fraction of the accompanying design for tile flooring is made up of the black tiles? Show how to construct this design by marking off points along the border and drawing parallel lines.

3. In the accompanying design for a parquet border two strips of wood appear to be intertwined.



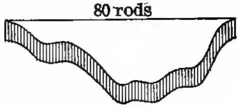
(*a*) If the border is 8 inches wide and 3 feet 4 inches long, find the area of one of these strips, including the part which appears to be obscured by the other strip.

(*b*) If the figure consists of squares, find the angle at which the strips meet the sides. Use the table on page 139.

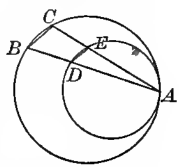
(*c*) If the width of the border is a and its length b, find the combined area of these strips.

Compare the total area of the obscured part of these strips with the sum of the areas of the small triangles along the edge of the border.

4. A solid board fence 5 feet in vertical height running due north and south is to be built across a valley, connecting two points of the same elevation. Find the number of square feet in the fence if the horizontal distance is 80 rods.



5. Are the data given in the preceding problem sufficient to solve it if the fence required is to be an ordinary four-board fence, each board 6 inches wide?



6. Two circles are tangent internally at a point A. Chords AB and AC of the larger circle are drawn meeting the smaller circle in D and E respectively. Prove that BC and DE are parallel.

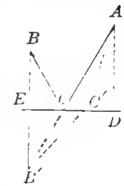
7. Two circles, radii r and r', are tangent internally. Find the length of a chord of the larger circle tangent to the smaller if:

(*a*) The chord is parallel to the line of centers,

(*b*) The chord is perpendicular to the line of centers,

(*c*) Meets the larger circle at the same point as the line of centers.

8. Given a straight line and two points A and B on the same side of it. Find a point C on the line such that the sum of the segments AC and BC shall be the least possible.

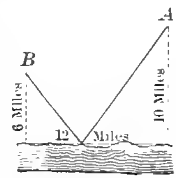


Solution. In the figure let B' be symmetric to B with respect to the line. Draw AB' meeting the line in C. Then C is the required point. For let C' be any other point on the line. Then $AC' + C'B > AC + CB$. The proof depends upon § 128 and Ax. III, § 61. Give it in full detail.

9. If in the figure preceding, AD is perpendicular to the line, prove that $\triangle ADC \sim \triangle CBE$ and hence $\dfrac{AD}{BE} = \dfrac{DC}{CE}$.

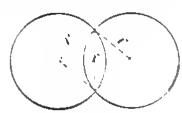
10. If in Ex. 8, $DE = a$, $AD = b$, $BE = c$, find CD and CE.

11. Two towns, A and B, are 10 and 6 miles respectively from a river and A is 12 miles farther up the river than B. A pumping station is to be built which shall serve both towns. Where must it be located so that the total length of water main to the two towns shall be the least possible?

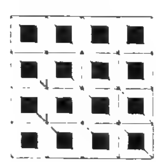


12. Two factories are situated on the same side of a railway at different distances from it. A spur is to be built to each factory and these are to join the railway at the same point. State just what measurements must be made and how to locate the point where these spurs should join the main line in order to permit the shortest length of road to be built.

13. Two equal circles of radius r intersect so that their common chord is equal to r. Find the area of the figure which lies within both circles.



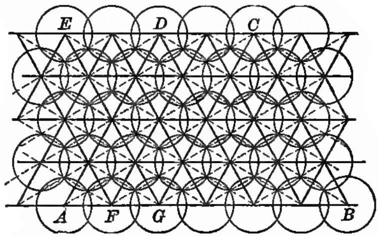
14. In the accompanying design for oak and mahogany parquet flooring the large squares are 6 inches and the small black ones $2\frac{1}{2}$ inches on a side. What fraction of the whole is the mahogany (the black squares)?



15. Construct circles on the three sides of a right triangle as diameters. Compare the area of the circle constructed on the hypotenuse with the sum of the areas of the other two. Prove.

16. In the accompanying design for grill work:

(*a*) Find the angles ABC, BAD, and AGC.



(*b*) If the radius of each circle is r, find the distance AF.

(*c*) What fraction of the area of the parallelogram $GBCE$ lies within one circle only?

(*d*) If the radius of each circle is r, find the distance between two horizontal lines.

(*e*) Construct the whole figure.

SUGGESTIONS. (1) Find AF and lay off points on AB.

(2) Find $\angle BAD$ and construct it.

(3) Through the points of division on AB draw lines parallel to AD.

(4) From A along AD lay off segments equal to AF and through these division points construct lines parallel to AB.

(5) Along DC lay off segments equal to AF. Connect points as shown in the figure.

(*f*) From the construction of the figure does it follow that

$$\angle DAB = \angle EGC?$$

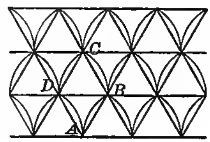
17. Prove that the sum of three altitudes of a triangle is less than its perimeter.

18. In the accompanying design for grill work, the arcs are constructed from the vertices of the equilateral triangles as centers.

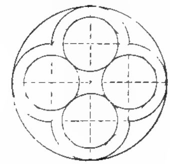
(*a*) Prove that two arcs are tangent to each other at each vertex of a triangle.

(*b*) Find the area bounded by the arcs AB, BC, CD, DA.

(*c*) Find the ratio between the area in (*b*) and the area of the triangle DBC.



19. The character of the accompanying design for a window is obvious from the figure. Denote the radius of the large circle by R, of the semicircles by R', and of the small circles by r.



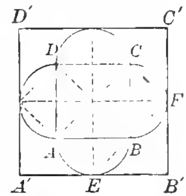
(a) If $R = 8$ feet find R' and r.

(b) Find R' and r in terms of R.

(c) What fraction of the area of the large circle lies within the four small circles?

(d) What fraction of the area of the large circle lies outside the four semicircles?

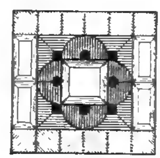
(e) If $R = 10$, find the area inclosed within the four small circles.



20. In the accompanying design for a stained glass window:

(a) What part of the square $A'B'C'D'$ lies within $ABCD$?

(b) If $A'B' = 4$ feet, find the sum of the areas of the semicircles.



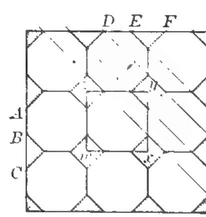
(c) Find the area inclosed by the line-segments FB', $B'E$ and the arcs FB and BE, if EB' is $1\frac{1}{2}$ feet.

(d) Find the areas required in (b) and (c) if $A'B' = a$.

21. The accompanying design for tile flooring consists of regular octagons and squares. The design can be constructed by drawing parallel lines as shown in the figure.

(a) If a side of the octagon AB is given, find BC, DE, and EF by construction.

Find the ratio of any two of these segments.



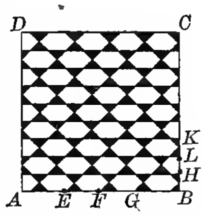
(b) If $AB = a$, find BC.

(c) If $AB = a$ find the area of the square $xyzw$.

(d) At what angles do the oblique lines meet the horizontal?

(e) Construct the figure by laying off the required points on the sides, drawing parallel lines in pencil, inking in the sides of the octagons and erasing the remainder of the lines.

22. This design for tile flooring is constructed by first making a network of squares and then drawing horizontal lines cutting off equal triangles from the squares.



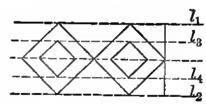
(*a*) At what angle to the base of the design are the oblique lines?

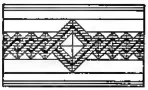
(*b*) If each of the small squares is 6 inches on a side, find EF and HL.

(*c*) Find EF and HL if the side of a small square is a.

(*d*) What fraction of the whole area is occupied by the black triangles?

23. Five parallel lines are drawn at uniform distances apart, as shown in the figure.

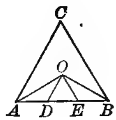




(*a*) If these lines are 4 inches apart, find the width of the strip from which the squares are made, so that their outer vertices shall just touch l_1 and l_2, and the corresponding inner vertices shall touch l_3 and l_4.

(*b*) What part of the area between l_1 and l_2 would be occupied by a series of such squares arranged as shown in the figure?

24. ABC is an equilateral triangle. AO and BO bisect its base angles. OD and OE are drawn parallel to CA and CB, respectively. Show that $AD = DE = EB$.



25. If one base of a trapezoid is twice the other, then each diagonal divides the other into two segments which are in the ratio 1 : 2.

26. If one base of a trapezoid is n times the other, show that each diagonal divides the other into two segments which are in the ratio $1 : n$.

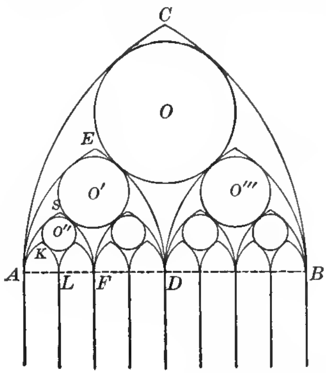
27. Prove that if an angle of a parallelogram is bisected, and the bisector extended to meet an opposite side, an isosceles triangle is formed.

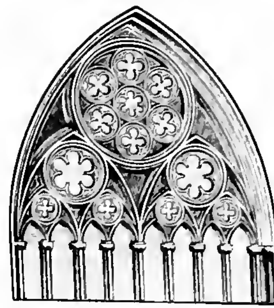
Is there any exception to this proposition? Are two isosceles triangles formed in any case?

28. Prove that two circles cannot bisect each other.

29. Find the locus of all points from which a given line-segment subtends a constant angle.

30. In the figure, equilateral arches are constructed on the base AB, and on its subdivisions into halves, fourths, and eighths.





From Lincoln Cathedral, England.

(a) Show how to construct the circle O tangent to the arcs as shown in the figure.

Suggestion. The point O is determined by drawing arcs from A and B as centers with BF as a radius (Why?).

(b) Show how to complete the construction of the figure.

(c) If $AB = 12$ feet, find the radii of the circles O, O', O''.

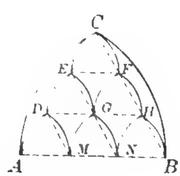
(d) Find these radii if $AB = s$ (*span* of the arch).

(e) What part of the area of the arch ABC is occupied by the arch ADE? by the arch AFS? by ALK?

(f) The sum of the areas of the seven circles is what part of the area of the whole arch?

(g) The sum of the areas of the two equal circles O' and O''' is what part of the area of the circle O?

31. The accompanying church window design consists of the equilateral arch ABC and the six smaller equal equilateral arches.

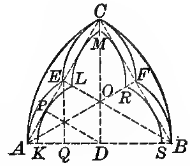


(a) If $AB = 8$ feet, find the area bounded by the arcs MG, GE, ED, DM.

(b) If $AB = 8$ feet, find the area bounded by the arcs AC, CE, ED, DA.

(c) Find the areas required under (a) and (b) if $AB = a$.

32. In the figure, ABC is an equilateral arch. D, E, and F, the middle points of the sides of the triangle ABC, are centers of the arcs AE, KL, BF, and SR; CF and MR; EC and LM respectively.



(*a*) Prove that the arc with center D and radius DA passes through the point E.

(*b*) Prove that arcs with centers D and F, and tangent to the segment AC, meet on the segment BE.

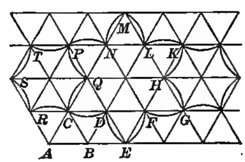
(*c*) If $AB = a$, find KS.

(*d*) Can we find the area bounded by the segment AB and the arcs BF, FC, CE, and EA when AB is given? If so, find this area when $AB = a$.

(*e*) Can we find the area bounded by KS and the arcs SR, RM, ML, LK, when AB is given? If so, find this area when $AB = 6$ feet.

33. Prove that the altitude of an equilateral triangle is three times the radius of its inscribed circle.

34. The accompanying grill design is based on a network of congruent equilateral triangles. Arcs are constructed with vertices of the triangles as centers.



(*a*) If $AB = 6$ inches, find the area bounded by $\widehat{CQ}$, $\widehat{QP}$, $\widehat{PT}$, $\widehat{TS}$, $\widehat{SR}$, $\widehat{RC}$.

(*b*) Has the figure consisting of these arcs a center of symmetry? How many axes of symmetry has it?

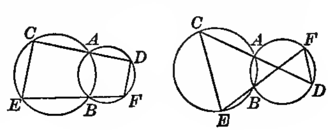
(*c*) Find the area required under (*a*) if $AB = a$.

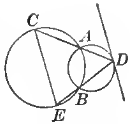
(*d*) If $AB = 4$ inches, find the area bounded by $\widehat{CD}$, $\widehat{DE}$, $\widehat{EF}$, $\widehat{FG}$, $\widehat{GH}$, $\widehat{HK}$, etc.

(*e*) Has the figure consisting of these arcs a center of symmetry? How many axes of symmetry has it?

(*f*) Find the area required under (*d*) if $AB = a$.

35. Two circles intersect in the points A and B. Through A a line is drawn, meeting the two circles in C and D respectively, and through B one is drawn meeting the circles in E and F respectively. Prove that CE and DF are parallel.



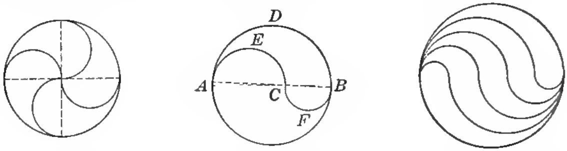


36. Prove that if the points D and F coincide in the preceding example the tangent at D is parallel to CE.

37. Two circles are tangent internally at A. Prove that all chords of the larger circle through A are divided proportionally by the smaller circle.

38. Chords are drawn through a fixed point on a circle. Find the locus of points which divide them into a fixed ratio.

39. Squares are inscribed in a circle, a semicircle, and a quadrant of the same circle. Compare their areas.



40. In a given circle two diameters are drawn at right angles to each other. On the radii thus formed as diameters semicircles are constructed. Show that the four figures thus formed are congruent.

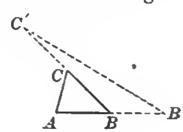
41. Let C be any point on the diameter AB of a circle.

(a) Compare the length of the arc ADB with the sum of the lengths of the arcs AEC and CFB.

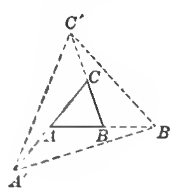
(b) Show that if $AB = 3\ CB$, then the area inclosed by the arcs BFC, CEA, ADB, is one third the area of the circle.

(c) Show that if $AB = m \cdot CB$, then the area inclosed by these arcs is one mth of the area of the circle.

42. By means of arcs constructed as shown in the third figure divide the area of a circle into any given number of equal parts. Make the construction.



43. Two sides AB and BC of a triangle are extended their own lengths to B' and C' respectively. Compare the areas of the triangles ABC and $BB'C'$.



44. The three sides of a triangle ABC are extended to A', B', C' as shown in the figure. Compare the areas of the triangles ABC and $A'B'C'$:

(a) if $BB' = AB$, $CC' = BC$, and $AA' = CA$;

(b) if $BB' = l \cdot AB$, $CC' = m \cdot BC$, and $AA' = n \cdot CA$.

CHAPTER VI.

VARIABLE GEOMETRIC MAGNITUDES.

GRAPHIC REPRESENTATION.

366. It is often useful to think of a geometric figure as **continuously varying** in size and shape.

E.g. if a rectangle has a fixed base, say 10 inches long, but an altitude which varies continuously from 3 inches to 5 inches, then the area varies continuously from $3 \cdot 10 = 30$ to $5 \cdot 10 = 50$ square inches.

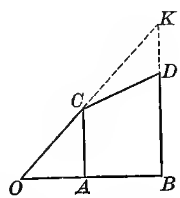
We may even think of the altitude as starting at zero inches and increasing continuously, in which case the area starts at zero and increases continuously.

From this point of view many theorems may be represented **graphically.** The graph has the advantage of exhibiting the theorem for all cases at once.

For a description of graphic representation see Chapter V of the authors' High School Algebra, Elementary Course.

367. *If in the figure* $AC \parallel BD$ *and* OA *and* OB *are commensurable, and if* $\frac{OA}{OB} = \frac{AC}{BD}$, *then* O, C, *and* D *lie in a straight line.*

For suppose D not in a line with OC. Produce OC and BD to meet at K.



Then $\frac{OA}{OB} = \frac{AC}{BK}$.

But by hypothesis $\frac{OA}{OB} = \frac{AC}{BD}$.

Hence, $BD = BK$. (Why?)

Therefore D coincides with K, and O, C and D are in a straight line.

368. THEOREM. *The areas of two rectangles having equal bases are in the same ratio as their altitudes.*

Graphic Representation. For rectangles with commensurable bases and altitudes we have

$$Area = base \times altitude. \qquad (\S\ 303)$$

Consider rectangles each with a base equal to b, altitudes h_1, h_2, h_3, etc., each commensurable with b, and areas A_1, A_2, A_3, etc.

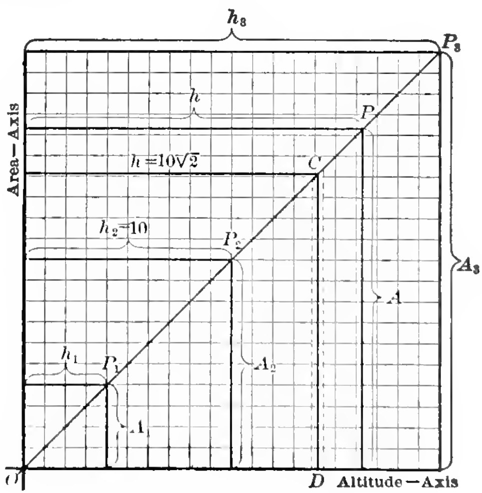
Then, $$\frac{A_1}{A_2} = \frac{bh_1}{bh_2} = \frac{h_1}{h_2}, \quad \frac{A_2}{A_3} = \frac{bh_2}{bh_3} = \frac{h_2}{h_3}, \text{ etc.} \qquad (1)$$

We exhibit graphically the special case where $b = 10$. Let one horizontal space represent one unit of altitude and one vertical space ten units of area.

Thus, the point P_2 has the ordinate $A = 10$ vertical units (representing 100 units of area) and the abscissa $h_2 = 10$ horizontal units.

Similarly locate P_1 and P_3 whose abscissas are h_1 and h_3 and ordinates A_1 and A_2.

Using equation (1) and § 367, show that O, P_1, P_2, P_3 lie in the same straight line.



If we suppose that while the base of the rectangle remains fixed, the altitude varies continuously through all values from $h_2 = 10$ to $h_3 = 20 = 2 \times 10$, then it must take among other values the value $10\sqrt{2}$.

Using $10\sqrt{2}$ as an abscissa, the question is, whether CD is the area ordinate corresponding to it.

This area ordinate is not less than CD and commensurable with the base, for in that case the altitude would be less than $10\sqrt{2}$, and for the same reason it is not greater than CD and commensurable with the base.

But we can find line-segments less than $10\sqrt{2}$ or greater than $10\sqrt{2}$ and *as near* to $10\sqrt{2}$ as we please.

Hence we conclude that the area ordinate for the rectangle whose altitude is $10\sqrt{2}$ is CD, that is, the point C lies in the line OP_2.

In like manner, the point determined by any other abscissa incommensurable with the base is shown to lie on the line OP_2.

Since the abscissa and ordinate of any point on OP are equal, we have for any altitude,

$$\frac{A}{A_1} = \frac{h}{h_1}.$$

369. The preceding theorem may also be stated:

The area of a rectangle with a fixed base varies directly as its altitude.

This means that if A and h are the varying area and altitude respectively, and if A_1 and h_1 are the area and altitude at any given instant, then

$$\frac{A}{A_1} = \frac{h}{h_1} \text{ or } A = \frac{A_1}{h_1} \cdot h \text{ or } A = kh, \text{ where } k \text{ is the \textit{fixed ratio} } \frac{A_1}{h_1}.$$

The graph representing the relations of two variables when one varies *directly* as the other is always a straight line.

370. EXERCISES.

1. Make a graph to show that the area of two rectangles having equal altitudes are in the same ratio as their bases.

2. Show by a graph that the area of a triangle having a fixed altitude varies as the base, and having a fixed base varies as the altitude.

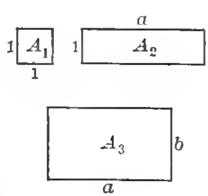
3. Represent graphically the relation between two line-segments both of which begin at zero, and one of which increases three times as fast as the other. Five times as fast. One half as fast.

4. If areas be represented by the length of a line-segment as in § 368, which question in Ex. 3 applies to the altitude and area of a parallelogram having a fixed base and varying altitude? Which applies to a triangle having a fixed base and varying altitude?

371. THEOREM. *The area of a rectangle is equal to the product of its base and altitude.*

Proof: Using the graphic representations of §§ 368, 370, Ex. 1, we have for all cases,

$$\frac{A_2}{A_1}=\frac{a}{1} \text{ and } \frac{A_3}{A_2}=\frac{b}{1}.$$



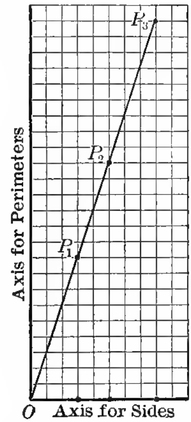
Hence $$\frac{A_3}{A_1}=\frac{A_2}{A_1}\times\frac{A_3}{A_2}=\frac{a}{1}\times\frac{b}{1}=ab.$$

But A_1 is the *unit of area.* Hence ab represents the numerical measure of A_3 by the area unit.

That is, *Area = base* × *altitude.*

372. PROBLEM. Make a graphic representation of the theorem: *The perimeters of similar polygons are in the same ratio as any two corresponding sides.*

SOLUTION. First consider the special case of equilateral triangles. On the horizontal axis lay off the lengths of one side of several such triangles, and on the vertical axis lay off the lengths of the corresponding perimeters. Show that the points so obtained lie in a straight line.

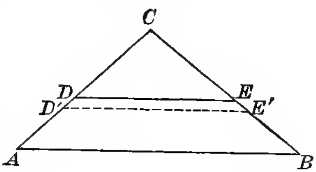


373. **EXERCISES.**

1. In the manner above graph the relation between the perimeters and sides of squares. Of regular pentagons. Of regular hexagons. Of rhombuses.

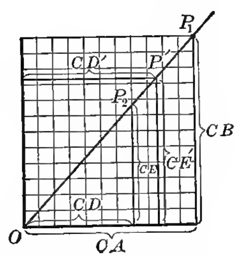
2. If a side of a given regular polygon is a and its perimeter p, graph the relation of the perimeters and corresponding sides of polygons similar to the given polygon.

374. THEOREM. *If a line is parallel to one side of a triangle and cuts the other two sides, then it divides these two sides in the same ratio.*



Graphic representation:

Lay off CA along the horizontal axis and CB along the vertical axis, thus locating the point P_1.



In like manner find P_2 with the coördinates CD and CE. If CD and CA are commensurable, we know that

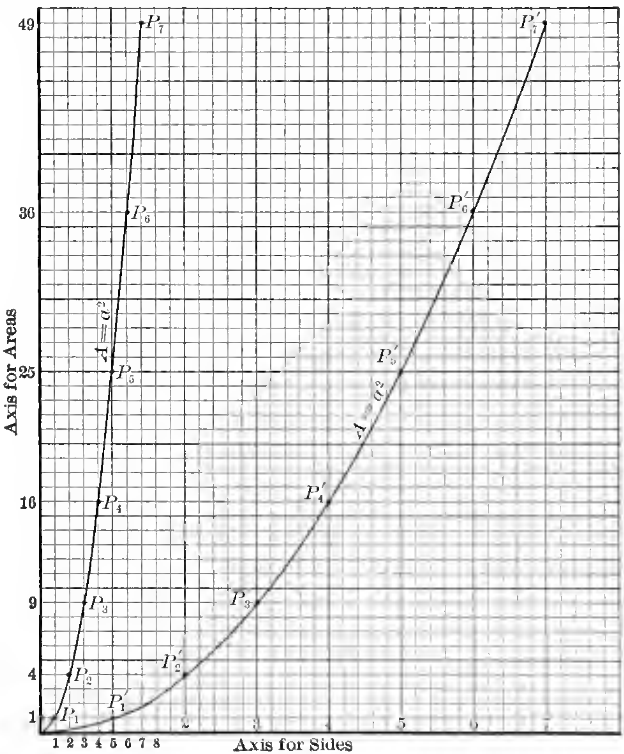
$$\frac{CD}{CA}=\frac{CE}{CB}.$$

Hence O, P_2 and P_1 are collinear.

If CD' and CA are incommensurable, show, as in § 368, that the corresponding point P' lies on the line OP_1, and hence, as in that case, also

$$\frac{CD'}{CA}=\frac{CE'}{CB}.$$

375. PROBLEM. *To represent graphically the relation between the area and side of a square as the side varies continuously.*



SOLUTION. On the horizontal axis lay off the segments equal to various values of the side *s*, and on the vertical axis lay off segments equal to the corresponding areas *A*.

(1) If one horizontal space represents one unit of length of side, and one vertical space one unit of area, then the points P_1, P_2, P_3, etc., are found to lie on the *steep* curve.

(2) If five horizontal spaces are taken for one unit of length of side and one vertical space for one unit of area, then the points P_1', P_2', P_3', etc., are found and the less steep curve is the result.

The student should locate many more points between those here shown and see that a *smooth curve* can be drawn through them all in each case.

The graph of the relation between two variables, one of which varies as the square of the other, is always similar to the one here given.

376. The area of a square is said to vary as the *square* of one of its sides, that is, $A = s^2$.

For example, the theorem: *The areas of two similar polygons are in the same ratio as the squares of any two corresponding sides*, means that if a given side of a polygon is made to vary continuously while the polygon remains similar to itself, the area of the polygon varies continuously as the square of the side.

377. EXERCISES.

1. From the last graph find approximately the areas of squares whose sides are 3.4; 5.25; 6.35.

2. Find approximately from the graph the side of a square whose area is 28 square units; 21 square units; 41.5 square units.

3. Construct a graph showing the relation between the areas and sides of equilateral triangles.

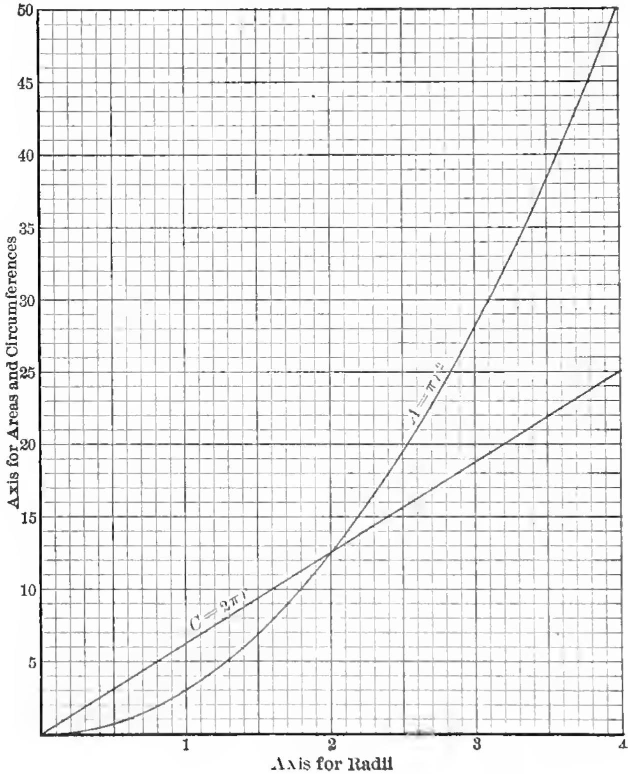
4. Given a polygon with area A and a side a. Construct a graph showing the relation between the areas and the sides corresponding to a in polygons similar to the one given.

5. From the graph constructed in Ex. 3, find the area of an equilateral triangle whose sides are 4. Also of one whose sides are 6. Compute these areas and compare results.

6. Construct a graph showing the relation between a side and the area of a regular hexagon. By means of it find the area of a regular hexagon whose sides are 6. Compare with the computed area.

378. PROBLEM. *To construct a graph showing the relation between the radius and the circumference of a circle, and between the radius and area of a circle, as the radius varies continuously.*

SOLUTION. Taking ten horizontal spaces to represent one unit of length of radius, and one vertical space for one unit of circumference in one graph and one unit of area in the other, we find the results as shown in the figure.



379. EXERCISES.

1. From the graph find approximately the circumference of a circle whose radius is 2.7, also of one whose radius is 3.4.

2. Find the radius of a circle whose circumference is 17, also of one whose circumference is 23.

3. Find the areas of circles whose radii are 1.9, 2.8, 3.6.

4. Find the radii of circles whose areas are 13.5, 25.5, 37, 45.

5. How does the circumference of a circle vary with respect to the radius?

6. How does the area of a circle vary with respect to the radius?

7. Find the radius of that circle whose area in square units equals its circumference in linear units.

DEPENDENCE OF VARIABLES.

380. In the preceding pages we have considered certain areas or perimeters of polygons as varying through a series of values. For example, if a rectangle has a fixed base and varying altitude, then the area also varies depending on the altitudes. The fixed base is called a **constant**, while the altitude and area are called **variables.**

The altitude which we think of as varying at our pleasure is called the **independent variable**, while the area, being dependent upon the altitude, is called the **dependent variable.**

381. The dependent variable is sometimes called a **function** of the independent variable, meaning that the two are connected by a definite relation such that for any definite value of the independent variable, the dependent variable also has a definite value.

Thus, in $A = s^2$ (§ 376), A is a function of s, since giving s any definite value also assigns a definite value to A.

Similarly, C is a function of r in $C = 2\pi r$, and A is a function of r in $A = \pi r^2$.

382. **EXERCISES.**

Justify each of the following statements, remembering that a variable y varies *directly* as another variable x if $y = kx$, and *directly as the square* of x if $y = kx^2$, where k is some constant. Find k in each case.

1. The area of a rectangle with constant base varies directly as its altitude. Find the value of k.

SUGGESTION. By § 368, $\frac{A}{A_1} = \frac{h}{h_1}$ or $A = \frac{A_1}{h_1} \cdot h$. Hence $A = kh$. In this case $k = \frac{A_1}{h_1} = b$, the constant base. See also § 369.

2. The area of a square varies directly as the square of its side.

SUGGESTION. Since $A = s^2$ and $A_1 = s_1^2$, we have $\frac{A}{A_1} = \frac{s^2}{s_1^2}$ or $A = \frac{A_1}{s_1^2} \cdot s^2$. That is, $A = ks^2$ where $k = \frac{A_1}{s_1^2} = 1$. See § 376.

3. An angle inscribed in a circle varies directly as the intercepted arc. Show that in this case $k = \frac{1}{2}$.

4. A central angle in a circle varies as the intercepted arc. Show that in this case $k = 1$.

5. In the figure of § 374, show that DE varies directly as CD if DE moves, remaining parallel to AB.

6. An angle formed by two chords intersecting within a circle varies directly as the *sum* of the two arcs intercepted by the angle and its vertical angle.

7. An angle formed by two secants intersecting outside a circle varies directly as the *difference* of the two intercepted arcs.

8. If a polygon varies so as to remain similar to a fixed polygon, then its perimeter varies directly as any one of its sides. Show that $k = \frac{P_1}{s_1}$, where P_1 and s_1 are the perimeter and side of the fixed polygon.

9. In the preceding, the area of the polygon varies directly as the square of any one of its sides.

10. The circumference of a circle varies directly as the radius, and its area varies directly as the square of its radius.

11. Plot $y = kx$ for $k = 1, \frac{1}{2}, 3, 4$.

12. Plot $y = kx^2$ for $k = 1, \frac{1}{2}, 3, 4$.

Notice that the graphs in Ex. 11 are all straight lines, while those in Ex. 12 are curves which rise more and more rapidly as the independent variable increases. See also § 378.

LIMIT OF A VARIABLE.

383. If a regular polygon (§ 357) is inscribed in a circle of fixed radius, and if the number of sides of the polygon be continually increased, for instance by repeatedly doubling the number, then the apothem, perimeter, and area are all variables depending upon the number of sides. That is, each of these is a function of the number of sides.

Now the greater the number of sides the more nearly does the apothem equal the radius in length. Indeed, it is evident that the difference between the apothem and the radius will ultimately become less than any fixed number, however small. Hence we say that the apothem **approaches the radius as a limit** as the number of sides increases indefinitely.

384. Similarly by § 352 the perimeters of the polygons considered in the preceding paragraph may be made as nearly equal to the circumference as we please by making the number of sides sufficiently great.

Hence we define the circumference of a circle as the **limit of the perimeter** of a regular inscribed polygon as the number of sides increases indefinitely.

It also follows from § 353 that the circumference of a circle may be defined as the limit of the perimeter of a circumscribed polygon as the number of sides is increased.

Likewise we may define the area of a circle as the **limit of the area** of the inscribed or the circumscribed polygon as the number of sides increases indefinitely. See § 361.

385. The notion of a limit may be used to define the length of a line-segment which is incommensurable with a given unit segment.

Thus, the diagonal d of a square whose side is unity is $d = \sqrt{2}$. Hence d may be defined as the *limit of the variable line-segment* whose successive lengths are 1, 1.4, 1.41, 1.414···. See §§ 234, 240.

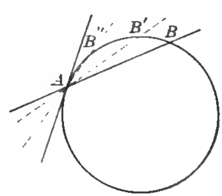
In like manner, the length of any line-segment, whether commensurable or incommensurable with the unit segment, may be defined in terms of a limit.

Thus, if a variable segment is increased by successively adding to it one half the length previously added, then the segment will approach a limit. If the initial length is 1, then the successive additions are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., and the successive lengths are 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $1\frac{15}{16}$, $1\frac{31}{32}$, etc. Evidently this segment approaches the limit 2.

Hence 2 may be defined as the limit of the variable segment, whose successive lengths are 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, etc., as the number of successive additions is increased indefinitely.

386. A useful definition of a tangent to a circle, or to any other smooth curve, may be given in terms of a limit.

Let a secant cut the curve in a fixed point A and a variable point B, and let the point B move along the curve and approach coincidence with A, thus making the secant continually vary its direction.



Then the tangent is defined to be the **limiting position of the secant** as B approaches A indefinitely.

This definition of a tangent is used in all higher mathematical work. It includes the definition given in the case of the circle in § 183.

387. The functional relation between variables and the idea of a limit as illustrated above are two of the most important concepts in all mathematics. The whole subject is much too difficult for rigorous consideration in this course.

388. **EXERCISES.**

1. Find the limit of a variable line-segment whose initial length is 6 inches, and which varies by successive additions each equal to one half the preceding.

2. Find the limit of a variable line-segment whose initial length is 1 and whose successive additions are .3, .03, .003, etc.

3. Construct a right triangle whose sides are 1 and 2. By approximating a square root, find five successive lengths of a segment which approaches the length of the hypotenuse as a limit.

4. If one tangent to a circle is fixed and another is made to move so that their intersection point approaches the circle, what is the limiting position of the moving tangent? What is the limit of the measure of the angle formed by the tangents?

5. The arc AB of 74° is the greater of the two arcs intercepted between two secants meeting at C outside the circle. The points A and B remain fixed while C moves up to the circle. What is the limit of the angle formed by the secants? The limit of the measure of their angle?

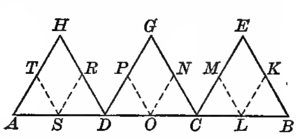
6. If in the preceding the secants meet within the circle, what is the limit of their angle and also of the measure of this angle?

7. If in Ex. 5 one secant and the intersection point remain fixed, while the other secant approaches the limiting position of a tangent at the point A, find the limit of the measure of the included angle.

8. If in Ex. 6 one secant and the point of intersection remain fixed while the other secant swings so as to make the included angle approach a straight angle, find the limit of the measure of the angle.

9. If in Exs. 5 and 6 the moving point crosses the circle, state the theorem on the measurement of the angle in question so as to apply equally well whether the point is inside or outside the circle.

10. A fixed segment AB is divided into equal parts, and equilateral triangles are constructed on each part as a base, as $\triangle s\ BEC$, CGD, DHA. Then each base is divided into equal parts and equilateral triangles are constructed on these parts as bases. What is the limit of the sum of the perimeters of these triangles as the number of them is increased indefinitely? What is the limit of the sum of their areas?



CHAPTER VII.

REVIEW AND FURTHER APPLICATIONS.

ON DEFINITIONS AND PROOFS.

389. A definition is a statement that a certain word or phrase is to be used in place of a more complicated expression.

Thus the word "triangle" is used instead of "the figure formed by three segments connecting three non-collinear points."

In geometry we may distinguish two classes of words:

(*a*) **Technical words** representing geometric concepts such as line, plane, polygon, circle, etc.

(*b*) **Words of ordinary speech** not included in the first class.

The meaning of words in the second class is taken for granted without any definition whatever. It is not possible to define *every* word of the first class, for every definition brings in new technical terms which in turn require definition.

Thus, one of the many definitions of "point" is "that which separates one part of a line from the adjoining part."

In this definition the technical terms "separate," "line," "adjoining" are used. If we try to define these, still other terms are brought in which need to be defined, and so on. The only escape is defining in a circle, which is not permitted in a logical science.

Since it is thus evident that some terms must be used without being defined, it is best to state which ones are left undefined.

390. In this book *point*, *straight line*, or simply *line*, *plane*, *size* and *shape* of figures are not defined. Nor do we define the expression, *a point is between two other points*, or *a point lies on a segment.*

Descriptions of some of these terms which do not, however, constitute definitions in the logical sense are given in §§ 1–5.

Other technical terms are defined by means of these simple undefined words with which we start.

Thus, "a segment is that part of a *line* which lies *between* two of its *points*" is a definition of *segment* in terms of the three undefined words, *point*, *line*, and *between*.

391. A geometric proposition consists of an affirmation that a geometric figure has some property not explicitly specified in its definition.

A proposition is said to be **proved** if it is shown to follow from other propositions which are admitted to be true. Hence every geometric proposition demands for its proof certain other propositions.

392. It is obvious, therefore, that **certain propositions must be admitted without proof.** Such unproved propositions are called **axioms.**

In order that a set of axioms for geometry shall be complete it must be possible to prove that every theorem of geometry follows from them. The set of axioms used in this text is not complete.

For instance, it is assumed without formal statement that if A, B, C, are three points on a segment we cannot have at the same time B between A and C and C between A and B; that the diagonals of a convex quadrilateral intersect each other; that a ray drawn from the vertex of an angle and included between its sides intersects every segment determined by two points, one on each side of the angle. In like manner many other tacit assumptions are made.

393. In a **complete logical treatment** every undefined term must occur in one or more axioms, since all knowledge of this term *in a logical sense* comes from the axioms in which it is found. In this text not every undefined term occurs in an axiom, for instance, the word *between.*

The axioms are, of course, based on our *space intuition*, or on our experience with the space in which we live. It is interesting to notice, however, that the axioms transcend that experience both as to *exactness* and *extent.* For instance, we have had no experience with *endless lines*, and hence we cannot know directly from experience whether or not there are complete lines which have no point in common. See §§ 89, 96.

394. **Proofs** are of two kinds, **direct** and **indirect.** A *direct proof* starts with the hypothesis and leads step by step to the conclusion.

An *indirect proof* starts with the hypothesis and with the assumption that the conclusion does not hold, and shows that this leads to a contradiction with some known proposition. Or it starts simply with the assumption that the conclusion does not hold and shows that this leads to a contradiction with the hypothesis. This kind of proof is based upon the logical assumption that a proposition must either be true or not true. The proof consists in showing that if the proposition were not true, impossible consequences would follow. Hence the only remaining possibility is that it must be true.

395. Every proposition in geometry refers to some **figure.** See § 12. The essential characteristic of a figure is its *description* in words and not the *drawing* that represents it. Each drawing represents just one figure from a class of figures defined by the description. Thus we say

"let *ABCD* be a convex quadrilateral," and we construct a particular quadrilateral. We must then take care that all we say about it applies to *any* figure whatever so long as it is a convex quadrilateral. The logic of the proof must be entirely independent of the *appearance* of the constructed figure.

The description of the figure must contain all the conditions given by the hypothesis.

A good way to show that the *description* of the figure is what really enters into the proof, is to let one pupil describe the figure in words and each of the others draw a figure of his own to correspond to that description. The proof must then be such as to apply to every one of these figures though there may not be two of them exactly alike.

396. EXERCISES.

1. Every word in the language is defined in the dictionary. How is this possible in view of what has been said about the impossibility of defining every word?

2. Can we determine experimentally whether or not the space in which we live satisfies the parallel line axiom (§ 96)?

3. Can we determine experimentally whether or not there can be more than one straight line through two given points?

4. Which theorems of Chapter I are found by direct proof and which by indirect proof?

5. If two triangles have two angles of the one equal to two angles of the other, and also any pair of corresponding sides equal, the triangles are congruent.

6. If two triangles have two sides of the one equal to two sides of the other, and also any pair of corresponding angles equal, the triangles are congruent in all cases except one. Discuss the various cases according as the given equal angles are greater than, equal to, or less than a right angle, and are, or are not, included between the equal sides, and thus discover the exceptional case.

7. State a theorem on the congruence of right triangles which is included in the preceding theorem.

8. What theorems of Chapter I on parallel lines can be proved without the parallel line axiom?

9. What theorems of Chapter I on parallelograms can be proved without the parallel line axiom?

10. What regular figures of the same kind can be used to exactly cover the plane about a point used as a vertex?

11. What combinations of the same or different regular figures can be used to exactly cover the plane about a point used as a vertex?

12. Suppose it has been proved that the base angles of an isosceles triangle are equal but that the converse has not been proved.

On this basis can it be decided whether or not the base angles are equal by simply measuring the sides?

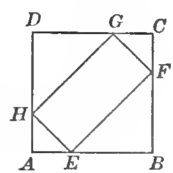
Can it be decided on the same basis whether or not the sides are equal by simply measuring the base angles? Discuss fully.

13. The sum of the three medians of a triangle is less than the sum of the sides. See Ex. 34, p. 83.

14. The sum of the three altitudes of a triangle is less than the sum of the sides.

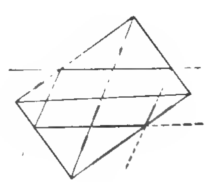
15. A triangle is isosceles (1) if an altitude and an angle-bisector coincide, (2) if an altitude and a median coincide, (3) if a median and an angle-bisector coincide.

16. If two sides of a triangle are unequal, the medians upon these sides are unequal and also the altitudes.



17. An isosceles triangle has two equal altitudes, two equal medians, and two equal angle-bisectors.

18. $ABCD$ is a square and the points E, F, G, H, are so taken that $AE = AH = CF = CG$. Prove that $EFGH$ is a rectangle of constant perimeter, whatever the length of AE.

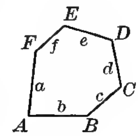


19. The bisectors of the exterior angles of a parallelogram form a rectangle the sum of the diagonals of which is the same as the sum of the sides of the parallelogram.

20. Prove that the perpendicular bisectors of the sides of a polygon inscribed in a circle meet in a point. Use this theorem to show that the statement is true of *any* triangle.

21. The bisectors of the exterior angles of any quadrilateral form a quadrilateral whose opposite angles are supplementary.

Definition. In a polygon of n sides there are n angles and hence $2n$ parts. Parts a, $\angle A$, b, $\angle B$, etc., are said to be **consecutive** if a lies on a side of $\angle A$, b lies on the other side of $\angle A$, and also on a side of $\angle B$, etc.



22. Is the following proposition true? If in two polygons each of n sides $2n-3$ consecutive parts of one are equal respectively to $2n-3$ consecutive parts of the other, the polygons are congruent.

Suggestion. Try to prove this proposition for $n = 3$, then for $n = 4$, and finally for the general polygon.

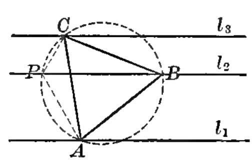
23. What theorems on the congruence of triangles are included in the preceding proposition?

A proposition may be proved *not true* by giving one example in which it does not hold.

24. Is the following proposition true? If in two polygons each of n sides $2n - 3$ parts of one are equal respectively to $2n - 3$ corresponding parts of the other, the polygons are congruent provided at least one of the equal parts is a side.

25. Given three parallel lines, to construct an equilateral triangle whose vertices shall lie on these lines.

Solution. Let P be any point on the middle line l_2. Draw PC and PA, making an angle of 60° with l_2. Through the points A, P, C construct a circle meeting l_2 in B. Then ABC is the required triangle.



Suggestion for Proof. Compare ⦞ BPA and BCA also ⦞ BPC and BAC.

26. Show how to modify the construction of the preceding example so as to make ABC similar to any given triangle.

27. If tangents are drawn to a circle at the extremities of a diameter and if another line tangent to the circle at P meets these two tangents in A and B respectively, show that $AP \cdot PB = r^2$, where r is the radius of the circle.

LOCI CONSIDERATIONS.

397. Two methods are available to show that a certain geometric figure is the locus of points satisfying a given condition.

First method:

Prove (*a*) *Every point satisfying the condition lies on the figure.*
(*b*) *Every point on the figure satisfies the condition.*

Second method:

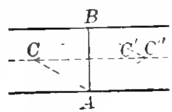
Prove (*a'*) *every point not on the figure fails to satisfy the condition.*
(*b'*) *Every point on the figure satisfies the condition.*

The first of the methods is more direct and usually more simple. See § 127.

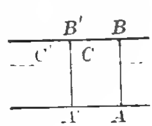
The second is likely to lead to proofs that are not general.

PROBLEMS AND APPLICATIONS.

1. AB is a fixed segment connecting two parallel lines and perpendicular to each of them. Find the locus of the vertices of all isosceles triangles whose common base is AB. Is the middle point of AB a part of this locus?



2. If in Ex. 1 AB is allowed to move always remaining perpendicular to the given lines, and if ABC is *any* triangle remaining fixed in shape, find the locus of the point C.



3. Find the locus of the centers of all parallelograms which have the same base and equal altitudes.

4. Find the locus of the centers of parallelograms obtained by cutting two parallel lines by parallel secant lines.

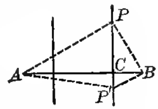
5. Find the locus of the vertices of all triangles which have the same base and equal areas.

6. Find the locus of a point whose distances from two intersecting lines are in a fixed ratio.

Note that the whole figure is symmetrical with respect to the bisector of each pair of vertical angles formed by the two given lines.

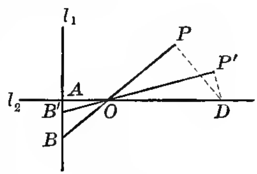
7. Find the locus of a point such that the difference of the squares of its distances from two fixed points is a constant.

Note that the locus must be symmetrical with respect to the perpendicular bisector of the segment connecting the two given points A and B.



Show that in the figure $\overline{AP}^2 - \overline{PB}^2 = \overline{AC}^2 - \overline{CB}^2$.

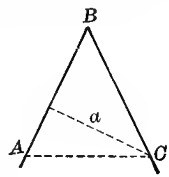
8. The lines l_1 and l_2 meet at right angles in a point A. O is any fixed point on l_2. Through O draw a line meeting l_1 in B. P is a varying point on this line such that $OB \cdot OP$ is fixed. Find the locus of P as the line swings about O as a pivot.



SUGGESTION. Draw $PD \perp$ to BP. Show that $BO \cdot OP = AO \cdot OD$. Hence we obtain a set of right triangles whose common hypotenuse is OD. Find the locus of the vertex P.

9. Find the locus of all points the sum of whose distances from two intersecting lines is equal to a fixed segment a.

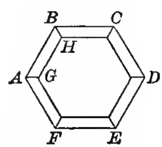
SUGGESTION. If AB and CB are the given lines, investigate points on the base AC of the isosceles triangle ABC in case the perpendicular from C on AB is equal to a.



10. The figure $ABCDEF$ is a regular hexagon. AG, BH, etc., are equal segments bisecting the angles A, B, etc.

(*a*) Prove $GH \parallel AB$.

(*b*) Prove that the inner figure is a regular hexagon.

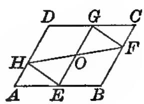


SUGGESTION. Show $ABHG$ congruent to the other similar parts by superposition.

(*c*) Find how many degrees in $\angle AGH$ and $\angle BHG$, as needed by a carpenter in making a hexagonal frame.

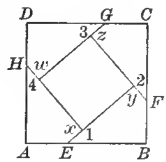
11. The figure $ABCD$ is a parallelogram. The points E, F, G, H are taken so that $AE = CG$ and $AH = CF$.

Prove $\triangle FCG \cong \triangle AEH$ and $\triangle EOH \cong \triangle FOG$.



This figure is found in an old Roman pavement in Sussex, England.

12. The figure $ABCD$ is a square. $AE = BF = CG = DH$, and Ey, Fz, Gw, Hx are so drawn that $\angle 1 = \angle 2 = \angle 3 = \angle 4$.



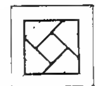
(*a*) Find $\measuredangle EyF$, FzG, etc.

Suggestion. Through B draw a line parallel to Fz and make use of the $\triangle$ thus formed.

(*b*) Prove $EBFy \cong FCGz \cong GDHw \cong HAEx$.

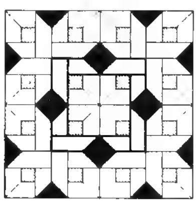
(*c*) What kind of figure is $xyzw$? Prove.

See the accompanying design.

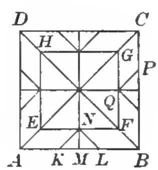


Parquet Pattern.

13. In a square $ABCD$ diagonals and diameters are drawn as shown in the figure. (A line connecting the middle points of opposite sides of a quadrilateral is a diameter.) The points K, L, N are laid off so that $MN = MK = ML$. The small triangles on the other sides are constructed congruent to KLN. Through the vertices of the triangles lines EF, FG, etc., are drawn parallel to the sides of the given square.



Tile Pattern.

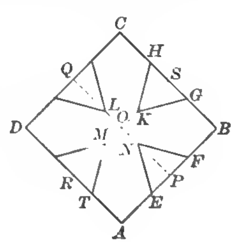


Prove that (*a*) $AKNE$, $LBFN$, etc., are congruent parallelograms and hence that EF, FG, etc., meet in points on the diagonals. (See Ex. 6, § 123.)

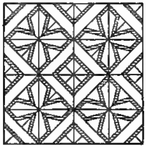
(*b*) $ABFE \cong BCGF$.

(*c*) $EFGH$ is a square.

(*d*) KN and QP lie on the same straight line.



14. The figure $ABCD$ is a square. PQ and RS are diameters. The points E, F, G, H, $\cdots$ are so taken that $AE = BF = BG = HC = \cdots$. Also $ON = OK = OL = OM$.



(*a*) Prove that $\triangle ENF \cong \triangle GKH$.

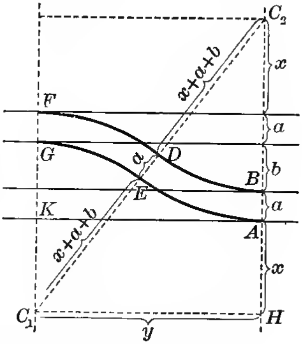
(*b*) Prove that $TMONEA \cong FNOKGB$.

(*c*) How many axes of symmetry has the figure $TAENFBGKH$ ...?

(*d*) Has this figure a center of symmetry?

See the four squares in the accompanying design.

15. A cross-over track is constructed as shown in the figure. The rails of the curved track are tangent to those of the main line at A, B, F and G. The curves are tangent to each other at D and E. The arcs GE and BD have equal radii as have the arcs AE and DF. C_1 is the center of the arcs GE and FD and C_2 is the center of the arcs BD and AE.



(*a*) Prove that C_2HC_1 is a right triangle.

(*b*) If the distance between the tracks is b feet and the distance between the rails in each track (the gauge) is a feet, show that

$$(2x + 3a + 2b)^2 = y^2 + (2x + 2a + b)^2.$$

Since a and b are given, this equation may be solved for x in terms of y or for y in terms of x.

Hence if the distance AK is known, we may use this equation to compute the radii of the arcs used in constructing the figure.

On the other hand, if the radii of the arcs are known, we may compute the distance AK.

This is a very common problem in railway construction. The construction is also used in laying out a curved street to connect two parallel streets.

16. If in the preceding problem $a = 4$ feet $8\frac{3}{4}$ inches, $b = 9$ feet, and $AK = 200$ feet (an actual case), find the radii of the arcs.

Using the same values for a and b as in the preceding example, find AK if $C_1E = 300$ feet.

FURTHER DATA CONCERNING CIRCLES.

398. In Chapter II numerous theorems were proved concerning the *equality* of arcs, angles, chords, etc.

The three following theorems involve *inequalities* of these elements. The student should construct figures in each case and give the proof in full.

399. THEOREM. *In the same circle or in equal circles, of two unequal angles at the center the greater is subtended by the greater arc; and of two unequal arcs the greater subtends the greater angle at the center.*

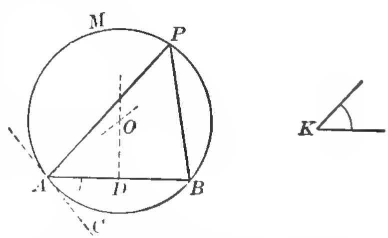
The proof is made by superposition.

400. THEOREM. *In the same circle or in equal circles, of two unequal minor arcs the greater is subtended by the greater chord; and of two unequal chords the greater subtends the greater minor arc.*

The proof depends upon the theorem of § 117.

401. THEOREM. *In the same circle or in equal circles, of two unequal angles at the center, both less than straight angles, the greater is subtended by the greater chord; and of two unequal chords the greater subtends the greater angle at the center.*

402. PROBLEM. *On a given line-segment as a chord to construct an arc of a circle in which a given angle may be inscribed.*



Construction. Let K be the given angle and AB the given segment.

It is required to construct a circle in which AB shall be a chord and such that an angle inscribed in the arc AMB shall be equal to the given angle K.

At A construct an angle BAC equal to $\angle K$.

If now a circle were passed through A and B so as to be tangent to AC at A, then one half the arc AB would be the measure of the angle BAC. See § 219.

Then any angle inscribed in the arc AMB as $\angle APB$ would be equal to $\angle BAC = \angle K$.

Hence the problem is to find the center of a circle tangent to AC at A and passing through A and B.

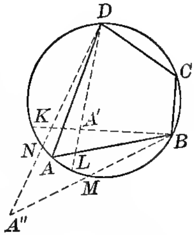
Let the student complete the construction and proof.

403. EXERCISES.

1. Prove that the problem of § 402 may be solved as follows: With A, any point on one side of the angle K, as center and AB as radius strike an arc meeting the other side of the angle at B. Circumscribe a circle about the triangle ABK.

2. Show that from any point within or outside a circle two equal line-segments can be drawn to meet the circle and that these make equal angles with the line joining the given point to the center.

3. Show that if two opposite angles of any quadrilateral are supplementary, it can be inscribed in a circle.



SUGGESTION. Let $ABCD$ be the quadrilateral in which $\angle A + \angle C = 2$ rt. $\angle$s.

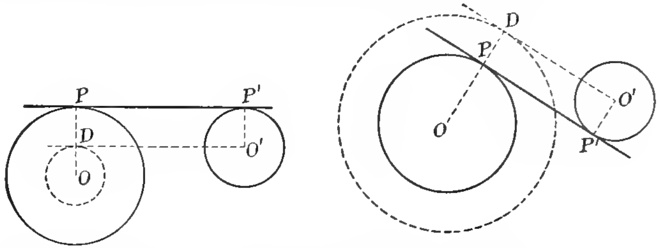
Pass a circle through B, C, and D. To prove that A lies also on the circle, and not at some inside or outside point as A' or A''.

(1) Show that $\angle A' > \angle A$ and $\angle A'' < \angle A$. See §§ 217, 222.

(2) Hence $\angle A' + \angle C > 2$ rt. $\angle$s if A' is within the circle and $\angle A'' + \angle C < 2$ rt. $\angle$s if A'' is outside the circle, both of which are contrary to the hypothesis. Hence the fourth vertex must lie on the circle.

Show that the condition that the polygon is *convex* follows from the hypothesis.

404. PROBLEM. *To draw a common tangent to two circles which lie wholly outside of each other.*



Construction. Let the given circles be O and O' of which the radius of the first is the greater.

Required to draw a tangent common to both circles.

Draw an auxiliary circle with center O and radius equal to the *difference* of the two given radii in one figure and equal to the *sum* of these radii in the other figure.

In each case draw a tangent to this auxiliary circle from O', thus fixing the point D. See § 230.

Draw the radius OD, thus fixing the point P.

Draw $O'P' \parallel DP$, thus fixing the point P'.

Then PP' is the required tangent.

Proof: Show that, in each case, $PP'O'D$ is a rectangle, thus making PP' perpendicular to the radii OP and $O'P'$, that is, tangent to each circle.

Definition. A common tangent to two circles is called **direct** if it does not cross the segment connecting the centers, and **transverse** if it does cross it.

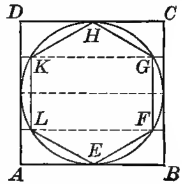
405. EXERCISES.

1. Describe the relative positions of two circles if they have two direct common tangents. Also if they have only one.

2. Describe the positions of two circles if they have two transverse common tangents; one; none.

PROBLEMS AND APPLICATIONS.

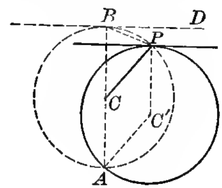
1. Find the locus of the middle points of all chords of a circle drawn through a fixed point within it.



2. The sides AD and BC of the square $ABCD$ are each divided into four equal parts and a circle inscribed in the square. Lines are drawn as shown in the figure. Show that $EFGHKL$ is a regular hexagon.

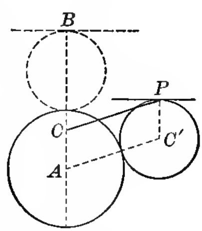
This is the construction by means of which a regular hexagonal tile is cut from a square tile.

3. In the preceding example what fraction of the area of the square is covered by the hexagon.



4. A circle of constant radius passes through a fixed point A. A line tangent to it at the point P remains parallel to a fixed line BD. Find the locus of P.

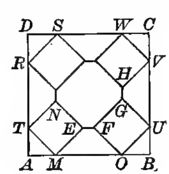
SUGGESTION. In the figure prove $AC'PC$ a parallelogram. Show that the locus consists of two circles.



5. The same as the preceding except that the circle of constant radius remains tangent to a fixed circle instead of passing through a fixed point.

SUGGESTION. On that diameter of the fixed circle which is perpendicular to the fixed line lay off $AC = C'P$. Prove $AC'PC$ a parallelogram. Show that the locus consists of two circles.

6. In the square $ABCD$, $AM = OB = BU = VC$, etc. The point E is the intersection of the segments MV and TU, F' is the intersection of OR and TU, G is the intersection of OW and US, etc. When these segments are partially erased, we have the figure.

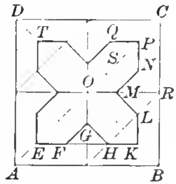


(*a*) Prove that $TMEN$, $OUGF$, etc., are squares.

(*b*) What kind of a figure is $MOFE$? Prove.

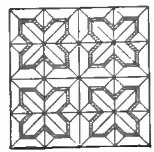
(*c*) How many axes of symmetry has the figure $NEFGH\cdots$? Has it a center of symmetry?

7. $ABCD$ is a square the mid-points of whose sides are joined. On its diagonals points E, K, P, T are taken so that $AE = BK = CP = DT$. The construction is then completed as shown in the figure. (FG and MN lie in the same straight line.)



(*a*) Prove that E, K, P, T are the vertices of a square.

(*b*) If $AB = 6$ find AE so that the area of the square $EKPT$ shall be half that of the square $ABCD$.



From the Alhambra.

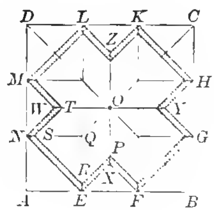
(*c*) If $AB = 6$, and if $CP = PS$, find the area of the figure $EFGHKLM \cdots$. Also find the area of the trapezoid $BKLR$.

(*d*) If $CP = \frac{2}{3}\ CS$, find the area of the inside figure and also of $BKLR$.

(*e*) What fraction of CS must CP be in order that the inside figure shall be half the square?

(*f*) If the inside figure is $\frac{1}{4}$ of the square and if $AB = 8$ inches, find CP.

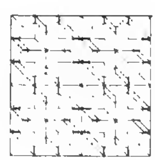
8. In the figure $ABCD$ is a square. Each of its sides is divided into three equal parts by the points E, F, G, H, ... The points E, X, Y, H; F, X, W, M; etc., lie in straight lines.



(*a*) If $AB = 6$ inches, find the area of that part of the figure which lies outside the shaded band.

(*b*) If $AB = 6$ inches, and if the width of the shaded band is $\frac{1}{4}$ inch, find the area of the band.

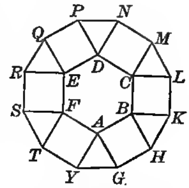
(*c*) If $AB = 8$ inches and $XP = \frac{1}{2}$ inch, find the area of that part of the figure which lies inside the band.



Parquet Flooring.

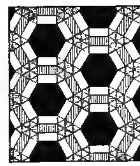
(*d*) If $AB = 8$ inches, what must be the width of the band in order that it shall occupy 10% of the area of the whole design?

(*e*) If $AB = a$ inches find the width of the band if it occupies n per cent of the area of the whole design.



9. By means of the accompanying figure find the area of a regular dodecagon whose sides are 6 inches. Notice that the dodecagon consists of the regular hexagon in the center, the six equilateral triangles, and the six squares.

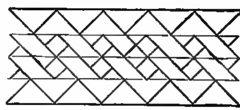
Note how this figure enters into the accompanying tile design.



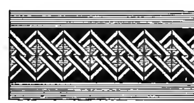
10. Find an expression for the area of a regular dodecagon whose sides are a.

11. Find the apothem of a regular dodecagon by dividing its area by half the perimeter. Also find it by finding the apothem of the hexagon and then adding a to this. Compare results.

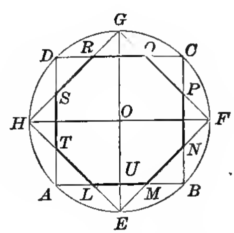
12. Find the radius of a circle circumscribed about a regular dodecagon whose side is a.



13. If the accompanying design for parquet border is 10 inches wide, find the width of the strips from which the small squares are made. Also find this result if the width of the border is a.



14. In the figure of the preceding example find the dimensions of the small triangles along the two edges if the width of the border is a inches.

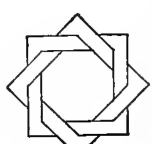


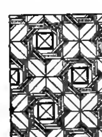
15. Show that the figure given below may be used as the basis for the design shown beside it. (See Ex. 7, p. 147.)

16. If an outside of each of the two squares shown in the figure is 6 inches, find the width of the strips of which they are made in order that each shall fit closely into the corner of the other?

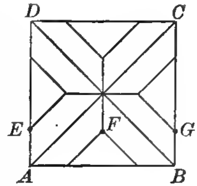
SUGGESTION. Find the altitude on EM in $\triangle UEM$ above if $AB = 6$ inches.

This is a common Arabic ornament.





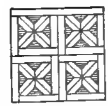
17. In the figure $ABCD$ is a square. On each side a triangle, two of whose sides are parallel to the diagonals of the square, is so constructed that the points E, F, and G lie in the same straight line.



(a) Prove that the triangles are right isosceles triangles.

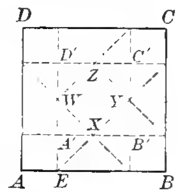
(b) What part of the square lies within the triangles?

(c) If $AB = a$, find AE so that the triangles shall occupy $\frac{1}{n}$ of the area of the square.



See the accompanying tile pattern.

18. $ABCD$ is a square and the small figures in the corners are squares.

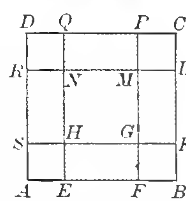


(a) Show that if the lines intersect as shown in the figure AE must be $\frac{1}{4}$ of AB.

(b) If $AB = 6$ inches find the areas of the squares $A'B'C'D'$ and $XYZW$.

19. In the tile design show that the figure within the square is the same as that of Ex. 18. What part of the large square is occupied by each shaded part?

20. $ABCD$ is a square. $AE = FB = BK = LC = CP = QD = DR = SA$.

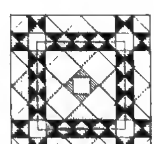


(a) If $AB = a$ and if $AE = b$, find the sum of the areas of the four shaded rectangles.

(b) If $AB = 8$ inches, find AE so that the sum of these rectangles shall be $\frac{1}{8}$ of the whole square. Interpret the two solutions.

(c) If $AB = a$ inches, find AE so that the sum of the rectangles shall be $\frac{1}{n}$ of the square.

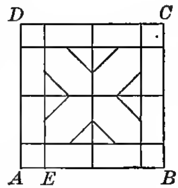
21. (a) Show that in the accompanying design for tile flooring the size of any one piece determines the size of every piece in the figure.

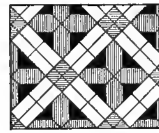


(b) What fraction of the figure is occupied by each color?

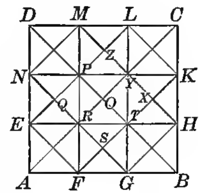
22. This tile floor design is based on the plane figure and this in turn is based on the figure of Ex. 17 where

$$AE = \frac{AB}{6}.$$





What part of the whole design is occupied by tiles of the various colors?



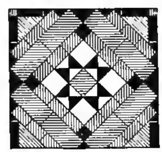
23. *ABCD* is a square. Lines are drawn parallel to the sides and intersecting as shown in the figure.

(*a*) Show how the sides of the square must be divided in order to make the lines intersect as they do.

(*b*) If $AB = 6$ inches, find the areas of the squares *SXZQ* and *RTYP*.

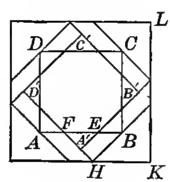
(*c*) Find these areas if $AB = a$.

(*d*) Show that the outline figure is the basis of the tile design here given.



24. *ABCD* and $A'B'C'D'$ are equal squares. $A'B'C'D'$ is surrounded by strips of equal width.

(*a*) If $AB = a$, find *AF*, *FE*, and *EB*.



(*b*) Find the width of the strips if their outer edge passes through the points *A*, *B*, *C*, *D*, when $AB = 8$ inches. Also when $AB = a$.

(*c*) If $AB = a$, find $A'H$ and hence *KL*.

(*d*) If the width of the strip is 1 inch, find *KL*.

(*e*) If the width of the strip is *b*, find *KL*.

(*f*) If $KL = c$, find the width of the strip.

When the width of the parquet floor border shown in the figure is given, (*f*) is the problem one needs to solve to know how wide a dark strip to use. Compare Ex. 16, page 235.

THE INCOMMENSURABLE CASES.

406. We have seen, in § 234, that there are segments which are **incommensurable**; that is, which have no common unit of measure, — for instance, the side and the diagonal of a square.

For **practical purposes** the lengths of such segments are approximated to any desired degree of accuracy, and their ratios are understood to be the ratios of these approximate numerical measures. See §§ 238–240.

All theorems involving the ratios of incommensurable segments, and the lengths and areas of circles, have thus far been proved only for such approximations, and these are quite sufficient for any refinements of measurement which it is possible to make.

But for **theoretical purposes** it is important to consider these incommensurable cases further, just as in algebra we not only *approximate* such roots as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., but we also deal with these surds as *exact* numbers.

For instance, in such an operation as

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 3 - 2 = 1.$$

While the length of the diagonal of a unit square cannot be expressed as an integer or as a rational fraction, that is, the quotient of two integers, we nevertheless think of such a segment as having a *definite length*, or what is the same thing, a *definite ratio* with the unit segment forming the side of the square.

E.g. If d is the diagonal of the square whose side is 1, then $d^2 = 1 + 1$ or $d = \sqrt{2}$. Now suppose $\sqrt{2} = \frac{a}{b}$, a fraction in its lowest terms. Then $2 = \frac{a^2}{b^2}$, a fraction also in its lowest terms. But a fraction in its lowest terms cannot be equal to 2. Hence $\sqrt{2}$ is neither an integer nor the quotient of two integers.

407. The following axiom and that of § 409 are *fundamental* in the consideration of incommensurable line-segments.

Axiom X. *Every line-segment* AB *has a definite length, which is greater than* AC *if* C *lies between* A *and* B.

The length of a line-segment is in every case a **number**, which is **rational** (an integer or the quotient of two integers) in case the segment is commensurable with the unit segment, but which otherwise is **irrational**. Under the operations of arithmetic these irrational numbers obey the same laws as the rational numbers.

The length of a line-segment is often called its **numerical measure.**

408. The exact ratio, or simply **the ratio,** of two line-segments is the quotient of their numerical measures, whether these are rational or irrational. That is, every such ratio is a number.

It is obvious that a segment may be constructed whose length is any given rational number. We have also seen how to construct with ruler and compasses segments whose lengths are certain irrational numbers, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. See Ex. 3, § 295.

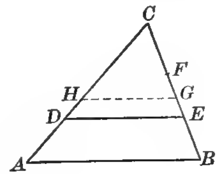
409. We now assume the following

Axiom XI. *For any given number* K *there exists a line segment whose length is* K.

This does not imply that it is possible by means of the ruler and compasses to *construct* a segment whose length is any given *irrational* number. For instance, we cannot thus construct a segment whose length is $\sqrt[3]{2}$.

We now prove the fundamental theorem on the proportionality of sides of triangles.

410. THEOREM. *A line parallel to the base of a triangle, and meeting the other two sides, divides these sides proportionally.*



Given $\triangle ABC$ with $DE \parallel AB$.

To prove $\frac{CD}{CA} = \frac{CE}{CB}$.

Proof: Whether CD and CA are commensurable or not, we know by §§ 407, 408 that $\frac{CD}{CA}$ and $\frac{CE}{CB}$ are definite numbers. We prove that these numbers cannot be different.

First, suppose $\frac{CD}{CA} < \frac{CE}{CB}$.

Take F between C and E so that $\frac{CD}{CA} = \frac{CF}{CB}$. Ax. XI (1)

Divide CB into equal parts, each less than EF. Then at least one of the division points, as G, lies between E and F. Draw $GH \parallel AB$.

Since CG and CB are commensurable, we have, by § 243,

$$\frac{CH}{CA} = \frac{CG}{CB}. \qquad (2)$$

Dividing (1) by (2), we have $\frac{CD}{CH} = \frac{CF}{CG}$. But this cannot be true, since $CD > CH$ and $CF < CG$.

Hence $\frac{CD}{CA}$ cannot be less than $\frac{CE}{CB}$. Why?

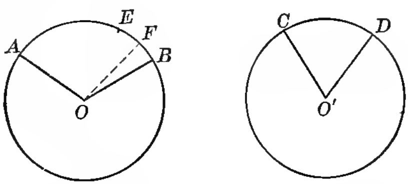
Secondly, prove in the same manner that $\frac{CD}{CA}$ cannot be greater than $\frac{CE}{CB}$. Hence, since the one is neither less than nor greater than the other, these ratios must be equal.

The following treatment of incommensurable arcs and angles is exactly similar to the above.

411. Axiom XII. *Any given arc* AB *has a definite ratio with a unit arc, which is greater than that of an arc* AC, *if* C *lies on the arc* AB.

412. Axiom XIII. *Any given angle* ABC *has a definite ratio with a unit angle, which is greater than that of* ABD *if* BD *lies within the angle* ABC.

413. THEOREM. *In the same or equal circles the ratio of two central angles is the same as the ratio of their intercepted arcs.*



Outline of Proof: We show that in the figure $\frac{\angle AOB}{\angle CO'D}$ can neither be less than nor greater than $\frac{\text{arc } AB}{\text{arc } CD}$.

Suppose $$\frac{\angle AOB}{\angle CO'D} < \frac{\text{arc } AB}{\text{arc } CD}.$$

Then take E so that $$\frac{\angle AOB}{\angle CO'D} = \frac{\text{arc } AE}{\text{arc } CD}. \quad (1)$$

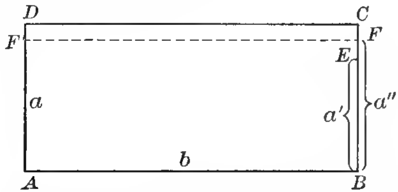
Divide arc CD into equal parts each less than arc EB. Lay off this unit arc successively on AB reaching a point F between E and B. Then arcs AF and CD are commensurable and by a proof exactly similar to that of § 243, making use of § 199, we can show that

$$\frac{\angle AOF}{\angle CO'D} = \frac{\text{arc } AF}{\text{arc } CD}. \quad (2)$$

Complete the proof as in § 410.

414. **Axiom XIV**. *Any given rectangle with base b and altitude a has a definite area which is greater than that of another rectangle with base b′ and altitude a′ if* $a \geqq a'$ *and* $b > b'$ *or if* $a > a'$ *and* $b \geqq b'$.

415. THEOREM. *The area of a rectangle is the product of its base and altitude.*



Proof: Denote the base and altitude by b and a, respectively, and the area by A.

Suppose $A < ab$, and let a' be a number such that $A = a'b$. Lay off $BE = a'$.

Consider first the case where b is commensurable with the unit segment and a is not.

Divide the unit segment into equal parts each less than CE and lay off one of these parts successively on BC reaching a point F between E and C.

Denote the length of BF by a'', and draw $FF' \parallel AB$.

Then by § 307 the area of $ABFF'$ is $a''b$.

By hypothesis $A = a'b$, but $a'b < a''b$ since $a' < a''$.

Hence $\qquad A < a''b. \qquad (1)$

But by Ax. XIV $\qquad A > a''b. \qquad (2)$

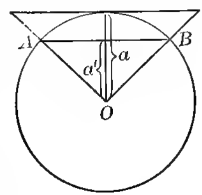
Hence the assumption that $A < ab$ cannot hold.

In the same manner prove that the $A > ab$ cannot hold.

The proof in case both sides are incommensurable with the unit segment is now exactly like the above and is left to the student.

416. Axiom XV. *A circle has a definite length and incloses a definite area which are greater than those of any inscribed polygon and less than those of any circumscribed polygon.*

417. THEOREM. *For a given circle and for any number* K, *however small, it is possible to inscribe and to circumscribe similar polygons such that their perimeters or their areas shall differ by less than* K.



Proof: First, Let p and p' be the perimeters of two similar polygons, the first circumscribed and the second inscribed, and let a and a' be their apothems.

Then $\frac{p}{p'}=\frac{a}{a'}$, or $\frac{p-p'}{p'}=\frac{a-a'}{a'}$. § 245

Hence $p-p'=p'\cdot\frac{a-a'}{a'}$.

Now $a-a'$ can be taken as small as we like.

Hence $p\cdot\frac{a-a'}{a'}$ can be made as small as we please; that is, $p-p'$ can be made smaller than any given number K.

Second, letting P and P′ be the areas of the circumscribed and inscribed polygons respectively, we have $\frac{P}{P'}=\frac{a^2}{a'^2}$, and the proof proceeds exactly as before.

418. Since the length C of the circle is greater than p' and less than p, it follows that C is thus made to differ from either p or p' by less than K.

And since the area A of the circle is greater than P′ and less than P, it follows that A is made to differ from either P or P′ by less than K.

419. THEOREM. *The lengths of two circles are in the same ratio as their radii.*

Proof: Let c and c' be the lengths of two circles whose centers are O and O' and whose radii are r and r'.

We shall prove that $\frac{c}{c'}=\frac{r}{r'}$ by showing that $\frac{c}{c'}$ is neither less than nor greater than $\frac{r}{r'}$.

First, suppose that $\frac{c}{c'}<\frac{r}{r'}$ or $c<c'\cdot\frac{r}{r'}$.

And let c differ from $c'\cdot\frac{r}{r'}$ by some number K.

Now circumscribe a regular polygon P about $\odot O$ with perimeter p such that

$$p<c'\cdot\frac{r}{r'}. \tag{1}$$

This is possible since p can be made to differ from c by less than K (§ 418).

Also circumscribe a polygon P' similar to P about $\odot O'$ with perimeter p'.

Then $$\frac{p}{p'}=\frac{r}{r'}, \text{ or } p=p'\cdot\frac{r}{r'}.$$

But $p'>c'$ and hence $p>c'\cdot\frac{r'}{r}$. (2)

Hence the supposition that $\frac{c}{c'}<\frac{r}{r'}$ leads to the contradiction expressed in (1) and (2) and is untenable.

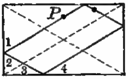
Now prove in same way that $\frac{c}{c'}>\frac{r}{r'}$ is untenable.

420. THEOREM. *The areas of two circles are in the same ratio as the squares of their radii.*

Using §§ 348 and 418, the proof is exactly similar to that of the preceding theorem.

PROBLEMS AND APPLICATIONS.

1. A billiard ball is placed at a point P on a billiard table. In what direction must it be shot in order to return to the same point after hitting all four sides?



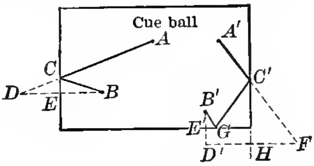
(The angle at which the ball is reflected from a side is equal to the angle at which it meets the side, that is, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.)

SUGGESTION. (*a*) Show that the opposite sides of the quadrilateral along which the ball travels are parallel.

(*b*) If the ball is started parallel to a diagonal of the table, show that it will return to the starting point.

2. Show that in the preceding problem the length of the path traveled by the ball is equal to the sum of the diagonals of the table.

3. Find the direction in which a billiard ball must be shot from a given point on the table so as to strike another ball at a given point after first striking one side of the table.



SUGGESTION. Construct $BE \perp$ to that side of the table which the ball is to strike and make $ED = BE$.

4. The same as the preceding problem except that the cue ball is to strike two sides of the table before striking the other ball.

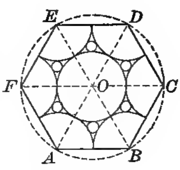
SUGGESTION. $B'E' = E'D'$, $D'H = HF$.

5. Solve Ex. 4, if the cue ball is to strike three sides before striking the other ball, — also if it is to strike all four sides.

6. In the figure $ABCDEF$ is a regular hexagon.

Prove that: (*a*) AD, BE, and CF meet in a point.

(*b*) $ABCO$ is a rhombus.



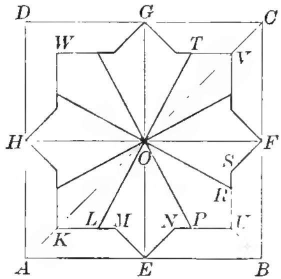
(*c*) The inner circle with center at O and the arcs with centers at A, B, C, etc., have equal radii.

(*d*) The straight line connecting A and C is tangent to the inner circle and to the arc with center at B.

(*e*) The centers of two of the small circles lie on the line connecting A and C.

(*f*) Find by construction the centers of the small circles.

7. In the figure EG and FH are diameters of the square $ABCD$. On the diagonals, points K, U, V, W are laid off so that $AK = BU = CV = DW$. Also $LM = NP = RS =$ etc. EN and SF are in the same straight line, and so on around the figure. Prove that:

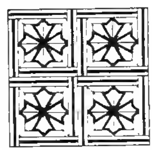


(*a*) $KUVW$ is a square.

(*b*) $AKME$ and $ENUB$ are equal trapezoids.

(*c*) L, O, T lie in a straight line.

(*d*) The four heavy six-sided figures are congruent.



(*e*) If $AB = a$, $AK = \frac{AO}{6}$ and $ML = KL$, find the areas of the figures $KUVW$, $AKME$, $LOPNEM$.

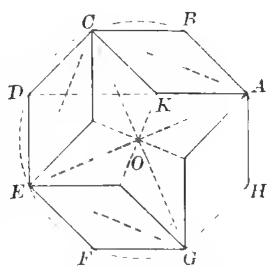
(*f*) Find the areas required under (*e*) if $AK = \frac{AO}{n}$ and $LM = m \cdot LK$.

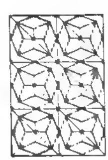
(*g*) If $AK = \frac{AO}{6}$, what is the length of ML if the four heavy figures occupy half the square $ABCD$?

8. Find a side of a regular octagon of radius r. See Ex. 3, p. 146.

9. Find the area of a regular octagon of radius r.

10. In the figure $ABCD\cdots$ is a regular octagon inscribed in a circle of radius r. Find the area of the triangle ABC.





SUGGESTION. Find first the altitude on AC in the $\triangle AOC$ and thus the altitude of $\triangle ABC$.

11. In the same figure the lines CH and AD are drawn meeting at K. Prove that $ABCK$ is a parallelogram and find its area if the radius of the circle is r.

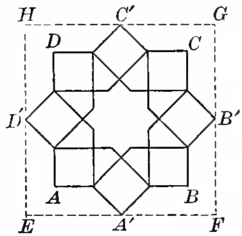
12. In the parquet floor design given with Ex. 10, the darker parts are parallelograms constructed as under Ex. 11. What part of the area is of white wood?

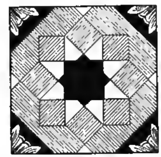
13. In the figure $ABCD$ and $A'B'C'D'$ are equal squares, placed as shown. Lines are drawn through A', B', C', and D' parallel to AB, BC, CD, and DA, forming a quadrilateral $EFGH$.

(*a*) Prove $EFGH$ a square.

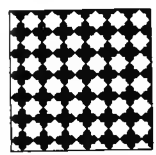
(*b*) What part of the large square is inclosed by the outside heavy figure?

(*c*) If $AB = a$, find the area of the inside heavy figure.



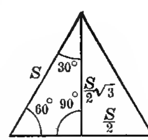


14. The design opposite consists of white figures constructed like the inner figure preceding, together with the remaining black figures. What part of the figure is white?


Design from the Alhambra.

15. Show that the altitude of an equilateral triangle with side s is $\frac{s}{2}\sqrt{3}$.

16. If the angles of a triangle are 30°, 60°, 90°, and if the side opposite the 60° angle is r, show that the other sides are $\frac{r}{3}\sqrt{3}$ and $\frac{2r}{3}\sqrt{3}$. See § 159, Ex. 14.

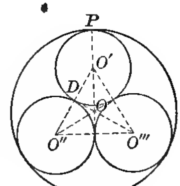


17. Three equal circles of radius 2 are inscribed in a circle as shown in the figure. Find the radius of the large circle.

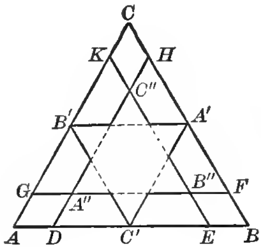
SUGGESTION. The center O of the large circle is at the intersection of the altitudes of the equilateral triangle $O'O''O'''$. (Why?)

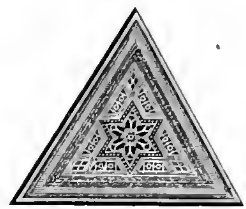
Hence $O'OD$ is a triangle with angles 30°, 60°, 90° as in Ex. 16.

Solve this problem for any radius r of the small circles. *Ans.* $R = r + \frac{2r}{3}\sqrt{3}$.



18. A', B', C', are the middle points of the sides of the equilateral triangle ABC. The sides of the triangle $A'B'C'$ are trisected and segments drawn as shown in the figure.





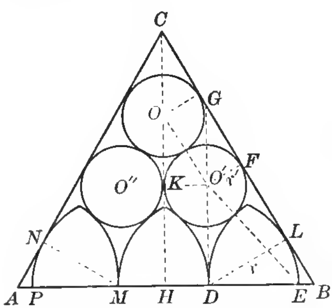
From Church of Or San Michele.

If $AB = a$,

(*a*) Prove $A''B''C''$ an equilateral triangle.

(*b*) Find the area of $A''B''C''$ and of the dotted hexagon.

(*c*) Find the area of the triangle GFC and of the trapezoid $DEB''A''$.



From Westminster Abbey.

19. In the figure ABC is an equilateral triangle. On the equal bases PM, MD, DE equilateral arches are constructed, two of them tangent to the sides of the triangle at L and N respectively. Circles O' and O'' are each tangent to a side of the triangle and to two of the arches. Circle O is tangent to circles O' and O'' and to both sides of the triangle.

(*a*) If $AB = a$, find DE.

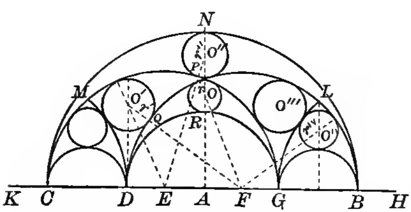
SUGGESTION. In the right triangle DBL one acute angle is 60°.

(*b*) Find the ratio $r : r'$, r being the base DE of the arch and r' the radius of the circle O'.

SUGGESTION. $GD = 2\,DL$, $GO' = 2\,r'$, and $O'D = \sqrt{(r+r')^2 - r^2} = \sqrt{2\,rr' + r'^2}$.

(*c*) By what fraction of a will the circles O' and O'' fail to touch each other?

20. In the figure $CD = DA = AG = GB$, and $DE = EA = AF = FG$. Semicircles are constructed on the diameters CB, CG, CD, DG, DB, GB. $HG = KD = CE$. Arcs GL and DM have centers H and K respectively.



From Church of Or San Michele.

Circles are constructed tangent to the various arcs as shown in the figure. Thus $\odot\, O''$ is tangent to semicircles on the diameters CB, CG, and DB. $\odot\, O'$ is tangent to the semicircles on the diameters CG and DB and to the arc DM. Let $CB = a$.

(*a*) Find the areas of each of the six semicircles.

(*b*) Find the radius r''.

SUGGESTION. EA, EP and AN are known.

(*c*) Find the radius r'.

SUGGESTION. Enumerate the known parts in $\triangle$ EDO' and FDO'.

(*d*) Find r and r^{IV}.

(*e*) Having determined the radii of the various circles, show how to construct the whole figure.

(*f*) What fraction of the area of the whole figure is occupied by the six circles?

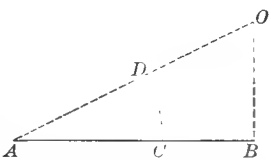
21. Prove that two segments drawn from vertices of a triangle to points on the opposite sides cannot bisect each other.

FURTHER APPLICATIONS OF PROPORTION.

421. Definition. A segment is said to be divided in **extreme and mean ratio** by a point on it, provided the ratio of the whole segment to the larger part equals the ratio of the larger part to the smaller.

E.g. the segment AB is divided in extreme and mean ratio by the point C if $\frac{AB}{AC}=\frac{AC}{CB}$.

422. PROBLEM. *To divide a given line-segment in extreme and mean ratio.*



Solution. At one extremity B erect $BO \perp AB$ and make $BO = \frac{1}{2}AB$. Draw OA.

On OA take $OD = OB$ and with A as a center and AD as a radius draw the arc DC. Then C is the required point of division.

Proof: Let a be the length of AB. Then $OB = \frac{a}{2}$.

Hence $\overline{AO}^2 = \overline{AB}^2 + \overline{OB}^2 = a^2 + \frac{a^2}{4} = \frac{5}{4}a^2$, or $AO = \frac{a}{2}\sqrt{5}$.

Then $AC = AD = AO - DO = \frac{a}{2}\sqrt{5} - \frac{a}{2} = \frac{a}{2}\left(\sqrt{5} - 1\right)$,

and $BC = AB - AC = a - \frac{a}{2}\left(\sqrt{5} - 1\right) = \frac{a}{2}\left(3 - \sqrt{5}\right)$.

Substituting these values in $\frac{AB}{AC}$ and $\frac{AC}{CB}$, and simplifying,

we have $\frac{2}{\sqrt{5} - 1}$ and $\frac{\sqrt{5} - 1}{3 - \sqrt{5}}$.

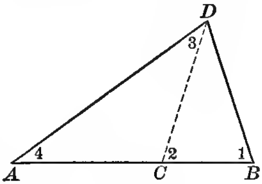
By rationalizing denominators these fractions are shown to be equal, and hence $\frac{AB}{AC} = \frac{AC}{CB}$.

423. PROBLEM. *To construct with ruler and compasses angles of* 36° *and* 72°.

Solution. On a given segment AB, determine a point C such that $\frac{AB}{AC} = \frac{AC}{BC}$. (See § 422.)

Using $BD = AC$ as a base and AB as one leg, construct the isosceles triangle ABD.

Then $\angle 1 = 72°$ and $\angle 4 = 36°$.



Proof: Draw DC.

Then $\triangle ABD \sim \triangle DBC$, (§ 259)

since $\frac{AB}{BD} = \frac{BD}{BC}$ and $\angle 1$ is common.

Hence $\triangle$s ABD and BDC are both isosceles (Why?),

and $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.

Also $\angle 2 = \angle 3 + \angle 4 = 2\angle 4 = \angle 1$. (Why?)

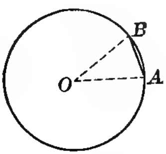
Thus in $\triangle ABD$ each base angle is double $\angle 4$,

making $\angle 4 = \frac{1}{5}$ of 2 rt. $\angle$s $= 36°$, (Why?)

and $\angle 1 = 2\angle 4 = 72°$.

424. EXERCISES.

1. Inscribe a regular decagon in a circle.



SUGGESTION. Construct the central angle $AOB = 36°$, thus determining the side AB of the decagon.

2. Inscribe a regular pentagon in a circle.

SUGGESTION. Join alternate vertices of a decagon, or construct a central angle equal to 72°.

3. Inscribe a regular polygon of fifteen sides in a circle.

SUGGESTION. Since $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$, it follows that the arc subtended by one side of a hexagon minus that subtended by one side of a decagon is the one which subtends one side of a polygon of fifteen sides.

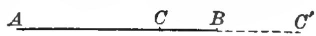
4. The radius of an inscribed regular polygon is a mean proportional between its apothem and the radius of a similar circumscribed polygon.

5. A circular pond is surrounded by a gravel walk, such that the area of the walk is equal to the area of the pond. What is the ratio of the radius of the pond to the width of the walk?

6. If a, b, c are three line-segments such as $\frac{a}{b} = \frac{b}{c}$ show that $\frac{a}{c}$ equals the ratio of the area of any triangle described on a as a base to the similar triangle described on b as a base.

425. Definitions. A line-segment AB is said to be divided **internally** in a given ratio $\frac{m}{n}$ by a point C lying on the segment, if $\frac{AC}{CB} = \frac{m}{n}$. See § 252.

A line-segment is said to be divided **externally** in a given ratio $\frac{r}{s}$ by a point C' lying on AB produced, if $\frac{AC'}{C'B} = \frac{r}{s}$.



A line-segment AB is said to be divided **harmonically** if the points C and C', lying respectively on AB and on AB produced, are such that $\frac{AC}{CB} = \frac{AC'}{C'B}$.

426. **EXERCISES.**

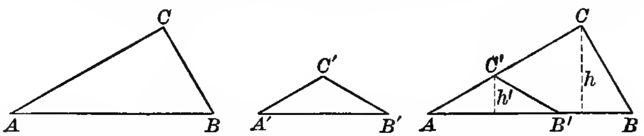
1. Show that if AB is divided harmonically by C and C', then CC' is divided harmonically by A and B.

2. Show that the base of any triangle is cut harmonically by the bisectors of the internal and external vertex angles.

3. Show how to divide a line-segment *externally* in extreme and mean ratio, that is, in the figure below, so that $\frac{AB}{C'A} = \frac{C'A}{C'B}$.

SUGGESTION. In the figure of § 422 produce BA to a point C' such that $C'A = AO + OB$. Then use the method there used, to show that $\frac{AB}{C'A}$ and $\frac{C'A}{C'B}$ each reduces to $\frac{\sqrt{5}-1}{2}$.

427. THEOREM. *If an angle of one triangle is equal to an angle of another, their areas are in the same ratio as the products of the sides including the equal angles.*



Given ⊿s ABC and $A'B'C'$ in which $\angle A = \angle A'$.

To prove that $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \cdot AC}{A'B' \cdot A'C'}$.

Proof: Place $\triangle A'B'C'$ so that $\angle A'$ coincides with $\angle A$.

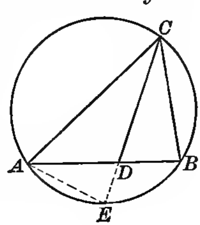
Then $\triangle ABC = \frac{1}{2} h \cdot AB$ and $\triangle A'B'C' = \frac{1}{2} h' \cdot A'B'$.

Hence, $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\frac{1}{2} h \quad AB}{\frac{1}{2} h' \cdot A'B'} = \dfrac{h}{h'} \cdot \dfrac{AB}{A'B'}$.

Now show that $\dfrac{h}{h'} = \dfrac{AC}{A'C'}$,

and hence that $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AC}{A'C'} \cdot \dfrac{AB}{A'B'} = \dfrac{AB \cdot AC}{A'B' \cdot A'C'}$.

428. THEOREM. *The square on the bisector of an angle of a triangle is equal to the product of the two adjacent sides minus the product of the segments of the opposite side.*



Outline of proof: Produce the bisector of the given angle to meet the circumscribed circle.

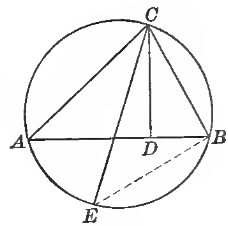
Since $\triangle BDC \sim \triangle AEC \quad AC \cdot BC = CD \cdot CE.$ (1)

But $CD \cdot CE = CD(CD + DE) = \overline{CD}^2 + CD \cdot DE$ (2)

and $CD \cdot DE = AD \cdot DB.$ (Why?) (3)

Using (1), (2), and (3), complete the proof.

429. THEOREM. *The product of two sides of a triangle is equal to the product of the altitude from the vertex in which these sides meet and the diameter of the circumscribed circle.*



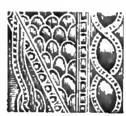
Outline of Proof: Using the figure, show from the similar triangles ACD and EBC that

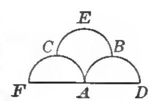
$$AC \cdot BC = CE \cdot CD.$$

430. **EXERCISES.**

1. The areas of two parallelograms having an angle of the one equal to an angle of the other are in the same ratio as the product of the sides including the eqnal angles.

2. Three semicircles of equal diameter are arranged as shown in the figure.

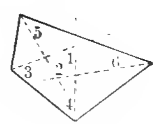




(*a*) If $AD = a$, find the area bonnded by the arcs AB, BEC, and CA.

(*b*) If the area just fonnd is 2 square feet, find AD.

3. Prove that the bisectors of the angles of any quadrilateral form a quadrilateral whose opposite angles are supplementary.



SUGGESTION. Show that $\angle 3 + \angle 4 + \angle 5 + \angle 6 = 2$ rt. $\angle$s, and hence that $\angle 1 + \angle 2 = 2$ rt. $\angle$s.

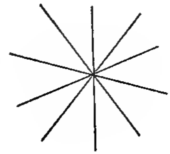
4. On each of two sides of a given triangle ABC as chords construct arcs in which an angle of 120° may be inscribed. If these arcs meet in a point O inside the triangle, show that the three sides of the triangle subtend the same angle from the point O.

5. If $\angle 1 + \angle 2 + \angle 3 = 4$ rt. $\angle$s, show how to find a point O within a given triangle ABC so that $\angle AOB = \angle 1$, $\angle BOC = \angle 2$, and $\angle COA = \angle 3$.

6. Given any two segments AB and CD and any two angles a and b, find a point O such that $\angle AOB = \angle a$ and $\angle COD = \angle b$. Discuss the various possible cases and the number of points O in each case.

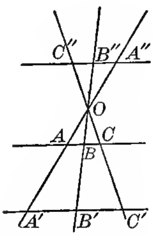
7. If $\angle AOB$ is a central angle of a circle and if $\angle CDE$ is inscribed in an arc of the same circle and if $\angle AOB = \angle CDE = \frac{4}{3}$ rt. $\angle$s, then the chords AB and CE are equal.

431. Definition. A set of lines which all pass through a common point is called a **pencil** of lines, and the point is called the **center of the pencil.**



432. EXERCISES.

1. The lines of a pencil intercept proportional segments on parallel transversals.



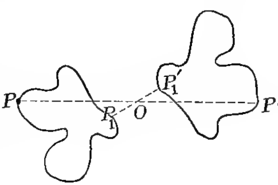
Given three lines meeting in O cut by three parallel transversals.

To prove that the corresponding segments are proportional, that is, to show that

$$\frac{AB}{BC} = \frac{A'B'}{B'C'} = \frac{A''B''}{B''C''}.$$

2. If two polygons are symmetrical with respect to a point, they are congruent. (See § 170, Ex. 2.)

3. Any two figures symmetrical with respect to a point are congruent.

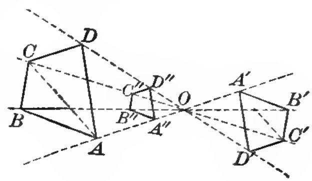


SUGGESTION. About the point O as a pivot swing one of the figures through a straight angle. Then any point P' of the right-side figure will fall on its symmetrical point P of the left-side figure.

4. By means of the theorem in the preceding example show how to make an accurate copy of a map.

SUGGESTION. Fasten the map to be copied on a drawing board. A long graduated ruler is made to swing freely about a fixed point O, and by means of it construct a figure symmetrical to the map with respect to the point O. How are the distances from O measured off?

433. THEOREM. *If corresponding vertices of two polygons lie on the same lines of a pencil and if they cut off proportional distances on these lines from the center, then the polygons are similar.*



Prove the theorem first for triangles.

Given. $\frac{OB}{OB'} = \frac{OC}{OC'} = \frac{OA}{OA'}$ and $\frac{OA}{OA''} = \frac{OB}{OB''} = \frac{OC}{OC''}$.

To prove that $\triangle$ ABC, $A'B'C'$, $A''B''C''$ are similar.

Prove the theorem for polygons of any number of sides.

434. Definition. Any two figures are said to have a **center of similitude** O, if for any two points P_1 and P_2 the lines P_1O and P_2O meet the other figure in points P'_1 and P'_2 such that

$$\frac{P_1O}{P'_1O} = \frac{P_2O}{P'_2O}.$$

Then P'_1 and P'_2 are said to **correspond** to the points P_1 and P_2.

Thus in the figure of § 433 O is called the center of similitude of the two polygons.

Any two figures which have a center of similitude are similar.

This affords a ready means of constructing a figure similar to a given figure and having some other required property that is sufficient to determine it.

Definition. The ratio of any two corresponding sides of similar polygons is called their **ratio of similitude.**

435. **EXERCISES.**

1. Construct a polygon similar to a given polygon such that they shall have a given ratio of similitude.

SUGGESTION. Let $ABCDEF$ be the given polygon and $\frac{m}{n}$ the given ratio of similitude. Select any convenient point O, such that $OA = m$. Draw lines OA, OB, etc. On OA lay off OA', n units. Lay off points B', C', etc., so that

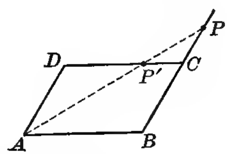
$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \frac{OD}{OD'} = \frac{OE}{OE'} = \frac{OF}{OF'}.$$

Prove that $A'B'C'D'E'F'$ is the required polygon.

2. Show how the preceding may be used to enlarge or reduce a map to any required size.

SUGGESTION. Arrange apparatus as under Ex. 4, § 432.

3. In the figure $ABCD$ is a parallelogram whose sides are of constant length. The point A is fixed, while the remainder of the figure is free to move. Show that the points P and P' trace out similar figures and that their ratio of similitude is $\frac{AD}{AD + CP}$.

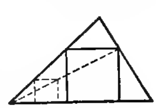


This shows the essential parts of an instrument called the pantograph, which is much used by engravers to transfer figures and to increase or decrease their size. The point P is made to trace out the figure which is to be copied. Hence P' traces a figure similar to it. The scale or ratio of similitude is regulated by adjusting the length of CP.

4. Construct a triangle having given two angles and the median a from one specified angle.

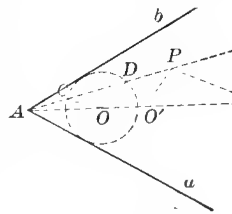
SOLUTION. Construct any triangle ABC having the required angles and construct a median AD. Prolong DA to D', making $AD' = a$. Extend BA and CA and through D' draw $B'C' \parallel BC$, making $\triangle AB'C'$. Prove that this is the required triangle. (Notice that A is the center of similitude.)

5. Inscribe a square in a given triangle using the figure given here. Compare this method with that given on page 147.



6. Construct a circle through a given point tangent to two given straight lines.

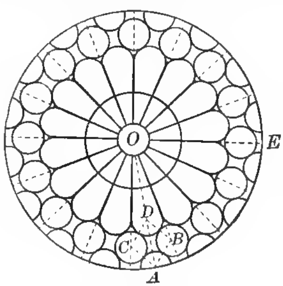
SOLUTION. Let a and b be the given lines and P the given point. Construct any circle O tangent to a and b. Draw AP meeting the circle O in C and D. Draw CO and OD and through P draw lines parallel to these meeting the bisector of the angle formed by a and b in O' and O''. Prove that O and O'' are centers of the required circles. Observe that A is a center of similitude.

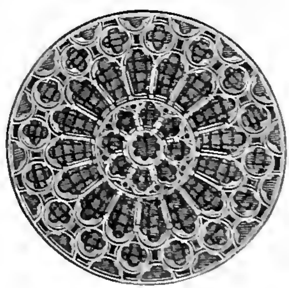


This method is used to construct a railway curve through a fixed point connecting two straight stretches of road.

7. On a line find a point which is equidistant from a given point and a given line.

The following is another instance of the use of this very important device in constructing figures that resist other methods of attack. It consists essentially in first constructing a figure similar to the one required and then constructing one similar to this and of the proper size.

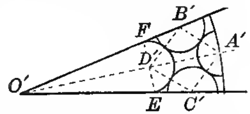




From Westminster Abbey.

8. Given a circle with radius OE. Construct within it the design shown in the figure. That is, the inner semicircles have as diameters the sides of a regular sixteen-sided polygon. Each of the small circles is tangent to two semicircles. The outer arcs have their centers on the given circle and each is tangent to two small circles. All these arcs and circles have equal radii.

SOLUTION. Construct an angle equal to one sixteenth of a perigon. Through any point D' in the bisector of this angle draw a segment EF perpendicular to the bisector and terminated by the sides of the angle. On EF as a diameter construct a semicircle. Make $B'D' = EF = B'A' = D'C' = C'A'$ and construct the small figure.



Now draw radii of the given circle dividing it into sixteen equal parts and bisect one of the central angles by a radius OA. Construct $\angle DAB = \angle D'A'B'$. Then AB is twice the radius of the required arcs and circles. The whole figure may now be constructed.

9. Given any three non-collinear points A', B', C', to construct an equilateral triangle such that A', B', C' shall lie on the sides of the triangle, one point on each side.

SUGGESTION. Through one of the points as A' draw a line such that B' and C' lie on the same side of it.

10. Given an equilateral triangle ABC, to construct a triangle similar to a given triangle $A'B'C'$ with its vertices on the sides of ABC.

SUGGESTION. Construct an equilateral triangle such that A', B', C' lie on its sides. Then construct a figure similar to this and of the required size.

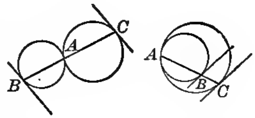
11. If p and p' are similar polygons inscribed in and circumscribed about the same circle, and if $2s$ is a side of the circumscribed polygon p', show that the difference of the areas of p and p' is equal to the area of a polygon similar to these and having a radius s.

SUGGESTION. Let the areas of the polygons whose radii are r, r', and s be A, A', A''. Prove that

$$\frac{A' - A}{A'} = \frac{r'^2 - r^2}{r'^2} \text{ and } \frac{A''}{A'} = \frac{s^2}{r'^2} = \frac{r'^2 - r^2}{r'^2}.$$

Complete the proof.

12. Two circles are tangent at A. A secant through A meets the circles at B and C respectively. Prove that the tangents at B and C are parallel to each other.

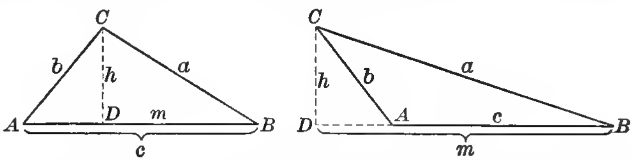


13. Prove that the segments joining one vertex of a regular polygon of n sides to the remaining vertices divide the angle at that vertex into n–2 equal parts.

FURTHER PROPERTIES OF TRIANGLES.

436. Definition. A segment AB is said to be **projected** upon a line l if perpendiculars from A and B are drawn to l. If these meet l in points C and D, then CD is the **projection** of AB upon l.

437. THEOREM. *The square of a side opposite an acute angle of a triangle is equal to the sum of the squares of the other two sides minus twice the product of one of these sides and the projection of the other upon it.*



Outline of Proof: In either figure let $\angle B$ be the given acute angle, and in each case BD is the projection of BC upon AB. Call this projection m.

We are to prove that $b^2 = a^2 + c^2 - 2\,cm$.

In the left figure, $b^2 = h^2 + (c - m)^2$. (1)

In the right figure, $b^2 = h^2 + (m - c)^2$. (2)

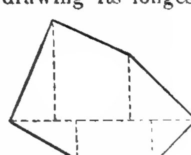
In either case $h^2 = a^2 - m^2$. (3)

Substitute (3) in (1) or in (2), and complete the proof.

Modify each figure so as to draw the projection of AB upon BC and call this n. Then give the proof to show that $b^2 = a^2 + c^2 - 2\,an$.

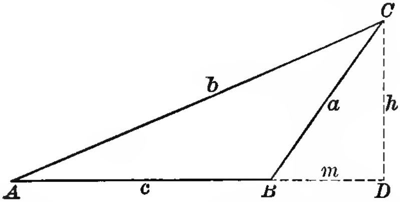
438. EXERCISE.

1. The area of a polygon may be found by drawing its longest diagonal and letting fall perpendiculars upon this diagonal from each of the remaining vertices.



Draw a figure like the one in the margin, only on a much larger scale, measure the necessary lines, compute the areas of the various parts (see § 314), and thus find its total area.

439. THEOREM. *The square of the side opposite an obtuse angle of a triangle is equal to the sum of the squares of the other two sides plus twice the product of one of these sides and the projection of the other upon it.*



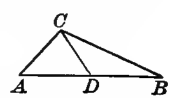
Outline of Proof: Let $\angle B$ be the given obtuse angle and BD the projection of BC upon AB. Call this projection m. As in the preceding theorem show that

$$b^2 = a^2 + c^2 + 2\,cm.$$

Also modify the figure so as to show the projection of AB on BC and call this n. Then show that

$$b^2 = a^2 + c^2 + 2\,an.$$

440. THEOREM. *The sum of the squares of two sides of any triangle is equal to twice the square of half the third side plus twice the square of the median drawn to that side.*



Suggestion. Make use of the two preceding theorems.

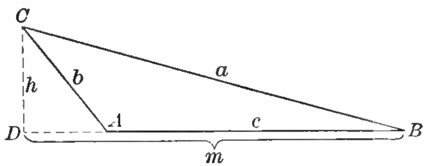
441. EXERCISES.

1. Compute the medians of a triangle in terms of the sides.

2. Show that the difference of the squares of two sides of a triangle is equal to twice the product of the third side and the projection of the median upon that side.

3. The base of a triangle is 40 feet and the altitude is 30 feet. Find the area of the triangle cut off by a line parallel to the base and 10 feet from the vertex.

442. PROBLEM. *To express the area of a triangle in terms of its three sides.*



Solution. The area of $\triangle ABC = \frac{1}{2} AB \times CD = \frac{1}{2} hc.$

It is first necessary to express h in terms of a, b, c.

We have $$b^2 = a^2 + c^2 - 2\,cm, \quad \text{(Why?)}$$

or $$m = \frac{a^2 + c^2 - b^2}{2\,c}.$$

Also $$h^2 = a^2 - m^2. \quad \text{(Why?)}$$

Hence $$h^2 = a^2 - \left(\frac{a^2 + c^2 - b^2}{2\,c}\right)^2 = \frac{4\,a^2c^2 - (a^2 + c^2 - b^2)^2}{4\,c^2}$$

$$= \frac{(2\,ac + a^2 + c^2 - b^2)(2\,ac - a^2 - c^2 + b^2)}{4\,c^2}$$

$$= \frac{[(a + c)^2 - b^2][b^2 - (a - c)^2]}{4\,c^2}$$

$$= \frac{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}{4\,c^2}.$$

Now call $a + b + c = 2\,s,$ or $a + c - b = 2\,s - 2\,b.$

Then $$a + c - b = 2(s - b)$$
$$b + a - c = 2(s - c)$$
$$b + c - a = 2(s - a).$$

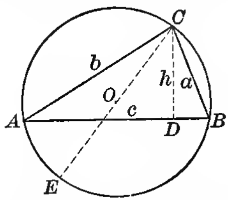
Hence $$h^2 = \frac{2\,s \cdot 2(s - b) \cdot 2(s - c) \cdot 2(s - a)}{4\,c^2},$$

or $$h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

Then $\quad \frac{1}{2} hc = \frac{1}{2} c \cdot \frac{2}{c} \sqrt{s(s-a)(s-b)(s-c)}.$

Hence area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}.$

443. PROBLEM. *To express the area of a triangle in terms of the three sides and the radius of the circumscribed circle.*



In the figure $CE = 2r$, where r is the radius of the circumscribed circle.

Then $\quad ab = 2r \cdot h. \quad$ (§ 429)

But $\quad h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}. \quad$ (§ 442)

Hence, $\quad ab = \frac{4r}{c}\sqrt{s(s-a)(s-b)(s-c)},$

or $\quad \frac{abc}{4r} = \sqrt{s(s-a)(s-b)(s-c)}.$

But area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}. \quad$ (§ 442)

Hence,

$$\text{area of } \triangle ABC = \frac{abc}{4r}.$$

444. EXERCISES.

1. Compute the areas of the triangles whose sides are (1) 7, 9, 12; (2) 11, 9, 7; (3) 3, 4, 5.

2. Express the radius of the circumscribed circle of a triangle in terms of the three sides.

3. Find the area of an equilateral triangle whose side is a by § 443, and also without this theorem, and compare results.

PROBLEMS AND APPLICATIONS.

1. Given a regular dodecagon (twelve-sided polygon) with radius r. Connect alternate vertices.

(*a*) Prove that the resulting figure is a regular hexagon.

(*b*) Find the apothem of the hexagon.

(*c*) Find the area of each of the triangles formed by joining the alternate vertices of the dodecagon.

2. Find the area of a regular dodecagon of radius r.

3. Find the side of a regular dodecagon of radius r.

4. Find the apothem of a regular dodecagon of radius r,

(*a*) by means of Exs. 2 and 3 and § 343.

(*b*) by means of Ex. 3 and § 319.

5. Given the side 8 of a regular dodecagon. Find the apothem. Also find the apothem if the side is s. See Exs. 9–12, page 235.

6. Given a circle of radius 6:

(*a*) Find the area of a regular hexagon circumscribed about it.

(*b*) Find the area of a regular octagon inscribed in it, also of one circumscribed about it.

7. Using the formula obtained in Ex. 5, find the side of a regular dodecagon whose apothem is b.

8. Find the radius of a circle circumscribed about a dodecagon whose apothem is b.

9. Find the difference between the radii of the regular dodecagons inscribed in and circumscribed about a circle of radius 10 inches.

10. Solve Ex. 9 if the radius of the circle is r.

11. What is the radius of a circle if the difference between the areas of the inscribed and circumscribed regular dodecagons is 12 square inches?

12. What is the radius of a circle if the difference between the areas of the inscribed and circumscribed regular hexagons is 8 square inches? See Ex. 11, page 259.

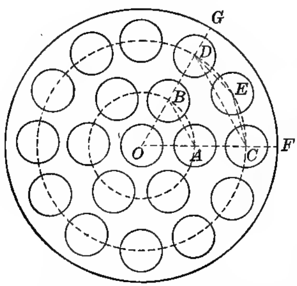
13. A regular hexagon and a regular dodecagon have equal sides. Find these sides if the area of the dodecagon is six square inches more than twice the area of the hexagon.

14. Solve Ex. 13 if the area of the dodecagon exceeds twice the area of the hexagon by b square inches.

15. Is there any *common* length of side for which the area of a regular dodecagon is twice that of a regular hexagon? Three times? Four times? Prove your answer.

16. Solve a problem similar to that of Ex. 15 if the side of the dodecagon is twice that of the hexagon.

17. The nineteen small circles in the accompanying figure are of the same size. The centers of the twelve outer circles lie on a circle



with center at O, as do the six circles between these and the innermost one. The centers of these small circles divide the large circles on which they lie into equal arcs.

If r is the radius of the small circles, then $OB = 2\frac{1}{2}r$, $BD = 2\frac{1}{2}r$, and $DG = 1\frac{1}{2}r$.

(*a*) Find AB and CD in terms of r.

(*b*) Find DE.

(*c*) What part of the circle OG is contained within the nineteen small circles?

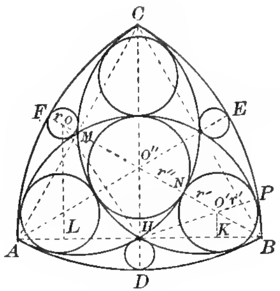
(*d*) If the radius of the large circle is 36 inches, find the radii of the small circles so that they shall occupy half its area.

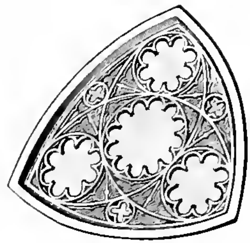
(*e*) If the radius of the large circle is r inches, find the radii of the small circles so they shall occupy one half the area of the large circle.

(*f*) Under (*d*) how far apart will the outer twelve centers be? The inner six?

(*g*) Under (*e*) how far apart will the outer twelve centers be? The inner six?

In Exs. (*f*) and (*g*) it is understood that the distances are measured along straight lines.





From First Congregational Church, Chicago.

18. In the figure ABC is an equilateral triangle. A, B, C are the centers of the arcs BC, CA, and AB. Semicircles are constructed on the diameters AB, BC, CA. Let $AB = a$. (See Ex. 11, p. 93.)

Circles are constructed tangent to the various arcs as in the figure.

(a) Find the radius r''.

Suggestion. Find in order BM, MN, BN, HO'', BO''.

(b) Find the radius r'.

Suggestion. BN is known from the solution of (a).

$$BN = BO' + r' \text{ and } BO' = 2\,KO'. \quad \text{Hence } KO' = \frac{BN - r'}{2}.$$

$$KB = \tfrac{1}{2}\sqrt{3}\cdot BO' = \tfrac{1}{2}\sqrt{3}\,(BN - r') \text{ and } HO' = \frac{a}{2} - r'.$$

But $$\overline{HO'}^2 = \overline{HK}^2 + \overline{KO'}^2 = \left(\frac{a}{2} - KB\right)^2 + \overline{KO'}^2. \qquad (1)$$

Substituting for HO', KB and KO' in (1) we may solve for r'.

(c) Find the radius r.

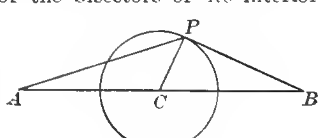
Suggestion. Use $\triangle BLO$ and HLO and proceed as under (b).

(d) Using the radii thus found, show how to construct the figure.

(e) What fraction of the whole area is contained in the circles?

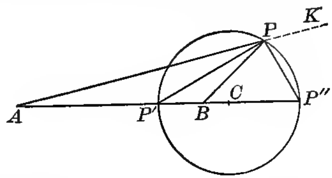
19. $ABCD$ is a parallelogram with fixed base and altitude. Find the locus of the intersection points of the bisectors of its interior base angles.

20. Find the locus of a point P such that the sum of the squares of its distances from two fixed points is constant.



Suggestion. By § 110, $\overline{AP}^2 + \overline{PB}^2 = 2\,\overline{AC}^2 + 2\,\overline{CP}^2$. Solve for CP.

21. Find the locus of a point P such that the ratio of its distances from two fixed points is equal to the constant ratio $m:n$.



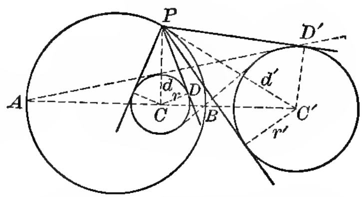
SUGGESTIONS. Let A and B be the fixed points. On the line AB there are two points P' and P'' on the locus, *i.e.* $AP'':P''B = AP':P'B = m:n$.

Let P be any other point on the locus.

Then $AP:BP = AP':P'B = AP'':P''B$.

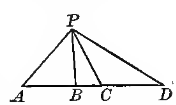
Show by the converses of §§ 250, 253 that PP' bisects $\angle APB$ and PP'' bisects $\angle BPK$, and hence $P'P$ and $P''P$ form a right angle.

22. Find the locus of a point P from which two circles subtend the same angle.



SUGGESTION. C and C' are the centers of the given circles. Prove $\triangle PDC \sim \triangle PD'C'$ and hence that $\frac{PC'}{PC} = \frac{r'}{r}$.

23. The points $ABCD$ are collinear. Find the locus of a point P from which the segments AB and CD subtend the same angle.

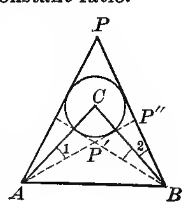


SUGGESTIONS. The $\triangle ABP$, ACP, BDP, and CDP have a common altitude, and hence their areas are to each other as their bases. Also $\triangle ABP$ and CDP have equal vertex angles, whence by § 427 their areas are to each other as the products of the sides forming these angles. Similarly for $\triangle ACP$ and BDP.

Hence $$\frac{AP \cdot PB}{CP \cdot PD} = \frac{AB}{CD} \text{ and } \frac{AP \cdot PC}{BP \cdot PD} = \frac{AC}{BD}.$$

From these equations show that $AP:PD$ is a constant ratio.

24. ABC is a fixed isosceles triangle. With center C and radius less than AC, construct a circle, and from A and B draw tangents to it meeting in P. Find the locus of P.



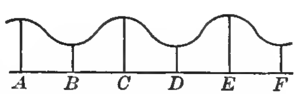
SUGGESTIONS. (*a*) Show that part of the locus is the straight line PP'. (*b*) Show that $\angle 1 = \angle 2$ and hence that $\angle AP''B$ is constant.

MAXIMA AND MINIMA.

445. Definitions. Of all geometric figures fulfilling certain conditions it often happens that some one is greater than any other, in which case it is called a **maximum.** Or it may happen that some one is less than any other, in which case it is called a **minimum.**

E.g. of all chords of a circle the diameter is the *maximum*, and of all segments drawn to a line from a point outside it the perpendicular is the *minimum.*

In the following theorems and exercises the terms maximum and minimum are used as above defined. However, a geometric figure is often thought of as *continuously varying in size*, in which case it is said to have a **maximum** at *any* position where it may *cease to increase and begin to decrease*, whether or not this is the greatest of all its *possible* values. Likewise it is said to have a **minimum** at any position where it may *cease to decrease and begin to increase.*



E.g. if in the figure a perpendicular from a point in the curve to the straight line be moved continuously parallel to itself, the length of this perpendicular will have maxima at A, C, and E, and minima at B, D, and F.

Certain simple cases of maxima and minima problems have already been given. Some of these will be recalled in the following exercises.

446. EXERCISES.

1. If from a point within a circle, not the center, a line-segment be drawn to meet the circle, show that this segment is a maximum when it passes through the center and a minimum when, if produced in the opposite direction, it would pass through the center.

2. Show that of all chords through a given point within a circle, not the center, the diameter is a maximum and the chord perpendicular to the diameter is a minimum.

3. Of all line-segments which may be drawn from a point outside a circle to meet the circle, that is a maximum which meets it after passing through the center, and that is a minimum which, if produced, would pass through the center.

4. Show that if a square and a rectangle have equal perimeters, the square has the greater area.

SUGGESTION. If s is the side of the square and a and b are the altitude and base of the rectangle respectively, then $2b + 2a = 4s$ or $s = \frac{b+a}{2}$.

Hence $$s^2 = \frac{b^2 + a^2 + 2ab}{4} = \frac{(b^2 - 2ab + b^2) + 4ab}{4} = \frac{(b-a)^2}{4} + ab.$$

That is, s^2 is greater than ab by $\frac{(b-a)^2}{4}$.

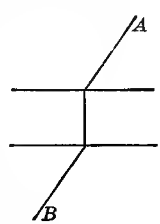
5. Use the preceding exercise to find that point in a given line-segment which divides it into two such parts that their product is a maximum.

6. Find the point in a given line-segment such that the sum of the squares on the two parts into which it divides the segment is a minimum.

SUGGESTION. If a and b are the two parts and k the length of the segment, then
$$a + b = k \text{ and } a^2 + 2ab + b^2 = k^2,$$
or,
$$a^2 + b^2 = k^2 - 2ab.$$

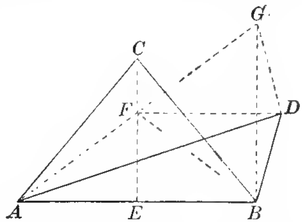
Hence, $a^2 + b^2$ is least when $2ab$ is greatest. Now apply Ex. 5.

7. In the preceding exercise show that $a^2 + b^2$ increases as b grows smaller, that is, as the point of division approaches one end. When is this sum a maximum?



8. Two points A and B, on opposite sides of a straight stream of uniform width, are to be connected by a road and a bridge crossing the stream at right angles. Find by construction the location of the bridge so as to make the total path from A to B a minimum.

447. THEOREM. *Of all triangles having equal perimeters and the same base, the isosceles triangle has the maximum area.*



Given $\triangle ABC$ **isosceles and having the same perimeter as** $\triangle ABD$.

To prove that area $\triangle ABC >$ area $\triangle ABD$.

Outline of proof: Draw $CE \perp AB$. Construct $\triangle AFB$ having its altitude FE the same as that of $\triangle ABD$. Prolong AF making $FG = AF$. Draw GB and GD. The object is to prove that EF, the altitude of $\triangle ABD$, is less than EC, the altitude of $\triangle ABC$.

(1) Show that $\triangle$s AFB, FBG, and GBD are all isosceles, for which purpose it must be shown that $GB \perp AB$ and $FD \parallel AB$.

Then $\quad AD + DB = AD + DG > AG.$

Or $\quad AF + FB < AD + DB.$

But $\quad AD + DB = AC + CB.$

Hence, $\quad AF + FB < AC + CB.$

(2) Show that $\quad AF < AC,$

and hence that $\quad EF < EC.$

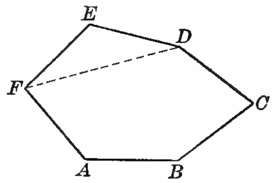
(3) Use the last step to show that

$$\text{area } \triangle ABC > \text{area } \triangle ABD.$$

Give all the steps and reasons in full.

448. COROLLARY. *Of all triangles having the same area and standing on the same base, that which is isosceles has the least perimeter.*

449. THEOREM. *Of all polygons having the same perimeter and the same number of sides, the one with maximum area is equilateral.*

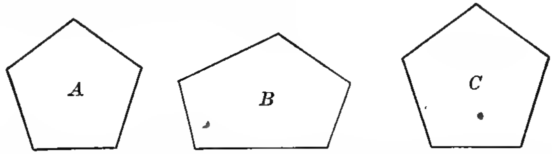


Given $ABCDEF$ **the maximum of polygons having a given perimeter and the same number of sides.**

To prove that $AB = BC = CD = DE = EF = FA$.

Suggestion. Show that every △, such as FDE, is isosceles by use of the preceding theorem.

450. THEOREM. *Of all polygons having the same area and the same number of sides, the regular polygon has the minimum perimeter.*



Given polygons A **and** B **with same number of sides and equal area,** A **being regular and** B **not.**

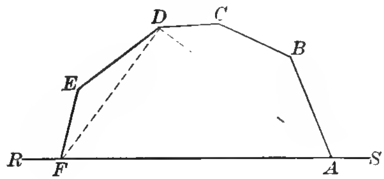
To prove that the perimeter of A is less than that of B.

Outline of Proof: Construct a regular polygon C having the same number of sides and same perimeter as B.

Then, area of $B <$ area of C, or area of $A <$ area of C.

But A and C are both regular and have the same number of sides. Hence the perimeter of A is less than that of C. That is, perimeter of $A <$ perimeter of B.

451. THEOREM. *The polygon with maximum area which can be formed by a series of line-segments of given lengths, starting and ending on a given line, is that one whose vertices all lie on a semicircle constructed on the intercepted part of the given line as a diameter.*



Given the line-segments *AB*, *BC*, *CD*, *DE* and *EF* meeting the line *RS* at *A* and *F* so as to form the polygon *ABCDEF* with maximum area.

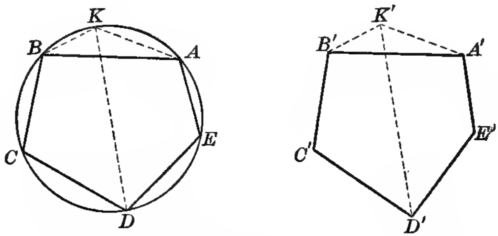
To prove that *B*, *C*, *D*, and *E* lie on the semicircle whose diameter is *AF*.

Proof: Suppose any vertex as *D* does not lie on this semicircle. Join *D* to *F* and to *A*. Then $\angle ADF$ is not a right angle, else it would be inscribable in a semicircle (§ 213).

If now the extremities *A* and *F* be moved in or out on the line *RS*, keeping *AD* and *DF* unchanged in length, till $\angle ADF$ becomes a right angle, then $\triangle ADF$ will be *increased* in area (Ex. 3, § 456), while the rest of the polygon is unchanged in area. This would increase the total area of the polygon *ABCDEF*, which is contrary to the hypothesis that this is the polygon with maximum area. Hence the vertex *D* must lie on the semicircle.

In the same manner it can be proved that each vertex lies on the semicircle.

452. THEOREM. *Of all polygons with the same number of sides equal in pairs and taken in the same order, the one with maximum area is that one which can be inscribed in a circle.*



Given a polygon *ABCDE* inscribed in a circle and *A'B'C'D'E'* not inscribable but with sides respectively equal to those of *ABCDE*.

To prove that $ABCDE > A'B'C'D'E'$.

Proof: In the first figure draw the diameter DK and draw AK and BK.

On $A'B'$ construct $\triangle A'B'K \cong \triangle ABK$ and draw $D'K'$.

The circle whose diameter is $D'K'$ cannot pass through all the points A', B', C', and E', else the polygon would be inscribed contrary to hypothesis.

If either A' or E' is not on this circle, then

$$AKDE > A'K'D'E'. \qquad \text{(Why?)} \quad (1)$$

Likewise if either B' or C' is not on the circle, then

$$KBCD > K'B'C'D'. \qquad \text{(Why?)} \quad (2)$$

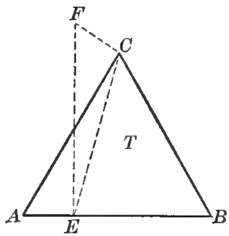
In any case, adding (1) and (2),

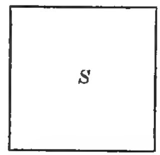
$$AKBCDE > A'K'B'C'D'E'. \quad (3)$$

By construction, $\triangle AKB \cong \triangle A'K'B'$. (4)

From (3) and (4), $ABCDE > A'B'C'D'E'$.

453. THEOREM. *Of all regular polygons having equal perimeters, that which has the greatest number of sides is the maximum.*





Given T a regular triangle and S a square having the same perimeter.

To prove that $S > T$.

Proof: Take any point E in AB. Draw CE and on CE construct $\triangle CEF \cong \triangle AEC$.

Then polygon $BCFE$ has a perimeter equal to that of T and the same area, and has the same number of sides as S.

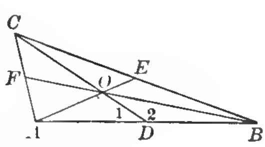
Hence, $S > BCFE$. (§ 449)

That is, $S > T$.

In like manner it can be shown that a regular polygon of five sides is greater than a square of equal perimeter, and so on for any number of sides.

454. **EXERCISE**

Of the three medians of a scalene triangle that one is the shortest which is drawn to the longest side.



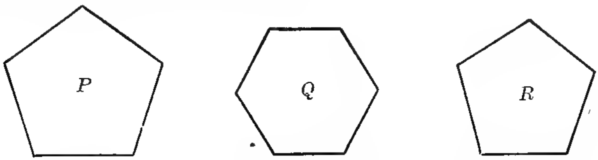
SUGGESTION. If $AC < BC$, to show that $BF > AE$.

In $\triangle\triangle$ ADC and BDC, $AD = DB$ and DC is common.

But $AC < BC$. Hence $\angle 1 < \angle 2$ (Why?).

Now use $\triangle\triangle$ ADO and BDO to show that $BO > AO$ and hence that $BF > AE$.

455. THEOREM. *Of all regular polygons having a given area, that one which has the greatest number of sides has the least perimeter.*



Given regular polygons P and Q such that P = Q while Q has the greater number of sides.

To prove that perimeter of $Q <$ perimeter of P.

Proof: Construct a regular polygon R having the same number of sides as P and the same perimeter as Q.

Then $R < Q$. (§ 453)

But $P = Q$. (By hypothesis)

Hence $R < P$.

Therefore, perimeter of $R <$ perimeter of P. (§ 323)

But R was constructed with a perimeter equal to that of Q

Hence perimeter of $Q <$ perimeter of P.

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SOLID GEOMETRY.

CHAPTER VIII.

LINES AND PLANES IN SPACE.

456. A figure in plane geometry is restricted so that all its points lie in the same plane. Such figures are called **two-dimensional figures.**

A straight line is a *one-dimensional figure,* and a point is of *zero dimension.*

457. A three-dimensional figure is a combination of points, lines, and surfaces not all parts of which lie in the same plane.

E.g. the six surfaces of the walls, floor, and ceiling of the schoolroom form a three-dimensional figure. This figure *is not the room itself.* The room is the space inclosed by the figure.

458. Solid geometry treats of the properties of three-dimensional figures.

DETERMINATION OF A PLANE.

459. Since we are to consider points and lines not all lying in the same plane, it is of first importance to be able to distinguish one plane from another.

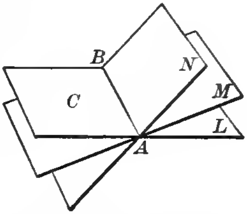
This is all the more important, since three-dimensional figures have to be represented by pencil or crayon drawings on the plane of the paper or blackboard. Models of paper, wire, etc., may be constructed by the pupils.

460. Axiom XVI. *If two points of a straight line lie in a plane, the whole line lies in the plane.*

Since a line is endless, it follows from this axiom that a plane is unlimited in all its directions.

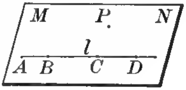
461. While two points determine a line, it is obvious that this is not sufficient to determine a plane.

E.g. suppose a plane L contains the points A and B which determine the line AB. If the plane L be revolved about the line AB as an axis, it may occupy indefinitely many positions, as L, M, N, but there is only one position in which it contains a third given point C outside of the line AB.

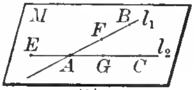


462. Axiom XVII. *Through three points not all in the same straight line one and only one plane can be passed.*

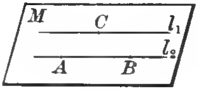
463. THEOREM. *A plane is determined by* (1) *a line and a point not in that line,* (2) *two intersecting lines,* (3) *two parallel lines.*



(1)



(2)



(3)

Proof: (1) **Let l be the given line and P the point outside.**

Take any two points A and B on l. Then A, B, and P determine a plane, which we call M (Ax. XVII).

Take any other two points C and D on l. Then C, D, and P determine a plane, which we call N (Ax. XVII).

By Ax. XVI, N contains the points A and B. Hence N is the plane determined by A, B, and P, since it contains these points. That is, N is the same plane as M.

Therefore one and only one plane is determined by a line and a point not on that line.

(2) **Let l_1 and l_2 be two intersecting lines.**

Take A the intersection of l_1 and l_2, and B and C any other points, one on l_1 and the other on l_2.

Then A, B, and C determine a plane M in which both l_1 and l_2 lie, since A and B lie on l_1 and A and C on l_2 (Ax. XVI).

Now M is the only plane in which both l_1 and l_2 lie, for any other three points E, F, G, on l_1 and l_2 (not all in the same line) determine the same plane M, since they all lie in it.

Hence, two intersecting lines determine a plane.

(3) **Let l_1 and l_2 be two parallel lines.**

By definition l_1 and l_2 lie in a plane M.

Now M is the only plane in which both l_1 and l_2 lie, for any three points A, B, C, on l_1 and l_2 (not all on the same line) determine this plane M, since they all lie in it.

Hence, two parallel lines determine a plane.

464. Axiom XVIII. *If two planes have a point in common, then they have at least another point in common.*

465. THEOREM. *Two intersecting planes have a straight line in common, and no points in common outside of this line.*

Proof: If two planes intersect, they must have at least two points in common (Ax. XVIII).

But these two points determine a straight line which lies wholly in each plane (Ax. XVI).

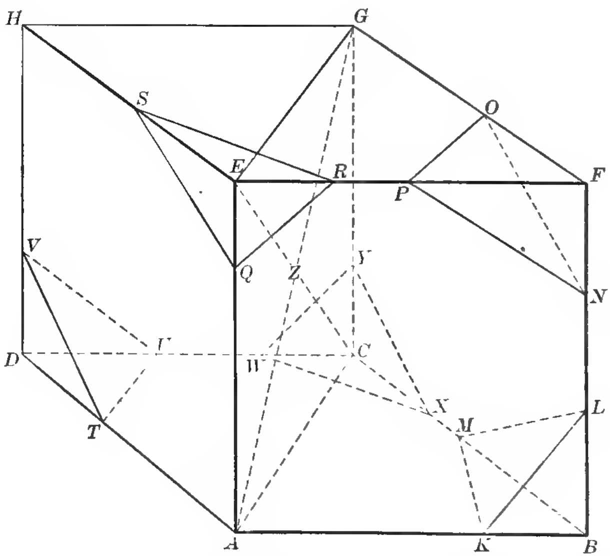
Hence, there is a straight line common to the two planes.

Now prove that these two planes can have no point in common outside of this line unless the planes are identical.

466. **EXERCISES.**

The diagram on this page represents a three-dimensional figure in the shape of an ordinary box. In this figure the points *A*, *K*, *B* do not determine a plane, since they all lie on the same straight line, while *A*, *C*, *E* do not lie in a straight line, and hence determine a plane.

1. In this figure pick out twelve sets of three points each which do not determine planes, and also twelve sets which do determine planes.



2. Describe the relation to the three-dimensional figure of each plane determined in Ex. 1. Thus, the plane *WXY* cuts off the lower right-hand corner at the back.

3. Read each of the six bounding planes in this figure, using various different sets of points or lines. Thus, the face *ABFE* is determined by the points *B*, *L*, *K*, or by *A* and the line *PN*, or by lines *AB* and *EF*.

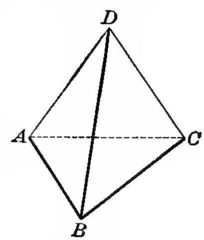
4. Describe the position of each of the following planes in the figure: *AEC*, *ACF*, *BCE*, *HCA*, *ADG*, *BDF*, *HEC*.

5. Pick out six planes, each determined by two parallel lines in the figure.

6. Decide whether each of the six planes in Ex. 5 may also be determined by two intersecting lines of the figure; by a line and a point outside of it.

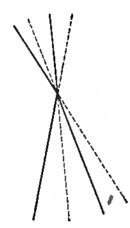
7. Using the point *Z*, pick out six planes, each determined by it and two other points of the figure, as *AZD*, *AZB*, etc.

8. Using the schoolroom or a room at home, imagine all the planes constructed which are involved in the foregoing questions. Do the same, using a small box or paper model.



9. Does a stool with three legs always stand firmly on a flat floor? One with four legs? Give reasons. On what condition does a stool with four legs stand firmly on a flat floor?

10. In the figure the point *D* is supposed not to lie in the same plane as *ABC*. Hence, does *C* lie in the plane of *ABD*? Does *B* lie in the plane of *ACD*?



11. How many planes are determined by four points which do not all lie in the same plane?

12. How many planes are determined by four lines which all meet in a point, but no three of which lie in the same plane?

13. What is the locus of all points common to two intersecting planes?

467. The demonstrations and constructions of solid geometry consist largely in applying the theorems already known in plane geometry to figures lying in various planes determined by points and lines in space, as illustrated in the preceding exercises.

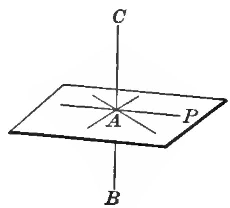
The next following problems are typical in this respect.

THEOREMS ON PERPENDICULARS.

468. Definitions. A line is said to be **perpendicular to a plane** if it is perpendicular to every line of the plane passing through the point in which it meets the plane. In this case the plane is also **perpendicular to the line.** The point in which the perpendicular meets the plane is the **foot of the perpendicular.**

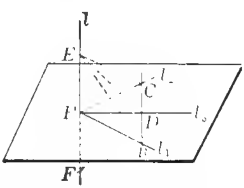
NOTE. It is obvious that at a point in a line there may be *many lines in space* which are perpendicular to it. For instance, if AP is $\perp BC$, rotate AP about BC as an axis, keeping $\angle PAC = \angle PAB$. Then AP remains $\perp BC$ in every one of its positions.

Thus, in some wheels all spokes are perpendicular to the axle.



469. A line or plane is said to be **constructed** whenever the points which determine it are constructed.

470. PROBLEM. *Through a given point to construct a plane perpendicular to a given line.*



Given: the line l and the point P.

To construct a plane through P perpendicular to l.

Construction. (a) Let P be on the given line l. Through P construct two lines l_1 and l_2, each perpendicular to l. Then the plane determined by l_1 and l_2 is the required plane.

Proof: Connect B and C, any two points different from P on l_1 and l_2 respectively.

Through P draw any line l_3 meeting BC in D.

On l lay off $PE = PF$ and complete the figure as shown.

Now prove (1) $\triangle EBC \cong \triangle FBC$, (2) $\angle EBD = \angle FBD$, (3) $\triangle EBD \cong \triangle FBD$, (4) $\triangle EPD \cong \triangle FPD$, (5) $\angle EPD = FPD$. Hence, $PD \perp l$.

This proof holds for every such line l_3 except the line through P parallel to BC. In this case we can select another point C' on l_2 so that l_3 shall not be parallel to BC'.

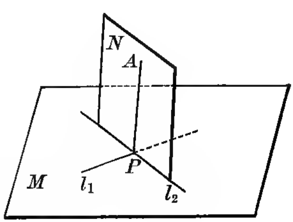
Since PD is any line through P in the plane of l_1 and l_2, it follows by definition that the plane determined by l_1 and l_2 is perpendicular to l.

(*b*) If the point P is not on the line l, draw a line from P perpendicular to l at some point Q, and then as above construct a plane through Q perpendicular to l.

471. EXERCISE.

Prove that through a point there is not more than one plane perpendicular to a given line.

472. PROBLEM. *At a point in a plane to construct a line perpendicular to the plane.*



Given: the point P in the plane M.

To construct a line perpendicular to M at P.

Construction. Let l_1 be any line in M through P.

Pass a plane N through $P \perp l_1$ (§ 470).

Let plane N cut plane M in l_2 (§ 465).

In plane N erect $PA \perp l_2$.

Then PA is the perpendicular required.

Proof: Show that PA is $\perp$ to both l_1 and l_2 and hence to their plane, that is, to the plane M.

473. **EXERCISES.**

1. At a point in a plane only one line can be erected perpendicular to the plane.

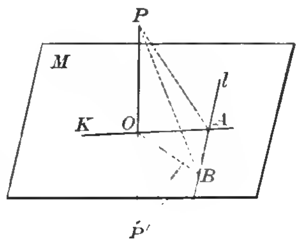
PROOF. Suppose a second perpendicular is possible. Then we shall have two lines as l_1 and l_2 perpendicular to M at P. Let the plane of l_1 and l_2 intersect M in a line l_3. Then l_1 and l_2 are both $\perp$ to l_3 and lie in the same plane with it, which is impossible.

2. Take two pieces of cardboard in one of which a line is drawn at random, and in the other a line is drawn perpendicular to an edge. Place them so as to illustrate the construction in § 472.

474. PROBLEM. *From a point outside a plane to construct a line perpendicular to the plane.*

Given: the plane *M* and the point *P* outside it.

To construct a line from $P \perp M$.



Construction. Let l be any line in plane M.

From P draw $PA \perp l$, and in plane M draw $AK \perp l$ at A.

Finally, draw $PO \perp AK$. Then PO is the required line perpendicular to M.

Proof: Through O in M draw any other line OB meeting l.

Prolong PO to P', making $OP' = OP$, and connect points as shown in the figure.

Then line $l \perp$ plane PAP'. (Why?)

Now show that (1) $PA = P'A$, (2) $\triangle APB \cong \triangle AP'B$, (3) $\triangle BOP \cong \triangle BOP'$.

Hence, $PO \perp M$. (Why?)

475. EXERCISES.

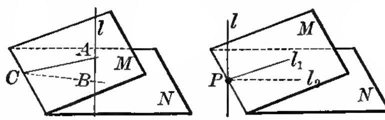
1. Only one perpendicular to a plane can be drawn from a point outside the plane.

SUGGESTION. Show that the hypothesis of two perpendiculars from an outside point to a plane leads to a contradiction.

2. A perpendicular is the shortest distance from a point to a plane.

3. A line cannot be perpendicular to each of two intersecting planes.

SUGGESTIONS. (1) If a line l is perpendicular to the planes M and N at the points A and B, and C is a point in their intersection, then $\triangle ABC$ would contain two right angles. (2) If l is perpendicular to M and N at a point P in their intersection, pass a plane through l, meeting M and N in l_1 and l_2.



4. A line which is perpendicular to each of two lines at their point of intersection is perpendicular to their plane. See § 470.

5. All perpendiculars to a line at a point in it lie in the same plane, namely, the plane determined by any two of them.

The following are theorems proved in the preceding constructions and exercises.

476. THEOREM. *Through any given point there is one and only one plane perpendicular to a given line.*

477. THEOREM. *Through any given point there is one and only one line perpendicular to a given plane.*

478. THEOREM. *All lines perpendicular to a line at a point lie in one and the same plane.*

479. THEOREM. *A line perpendicular to each of two intersecting lines is perpendicular to their plane.*

480. THEOREM. *The shortest distance from a point to a plane is the perpendicular to the plane.*

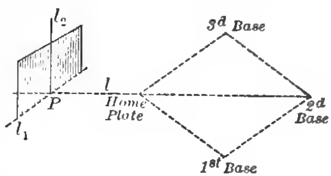
481. EXERCISES.

1. Show how a carpenter could use the theorem of § 479 to stand a post perpendicular to the floor, by use of two ordinary steel squares.

2. If a steel square be made to rotate about one of its edges as an axis, what kind of surface does the other edge describe?

3. In making long timbers perpendicular to each other, a carpenter may use simply a measuring rod, such as a ten-foot pole. Show how to make a timber perpendicular to the floor by this process.

4. Show how a back-stop on a ball field can be adjusted perpendicular to the line through second base and the home plate. What theorems of solid geometry are used?



5. If a plane is perpendicular to a line-segment PP' at its middle point, prove: (1) Every point in the plane is equally distant from P and P'; (2) every point equally distant from P and P' lies in this plane. What is the locus of all points in space equidistant from P and P'? See § 127.

6. Given the points A and B not in a plane M. Find the locus of all points in M equidistant from A and B.

SUGGESTION. All such points must lie in the plane M and also in the plane which is the perpendicular bisector of the segment AB.

Is there any position of the points A and B for which there is no such locus? For which the locus contains the whole plane M?

7. On a line find a point equidistant from two given points not on the line.

Is there any position of the points for which there is no such point on the line? For which all points on the line satisfy this condition?

8. Find the locus of all points in space equidistant from the three vertices of a triangle.

9. Find the locus of all points in a given plane equidistant from the vertices of a triangle not in the plane.

Is there any position of the triangle for which this locus contains more than one point? No point?

10. Ask and answer questions similar to the two preceding, using the vertices of a square instead of a triangle; the vertices of a rectangle; of any polygon which can be inscribed in a circle.

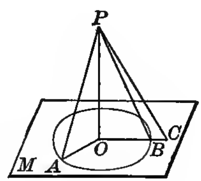
11. Find the locus of all points in space equidistant from the points of a circle and also of all points in a plane equidistant from the points of a circle not in that plane. Discuss as in Ex. 9.

12. Find the locus of all points equidistant from two given points A and B, and also equidistant from two points C and D. Discuss.

13. In geometry of two dimensions, how many lines may be perpendicular to a given line at a given point? In geometry of three dimensions?

14. In how many points can a straight line cut a plane? In how many points may it cut a *curved surface*, such as a stovepipe? In how many points can a straight line cut a circle? In how many points can a plane cut a circle, the plane being distinct from the plane of the circle?

15. State and prove a theorem of solid geometry corresponding to the theorem in the Plane Geometry, comparing the lengths of oblique segments cutting off equal distances from the foot of a perpendicular.

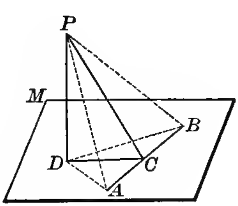


Also state and prove the converse.

State and prove a theorem comparing the lengths of segments cutting off unequal distances. See §§ 112, 115, and 116.

16. Find the locus of all points in a plane which are equally distant from a given point outside the plane. If a perpendicular be drawn to the plane from this point, how is its foot related to this locus?

17. If in the figure $PD \perp$ plane M, and $DC \perp AB$ any line of the plane, prove that $PC \perp AB$.



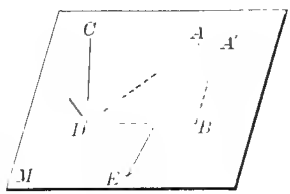
SUGGESTION. Lay off $CA = CB$, and compare triangles.

18. If in the same figure $PD \perp M$, and $PC \perp AB$ any line of the plane, prove that $DC \perp AB$.

THEOREMS ON PARALLELS.

482. THEOREM. *If two lines are perpendicular to the same plane, they are parallel.*

CONVERSELY. *If one of two parallel lines is perpendicular to a plane, the other is also.*



Given: (1) AB and CD each $\perp$ to the plane M.

To prove that $AB \parallel CD$.

Proof: Draw BD and make $DE \perp DB$.

Take points A and E so that $BA = DE$, and draw AD, AE, and BE.

Now prove (1) $\triangle ABD \cong \triangle BDE$. (2) $\triangle ADE \cong \triangle ABE$.

Hence, $\angle ADE$ is a right angle, and DC, DA, and DB all lie in the same plane. (Why?)

Therefore CD and AB are in the same plane and perpendicular to the same line BD. (Why?)

Hence, $AB \parallel CD$.

Given: (2) $AB \parallel CD$ and $CD \perp M$.

To prove that $AB \perp M$.

Proof: If AB is not $\perp M$, suppose $A'B \perp M$. Then $A'B \parallel CD$. But through B there is only one line $\parallel CD$.

Hence, $A'B$ and AB coincide.

But $A'B$ was taken $\perp M$. Hence, $AB \perp M$.

HISTORICAL NOTE. The above proof for the direct case is the one given by Euclid. Many proofs have been given, but this seems the most elegant.

483. COROLLARY. *Two lines in space, each parallel to the same line, are parallel to each other.*

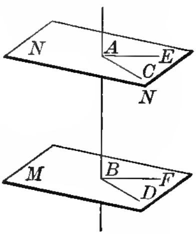
SUGGESTIONS. Let $l_1 \parallel l_3$ and $l_2 \parallel l_3$. To prove $l_1 \parallel l_2$. Take a plane $M \perp l_3$ and finish the proof.

484. Definitions. Two planes which do not meet are said to be **parallel.**

A straight line and a plane which do not meet are said to be **parallel.**

485. THEOREM. *If each of two planes is perpendicular to the same line, they are parallel.*

CONVERSELY. *If one of two parallel planes is perpendicular to a line, the other is also.*



Given: (1) $M \perp AB$ and $N \perp AB$.

To prove that $M \parallel N$.

Proof: Suppose M and N to meet in some point P, and show that this leads to a contradiction.

Given: (2) $M \parallel N$ and $M \perp AB$.

To prove that $N \perp AB$.

Proof: Through AB pass a plane cutting M in BD and N in AC, and also a second plane cutting M in BF and N in AE.

Then $BD \parallel AC$, for if they could meet, then M and N would meet, which is contrary to the hypothesis.

Likewise $BF \parallel AE$.

Complete the proof, showing that $AB \perp AC$, and $AB \perp AE$, and hence $AB \perp N$.

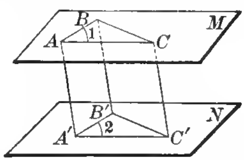
486. COROLLARY. *If a plane intersects two parallel planes, the lines of intersection are parallel.*

487. THEOREM. *If a straight line is parallel to a given plane, it is parallel to the intersection of any plane through it with the given plane.*

SUGGESTION. If l_1 is the given line, and l_2 the intersection of the plane through l_1 with the given plane, show that l_1 and l_2 lie in the same plane and cannot meet.

488. COROLLARY. *If a line outside a plane is parallel to some line in the plane, then the first line is parallel to the plane.*

489. THEOREM. *If two intersecting lines in one plane are parallel, respectively, to two intersecting lines in another plane, then the two planes are parallel, and the corresponding angles formed by the lines are equal.*



Given: the planes M and N in which $AB \parallel A'B'$, and $AC \parallel A'C'$.

To prove that $M \parallel N$ and $\angle 1 = \angle 2$.

Outline of proof: (1) *To prove* $M \parallel N$.

If M and N could meet, let l be their line of intersection. Then neither AB nor AC can meet l since each is $\parallel N$ by § 488. Hence $AB \parallel l$ and $AC \parallel l$, which is impossible. (Why ?)

(2) *To prove* $\angle 1 = \angle 2$, study the following analysis:

Lay off $AB = A'B'$, $AC = A'C'$, and draw BC, $B'C'$, AA', BB', and CC'.

Then $\angle 1 = \angle 2$ if $\triangle ABC \cong \triangle A'B'C'$, which is true if $BC = B'C'$. But $BC = B'C'$ if $BCC'B'$ is a ▱, which is so if $BB' = CC'$ and $BB' \parallel CC'$. This last is true if $ABB'A'$

and $ACC'A'$ are ▱, for then $BB' = CC' = AA'$ and $BB' \parallel AA' \parallel CC'$.

Now reverse the order of these steps and give the proof in full, with all the reasons.

490. COROLLARY. *If two angles in space have their sides respectively parallel, the angles are either equal or supplementary.*

State this in full detail as in §§ 106–108 and show how it applies to the above figure.

491. **EXERCISES.**

1. Show that parallel line-segments included between parallel planes are equal, and hence that parallel planes are everywhere equally distant.

2. Find the locus of all points at a given distance from a given plane; also of all points equally distant from two parallel planes.

3. Show how to determine a plane parallel to a given plane and at a given distance from it. How would you place three shelves parallel to each other and a foot apart?

4. Show that two straight lines in space may not meet and yet not be parallel.

5. Show that through any given line a plane may be passed parallel to any other given line in space.

SUGGESTION. If l_1 and l_2 are the given lines, through any point in l_1, draw $l_3 \parallel l_2$, and show that the plane determined by l_1 and l_3 is the required plane $\parallel l_2$.

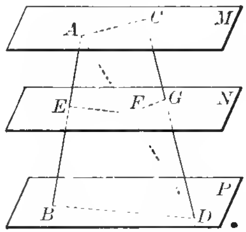
6. Through a given point pass a plane parallel to each of two given lines in space.

SUGGESTION. Through the given point pass lines $\parallel$ to each of the given lines, then use § 488.

7. Show that through a point outside a plane any number of lines can be drawn parallel to the plane. How are all these parallels situated?

8. Find the locus of all lines through a fixed point parallel to a fixed plane.

492. THEOREM. *If two straight lines are cut by three parallel planes, the intercepted segments on one line are in the same ratio as the corresponding segments on the other.*



Given: the lines AB and CD cut by the planes M, N, and P,

To prove that $\frac{AE}{EB}=\frac{CG}{GD}$.

Proof: Connect the points A and D, and let the plane determined by AD and CD cut the planes M and N in AC and FG respectively. And let the plane of AB and AD cut N and P in EF and BD respectively.

Now show that $AC \parallel FG$ and $EF \parallel BD$, and hence complete the proof, giving all the reasons.

493. COROLLARY. *Parallel planes which intercept equal segments on any transversal line intercept equal segments on every transversal line.*

494. EXERCISES.

1. Show how to find all the points on the floor of the schoolroom which are equally distant from one of the lower corners of the window sill and one of the upper corners of the opposite door.

2. If the spaces between four shelves are 5, 8, and 10 inches respectively, and a slanting rod intersecting them has a 7-inch segment between the first two shelves, find the other two segments of the rod.

3. If we attempt to stand up a ten-foot pole in a room eight feet high, find the locus of the foot of the pole on the floor when the top is kept at a fixed point on the ceiling.

4. Show that, if three line-segments not in the same plane are equal and parallel, the triangles formed by joining their extremities, as in the figure of § 489, are congruent, and their planes are parallel.

5. Given a plane M and a point P not in M. Find the locus of the middle points of all segments connecting P with points in M.

6. Given a plane M and a point P not in M. Find the locus of a point which divides in a given ratio each segment connecting P with a point in M: (*a*) if the segments are divided internally; (*b*) if they are divided externally.

7. The locus required in Ex. 6 consists of two planes, each parallel to the given plane M. Are these two planes equally distant from M?

8. Show that a plane containing one only of two parallel lines is parallel to the other.

9. If in two intersecting planes a line of one is parallel to a line of the other, then each of these lines is parallel to the line of intersection of the planes.

10. Show that three lines which are not concurrent must all lie in the same plane, if each intersects the other two.

11. Show that three planes, each of which intersects the other two, have a point in common unless their three lines of intersection are parallel.

SUGGESTION. Suppose two of the intersection lines are not parallel, and meet in some point O. Then show that the other line of intersection passes through O, and hence that O is the point common to all three planes.

12. Given two intersecting planes M and N. Find the locus of all points in M at a given distance from N.

13. Given two non-intersecting lines l_1 and l_2. Find the locus of all lines meeting l_1 and parallel to l_2.

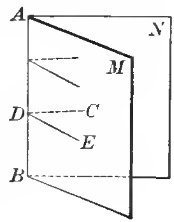
14. Prove that the middle points of the sides of any quadrilateral in space are the vertices of a parallelogram.

SUGGESTION. Use the fact that a line bisecting two sides of a triangle is parallel to the third side. Note that the four vertices of a quadrilateral in space do not necessarily all lie in the same plane.

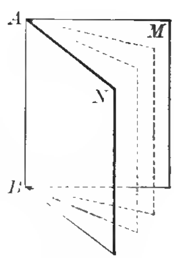
DIHEDRAL ANGLES.

495. Definitions. The part of a plane on one side of a line in it is called a **half-plane.** The line is called the **edge** of the half-plane. Two half-planes meeting in a common edge form a **dihedral angle.** The common edge is the **edge** of the angle and the half-planes are its **faces.**

Lines in the faces of a dihedral angle perpendicular to its edge at a common point form a plane angle, which is called **the plane angle of the dihedral angle.**



Thus, in the figure, the half-planes M and N have the common edge AB, and form the dihedral angle M–AB–N, read by naming the two faces and the edge. The $\angle CDE$, whose sides are $CD \perp AB$ in N and $ED \perp AB$ in M is the plane angle of the dihedral angle M–AB–N.

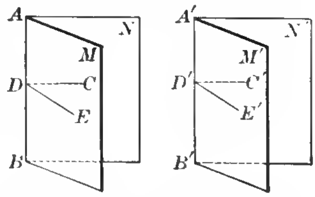


496. By § 489 all plane angles of a dihedral angle are equal to each other.

A dihedral angle may be thought of as **generated** by the rotation of a half-plane about its edge. The magnitude of the angle depends solely upon the *amount of rotation.*

497. Two dihedral angles are equal when they can be so placed that their faces coincide.

498. THEOREM. *Two dihedral angles are equal if their plane angles are equal.*



Given: the dihedral angles M–AB–N and M'–$A'B'$–N' in which the plane ∠s CDE and $C'D'E'$ are equal.

To prove that

M–AB–$N = M'$–$A'B'$–N'.

Proof: Place the equal ∠s CDE and $C'D'E'$ in coincidence.

Then the edges AB and $A'B'$ must also coincide, since they are both perpendicular to the plane CDE at the point D (§ 477).

Face M then coincides with M', since its determining lines, AB and DE, coincide respectively with those of M', namely, $A'B'$ and $D'E'$.

Likewise, face N coincides with N'.

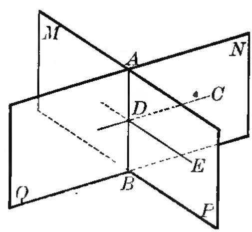
Hence, $M–AB–N = M'–A'B'–C'$ since their faces have been made to coincide (§ 497).

499. THEOREM. *State and prove the converse of the preceding theorem.*

500. Definitions. It follows from §§ 498 and 499 that the plane angle of a dihedral angle may be regarded as its **measure**.

A dihedral angle is **right**, **acute**, or **obtuse** according as its plane angle is right, acute, or obtuse.

Two dihedral angles are **adjacent**, **vertical**, **supplementary**, or **complementary**, according as their corresponding plane angles, with a common vertex, are adjacent, vertical, supplementary, or complementary.



In the figure pick out all the dihedral angles and describe them and their relations, (1) if $\angle CDE$ is acute, (2) if $\angle CDE$ is a right angle.

501. EXERCISES.

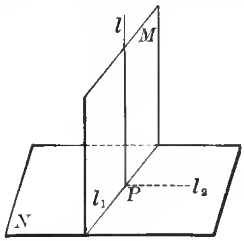
1. State theorems on dihedral angles corresponding to those on plane angles in §§ 64, 68, 69, 72–76.

2. State theorems concerning two planes cut by a transversal plane corresponding to those on lines in §§ 90, 92, 93, 97, 99.

Note that in Exs. 1 and 2 the proofs are exactly analogous to those in the Plane Geometry.

502. Definition. Two planes are **perpendicular to each other** if their dihedral angle is a right angle.

503. THEOREM. *If a line is perpendicular to a plane, every plane containing this line is perpendicular to the plane.*



Given: a line $l \perp$ plane N at P,

To prove that a plane M containing l is $\perp N$.

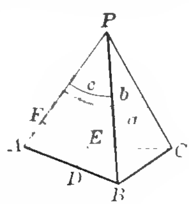
Proof: Let l_1 be the intersection of N and M. In N draw $l_2 \perp l_1$ at P.

Then $l \perp l_1$, $l_2 \perp l_1$, and $l_2 \perp l$. (Why?)

Hence, $M \perp N$. (Why?)

504. EXERCISES.

1. Show that the above theorem is equivalent to the following: If a plane is perpendicular to a line lying in another plane, then the first plane is perpendicular to the second.



2. Name all the dihedral angles in the accompanying figure.

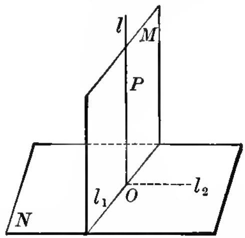
3. Find the locus of all points equidistant from two given parallel planes and also equidistant from two given points.

4. Find the locus of all points at a given distance from a given plane and equidistant from two given points.

Discuss Exs. 3 and 4 for the various cases possible.

5. How can the theorem of § 503 be used to erect a plane perpendicular to a given plane?

505. THEOREM. *If two planes* M *and* N *are mutually perpendicular, and if* P *is any point in* M, *then a line through* P *perpendicular to* N *lies wholly in* M *and is perpendicular to the line of intersection of* M *and* N.



Given: plane $M \perp$ plane N, and P any point in M. Let l_1 be the intersection of M and N and let l be a line through $P \perp N$.

To prove that l lies wholly in M and that $l \perp l_1$.

Proof: (1) **Let P be any point in M outside of l_1.**

Suppose l does not lie wholly in M.

Then through P draw a line l' in the plane M perpendicular to l_1 at O. Through O draw line l_2 in $N \perp l_1$. Then l_2 and l' form a right angle (§ 502). Hence $l' \perp N$ (§ 479).

But from P there can be but one perpendicular to N.

Hence, l coincides with l', and we have l lying wholly in M and $l \perp l_1$.

(2) **Let the point P be in the intersection l_1.**

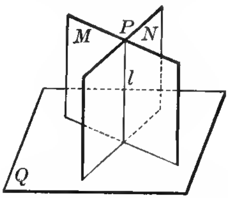
The proof is similar to case (1). Give it in full.

506. EXERCISES.

1. Show that if in the figure of § 505, $M \perp N$ and l is a line in M such that $l \perp l_1$, then $l \perp N$.

2. Through a given point, on or outside of a given plane, how many planes can be constructed perpendicular to the given plane?

507. THEOREM. *If a plane is perpendicular to each of two planes, it is perpendicular to their line of intersection.*



Given: $Q \perp M$ and $Q \perp N$ and l the intersection of M and N,

To prove that $Q \perp l$.

Proof: From a point P common to M and N draw a line $l' \perp Q$.

Then l' lies wholly in both M and N. (Why?)

Hence, l' and l are the same line.

That is, $l \perp Q$, or $Q \perp l$.

508. EXERCISES.

1. Any plane ⊥ to the edge of a dihedral angle is ⊥ to each of its faces.

2. If three lines are ⊥ to each other at a common point, then each is ⊥ to the plane of the other two.

3. Through a given line in space pass a plane ⊥ to a given plane. How many such planes can be constructed?

SUGGESTION. From some point in the given line draw a line ⊥ to the given plane. Then use § 503.

4. What is the answer to the question in Ex. 3 in case the given line is perpendicular to the given plane?

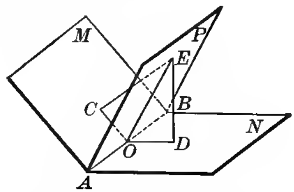
5. Through a given point in space pass a plane ⊥ to each of two given planes. How many such can be passed?

SUGGESTION. Consider the relation of the required plane to the intersection of the two given planes.

6. What is the answer to the question in Ex. 5 in case the given point is on the line of intersection of the given planes?

509. Definition. A plane through the edge of a dihedral angle **bisects the angle** if it forms equal angles with the faces of that angle.

510. THEOREM. *The locus of all points in space equally distant from the faces of a dihedral angle is the half-plane bisecting the angle.*



Given: the half-plane *P* bisecting the dihedral ∠ *M–AB–N*.

To prove: (1) that any point E in P is equally distant from M and N, and (2) that any point which is equally distant from M and N lies in P.

Outline of proof: (1) Draw $EC \perp M$ and $ED \perp N$. Then the plane CED cuts M, N, and P in OC, OD, and OE respectively and is ⊥ to both M and N. (Why?)

Then ∠ EOD is the measure of $P–AB–N$ and ∠ EOC is the measure of $P–AB–M$. (Why?)

Now use ⚠ EOC and EOD to show that $EC = ED$.

(2) Take E any point such that $EC = ED$, and let P' be the plane through AB containing E. Now argue as in § 125, to show that $\angle EOD = \angle EOC$, and hence that P' is the same plane as the bisector plane P.

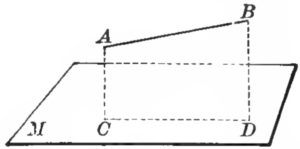
Give the proof in full.

511. COROLLARY. *If from any point within a dihedral angle perpendiculars are drawn to the faces, the angle between the perpendiculars is the supplement of the dihedral angle.*

512. Definition. The **projection of a point** on a plane is the foot of the perpendicular from the point to the plane.

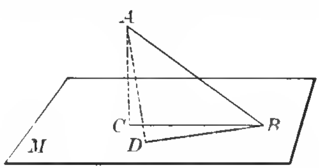
The **projection of any figure** on a plane is the locus of the projections of all points of the figure on the plane.

513. THEOREM. *The projection of any straight line on a plane is a straight line in that plane.*



Outline of proof: Pass a plane through the given line and ⊥ to the given plane. Is this always possible? (Why?) Now show that the intersection of these two planes contains the projection of every point of the given line upon the given plane.

514. THEOREM. *The acute angle formed by a straight line with its own projection on a plane is the least angle which it makes with any line in that plane.*



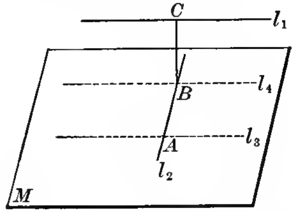
Outline of proof: Let BC be the projection of AB, and let BD be any other line in M through B.

Lay off $BD = BC$ and draw AD.

In the ⚠ ABD and ABC show that $\angle ABD > \angle ABC$.

515. Definition. The angle between a plane and a line oblique to it is understood to mean the acute angle formed by the line and its projection upon the plane.

516. PROBLEM. *To construct a common perpendicular to two non-parallel lines in space.*



Given: l_1 and l_2, two non-parallel lines, and also non-intersecting.

To construct a line BC perpendicular to each of them.

Construction: Through A, any point in l_2, draw $l_3 \parallel l_1$.

Let M be the plane determined by l_2 and l_3.

Through l_1 pass a plane $P \perp M$ and meeting M in l_4.

At B the intersection of l_2 and l_4, in the plane P, erect $BC \perp M$.

Then BC is the required perpendicular.

Proof: Give the proof in full.

517. EXERCISES.

1. If a line is $\perp$ to a plane, its projection on that plane is a point.

2. If a line-segment is $\parallel$ to a plane, show that its projection is a segment equal to the given segment.

3. If a line-segment is oblique to a plane, show that its projection is less than the given segment.

4. If a line-segment 6 in. long makes an angle of 30° with a plane, find the length of its projection on the plane. If it makes an angle of 45°; an angle of 60°.

5. If two parallel lines meet a plane, they make equal angles with it. (Why?) Is the converse true?

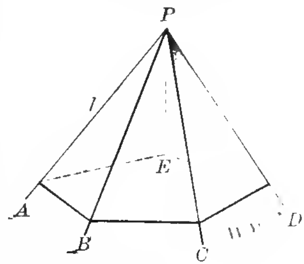
6. If a line cuts two parallel planes, it makes equal angles with them. (Why?) Is the converse true?

7. If two parallel line-segments are oblique to a plane, their projections on the plane are in the same ratio as the given segments.

POLYHEDRAL ANGLES.

518. Definitions. Given a convex polygon and a point P not in its plane. If a half-line l with end point fixed at P moves so that it always touches the polygon and is made to traverse it completely, it is said to generate a **convex polyhedral angle**.

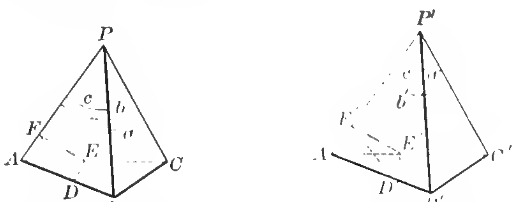
The fixed point is the **vertex** of the polyhedral angle, and the rays through the vertices of the polygon are the **edges** of the polyhedral angle.



Any two consecutive edges determine a plane, and the portion of such a plane included between these edges is called a **face** of the polyhedral angle.

The plane angles at the vertex are called the **face angles** of the polyhedral angle. A polyhedral angle having three faces is called a **trihedral angle.**

519. THEOREM. *If two trihedral angles have the three face angles of the one equal respectively to the three face angles of the other, the dihedral angles opposite the equal face angles are equal.*



Given: the trihedral angles P and P', in which $\angle a = \angle a'$, $\angle b = \angle b'$, $\angle c = \angle c'$.

To prove that the pairs of dihedral angles are equal whose edges are PA and $P'A'$, PB and $P'B'$, PC and $P'C'$.

Proof: Let P and P' be cut by the planes ABC and $A'B'C'$, making $PA = PB = PC = P'A' = P'B' = P'C'$.

On the edges PA and $P'A'$ lay off $AF = A'F'$, and through F and F' pass planes $\perp$ to PA and $P'A'$ respectively.

Let the first plane cut AC in some point E, and AB in some point D, and likewise the second plane cut $A'C'$ and $A'B'$ in corresponding points E' and D'.

Why must there be such intersection points? Must they be between A and C, A and B, A' and C', A' and B' respectively?

Then $\angle s$ DFE and $D'F'E'$ are the measures of the dihedral angles whose edges are AP and $A'P'$, and these are to be proved equal to each other.

Now use the pairs of face triangles APB, $A'P'B'$, etc., to show that $\triangle ABC \cong \triangle A'B'C'$. Then show in order

(1) $\triangle ADF \cong \triangle A'D'F'$. (2) $\triangle AEF \cong \triangle A'E'F'$.

(3) $\triangle ADE \cong \triangle A'D'E'$. (4) $\triangle DFE \cong \triangle D'F'E'$.

In like manner the other pairs of dihedral angles may be proved equal to each other.

520. Two polyhedral angles are **congruent** if they can be so placed that their vertices and edges coincide.

A polyhedral angle is read by naming the vertex and one letter in each edge, as P-$ABCDE$, or by the vertex alone where no ambiguity would arise.

521. COROLLARY. *If two trihedral angles have their face angles equal each to each and arranged in the same order, they are congruent.*

For by the theorem their dihedral angles are equal each to each, and if the equal faces are arranged in the same order, the trihedral angles may be made to coincide throughout.

SUMMARY OF CHAPTER VIII.

1. Describe the various ways of determining a plane.

2. State the axioms used in this chapter.

3. State the definitions and theorems on perpendicular lines and planes.

4. State the definitions and theorems on parallel lines and planes.

5. Name some applications in connection with perpendiculars and parallels which have impressed you.

6. State some facts in regard to perpendiculars and parallels in the plane which do not hold in space.

7. Give the definitions and theorems on dihedral angles.

8. What theorems on perpendicular planes are proved in connection with dihedral angles?

9. Give the definitions and theorems on projections used in this chapter.

10. Give the definitions and theorems on polyhedral angles thus far used.

11. Make a list of all the loci problems in this chapter.

12. Study the following collection of problems and applications, and then make a collection of those which impress you as most interesting or useful. Include in this list any applications in the chapter.

PROBLEMS AND APPLICATIONS.

1. A Christmas tree is made to stand on a cross-shaped base. If the tree is perpendicular to each piece of the cross is it perpendicular to the floor?

2. If the projections on a plane of a number of points outside the plane lie in the same straight line, do the points themselves necessarily lie in a straight line?

3. If A, B, and C do not lie in the same line, and if their projections on a plane M do lie in a straight line, what is the relation of the planes M and ABC?

4. Is it possible to project a circle upon a plane so that the projection shall be a straight line-segment? If so, how must the circle and the plane be related?

5. How many planes may be made to pass through a given point parallel to a given line? Discuss the mutual relations of all such planes.

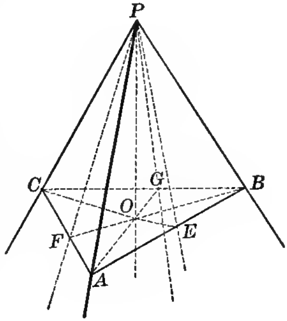
6. Through a point P construct a line meeting each of two lines l_1 and l_2.

SUGGESTION. Let M and N be the planes determined by P and l_1 and by P and l_2. Is this construction always possible?

7. Given a plane M and lines l_1, l_2. Construct a line perpendicular to M and meeting both l_1 and l_2.

SUGGESTION. Project l_1 and l_2 on M.

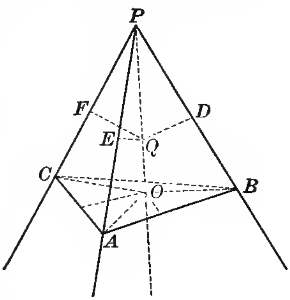
8. A line l is parallel to a plane M, and lines l_1 and l_2 in M are not parallel to l. Show that the shortest distance between l and l_1 is equal to the shortest distance between l and l_2.



9. Prove that the planes bisecting the dihedral angles of a trihedral angle meet in a line.

10. Find the locus of all points equidistant from the planes determined by the faces of a trihedral angle.

11. Find the locus of all points equidistant from the edges of a trihedral angle.



SUGGESTION. On the edges lay off $PA = PB = PC$. Let O be equidistant from A, B, and C. Then any point Q in PO is equidistant from the edges.

12. Planes determined by the edges of a trihedral angle and the bisectors of the opposite face angles meet in a line.

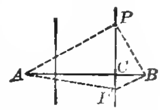
SUGGESTION. If, in the figure of Ex. 9, $PA = PB = PC$, and if PE, PF, PG bisect the face angles, then E, F, G are the middle points of the sides of the triangle ABC. Now apply the theorem that the medians of a triangle meet in a point.

13. Planes perpendicular to the faces of a trihedral angle and bisecting its face angles meet in a line.

14. Show how to locate a point which is at a distance of 2 feet from each face of a trihedral angle.

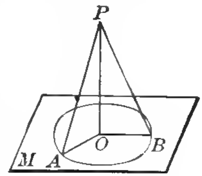
SUGGESTION. Pass a plane parallel to each face of the trihedral angle and at a distance of 2 feet from it.

15. Show how to locate a point which is 2 feet from one face of a trihedral angle, 3 feet from the second, and 4 feet from the third.



16. Find the locus of a point in space such that the difference of the squares of its distances from two fixed points is constant.

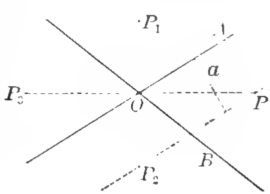
SUGGESTION. First solve the problem in the plane, obtaining as the required locus the line PP'. Then rotate the figure about the line AB as an axis. See page 227, Ex. 7.



17. Find the locus of all points in a plane at which lines from a fixed point P not in the plane meet the plane at equal angles.

18. Find the locus of all points which are at the same fixed distance from each of two intersecting planes.

SOLUTION. Pass a plane perpendicular to the intersection of the two given planes, making the plane cross section shown in the figure. Draw the bisector of $\angle AOB$ and a line at the required distance a from OA and parallel to it.



Then P is at the distance a from each of the lines OA and OB. In the same manner three other such points, P_1, P_2, P_3, are constructed.

Now let this figure move through space parallel to itself, the point O moving along the intersection of the given planes. The points P, P_1, P_2, P_3 will thus move along straight lines which constitute the required locus.

19. Find the locus of points 3 feet from one of two intersecting planes and 6 feet from the other.

20. It is required that a series of electric lights shall be 7 feet above the floor of a room and 3 feet from the walls. Find the locus of all points at which such lights may be placed.

21. Find the locus of all points in space equally distant from each of two intersecting straight lines.

22. Find the locus of all points in space equally distant from two parallel lines.

23. Given two points A and B on the same side of a plane M. Determine a point P in M such that $AP + PB$ shall be a minimum.

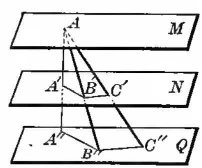
SUGGESTION. Pass a plane through A and B perpendicular to M, and proceed as in Ex. 8, page 200.

24. Show that if the edge of a dihedral angle is cut by two parallel planes, the sections which they make with the faces form equal angles.

25. Show that if all edges of a trihedral angle are cut by each of a series of parallel planes, the intersections form a series of similar triangles.

26. Find the locus of the intersection points of the medians of the triangles obtained in Ex. 25. Also of the altitudes.

27. If three planes are so related that the segments intercepted on any transversal line are in the same ratio as the segments intercepted on any other transversal then the planes are parallel.



SUGGESTION. Let M, N, Q be the three planes. From A any point in M draw three lines, not in the same plane, meeting N in A', B', C', and Q in A'', B'', C''. Use the hypothesis to show that $A'B' \parallel A''B''$ and $C'B' \parallel C''B''$. Hence $Q \parallel N$. Similarly show that $M \parallel N$.

28. A segment AB of fixed length is free to move so that its end-points lie in two fixed parallel planes. Find the locus of a point C on AB if AC is of fixed length.

29. If the projections of a set of points on each of two planes not parallel to each other lie in straight lines, show that the points themselves lie in a straight line.

30. If l_1 and l_2 are not parallel and non-intersecting, show that there is only one plane through l_1 parallel to l_2.

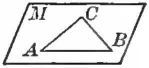
31. Show that if two lines are not parallel and do not lie in the same plane, they have only one common perpendicular, and that the shortest distance between the lines is measured along this perpendicular.

CHAPTER IX.

PRISMS AND CYLINDERS.

522. Definitions. Any portion of a plane entirely bounded by line-segments or curves is called a **plane-segment**.

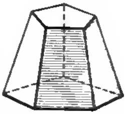
E.g. the portion of the plane M inclosed by the triangle ABC is the plane-segment ABC.



If the boundary is composed entirely of straight line-segments, the inclosed portion is called a **polygonal plane-segment**.

523. A **polyhedron** is a three-dimensional figure formed by polygonal plane-segments which entirely inclose a portion of space.

The line-segments which are common to two polygons are the **edges** of the polyhedron.



The plane-segments inclosed by the edges are the **faces**, and the intersections of the edges are the **vertices** of the polyhedron. A polyhedron is **convex** if every section of it made by a plane is a convex polygon.

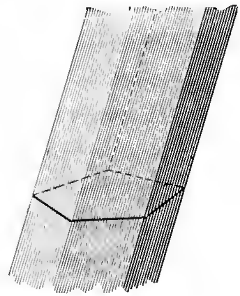
NOTE. The word *polyhedron*, as here defined, means the *surface* inclosing a portion of space and not that portion of space itself. However, it is sometimes convenient to use the word *polyhedron* when referring to the inclosed space. Thus, we speak of dividing a polyhedron into other polyhedrons when, strictly, we mean that smaller polyhedrons are constructed which divide into parts the space inclosed in the given polyhedron.

In the same manner the word *polygon* is sometimes used to indicate the plane-segment bounded by it, and the word *angle* to indicate the part of the plane within it.

Thus, a face of a polyhedron is sometimes called a polygon, and the face of a polyhedral angle is sometimes called an angle.

In all cases the context will clearly indicate what is meant.

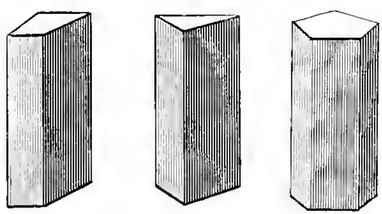
524. Definitions. Given a convex polygon and a straight line not in its plane. If the straight line move so as to remain parallel to itself while it always touches the polygon and is made to traverse it completely, the line is said to **generate a closed prismatic surface.**



The moving line is the **generator** of the surface, and the guiding polygon the **directrix.** The generator in any one of its positions is an **element** of the surface.

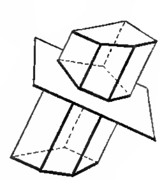
A **cross section** of a prismatic surface is made by any plane cutting all its elements.

525. A **prism** is that part of a closed prismatic surface included between two parallel cross sections, together with the intercepted plane-segments.



The parallel plane-segments are the **bases** of the prism, and the portion of the prismatic surface between the bases is called the **lateral surface** of the prism.

The lateral surface is composed of parallelograms (why?), and these are called the **lateral faces** of the prism. The sides of these parallelograms, not common to the bases, are the **lateral edges** of the prism.



A **right section** of a prism is made by a plane cutting each of its lateral edges, extended if necessary, and perpendicular to them.

526. Prisms are classified according to the form of their right sections, as **triangular, quadrangular, pentagonal, hexagonal**, etc. A **regular prism** is one whose right section is a regular polygon.

A prism is a **right prism** if its lateral edges are perpendicular to its bases; otherwise it is **oblique**.

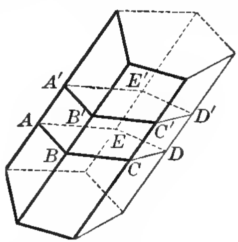
The **altitude** of a prism is the perpendicular distance between its bases. The altitude of a right prism is equal to its edge.

The **lateral area** of a prism is the sum of the areas of its lateral faces.

The **total area** is the sum of its lateral area and the area of its bases.

THEOREMS ON PRISMS.

527. THEOREM. *The cross sections of a prism made by parallel planes are congruent polygons.*



Given a prism cut by two parallel planes forming the polygons $ABCDE$ **and** $A'B'C'D'E'$.

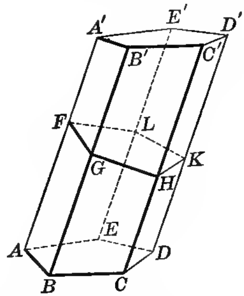
To prove that $ABCDE \cong A'B'C'D'E'$.

Outline of proof: (1) Show that $AB = A'B'$, $BC = B'C'$, etc., by proving that $ABB'A'$, $BCC'B'$, etc., are ▱.

(2) Show that $\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$, etc.

(3) Hence, show that $ABCDE$ and $A'B'C'D'E'$ can be made to coincide.

528. THEOREM. *The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.*



Suggestion. Show that the lateral edges are mutually equal and that the area of each face is the product of a lateral edge and one side of the right-section polygon.

Complete the proof.

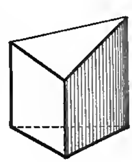
NOTE. The form of statement in this theorem is the usual abbreviation for the more *precise* form:

The lateral area of a prism is equal to the product of the *numerical measures* of a lateral edge and the perimeter of a right section.

Similar abbreviations are used throughout this text.

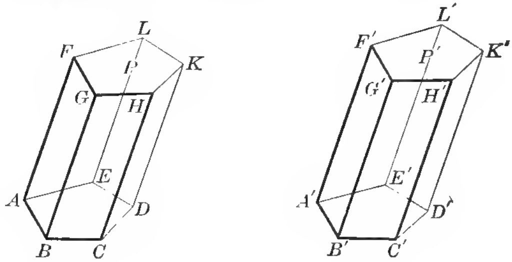
529. COROLLARY. *The lateral area of a right prism is equal to the product of its altitude and the perimeter of its base.*

530. **Definitions.** A polyhedron which is a part of a prism cut off by a plane meeting all the lateral edges, but not parallel to the base, is called a **truncated prism.**



Two polyhedrons are said to be **added** when they are placed so that a face of one coincides with a face of the other, but otherwise each lies outside the other.

531. THEOREM. *Two prisms are congruent if three faces having a common vertex in the one are congruent respectively to three faces having a common vertex in the other, and similarly placed.*



Given the three faces meeting at B in prism P congruent respectively to the three faces meeting at B' in prism P', and similarly placed.

To prove that P can be made to coincide with P'.

Outline of proof: Trihedral angles B and B' are congruent. (Why?)

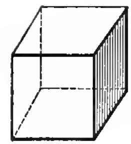
Now apply the two prisms, making B coincide with B', and thus show in detail that:

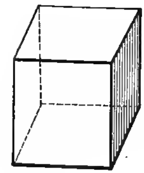
(1) The lower bases coincide.
(2) The lateral faces at B and B' coincide.
(3) The upper bases coincide.
(4) All the lateral faces coincide.

Hence P and P' coincide throughout.

532. COROLLARY. *Two right prisms are congruent if they have congruent bases and equal altitude.*

533. COROLLARY. *Two truncated prisms are congruent if the faces forming a trihedral angle of one are equal respectively to the corresponding faces of the other.*



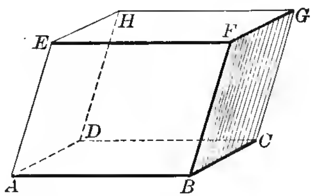


534. Definitions. A parallelopiped is a prism whose bases, as well as lateral faces, are parallelograms.

A **rectangular parallelopiped** has all its faces rectangles.

A **cube** is a parallelopiped whose faces are all squares.

535. THEOREM. *Any two opposite faces of a parallelopiped are parallel and congruent.*



Suggestions. Consider the opposite faces *ABFE* and *DCGH*.

(1) Show that the sides of the angles *ABF* and *DCG* are parallel, and hence the planes determined by them are parallel.

(2) Show that these faces are congruent.

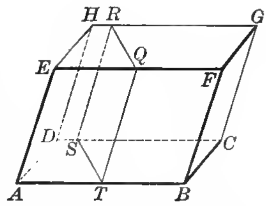
In like manner argue about any other pair of opposite faces.

536. **EXERCISES.**

1. Can a parallelopiped be a *right* prism without being a *rectangular* parallelopiped? Illustrate.

2. Show that in a rectangular parallelopiped each edge is perpendicular to all the other edges meeting it.

3. Is it possible to construct a prism which has no right section according to the definition of § 525?

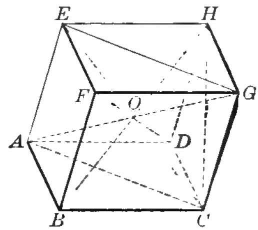


4. Show that any section of a parallelopiped made by a plane cutting four parallel edges is a parallelogram.

5. A section made by a plane passed through two diagonally opposite edges of a parallelopiped is a parallelogram.

E.g. the section through DH and BF in the figure of Ex. 4.

6. Show that any two of the four diagonals of a parallelopiped bisect each other.



E.g. AG and CE in the figure. Make use of the preceding example.

7. Show that the diagonals of a rectangular parallelopiped are all equal to each other.

8. Show that the square on the diagonal of a rectangular parallelopiped is equal to the sum of the squares on the sides meeting at a vertex from which it is drawn.

E.g. in the figure of Ex. 6, $\overline{AG}^2 = \overline{AC}^2 + \overline{CG}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CG}^2$.

9. Find the ratio of the diagonal to one edge of a cube.

10. Find the edge of a cube whose diagonal is 14 inches. Find the diagonal of a cube whose edge is 16 inches.

11. Find a diagonal of a rectangular parallelopiped whose edges are 6, 8, and 10 inches respectively.

12. Are the diagonals of a cube perpendicular to each other?

13. If two diagonals of a rectangular parallelopiped meet at right angles, and if two of its faces are squares, find the ratio of the sides of the remaining faces.

14. If two congruent right prisms whose bases are equilateral triangles are placed together so as to form one prism whose base is a parallelogram, compare the lateral area of the prism so formed with the sum of the lateral areas of the original prisms.

15. A right prism whose bases are regular hexagons is divided into six prisms whose bases are equilateral triangles. Compare the lateral area of the original prism with the sum of the lateral areas of the resulting prisms.

16. If the perimeter of a right section of a prism is 24 inches and its altitude 6 inches, what is the smallest possible lateral area? What can be said about the largest possible area of such a prism?

17. Any section of a prism made by a plane parallel to a lateral edge is a parallelogram.

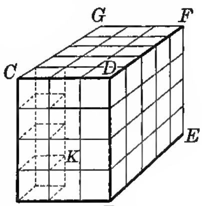
18. Show that for every prism there is at least one set of parallel planes each of which cuts the prism in a rectangular section. See suggestions Ex. 7, p. 327.

VOLUMES OF RECTANGULAR PARALLELOPIPEDS.

537. Thus far certain properties of prisms have been studied, but no attempt has been made to *measure the space inclosed* by such a solid. For this purpose we consider first a rectangular parallelopiped.

538. Definition. In case each edge of a rectangular parallelopiped is commensurable with a unit segment, the number of times which a unit cube is contained in it is the **numerical measure** or the **volume** of the parallelopiped.

539. In the case just described in the definition, the volume is easily computed.



E.g. if in the figure one edge AC is 4 units, and an adjoining edge AB is 3 units, then a cube as AK, whose edge is one unit, may be laid off 4 times along AC and a tier of $3 \cdot 4 = 12$ such cubes will adjoin the face AD, while 5 such tiers will exactly fill the space within the solid. That is, $5 \cdot 3 \cdot 4 = 60$ is the number of cubic units in the solid.

Again, if the given dimensions are 3.4, 2.6, 4.5 decimeters respectively, then unit cubes, with edge one *decimeter*, cannot be made to fill exactly the space inclosed by the figure, but cubes with edge each one *centimeter* will do so, giving 34, 26, and 45 respectively along the three edges of the solid.

Hence the volume is

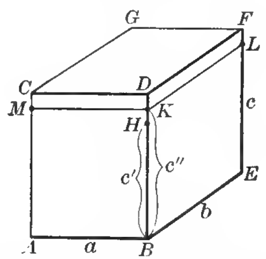
$34 \cdot 26 \cdot 45 = 39{,}780$ cubic centimeters, or 39.78 cubic decimeters.

540. Axiom XIX. *Any rectangular parallelopiped incloses a definite volume which is greater than that of another, provided no dimension of the first is less than the corresponding dimension of the second, and at least one dimension is greater.*

541. THEOREM. *The volume of any rectangular parallelopiped is equal to the product of the numerical measures of its linear dimensions.*

Proof: CASE 1. *If each dimension is commensurable with the unit segment.*

This is the case treated in § 539.



CASE 2. *If two dimensions are commensurable with the unit segment and the third is not.*

By Axiom XIX the parallelopiped has a definite volume V.

Suppose this is not equal to abc and that $V < abc$.

Let c' be a number such that $V = abc'$. Then $c' < c$. On BD lay off $BH = c'$.

Divide the unit segment into equal parts, each less than HD, and lay off one of these parts successively on BD, reaching a point K between H and D. Denote the length of BK by c'' and pass a plane through K parallel to ABE.

By Case 1 the volume of the parallelopiped AL is abc''.

By hypothesis, $V = abc'$, but $abc' < abc''$, since $c' < c''$.

Hence, $V < abc''$. (1)

But by Ax. XIX, $V > abc''$. (2)

Hence, the assumption that $V < abc$ cannot hold.

In the same manner show that $V > abc$ cannot hold.

Hence, we have $V = abc$.

CASE 3. *If one dimension is commensurable with the unit segment and two are not.*

CASE 4. *If all three dimensions are incommensurable with the unit segment.*

The proofs in these cases are similar to the above.

Thus, in Case 3, when b and c are both incommensurable with the unit segment, we obtain the parallelopiped AL, two of whose dimensions a and c'' are commensurable with the unit. Hence, by Case 2, its volume is abc''. Then the argument is identical with that given before.

Hence, in all cases $V = abc$.

542. COROLLARY 1. *The volume of a rectangular parallelopiped is equal to the product of the numerical measures of its base and altitude.*

543. COROLLARY 2. *If two rectangular parallelopipeds have two dimensions respectively equal to each other, their volumes are in the same ratio as their third dimensions; and if they have one dimension the same in each, their volumes are in the same ratio as the products of their other two dimensions.*

For if V and V' are the volumes, and a, b, c and a', b', c' the dimensions, then $\frac{V}{V'} = \frac{a \cdot b \cdot c}{a' \cdot b' \cdot c'} = \frac{a}{a'}$ if $b = b'$ and $c = c'$; or $\frac{V}{V'} = \frac{a \cdot b}{a' \cdot b'}$ if $c = c'$.

VOLUMES OF PRISMS IN GENERAL.

544. From the formula for rectangular parallelopipeds,

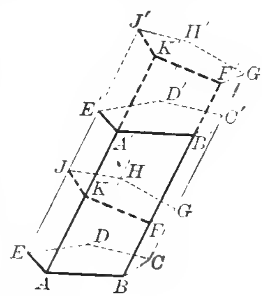
Volume = length × width × altitude, or $V = abc$

we deduce the volumes of prisms in general by means of the principle:

Two polyhedrons are equal (that is, have the same volume) if they are congruent, or if they can be divided into parts which are congruent in pairs.

The sign = between two polyhedrons means that they are equal in volume. The word *equivalent* is sometimes used to mean *equal in volume.*

545. THEOREM. *The volume of an oblique prism is equal to that of a right prism having for its base a right section of the oblique prism and for its altitude a lateral edge of the oblique prism.*



Given the oblique prism AD', with $FGHJK$ a right section, and $F'G'H'J'K'$ a right section of the prism extended so that the edge $AA' = KK'$.

To prove that the oblique prism AD' has the same volume as the right prism KH'.

Outline of proof: Show that

(1) $ABCDE \cong A'B'C'D'E'$;

(2) $ABFK \cong A'B'F'K'$;

(3) $EAKJ \cong E'A'K'J'$.

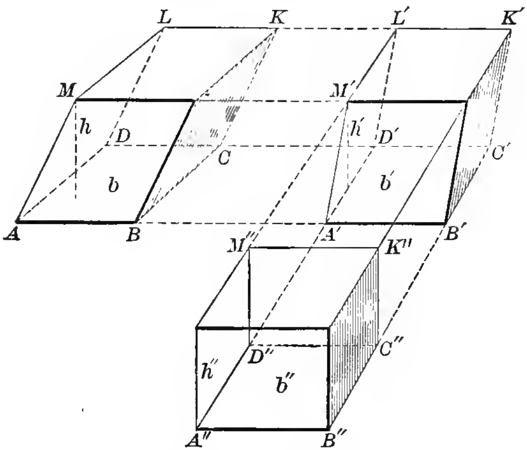
Hence, by § 533 AH and $A'H'$ are congruent.

Now the given prism $AD' = AH + KD'$ (§ 530)

and the right prism $KH' = A'H' + KD'$.

Hence, $AD' = KH'$. (Why?)

546. THEOREM. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Given the oblique parallelopiped AK, with base b and alt. h.

To prove that its volume is equal to $b \cdot h$.

Proof: Considering face AL as the base of prism AK, produce the four edges parallel to AB, and lay off $A'B' = AB$.

Through A' and B' erect planes $\perp$ to AB', thus cutting off the right prism $A'K'$ with $A'L'$ as one base.

Now considering $C'L'$ as the base of prism $A'K'$, produce the four edges $\parallel$ to $C'B'$ and lay off $C''B'' = C'B'$.

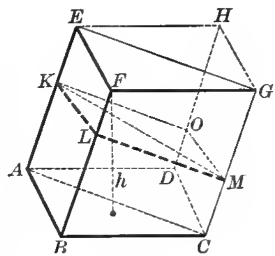
At C'' and B'' erect planes $\perp$ to $C'B''$, cutting off the right prism $A''K''$, which is a rectangular parallelopiped.

Then show that (1) $h = h' = h''$; (2) $b = b' = b''$; (3) prisms AK, $A'K'$, $A''K''$ are equal (§ 545).

But prism $A''K'' = b'' \cdot h''$. (Why?) Hence, prism $AK = b \cdot h$.

Write out this proof in full.

547. THEOREM. *The volume of a triangular prism is equal to the product of its base and altitude.*



Given the triangular prism whose base is ABC.

To prove that the volume of this prism is equal to the area of $\triangle ABC$ multiplied by the altitude h, that is, by the perpendicular distance between ABC and EFG.

Proof: Complete the ▱s $ABCD$ and $EFGH$ and draw DH. Now show that $CDHG$ and $ADHE$ are ▱s, and hence that BH is a parallelopiped.

Let $KLMO$ be a right section of BH, and let KM be the line in which the plane $ACGE$ cuts the plane $KLMO$.

Now (1) $KLMO$ is a ▱. Hence, $\triangle KLM \cong \triangle KMO$.

(2) Volume of prism $ABC\text{–}F$ = area $\triangle KLM$ times BF.

(3) Volume of prism BH = area ▱ $KLMO$ times BF.

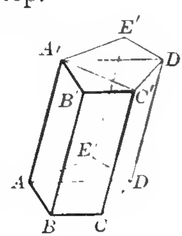
(4) Hence, prism $ABC\text{–}F = \frac{1}{2}$ prism BH.

(5) But prism $BH = h \times$ area of ▱ $ABCD$.

(6) Hence, prism $ABC\text{–}F = h \times \frac{1}{2}$ area of ▱ $ABCD$ $= h \times$ area of $\triangle ABC$.

Therefore prism $ABC\text{–}F$ is equal to the product of its base and altitude. Give reasons for each step.

548. COROLLARY 1. *The volume of any prism is equal to the product of its base and altitude.*



For any prism can be divided into triangular prisms by planes passing through one edge and each of the other non-adjacent edges.

549. COROLLARY 2. *Any two prisms of equal altitudes have the same volumes if their bases are equal.*

550. COROLLARY 3. *The volumes of two prisms have the same ratio as their altitudes if their bases are equal; and the same ratio as the areas of their bases if their altitudes are equal.*

551. EXERCISES.

1. The proposition that the volume of any prism is equal to the product of its base and altitude is of great importance. What theorems of this chapter were used directly or indirectly in proving it?

2. If the base of a prism is 36 square inches and its altitude 12 inches, what is its volume?

3. If the perimeter of the base of a prism and the length of a lateral edge are known, is the lateral area thereby determined?

4. If the area of the base and the length of an edge are known, can the volume be found?

5. What dimensions of a prism must be known in order to determine its lateral area by means of the theorems of this chapter? What dimensions must be known to determine its volume?

6. Parallel sections of a closed prismatic surface are congruent polygons. Prove.

7. Find the edge of a cube if its total area is equal numerically to its volume, an inch being used as a unit.

8. A side of the base of a regular right hexagonal prism is 3 inches. Find its altitude if its volume is $54\sqrt{3}$ cubic inches. What is the total area of this prism?

9. A side of one base of a regular right triangular prism is equal to the altitude of the prism. Find the length of the side if the total surface is numerically equal to the volume.

10. Solve the preceding problem in case the prism is a regular right hexagonal prism.

11. The volume of a regular right prism is equal to the lateral area multiplied by half the apothem of the base. Prove.

CYLINDERS.

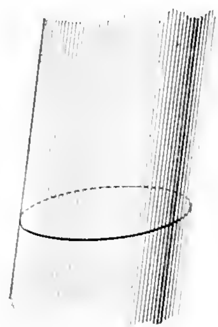
552. Definitions. A surface no segment of which is plane is called a **curved surface**.

E.g. the surface of an eggshell or of a stovepipe is a curved surface.

A **closed plane curve** is one which can be traced continuously by a point moving in a plane so as to return to its original position without crossing its path.

A **convex closed curve** is one which can be cut by a straight line in only two points.

553. If a straight line moves so as to remain parallel to itself, while it always touches a closed convex curve and is made to traverse it completely, the line is said to generate a **closed convex cylindrical surface.** The moving line is the **generator**, and the guiding curve the **directrix.** The generator in any one of its positions is an **element** of the surface.



A **cross section** of a cylindrical surface is made by a plane cutting all its **elements.**

554. A **cylinder** is that part of a closed cylindrical surface included between two parallel cross sections, together with the plane-segments thus intercepted. The plane-segments are the **bases** of the cylinder, and the part of the cylindrical surface between the bases is the **lateral surface.**

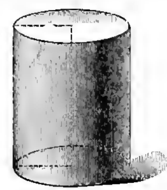
The portion of the generating line in any position which is included between the bases is an **element** of the cylinder. The **altitude** of a cylinder is the perpendicular distance between its bases.

A **right section of a cylinder** is made by a plane cutting each of its elements at right angles.

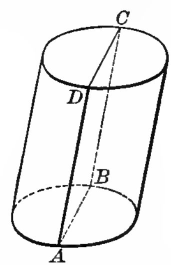
A **circular cylinder** is one whose right section is a circle.

The radius of a circular cylinder is the radius of its right section.

A cylinder whose bases and right sections are circles is called a **right circular cylinder.** A right circular cylinder is called a **cylinder of revolutions,** since it may be generated by revolving a rectangle about one of its sides as an axis. The side opposite the axis generates the lateral surface, and the sides adjacent to the axis generate the bases.



555. THEOREM. *If a plane contains an element of a cylinder and meets it in one other point, then it contains another element also, and the section is a parallelogram.*



Given the cylinder *AC* and a plane containing the element *AD* and one other point as *B*.

To prove that it contains another element *BC* and that *ABCD* is a parallelogram.

Proof: Through *B* pass a line *BC* ∥ *AD*.

Then *BC* is an element of the cylinder. (Why?)

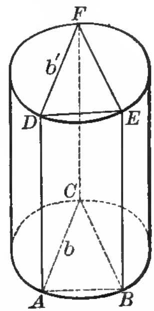
Also *BC* lies in the plane *ABD*.

Now show that *ABCD* is a parallelogram.

What is the section *ABCD* if *AC* is a *right* cylinder?

556. Definition. If a plane contains an element of a cylinder and no other point of it, the plane is said to be **tangent** to the cylinder, and the element is called the **element of contact.**

557. THEOREM. *The bases of a cylinder are congruent plane-segments.*



Given a cylinder with the bases b and b'.

To prove that $b \cong b'$.

Proof: Take any three points A, B, C in the rim of the base b and draw the elements at these points, meeting the base b' in D, E, F.

Show that $\triangle ABC \cong \triangle DEF$.

Now, while the elements AD and BE remain fixed, conceive CF to generate the cylinder.

Evidently $\triangle ABC \cong \triangle DEF$ in every position of CF.

Hence, if base b' is applied to base b with these triangles coinciding in *one* position, they will coincide in every position corresponding to the moving generator.

That is, $b \cong b'$.

558. EXERCISES.

1. Show that right sections of any cylinder are congruent, and any section parallel to the base is congruent to the base.

2. If two intersecting planes are each tangent to a cylinder, show that their line of intersection is parallel to an element of the cylinder and also parallel to the plane containing the two elements of contact.

3. What is the locus of all points at a perpendicular distance of 2 feet from a given line?

4. If the section of a cylinder made by every plane parallel to an element of it is a rectangle, what kind of cylinder is it?

5. If the radius r of a right circular cylinder is equal to its altitude, find the distance from the center of the base to a plane whose intersection with the cylinder is a square.

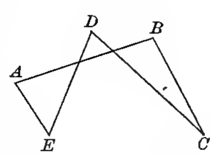
6. Roll a sheet of paper so as to form a circular cylinder, that is, one whose *right section is a circle.* Now determine by inspection the shape of the base if the paper is cut so as to let the cylinder stand in an oblique position. Also deform the cylinder so as to make the oblique base circular, and then determine the shape of the right section.

7. Show that for every cylinder there is at least one set of parallel planes which cut the cylinder in rectangular sections.

Suggestion. Project an element on the plane of the base, and draw lines in the base at right angles to this projection. Through these lines pass planes parallel to an element.

8. Is a polygon circumscribed about a convex closed curve necessarily a convex polygon? See § 160.

9. Is a polygon inscribed in a convex closed curve necessarily convex? (Note that a broken line AB, BC, CD, DE, EA which cuts itself as shown in the figure is not here regarded as a polygon.) The answers to Exs. 8 and 9 limit the polygons to be used under Ax. XX to convex polygons. Hence it is not necessary to state in that axiom that these polygons must be convex.



10. If polygons other than convex are permitted, is it possible to construct one within a convex closed curve whose perimeter is greater than the length of the curve? Can such polygons be inscribed in the curve?

MEASUREMENT OF THE SURFACE AND VOLUME OF A CYLINDER.

559. Thus far in geometry the word **area** has been used in connection with *plane-segments only*. In some cases the computation of an area has been possible by an approximation process only, as in the case of some rectangles and of the circle.

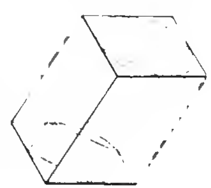
In the case of *any curved surface* it is evident that approximate measurement is the *only kind possible* in terms of a plane area unit, since no such unit, however small, can be made to coincide with a segment of a curved surface.

Theorems concerning the surface and the volume of a cylinder are based upon the following definitions and assumptions:

560. Axiom XX. *Any convex closed curve has a definite length and incloses a definite area, which are less respectively than the perimeter and area of any circumscribed polygon and greater than those of any inscribed polygon.*

Also the perimeter and area of either the inscribed or the circumscribed polygon may be made to differ from those of the curve by as little as we please by taking all the sides sufficiently small.

561. Definitions. A prism is said to be **inscribed in a cylinder** if its lateral edges are elements of the cylinder, and their bases lie in the same planes.



A prism is said to be **circumscribed about a cylinder** if its lateral faces are all tangent to the cylinder, and their bases lie in the same planes.

562 Axiom XXI. *The lateral surface of a convex cylinder has a definite area, and the cylinder incloses a definite volume, which are less respectively than those of any circumscribed prism and greater than those of any inscribed prism.*

563. THEOREM. *The lateral area of a convex cylinder is equal to the product of an element and the perimeter of a right section.*

Given a convex cylinder of which L is the lateral area, e an element, and p the perimeter of a right section.

To prove that $L = ep$.

Proof: It will be shown that L can be neither greater than nor less than ep.

First, suppose $L < ep$.

Let $L = eK$. Then $K < p$. (1)

Now inscribe a cylinder the perimeter p' of whose right section is greater than K. This is possible by § 560.

Then $ep' > eK$. (2)

Hence, ep' is greater than the supposed value eK of L.

But this is impossible, since ep' is the area of an *inscribed* prism (§ 562).

Hence, the supposition $L < ep$ leads to a contradiction.

Secondly, the supposition that $L > ep$ may be shown to be impossible by using a circumscribed prism.

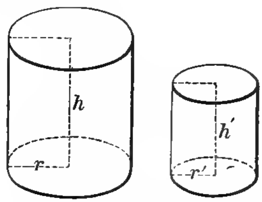
Hence, we have $L = ep$.

564. COROLLARY. *If r is the radius of any circular cylinder, and e is an element, then $L = 2\,\pi re$.*

In the case of a *right* circular cylinder $e = h$, the altitude, and we have $L = 2\,\pi rh$.

565. **Definition.** Two right circular cylinders are **similar** if they are generated by similar rectangles revolving about corresponding sides.

566. THEOREM. *The lateral areas or the entire areas of two similar right circular cylinders are in the same ratio as the squares of their altitudes or as the squares of the radii of their bases.*



Suggestion. If h and h' are the altitudes, r and r' the radii, L and L' the lateral areas, and A and A' the total areas, we are to show that

$$\frac{L}{L'} = \frac{A}{A'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

Make use of the following, giving each in detail.

$$(2)\ L = 2\pi rh, \quad (2)\ A = 2\pi r(r+h),$$

$$(3)\ \frac{r}{r'} = \frac{h}{h'}, \text{ and } (4)\ \frac{r+h}{r'+h'} = \frac{r}{r'} = \frac{h}{h'}.$$

567. THEOREM. *The volume of any convex cylinder is equal to the product of its altitude and the area of its base, or to the product of an element and the area of its right section.*

Suggestions. (1) If h is the altitude and b the base, show by an argument similar to that of § 563, that V, the volume, can be neither greater nor less than bh.

(2) If e is an element and c the area of a right section, give a similar argument to show that $V = ec$, using § 545. Give all the steps in full.

568. COROLLARY. *If a cylinder has a right circular section whose radius is r_1 and an element e, then* $V = \pi r_1^2 e$.

If a cylinder has a circular base of radius r_2, and an altitude h, then $V = \pi r_2^2 h$.

In the case of a right circular cylinder, $r_1 = r_2$ and $h = e$. Hence, the two formulas become identical.

569. THEOREM. *The volumes of two similar right circular cylinders are in the same ratio as the cubes of the radii of their bases or as the cubes of their altitudes.*

Give the proof in full, using suggestions similar to those given in § 566.

PROBLEMS AND APPLICATIONS.

1. What dimensions of a cylinder must be known in order that its lateral area may be computed? State fully for different kinds of cylinders.

2. What dimensions of a cylinder must be known in order that its volume may be computed?

3. If the base of a cylinder is a circle with radius 5 inches, find its volume if the altitude is 8 inches.

4. If the lateral surface of a cylinder and the length of an element are known, can the perimeter of a right section be found? If the lateral area is 400 π, and an element 15, find the perimeter of a right section.

5. The diameter of a right circular cylinder is 8, and the diagonal of the largest rectangle which can be cut from it is 16. Find its altitude.

6. The volume of a right circular cylinder is 128 π cubic inches and its altitude is equal to its diameter. Find the altitude and the diameter.

7. If the diameter of a right circular cylinder is equal to its altitude, determine the diameter so that the total area of the cylinder is equal numerically to its volume.

8. Find the diameter of a right circular cylinder if the total area of the inscribed regular triangular prism is equal numerically to the volume of the cylinder, the diameter of the cylinder being equal to its altitude.

9. In the preceding find the diameter of the cylinder if the volume of the prism is equal numerically to the total area of the cylinder.

10. Solve a problem like Ex. 8 if a regular hexagonal prism is used instead of a triangular prism.

11. Solve a problem like Ex. 9 if a regular hexagonal prism is used instead of a triangular prism.

12. A rectangle whose sides are a and b is turned about the side a as an axis and then about the side b. Find the ratio of the volumes of the two cylinders thus developed.

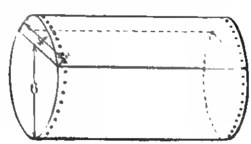
13. Compare the total surfaces of the two figures developed in Ex. 12.

14. Find the diameter of a right circular cylinder if its lateral area is equal numerically to its volume. Does the result depend upon the altitude of the cylinder?

15. If the altitude of a right circular cylinder is equal to its diameter, find the ratio of the numerical values of its total area and its volume. Does this depend on the radius?

16. A regular octagonal prism is inscribed in a right circular cylinder whose altitude is equal to the diameter. Find the difference between the volumes of the cylinder and the prism, if a side of the octagon is 4 inches.

17. A cylindrical tank 8 feet in diameter, partly filled with water, is lying on its side. If the greatest depth of the water is 6 feet, what fraction of the volume of the tank is filled with water?

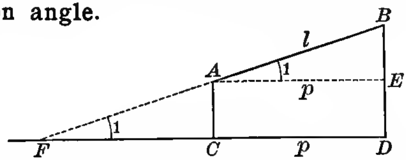


18. In the preceding problem find the fraction of the volume occupied by water if the width of the top of the water along a right cross section of the tank is 4 feet.

THEOREMS ON PROJECTION.

570. The projection of a line-segment on a given line was defined in § 512. The **length of the projection** will now be computed in terms of the given line-segment.

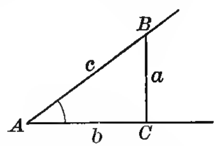
571. Definitions. The acute angle between a line-segment and a given line on which it is projected is called the **projection angle.**



E.g. $\angle AFC$ or its equal $\angle BAE$ found by drawing $AE \parallel CD$.

If l is the length of a line-segment and p the length of its projection, then the ratio $\frac{p}{l}$ is called the **cosine of the projection angle.**

E.g. in the figure, $\frac{p}{l} =$ cosine $\angle BAE$.



572. In any right triangle ABC, either acute angle, as $\angle A$, is the projection angle between the hypotenuse and the side adjacent to the angle.

Hence the **cosine of an acute angle of a right triangle** is the *ratio of the adjacent side to the hypotenuse.*

We have already defined the **sine of an acute angle of a right triangle** as the *ratio of the opposite side to the hypotenuse.* See § 279.

We now define the **tangent of an acute angle of a right triangle** as the *ratio of the opposite side to the adjacent side.*

Using the common abbreviations, sin, cos, and tan, we have in the figure:

$$\sin A = \frac{a}{c}, \ \cos A = \frac{b}{c}, \ \tan A = \frac{a}{b}.$$

573. The sine, cosine, and tangent are of great importance in many computations. By careful measurement (and in other ways) their values may be computed for any acute angle, and a table formed, like that on page 335.

E.g. if in the figure of § 572 $\angle A = 35°$ (measured with a protractor), we may measure AC, AB, and BC, and thus compute the ratios $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a}{b}$, and find the values of sin 35°, cos 35°, tan 35°.

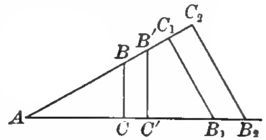
With an ordinary ruler it will not usually be possible to make these measurements with sufficient accuracy to obtain more than one decimal place.

Draw an angle of 35°, and make the measurements and computations, using an hypotenuse of various lengths (the longer the better), and show that the results do not depend upon the length of hypotenuse chosen.

574. EXERCISES.

1. Using a protractor, construct angles of 10°, 30°, 50°, 70°, and by measurement determine the sine, cosine, and tangent of each.

2. Prove that the cosine of any given angle is the same, no matter what point is taken in *either* side from which to let fall the perpendicular to the other side. Prove the same for the tangent.



3. Show that if the hypotenuse be taken *one decimeter* in length, then the length of the side adjacent, measured in decimeters, is the *cosine* of the angle, and the length of the side opposite is the *sine* of the angle.

4. Show that if the side adjacent be taken one decimeter in length, the length of the side opposite, measured in decimeters, is the *tangent* of the angle.

5. Without any direct measurement, show how to compute the three ratios for each of the angles, 30°, 45°, 60°.

SUGGESTION. Make use of the fact that if one acute angle in a right triangle is 30°, the side opposite it is one half the hypotenuse.

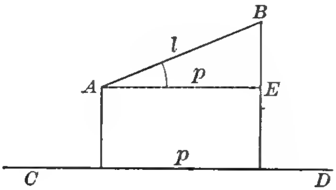
6. As the angle is made smaller and smaller, what are the values approached by the sine, cosine, and tangent?

7. As the angle is made more and more nearly 90°, what are the values approached by the sine and cosine? Discuss this case for the tangent.

TABLE OF SINES, COSINES, AND TANGENTS.

Ang.	Sin	Cos	Tan	Ang.	Sin	Cos	Tan	Ang.	Sin	Cos	Tan
0°	0	1	0	31°	.515	.857	.601	61°	.875	.485	1.80
1°	.017	1.000	.017	32°	.530	.848	.625	62°	.883	.469	1.88
2°	.034	.999	.035	33°	.545	.839	.649	63°	.891	.454	1.96
3°	.052	.999	.052	34°	.559	.829	.675	64°	.899	.438	2.05
4°	.070	.998	.070	35°	.574	.819	.700	65°	.906	.423	2.14
5°	.087	.996	.087	36°	.588	.809	.727	66°	.914	.407	2.25
6°	.105	.095	.105	37°	.602	.799	.754	67°	.921	.391	2.36
7°	.122	.993	.123	38°	.616	.788	.781	68°	.927	.375	2.48
8°	.139	.990	.141	39°	.629	.777	.810	69°	.934	.358	2.61
9°	.156	.988	.158	40°	.643	.766	.839	70°	.940	.342	2.75
10°	.174	.985	.176	41°	.656	.755	.869	71°	.946	.326	2.90
11°	.191	.982	.194	42°	.669	.743	.900	72°	.951	.309	3.08
12°	.208	.978	.213	43°	.682	.731	.933	73°	.956	.292	3.27
13°	.225	.974	.231	44°	.695	.719	.966	74°	.961	.276	3.49
14°	.242	.970	.249	45°	.707	.707	1.00	75°	.966	.259	3.73
15°	.259	.966	.268	46°	.719	.695	1.04	76°	.970	.242	4.01
16°	.276	.961	.287	47°	.731	.682	1.07	77°	.974	.225	4.33
17°	.292	.956	.306	48°	.743	.669	1.11	78°	.978	.208	4.70
18°	.309	.951	.325	49°	.755	.656	1.15	79°	.982	.191	5.14
19°	.326	.946	.344	50°	.766	.643	1.19	80°	.985	.174	5.67
20°	.342	.940	.364	51°	.777	.629	1.23	81°	.988	.156	6.31
21°	.358	.934	.384	52°	.788	.616	1.28	82°	.990	.139	7.12
22°	.375	.927	.404	53°	.799	.602	1.33	83°	.993	.122	8.14
23°	.391	.921	.424	54°	.809	.588	1.38	84°	.995	.105	9.51
24°	.407	.914	.445	55°	.819	.574	1.43	85°	.996	.087	11.43
25°	.423	.906	.466	56°	.829	.559	1.48	86°	.998	.070	14.30
26°	.438	.899	.488	57°	.839	.545	1.54	87°	.999	.052	19.08
27°	.454	.891	.510	58°	.848	.530	1.60	88°	.999	.035	28.64
28°	.469	.883	.532	59°	.857	.515	1.66	89°	1.000	.017	57.29
29°	.485	.875	.554	60°	.866	.500	1.73	90°	1.000	0	
30°	.500	.866	.577								

575. THEOREM. *The length of the projection of a line-segment upon a given line is equal to the length of the line-segment multiplied by the cosine of the projection angle.*



Given the projection p of the line-segment l on the line CD, with the projection angle A.

To prove that $p = l \cos A$.

Proof: By definition we have $\frac{p}{l} = \cos A$.

Hence, $p = l \cos A$.

576. **EXERCISES.**

1. Find the cosines of the angles 35° 30′, 54° 15′, 15° 45′.

SUGGESTION. The cosine of 35° 30′ lies between cos 35° and cos 36°. We assume that it lies halfway between these numbers. This assumption, while not quite correct, is very nearly so for small differences of angles, as in this case, where the total difference is only one degree. From the table cos 35° = .819, cos 36° = .809.

The number midway between these is .814, which we take as the cosine of 35° 30′.

This process is called **interpolation.** A similar process is used for sines and tangents.

2. Find the tangents of the angles 25° 20′, 47° 45′, 63° 40′.

3. Find the angle whose tangent is 1.74.

SOLUTION. From the table we have tan 60° = 1.73 and tan 61° = 1.80. Hence, the required angle must lie between 60° and 61°. Moreover, the number 1.74 is one seventh the way from 1.73 to 1.80. Hence, we assume the angle to lie one seventh the way from 60° to 61°, which gives $60° + \frac{1}{7} \times 1° = 60° + 9'$ nearly. The required angle is 60° 9′.

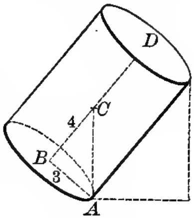
4. Find the angles whose sines are .276; .674; .137.

5. Find the angles whose cosines are .940; .094; .135.

6. Find the angles whose tangents are .781; 1.41; 3.64.

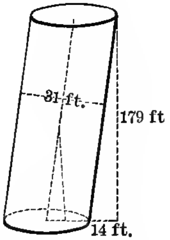
Notice that as an angle increases, its sine and tangent both *increase*, but its cosine *decreases*.

7. At what angle with the horizontal must the base of a right circular cylinder be tilted to make it just topple over if its diameter is 6 feet and its altitude 8 feet?



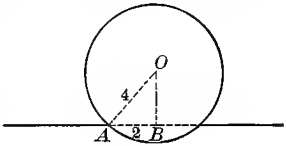
SUGGESTION. The center of gravity is at the middle point C of the axis of the cylinder. The base must be tilted so that the line AC becomes vertical.

8. The Leaning Tower of Pisa is 179 feet high and 31 feet in diameter. It now leans so that a plumb line from the top on the lower side reaches the ground 14 feet from the base.



At what angle is its side now inclined from the vertical? At what angle would its side have to incline from the vertical before it would topple over?

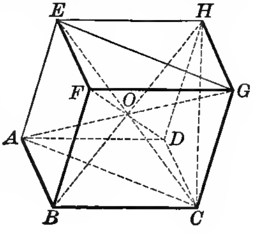
9. A four-inch hole is cut in a board, and a ball 8 in. in diameter is made to rest on it. At what angle must the board be held so that the ball will just roll out of the hole?



SUGGESTION. The board must be held so that the line OA becomes vertical; that is, the board must be tipped at an angle equal to $\angle BOA$.

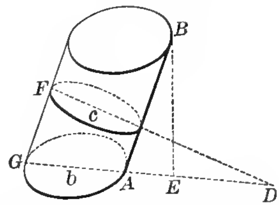
10. Using a ball 8 inches in diameter, what must be the radius of the hole in the board of the preceding problem so that the ball shall just roll out when the board is inclined at an angle of 45° to the horizontal?

11. If the figure $ABCD\text{–}H$ is a cube, find each of the following angles: $\angle ECA$, $\angle AEC$.



Check the results found by using the fact that the sum of the angles of a triangle is 180°.

577. THEOREM. *The altitude of an oblique prism or cylinder is equal to an element multiplied by the cosine of the angle between the plane of the base and that of a right section.*



Given the oblique cylinder with base *b* and right section *c*, and let *BE* be a perpendicular between the bases.

Consider the plane determined by *BE* and the element *AB*.

This plane is ⊥ to the plane of *b* and also to the plane of *c*. (Why?)

Hence, it is ⊥ to the line of intersection of the planes of *b* and *c*. (Why?)

Let this plane cut the planes of *b* and *c* in *GD* and *FD* respectively, *D* being the point in their line of intersection.

Then ∠ *D* is the measure of the dihedral angle between the planes of *b* and *c*. (Why?)

To prove that $BE = AB \cdot \cos D$.

Proof: We have $BE = AB \cdot \cos \angle ABE$. (Why?)

But $\angle D = \angle ABE$. (Why?)

Hence, $BE = AB \cdot \cos D$.

The argument is similar for any oblique prism.

578. COROLLARY. *The dihedral angle between the planes of the base and a right section of an oblique cylinder or prism is equal to the angle between an element and the altitude.*

579. EXERCISES.

1. Given a line-segment 10 inches long. Find the length of its projection on a plane if the projection angle is 20°. If the angle is 30°, 45°, 60°, 90°, 0°.

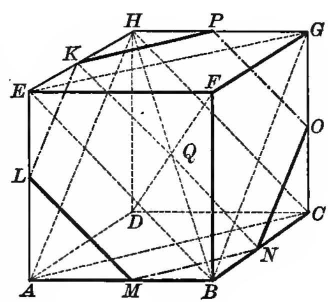
2. A kite string forms an angle of 40° with the ground. The distance from the end of the string to a point directly beneath the kite is 200 ft. Find the length of the string and the perpendicular height of the kite.

3. The altitude of an oblique prism is 15 inches. Find the length of an element if it makes an angle of 45° with the perpendicular between the bases.

4. A right section of a cylinder makes an angle of 20° with the plane of the base. Find the ratio between the altitude and an element.

5. Show that the theorem of § 577 holds for the special case of a right prism or cylinder.

6. Prove that by joining the middle points of six edges of a cube, as shown in the figure, a regular hexagon is formed.



7. Prove that in the preceding example the plane of the regular hexagon, *KLMNOP*, is perpendicular to the diagonal *DF*.

8. How large a cube will be required from which to cut a stopper for a hexagonal spout, each of whose sides is 4 inches?

9. In the figure find the angle *KQH*.

SUGGESTION. Let a be a side of the cube. Compute KH, KQ, and HQ in terms a. Note that $\angle QKH =$ rt. $\angle$.

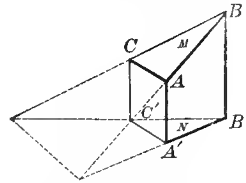
10. Find the area of the projection of the hexagon *KLMNOP* on the face *BCGF*. Note that this projection equals the whole square less $\triangle NCO + \triangle KEL$. See § 580.

11. Find the area of the hexagon in terms of the side a of the cube.

12. By means of the theorem of § 581 and the results in Exs. 10 and 11 find the dihedral angle formed by the planes *BCG* and *MOK*.

PROJECTION OF A PLANE-SEGMENT.

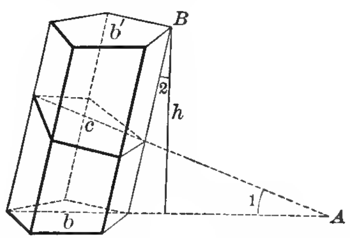
580. **Definition.** If from each point in the boundary of a plane-segment a perpendicular is drawn to a given plane, the locus of the feet of these perpendiculars will bound a portion of the plane, which is called the **projection of the plane-segment** on the given plane.



E.g. the plane-segment $A'B'C'$ in the plane N is the projection of the plane-segment ABC from the plane M upon N.

The **angle of projection** is the angle between the planes M and N.

581. THEOREM. *The area of the projection of a plane-segment on a plane is equal to the area of the plane-segment multiplied by the cosine of the projection angle.*



Proof: Let the boundary of the given plane-segment b be any convex polygon or closed curve.

Using this polygon or curve as a *directrix* and a line perpendicular to the given plane as a *generator*, develop a prismatic or cylindrical surface. The given plane will cut this surface in a right section whose area we denote by c.

Now cut the surface by a plane parallel to b, forming the lower base b of a prism or cylinder whose altitude is h, edge e, and volume V.

Then c is the projection of b upon the given plane, and $\angle 1 = \angle 2$ is the projection angle.

We are to show that $c = b \cos \angle 1$.

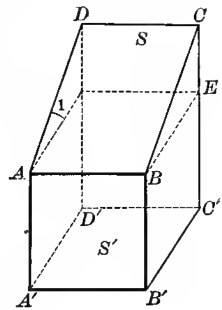
We know that $V = ce = bh$. (Why?)

But $h = e \cos \angle 2$.

Hence, $ce = be \cos \angle 2$.

That is, $c = b \cos \angle 2 = b \cos \angle 1$.

NOTE. The foregoing theorem may be proved *directly* in case the plane-segment is a rectangle with one side parallel to the line of intersection of the two planes. In the figure let S be the given rectangle and S' its projection, with $AB \parallel$ to the line of intersection of the planes in which S and S' lie, and $\angle 1$ the angle between them.



Then $S = AB \cdot BC$

and $S' = A'B' \cdot B'C'$.

But $AB = A'B'$

and $BE = B'C'$. (Why?)

But $BE = BC \cdot \cos \angle 1$ (Why?)

and $S' = A'B' \cdot B'C' = AB \cdot BC \cos \angle 1$.

That is, $S' = S \cos \angle 1$.

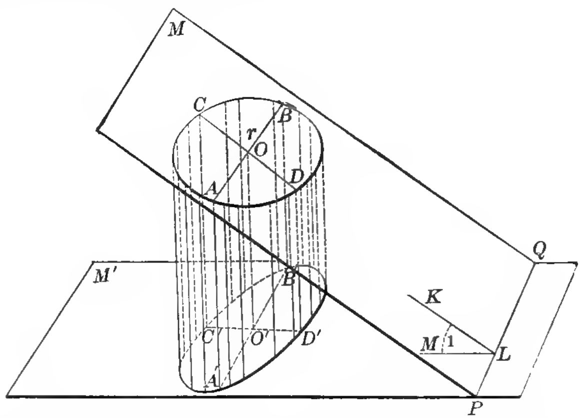
In the case of any plane-segment, rectangles may be inscribed in it in this position and their number increased indefinitely, so that their sum will approach more and more nearly to the area of the plane-segment, and in this way it may be shown to any desired degree of approximation that the projection of a plane-segment equals the given plane-segment multiplied by the cosine of the projection angle.

582. An important special case of the theorem of § 581 is the area of the figure obtained by projecting a circle upon a plane not parallel to the plane of the circle.

Definition: The figure obtained by projecting a circle upon a plane not parallel to the plane of the circle, nor at right angles to it, is called an **ellipse**.

583. Area of the Ellipse. — In the figure two planes, M and M', meet in a line PQ. The circle O in M has a diameter $AB \parallel PQ$ and a diameter $CD \perp PQ$.

In projecting the whole figure upon the plane M' the diameter AB projects into its equal $A'B'$, while CD projects into $C'D'$ so that $C'D' = CD \cos \angle 1$.



By theorem § 581 the area of the ellipse equals the area of the circle multiplied by $\cos \angle 1$.

Hence, $\pi r^2 \cdot \cos \angle 1$ is the area of the ellipse.

But $r \cos \angle 1 = O'C'$ and $r = O'B'$. (§ 575)

Hence, the area of the ellipse is $\pi \cdot O'C' \times O'B'$.

The segments $A'B'$ and $C'D'$ are called respectively the major and minor axes of the ellipse, and $O'B'$ and $O'C'$ the semimajor and the semiminor axes. These latter are usually denoted by a and b.

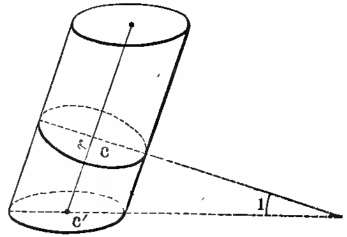
Hence, the area of the ellipse is πab.

Note that when a and b are equal, the ellipse becomes a circle, and this formula reduces to πa^2 as it should.

It may be of interest to note that the problem of finding the *length* of the ellipse is very much more difficult, and can be solved only by means of higher mathematics.

584. The figure of § 583 may also be regarded as representing a cylindrical surface of which the ellipse with center O' is a right section.

It is also true that if we start with a circular cylinder, that is, one whose right section c is a circle, then every oblique section of it, as c', is an ellipse.



The minor axis of such an ellipse will be the diameter $2r$ of the circle, and the major axis $2r \div \cos \angle 1$.

Thus, if a circular cylinder of diameter 6 in. is cut by a plane making an angle of 30° with the right section c, then the section c' thus made is an ellipse whose axes are 6 in. and $6 \div \cos 30° = 6 \div .866 = 6.93$ in. nearly.

Hence, the area of this elliptical section is

$$\pi ab = 3.14 \times 6.93 \times 6 = 130.56 \text{ sq. in.}$$

SUMMARY OF CHAPTER IX.

1. Make a list of the definitions on prisms and also a list of those on cylinders and compare them.

2. Make a list of the theorems on prisms and also of those on cylinders and compare them.

3. State the axioms of this chapter and note that they all refer to areas and volumes which involve *curved* surfaces. Compare these with the axioms on the circle in Plane Geometry.

4. Make a list of all the formulas given by the theorems and corollaries of this chapter.

5. Make an outline of the definitions and theorems concerning projections of lines and surfaces.

6. Explain how the area of an ellipse is obtained either by projecting it into a circle or by projecting a circle into it.

7. Make a list of the applications in this chapter which have appealed to you as interesting or practical or both. Return to this question, after studying the problems and applications which follow.

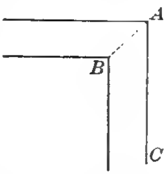
PROBLEMS AND APPLICATIONS.

1. Given a right circular cylinder the radius of whose base is 6 inches. Find the area of an oblique cross section inclined at an angle of 45° to the plane of the base.

2. Given an oblique circular cylinder the radius of whose right section is 10 inches. Find the area of the base if it is inclined at an angle of 60° to the right section.

3. If an oblique circular cylinder has an altitude h, an element e, radius of right section r, and $\angle A$ the inclination of the base to the right section, express the volume in two ways and show that these are equivalent.

4. A six-inch stovepipe has a 45° elbow, that is, it turns at right angles. (The angle CAB is called the elbow angle.) Find the area of the cross section at AB. Likewise if it has a 60° elbow angle.



5. At what angle must the damper in a circular stovepipe be turned in order to obstruct just half the right cross sectional area of the pipe?

SUGGESTION. The damper must be turned through an angle such that the area of the projection of the damper upon a right cross section is equal to half that cross section.

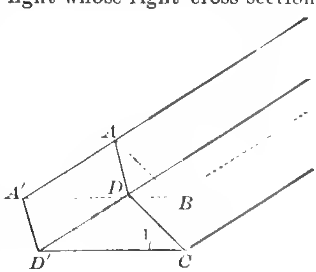
6. The comparatively low temperature of the earth's surface near the pole, even in summer, when the sun does not set for months, is due largely to the *obliqueness* with which the sun's rays strike the earth. That is, a given amount of sunlight is spread over a larger area than in lower latitudes.

Thus, if in the figure $D'C$ is a horizontal line, and $D'D$ the direction of the sun's rays, then a beam of light whose right cross section is $ABCD$ is spread over the rectangle $A'BCD'$. In other words, a patch of ground $A'BCD'$ receives only as much sunlight as a patch the size of $ABCD$ receives when the sun's rays strike it vertically.

Area $ABCD =$ area $A'BCD' \cos \angle 1$,

or

area $A'BCD' =$ area $ABCD \times \dfrac{1}{\cos \angle 1}$.



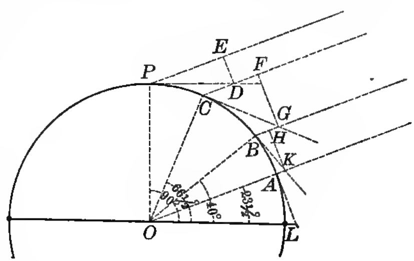
Hence, each unit of area in $A'BCD'$ receives $\cos \angle 1$ times as much light as a unit in $ABCD$.

Hence, to compare the heat-producing power of sunlight in any latitude with that at the place where the sun's rays fall vertically, we need to know how the projection angle, $\angle 1$, is related to the difference in latitude of the two places.

7. If $\angle 1 = 30°$, compare the amount of heat received by a unit of area in $ABCD$ and $A'BCD'$.

8. What must $\angle 1$ be in order that a unit of area in $A'BCD'$ shall receive only $\frac{1}{5}$ as much light as a unit in $ABCD$?

9. The figure represents a cross section of the earth with an indication of the direction of the rays of light as they strike it at the summer solstice when they are vertical at A, the tropic of Cancer. B represents the latitude of Chicago, C the polar circle, and P the north pole. The angles PDE, CGF, BKH represent the projection angle, $\angle 1$, for the various latitudes.



Prove that $\angle PDE = \angle POK$, $\angle CGF = \angle COK$, $\angle BKH = \angle BOK$.

That is, $\angle 1$ for each place is the latitude of that place minus the latitude of the place where the sun's rays are vertical.

10. Find the relative amount of sunlight received by a unit of area at the tropic of Cancer and at the north pole at the time of the summer solstice.

SUGGESTION. The required ratio is $\dfrac{1}{\cos \angle 1} = \dfrac{1}{\cos .66\frac{1}{2}°} = \dfrac{1}{.399}$.

11. Find the ratio between the amount of light received by a unit of the earth's area at Chicago and at the tropic of Caucer at the time of the summer solstice.

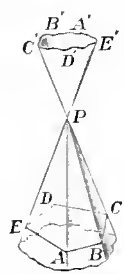
12. Find the same ratio for the polar circle and the tropic of Cancer at the spring equinox, when the sun is vertical over the equator.

13. Find the same ratio for the equator and Chicago at the winter solstice when the sun is vertical at latitude $23\frac{1}{2}°$ south.

CHAPTER X.

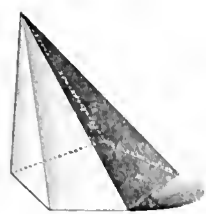
PYRAMIDS AND CONES.

585. Definitions. Given a convex polygon and a fixed point not in its plane. If a line through the fixed point moves so as always to touch the polygon and is made to traverse it completely, the line is said to generate a convex **pyramidal surface.**



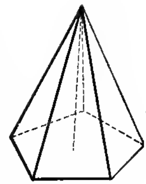
The moving line is the **generator** of the surface, and in any of its positions it is an **element** of the surface. The guiding polygon is the **directrix**, and the fixed point the **vertex.** A pyramidal surface has two parts, called **nappes**, on opposite sides of the fixed point. Compare definitions in § 603.

A polyhedral angle is a pyramidal surface of one nappe. See § 518.



586. That part of a pyramidal surface included between the fixed point and a plane cutting all its elements, together with the intercepted segment of the plane, is called a **pyramid.**

The intercepted plane-segment is the **base** of the pyramid, and the part of the pyramidal surface between the base and the vertex is the **lateral surface.**



The lateral surface is composed of **triangles** having a common vertex at the vertex of the pyramid, and having as bases the sides of the polygonal base. The sides common to two such triangles are the **edges** of the pyramid.

Pyramids are classified, according to the shape of the base, as **triangular, quadrangular, pentagonal,** etc.

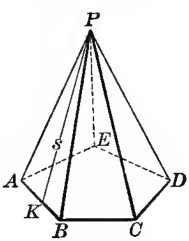
A pyramid having a triangular base has, in all, four faces, and is called a **tetrahedron.** In this case every face is a triangle, and any one may be taken as the base.

The **altitude of a pyramid** is the perpendicular distance from the vertex to the base.

A **regular right pyramid,** or simply a **regular pyramid,** is one whose base is a regular polygon such that the perpendicular from the vertex upon it meets it at the center.

EXERCISE. Show that the faces of a regular right pyramid are congruent isosceles triangles, and hence have equal altitudes.

The **slant height** of a regular right pyramid is the altitude of any one of its triangular faces.



587. THEOREM. *The lateral area of a regular right pyramid is equal to one half the product of its slant height and the perimeter of the base.*

Suggestion. Calling L the lateral area, $s = KP$ the slant height, and $p = AB + BC + CD + DE + EA$, the perimeter, show that

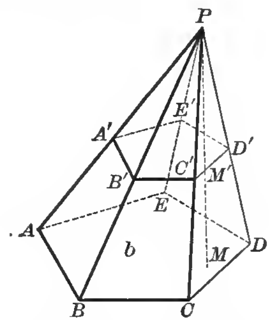
$$L = \tfrac{1}{2}\, sp.$$

588. THEOREM. *If a pyramid is cut by a plane parallel to the base:*

(1) *The edges and the altitude are divided in the same ratio.*

(2) *The polygonal section is similar to the base.*

(3) *The areas of this section and of the base are in the same ratio as the squares of their perpendicular distances from the vertex.*



Outline of proof: Given $ABCDE \parallel A'B'C'D'E'$.

(1) To prove that $\frac{PM'}{PM} = \frac{PA'}{PA} = \frac{PB'}{PB}$, etc., pass another plane through P parallel to the base and then use § 492.

(2) To prove that $ABCDE \sim A'B'C'D'E'$, we show that $\angle A = \angle A'$, $\angle B = \angle B'$, etc., and also $\frac{A'B'}{AB} = \frac{B'C'}{BC}$, etc.

(3) Calling the area of the cross section b' and that of the base b, we are to prove that $\frac{b'}{b} = \frac{\overline{PM'}^2}{\overline{PM}^2}$, and for this we need to show that $\frac{A'B'}{AB} = \frac{PA'}{PA} = \frac{PM'}{PM}$.

Give all the steps in detail.

589. COROLLARY. *If two pyramids have equal altitudes and bases of equal areas lying in the same plane, the sections made by a plane parallel to the plane of the bases have equal areas.*

Suggestion. If t and t' are the areas of the sections and b and b' those of the bases, show by the theorem that $\frac{t}{b}=\frac{t'}{b'}$, and hence that $t=t'$ if $b=b'$.

590. EXERCISES.

1. What is the slant height of a regular right pyramid if its lateral area is 160 sq. in. and the perimeter of its base is 20 in.?

2. What is the perimeter of the base of a regular right pyramid if its lateral area is 250 sq. in. and its slant height is 17 in.?

3. How could you find the lateral area of a *regular* right pyramid? Of any irregular pyramid? What measurements would be necessary in each case? Why are fewer measurements needed in the case of a regular right pyramid?

4. The base of a regular right pyramid is a regular hexagon whose side is 8 ft. Find the lateral area if the altitude of the pyramid is 6 ft.

5. The lateral area of a regular right hexagonal pyramid is 48 sq. ft. and the slant height is 12 ft. Find the altitude of the pyramid.

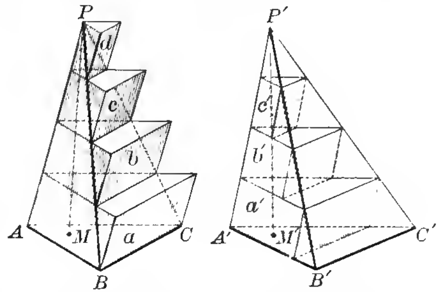
6. The base of a regular right pyramid is a square whose side is 16 ft., and the altitude of the pyramid is 6 ft. Find the lateral area.

7. A pyramid with altitude 8 and a base whose area is 36 is cut by a plane parallel to the base so that the area of the section is 18 sq. in. Find the distance from the base to the cutting plane.

8. If the altitude of a pyramid is h, how far from the base must a plane parallel to it be drawn so that the area of its cross section shall be half that of the base of the pyramid?

9. In a regular right pyramid a plane parallel to the base cuts it so as to make a section whose area is one half that of the base. Find the ratio between the lateral area of the pyramid and that of the small pyramid cut off by the plane.

591. Definition. A triangular pyramid is cut by a series of planes parallel to the base, including one through the vertex and also the one in which the base lies.



Through the lines of intersection of these planes with one of the faces, planes are constructed parallel to the opposite edge, thus forming a set of prisms all lying within the pyramid, as a', b', c', in pyramid P', or a set lying partly outside the pyramid, as a, b, c, d in pyramid P.

The inner prisms thus constructed are called **a set of inscribed prisms,** and the other **a set of circumscribed prisms.**

592. Axiom XXII. *A pyramid has a definite volume which is less than the combined volume of any set of circumscribed prisms and greater than that of any set of inscribed prisms.*

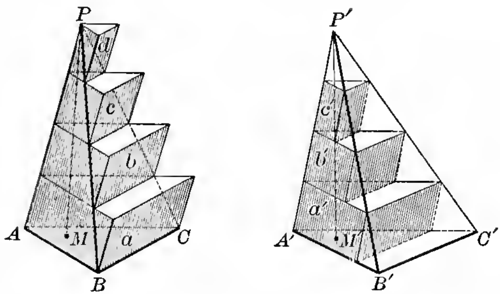
593. **EXERCISES.**

1. In the figure above prove that prisms d and c' are equal in volume. Also that $c = b'$, etc.

2. If the area of the base of pyramid P is 12 sq. in., what is the altitude of prism a if its volume is 1 cu. in.? If its volume is $\frac{1}{10}$ cu. in.?

3. If the altitude of the pyramid in the preceding exercise is 16 in., into at least how many equal parts must it be divided if the volume of prism a is to be less than 1 cu. in.? less than .01 cu. in.? Is it possible to divide the altitude of the pyramid into a sufficiently large number of equal parts to make the volume of prism a as small as we like?

594. THEOREM. *If two triangular pyramids have equal altitudes and bases of equal areas, their volumes are equal.*



Given the prisms P and P', in which $PM = P'M'$, and the bases ABC and $A'B'C'$ have equal areas.

To prove that P and P' have equal volumes.

Proof: Divide PM and $P'M'$ into the same number of equal parts, and using these division points, construct a set of circumscribed prisms for P and a set of inscribed prisms for P'.

Then $a' = b,\ b' = c,\ c' = d.$ (See § 549.)

Denote $a + b + c + d$ by V and $a' + b' + c'$ by V'.

Then $V - V' = a.$ (Why?) (1)

If P differs at all in volume from P', let P be the greater, and let the difference be some fixed number, K, so that

$$P - P' = K. \qquad (2)$$

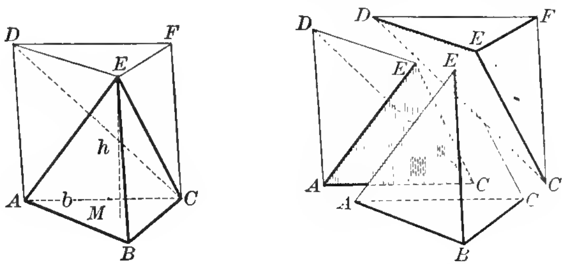
But from (1) $P - P' < a$, since $P < V$ and $P' > V'$.

Now a can be made less than K by taking the divisions on PM small enough.

Hence, $P - P' < K.$ (3)

Thus (3) contradicts (2), and hence the supposition that P and P' differ in volume is impossible.

595. THEOREM. *The volume of a triangular pyramid is one third of the product of its base and altitude.*



Given the triangular pyramid *E–ABC*. Let *h*, *b*, and *V* be the numerical measures respectively of the altitude *EM*, the base *ABC*, and the volume.

To prove that $V = \frac{1}{3}bh.$

Proof: Construct on the base *ABC* a triangular prism with altitude *h* and lateral edge *EB*.

This prism may be cut into three pyramids, as shown in the figure to the right, by the plane sections through *DEC* and *AEC*. See Note, § 523.

The pyramids *E–ABC* and *C–DEF* have the same volume (§ 594).

Likewise the pyramids *E–ACD* and *E–CFD* have the same volume (§ 594).

But *C–DEF* and *E–CFD* are only different notations for the same pyramid.

Hence, *E–ABC* = *C–DEF* = *E–ACD*.

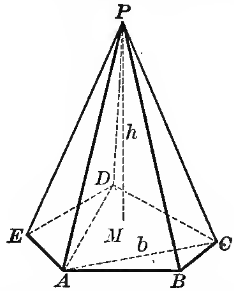
That is, *E–ABC* is one third of the prism.

But the volume of prism = bh. (Why?)

Hence, V = volume of pyramid = $\frac{1}{3}bh$.

State in detail the reasons for each step.

596. THEOREM. *The volume of any pyramid is one third of the product of its base and altitude.*



Given the pyramid $P\text{-}ABCDE$. **Let** V, b, **and** h **be the numerical measures respectively of the volume, base, and altitude.**

To prove that $$V = \tfrac{1}{3} bh.$$

Proof: By means of the diagonal planes PAC and PAD, divide the given pyramid into three triangular pyramids.

Complete the proof.

597. COROLLARIES. 1. *The volumes of any two pyramids having the same or equal altitudes are in the same ratio as the areas of their bases.*

2. *The volumes of any two pyramids having the same or equal bases are in the same ratio as their altitudes.*

598. EXERCISES.

1. The altitude of a certain pyramid is 14 in. and its volume is 380 cu. in. Find the area of its base.

2. The area of the base of a pyramid is 48 sq. ft. and its volume 260 cu. ft. Find its altitude.

3. Find the locus of the vertices of pyramids having the same base and equal volumes.

4. A diagonal of the square base of a regular right pyramid is $7\sqrt{2}$ in. and its volume 147 cu. in. Find its altitude and lateral area.

5. A flower bed is in the form of a regular right pyramid, with a square base 5 ft. on a side. The altitude is 2 ft. Find the number of cubic feet of soil in its construction.

6. A tent is to be made in the form of a right pyramid, with a regular hexagonal base. If the altitude is fixed at 15 ft., what must be the side of the base in order that the tent may inclose 350 cu. ft. of space?

7. Two marble ornaments of equal altitudes are pyramidal in form. One has a square base 2 in. on a side and the other a regular hexagonal base 1 in. on a side. Compare their volumes.

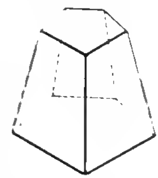
8. Two monuments having bases of equal areas are pyramidal in shape, one being 15 ft. high and the other 18 ft. Compare their volumes.

9. If the base and the volume of a pyramid are known, is it possible to determine its lateral area?

10. Given a pyramid with rectangular base. By how much is its volume multiplied if the length and width of the base and also the altitude are each multiplied by 2; by 3; by any number n?

11. Given a pyramid with altitude 10 and a regular hexagonal base, each of whose sides is 5. By how much is its volume multiplied if each side of the base and also the altitude is multiplied by 2; by 3; by 4; by any number n?

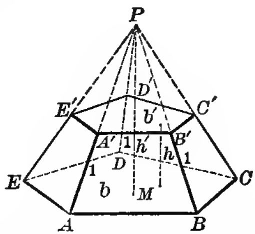
For a general statement of the law exemplified in these exercises, see § 650.



599. Definitions. The figure formed by the base of a pyramid, any cross section, and the portion of the lateral faces included between these planes, is called a **truncated pyramid.** If the cross section is parallel to the base, the figure is called a **frustum of a pyramid**, and this section is the **upper base.**

The **altitude** of a frustum is the perpendicular distance between its bases. The **slant height** of the frustum of a regular right pyramid is the common altitude of its trapezoidal faces.

600. THEOREM. *The volume of a frustum of a pyramid is equal to the combined volume of three pyramids whose common altitude is the same as that of the frustum, and whose bases are the upper and lower bases of the frustum and the mean proportional between these bases.*



Given the frustum AC' with lower base b, upper base b', and altitude h. Let h' be the altitude PM of the completed pyramid $P\text{–}ABCDE$. Then $\sqrt{bb'}$ is the mean proportional between b and b'.

To prove that the volume of AC' is

$$V = \tfrac{1}{3}h[b + b' + \sqrt{bb'}].$$

Proof: The altitude of the pyramid $P\text{–}A'B'C'D'E'$ is $h' - h$.

Hence, $$\frac{b}{b'} = \frac{h'^2}{(h'-h)^2}, \quad (\S\ 588)$$

from which $$h' = \frac{h\sqrt{b}}{\sqrt{b} - \sqrt{b'}}. \quad (1)$$

Now V is the difference between the pyramids whose altitudes are h' and $h' - h$.

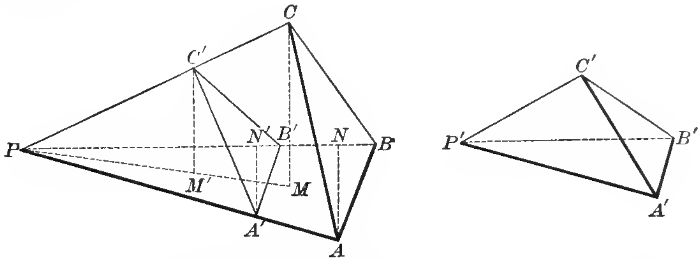
Hence, $$V = \tfrac{1}{3}bh' - \tfrac{1}{3}b'(h' - h),$$

or, rearranging, $$V = \tfrac{1}{3}b'h + \tfrac{1}{3}h'(b - b'). \quad (2)$$

Substituting (1) in (2), $$V = \tfrac{1}{3}h[b + b' + \sqrt{bb'}].$$

Show all the details.

601. THEOREM. *The volumes of two tetrahedrons, having a trihedral angle of the one congruent to a trihedral angle of the other, are in the same ratio as the products of the edges which meet in the vertices of these angles.*



Given the tetrahedrons $P\text{–}ABC$ and $P'\text{–}A'B'C'$ whose volumes are V and V' and in which Tri. $\angle P =$ Tri. $\angle P'$.

To prove that $\dfrac{V}{V'} = \dfrac{PA \cdot PB \cdot PC}{P'A' \cdot P'B' \cdot P'C'}$.

Proof: Place $P'\text{–}A'B'C'$ so that Tri. $\angle P'$ coincides with Tri. $\angle P'$.

Let CM and $C'M'$ be the altitudes of $P\text{–}ABC$ and $P\text{–}A'B'C'$ from the vertices C and C' upon the plane PAB.

Let AN and $A'N'$ be the altitudes of the $\triangle$ PAB and $PA'B'$.

Then $\dfrac{V}{V'} = \dfrac{\frac{1}{3} CM \cdot \text{area } PAB}{\frac{1}{3} C'M' \cdot \text{area } PA'B'} = \dfrac{CM}{C'M'} \cdot \dfrac{PB}{PB'} \cdot \dfrac{AN}{A'N'}$. (1)

Now prove $\dfrac{CM}{C'M'} = \dfrac{PC}{PC'}$ and $\dfrac{AN}{A'N'} = \dfrac{PA}{PA'}$.

Hence, substituting in (1),

we have $\dfrac{V}{V'} = \dfrac{PC \cdot PB \cdot PA}{P'C' \cdot P'B' \cdot P'A'}$.

Give all the steps and reasons in detail.

602. EXERCISES.

1. Show that the lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids. Hence find the area of its lateral surface in terms of the slant height and the perimeters of the bases.

2. Show that sections of a pyramid made by two planes parallel to the base are similar polygons whose areas are in the same ratio as the squares of the distances from the vertex.

3. Show that any two pyramids standing on the same base, or on equal bases in the same plane, have the same volume if their vertices coincide or lie in a plane parallel to the base.

4. Show that the volumes of two pyramids have the same ratio as the areas of their bases if they have equal altitudes, and the same ratio as their altitudes if they have equal bases.

5. A frustum of a pyramid is cut from a pyramid the perimeter of whose base is 60 inches and whose altitude is 15 inches. What is the altitude of the frustum, if the perimeter of its upper base is 40 inches?

Does the result depend upon the number of sides of the pyramid?

6. Solve the preceding problem if the perimeter of the upper base of the frustum is one nth that of the lower base. Does this result depend upon the number of sides of the pyramid?

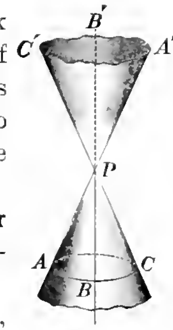
7. The area of the base of a pyramid is 180 square inches and its altitude is 20 inches. Cut from it a frustum, the area of whose upper base is 45 square inches; also one the area of whose upper base is one nth of 180 square inches. Do these results depend upon the number of sides of the pyramid?

8. Two triangular pyramids have equal trihedral angles at the vertex. The lateral edges of one pyramid are 14, 16, and 18, and those of the other 7, 8, and 9. Find the ratio between their volumes. Are the data given sufficient to find the volume of each pyramid?

9. If two triangular pyramids have equal trihedral angles at the vertex and if the lateral edges of one are a, b, and c, and two lateral edges of the others are a' and b', find the third lateral edge of the second pyramid so that their volumes shall be equal.

10. The slant height of a frustum of a regular pyramid is 10 inches and the apothems of its bases 8 and 6 inches respectively. Find its altitude.

CONES.

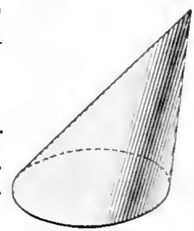


603. Definition. Given a closed convex curve and a fixed point not in its plane. If a line through the fixed point moves so as always to touch the curve and is made to traverse it completely, it is said to generate a **convex conical surface.**

The moving line is called the **generator** of the surface, and in any particular position it is an **element** of the surface.

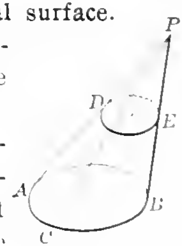
The fixed curve is called the **directrix**, and the fixed point the **vertex.**

A conical surface consists of two parts, called **nappes,** on opposite sides of the fixed point.



604. That part of a convex conical surface included between its vertex and a plane cutting all its elements, together with the intercepted portion of the plane, is called a **cone.**

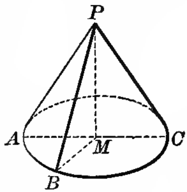
The intercepted part of the plane is the **base** of the cone, and the curved surface is its **lateral** surface.

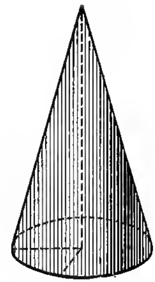


The **altitude** of a cone is the perpendicular distance from the vertex to the plane of the base.

A **circular** cone is one which has a circular cross section such that the perpendicular upon it from the vertex meets it at the center. If the base is such a circle, the cone is then called a **right circular cone.** Otherwise, it is an **oblique circular cone.**

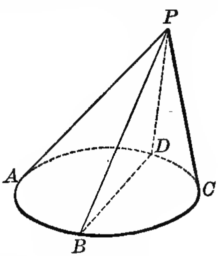
A right circular cone may be generated by rotating a right triangle *PMB* about one of its legs, *PM*, as an axis. The hypotenuse *PB* generates the conical surface, and the other leg, *MB*, generates the base.





The generator of a right circular cone in any position is called the slant height.

605. THEOREM. *If a plane contains an element of a cone and meets it in one other point, then it contains another element also, and the section is a triangle.*



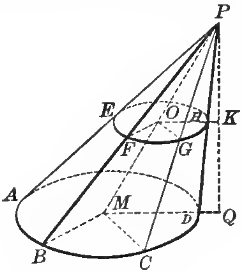
Let a plane contain the element *PD* of the cone *P–ABC*, and also one other point *B*.

To prove that this plane contains another element *PB*, and that the section is a triangle *PBD*.

Suggestion. Connect *P* and *B*. This segment lies in the conical surface. (Why?) Complete the proof by showing that *BD* lies in the base of the cone. Compare this proof with that of § 555.

606. **Definition.** If a plane contains an element of a cone and no other point of the cone, the plane is **tangent** to the cone, and the element is called the **element of contact.**

607. THEOREM. *If the base of a cone is circular, every plane section parallel to the base is also circular.*



Given a cone with a circular base AD.

To prove that the ∥ section EH is also circular.

Proof: Draw the straight line from P to the center M of the base, and let it meet the section EH in the point O. Let F and G be any two points on the perimeter of the section EH.

Pass planes containing PM through the points F and G, and let them cut the base in MB and MC respectively.

Now in the △ PMB and PMC prove that $OF = OG$.

Hence, as F and G, *any* two points on the perimeter of this section, are equally distant from O, this shows that EH is a circle whose center is O.

608. COROLLARY. *If a cone has a circular base, the areas of two parallel cross sections are in the same ratio as the squares of their perpendicular distances from the vertex and also as the squares of the distances of their centers from the vertex.*

Suggestion. Use the figure of § 607, and let PQ be the altitude of the cone. Then show that

$$\frac{\text{Area } AD}{\text{Area } EH} = \frac{\overline{MD}^2}{\overline{OH}^2} = \frac{\overline{PM}^2}{\overline{PO}^2} = \frac{\overline{PQ}^2}{\overline{PK}^2}$$

609. **EXERCISES.**

1. Into how many parts do the two nappes of a conical surface or of a pyramidal surface divide the remaining points of space?

2. If in constructing a conical surface a polygon is used as a directrix instead of a closed convex curve, what kind of surface is obtained?

3. Why is it specified in the definition of the convex conical surface that the vertex must not lie in the same plane as the directrix?

4. How many cones may be cut from a conical surface if they are to have no point in common except the vertex?

5. If a triangle which is not a right triangle is made to revolve about one of its sides, does it generate a cone?

6. Can every circular cone be developed by revolving a right angled triangle about one of its sides?

7. If a cone has a circular base, the line from the vertex to the center passes through the center of every plane section parallel to the base.

8. If a cone has a circular base, the plane determined by a tangent to the base and the element at the point of tangency is a tangent plane to the cone.

9. Through a point outside a cone with a circular base, how many planes are there which are tangent to the cone?

10. The diameter of the circular base of a cone is 8 in. and the altitude of the cone 9 in. A plane parallel to the base cuts the cone in a section whose diameter is 3 in. Find the distance from the vertex of the cone to this plane.

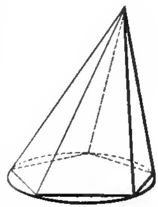
11. If the area of the circular base of a cone is 16π sq. in. and its altitude 6 in., find the distance from the vertex to a plane, parallel to the base, which cuts the cone in a section with area 9π sq. in.

12. The area of the circular base of a cone is b sq. in. and its altitude h in. Find the distance from the base to a plane parallel to it, which cuts off a cone the area of whose base is one nth that of the base of the original cone.

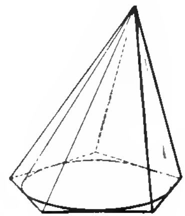
13. Compare the exercises on the cone thus far studied with those on the pyramid given on pages 354, 357. Note that a pyramid can be made to approximate very closely to a cone by making its number of faces very large.

MEASUREMENT OF THE SURFACE AND VOLUME OF A CONE.

610. **Definitions.** A pyramid is said to be **inscribed in a cone** if its lateral edges are elements of the cone, and the bases of the cone and the pyramid lie in the same plane, as in Fig. 1.



(1)



(2)

A pyramid is said to be **circumscribed about a cone** if its lateral faces are all tangent to the cone, and the bases of the cone and the pyramid lie in the same plane, as in Fig. 2.

611. Theorem. *In a right circular cone a pyramid may be inscribed whose slant height differs from the slant height of the cone by less than any given fixed number.*

Proof: Let d be the given fixed number. From plane geometry we know that a regular polygon may be inscribed in the base of the cone whose apothem differs from the radius of the base by less than d. Let this regular polygon be the base of an inscribed pyramid. Then the slant height of this pyramid differs from that of the cone by less than d, since the difference of two sides of a triangle is less than the third side.

612. **Axiom XXIII**. *The lateral surface of a convex cone has a definite area, and the cone incloses a definite volume, which are less respectively than those of any circumscribed pyramid and greater than those of any inscribed pyramid.*

613. THEOREM. *The area of the lateral surface of a right circular cone is equal to one half the product of its slant height and the circumference of its base.*

Given a right circular cone of which s is the slant height, c is the circumference of the base, and L the lateral area.

To prove that $L = \frac{1}{2} sc$.

Proof: Suppose that $L > \frac{1}{2} sc$. Then $L = \frac{1}{2} sK$ (1)

where $K > c$.

Circumscribe about the cone a pyramid the perimeter of whose base is p, such that $p < K$. (Why is this possible?)

Hence, $\frac{1}{2} sp < \frac{1}{2} sK$. (2)

That is, (2) contradicts (1) because of § 612.

Next suppose $L < \frac{1}{2} sc$. Then $L = \frac{1}{2} sK'$ (3)

where $K' < c$.

Let $c - K' = d$. By § 352, a polygon may be inscribed in the base of the cone whose perimeter p differs from c by as little as we please, and by § 611 a pyramid may be inscribed in the cone whose slant height s' differs from the slant height s of the cone by as little as we please. Hence a pyramid may be inscribed such that $\frac{1}{2} s'p$ differs from $\frac{1}{2} sc$ by less than d.

That is $\frac{1}{2} s'p > \frac{1}{2} sK$. (4)

But (4) contradicts (3) because of § 612.

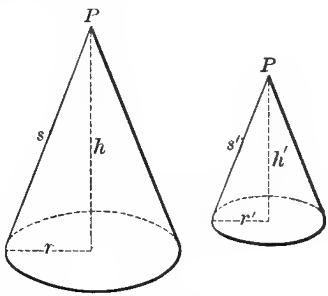
Since therefore L is neither less than nor greater than $\frac{1}{2} sc$, it must be equal to $\frac{1}{2} sc$.

614. COROLLARY. *If r is the radius of the base of a right circular cone and s the slant height, then*

$$L = \frac{1}{2} \cdot 2 \pi r S = \pi r S.$$

615. Definition. Two right circular cones are **similar** if they are generated by two similar right triangles revolving about corresponding sides.

616. THEOREM. *The lateral areas or the entire areas of two similar right circular cones are in the same ratio as the squares of their altitudes, their slant heights, or the radii of their bases.*



Given the cones P and P', with altitudes h, h', slant heights s, s', radii of bases r, r', lateral areas L, L', and entire areas A, A'.

To prove that

$$\frac{A}{A'} = \frac{L}{L'} = \frac{h^2}{h'^2} = \frac{s^2}{s'^2} = \frac{r^2}{r'^2}.$$

Proof: See the suggestions under § 566. Give all the steps.

617. EXERCISES.

1. The lateral area of a cone is 36 square inches. What is the lateral area of a similar cone whose altitude is ¾ that of the given cone?

2. The total area of one of two similar cones is three times that of the other. Compare their altitudes and also their radii.

3. The sum of the total areas of two similar cones is 144 square inches. Find the area of each cone if one is 1⅓ times as high as the other.

4. Prove that if in two tetrahedrons three faces of one are congruent respectively to three faces of the other and similarly placed about a vertex, the tetrahedrons are congruent.

5. Prove that if in two tetrahedrons two faces and the included dihedral angle are congruent and similarly placed, the tetrahedrons are congruent.

6. A pedestal for a monument is in the shape of a frustum of a regular hexagonal pyramid, the radius of the upper base being 4 ft., that of the lower base 6 ft., and the altitude of the frustum 8 ft. Find its volume, slant height, and lateral surface.

7. Find the volume of a frustum of a pyramid the areas of whose bases are 25 sq. in. and 18 sq. in. and whose altitude is 6 in.

8. The area of the lower base of a frustum is 42 sq. ft., its altitude 8 ft., and volume 200 cu. ft. Find the area of the upper base.

9. The area of the base of a pyramid is 480 sq. ft. and its altitude 30 ft. Find the volume of the frustum remaining after a pyramid with altitude 10 ft. has been cut off by a plane parallel to the base.

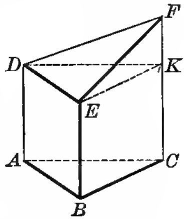
10. The area of the base of a pyramid is 250 sq. in. If a plane section of the pyramid parallel to the base and at a distance of 5 in. from it has an area of 175 sq. in., find the altitude of the pyramid.

11. The sides of the base of a triangular pyramid are 6 ft., 8 ft., 10 ft., and its volume 96 cu. ft. Find its altitude.

12. What part of the volume of a cube is a frustum of a pyramid cut from a pyramid whose base is one face of the cube and whose vertex lies in the opposite face, if the altitude of the frustum is one half the edge of the cube?

13. Find the dihedral angle at the base of a regular pyramid if the altitude is one half the slant height.

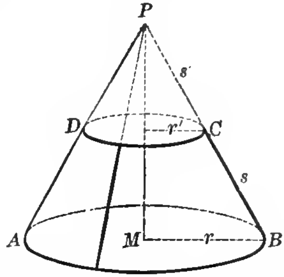
14. A right triangular prism is cut by a plane not parallel to the base, but such that its intersection DE is parallel to the base segment AB. Show that the volume of the part thus cut off is one third the product of the sum of the three vertical edges and the area of the base.



SUGGESTION. Draw plane $DEK \parallel ABC$.

15. Find the volume of a truncated triangular prism, the area of whose base is 25 square inches and whose lateral edges are 8, 7, 8.

618. THEOREM. *The lateral area of a frustum of a right circular cone is equal to one half of the sum of the circumferences of the bases multiplied by the slant height.*



Given the frustum $ABCD$, with slant height s and radii r and Let L represent its lateral area.

To prove that $L = \frac{1}{2}(2\pi r + 2\pi r')s = \pi s(r + r')$.

Proof: Complete the cone, and let $PC = s'$.

Then $\quad L = \frac{1}{2}[2\pi r(s + s') - 2\pi r' s']$

$$= \pi r s + \pi s'(r - r'). \tag{1}$$

But $\quad \frac{r}{r'} = \frac{s + s'}{s'}$, from which $s' = \frac{r's}{r - r'}$. $\quad$ (2)

Substituting s' from (2) in (1),

$$L = \pi r s + \pi r' s = \pi s(r + r').$$

619. COROLLARY. *The lateral area of a frustum of a right circular cone is equal to the circumference of a section midway between the bases multiplied by the slant height.*

Suggestion. From the theorem

$$L = \pi s(r + r') = 2\pi \frac{(r + r')}{2} s.$$

Now show that $\frac{r + r'}{2}$ is the radius of the section midway between the two bases.

620. **EXERCISES.**

1. The lateral surface of a right circular cone is 75 sq. ft. Find the altitude if the radius of the base is 4 ft.

2. A circular chimney 100 ft. high is in the form of a frustum of a right cone whose lower base is 10 ft. in diameter and upper base 8 ft. Find the lateral surface.

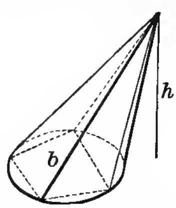
3. The lateral area of a frustum of a right circular cone is 60π sq. in., the radii of the two bases are 6 in. and 4 in. Find the slant height of the frustum.

4. Find the altitude of the frustum in the preceding example, and also the altitude of the cone from which it is cut.

5. A frustum of a right circular cone has an altitude one half that of the cone from which it is cut. If its slant height is 8 ft. and lateral area 64π sq. ft., find the diameters of its bases.

6. Find the altitude of the frustum of the cone in the preceding example; also the lateral area of the cone from which it was cut.

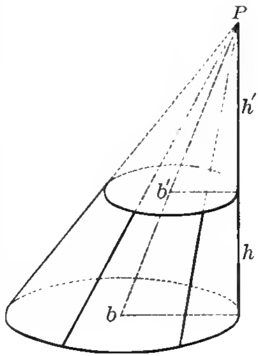
621. THEOREM. *The volume of any convex cone is equal to one third the product of its base and altitude.*



Suggestion. Let h be the altitude, b the area of the base, and V the volume.

Show that V cannot be different from $\frac{1}{3}bh$ by an argument similar to that of § 563, making use of § 596.

622. COROLLARY. *If a cone has a circular base of radius r and altitude h, then* $V = \frac{1}{3}\pi r^2 h$.



623. THEOREM. *The volume of the frustum of a convex cone is equal to the combined volumes of three cones whose common altitude is the altitude of the frustum and whose bases are the upper and lower bases of the frustum and a mean proportional between these bases.*

Suggestion. The proof is exactly like that of § 600.

NOTE. Observe that the two preceding theorems apply to *any* convex cone because the *altitude h is constant.* The actual computation is *practicable* only when the areas of the bases can be found, as in the case of the circle or ellipse.

SUMMARY OF CHAPTER X.

1. Make a list of definitions on pyramids and also one on cones and compare them.

2. Make a list of theorems on pyramids and also one on cones and compare them.

3. What axioms have been used in this chapter? Compare these with the axioms in Chapter IX.

4. Make a list of all the formulas given by the theorems of this chapter and compare them with the corresponding formulas in Chapter IX.

5. What theorems on cylinders have no corresponding theorems for cones?

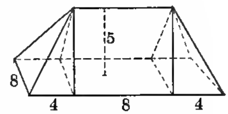
6. Show that a frustum of a cone becomes more and more nearly identical with a cylinder if the vertex of the cone is removed farther and farther from the base.

7. Make a list of the applications in this chapter which have impressed you as interesting or practical or both. Return to this question again after studying the problems and applications which follow.

PROBLEMS AND APPLICATIONS.

1. Show that the volume of a right triangular prism is equal to one half the product of the area of one face and the distance from the opposite edge to that face.

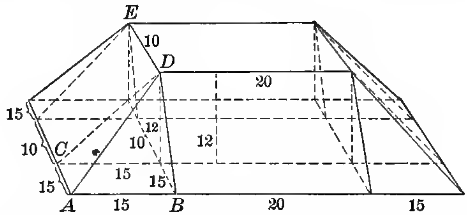
2. A mound of earth in the shape shown in the figure has a rectangular base 16 yards long and 8 yards wide. Its perpendicular height is 5 yards, and the length on top is 8 yards. Find the number of cubic yards of earth in the mound.



SUGGESTION. If from each end a pyramid with a base 8 yd. by 4 yd. is removed, the remaining part is a triangular prism.

3. Given a figure in general shape the same as the preceding, with a rectangular base of length 24 ft. and width 6 ft. Find its volume and lateral area if the dihedral angles around the base are each 45°.

4. The accompanying figure represents a solid whose base is a



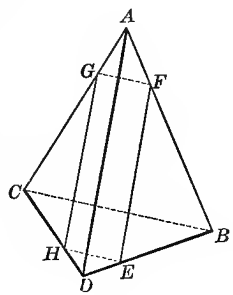
rectangle 50 units long and 40 units wide. Its height is 12 units and its top a rectangle 20 units by 10 units. Find its volume.

SUGGESTION. Divide the solid as indicated in the figure. Notice that this is not a frustum of a pyramid.

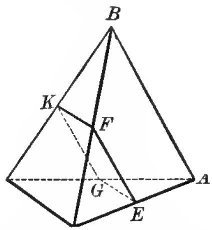
5. In a figure like the foregoing, how can we determine whether or not it represents the frustum of a pyramid?

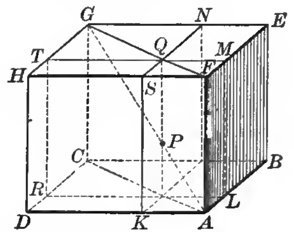
6. Show how to pass a plane through a tetrahedron so that the section shall be a parallelogram.

SUGGESTION. Pass a plane parallel to each of two opposite edges. See Ex. 6, page 293.



7. If the middle points of four edges of a tetrahedron, no three of which meet at the same vertex, are joined, a parallelogram is formed. Prove.





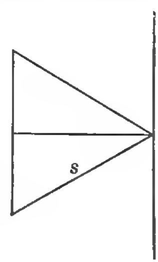
8. If through any point P in a diagonal of a parallelopiped planes KN and RM are drawn parallel to two faces, show that the parallelopipeds DQ and LN thus formed have equal volumes.

9. Find the volume and area of a figure formed by revolving an equilateral triangle about an altitude, the sides of the triangle being s.

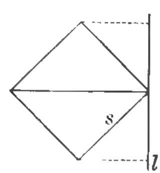
10. Find the area and volume of the figure developed by an equilateral triangle with sides s if it is revolved about one of its sides.

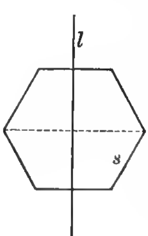
11. Find the area and volume of the figure developed by revolving a square whose side is s about one of its diagonals.

12. Through one vertex of an equilateral triangle with sides s draw a line l perpendicular to the altitude upon the opposite side. Find the volume and area of the figure developed by revolving the triangle about the line l.



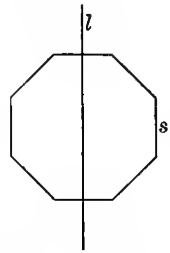
13. Through a vertex of a square with sides s draw a line l perpendicular to the diagonal through that vertex. Find the area and volume of the figure developed by turning the square around the line l.





14. In a regular hexagon with sides s draw a line l bisecting two opposite sides. Find the area and volume of the figure developed by turning the hexagon about l as an axis.

15. Solve a problem like the preceding, using a regular octagon instead of a hexagon.



16. If several planes are tangent to the same cone, find one point common to them all.

17. Find the locus of all lines which make a given angle with a given line at a given point in it.

18. Find the locus of all lines which make a given angle with a given plane at a given point.

19. One angle of a right triangle is 30°. Find the ratios between the surfaces of the solids developed by revolving this triangle around each of its three sides.

20. Find the ratios between the volumes of the solids developed in the preceding example.

21. Find the total area and the volume of a regular tetrahedron each of whose edges is e. (See § 625.)

22. If the numerical values of the volume and of the total area of a regular tetrahedron are equal, what is the length of its edge?

23. Find the length of an edge of a regular tetrahedron if its volume is numerically equal to the square of an edge.

24. Cut a pyramid of altitude h by means of a plane parallel to the base so that the perimeter of the section shall be half that of the base. Also cut it so that the perimeter of the section shall be $\frac{1}{3}$ that of the base.

25. Cut a right circular cone by 3 planes, each parallel to the base, so that the perimeters of the sections shall be $\frac{p}{4}, \frac{2p}{4}, \frac{3p}{4}$, p being the perimeter of the base. Find the distances from the vertex to the planes.

26. Cut a right circular cone of altitude h by a plane parallel to the base so that the area of the section shall be half that of the base. Find the distance from the vertex to the plane.

27. Show that the lateral area of the small cone cut off in the preceding example is one half the lateral area of the original cone.

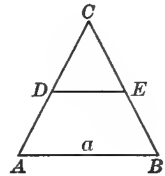
28. Cut a pyramid of altitude h by n planes, each parallel to the base, so that the areas of the sections shall be $\frac{A}{n+1}, \frac{2A}{n+1}, \frac{3A}{n+1}, \cdots,$

$\frac{(n-1)A}{n+1}$, A being the area of the base. Show that the distances from the vertex to the planes are $h\sqrt{\frac{1}{n+1}}$, $h\sqrt{\frac{2}{n+1}}$, $h\sqrt{\frac{3}{n+1}}$, $\cdots$.

29. Cut a cone with altitude h by a plane parallel to the base so that the volume of the frustum formed shall equal half that of the cone. Find the distance from the vertex of the cone to the plane.

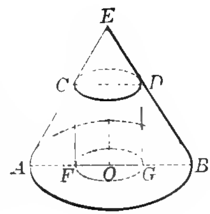
30. Cut a cone of altitude h by n planes, each parallel to the base, so that the frustums formed and the one pyramid cut off at the top shall all have equal volumes.

31. An equilateral triangle ABC is swung around the line DE as an axis, D and E being middle points of the sides of the triangle. Find the volume of the figure thus developed by the trapezoid $ABED$ if $AB = a$.



32. Find the total surface of the figure in the preceding example.

33. In a right circular cone, with altitude h, and r the radius of its base, a cylinder is inscribed as shown in the figure. Find the radius OF of the cylinder if the area of the ring bounded by the circles OF and OA is equal to the lateral area of the small cone cut off by the upper base of the cylinder.



34. The same as the preceding, except that the lateral area of the small cone is to equal the lateral area of the cylinder.

35. Find the dihedral angles of a regular tetrahedron.

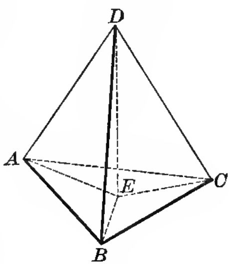
CHAPTER XI.

REGULAR AND SIMILAR POLYHEDRONS.

REGULAR POLYHEDRONS.

624. Definitions. A polyhedron is said to be **regular** if its polyhedral angles are all congruent and its faces are congruent regular polygons.

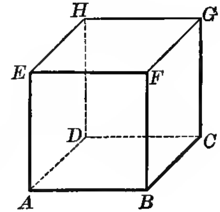
625. Construction of regular polyhedrons. Certain regular polyhedrons are very simple of construction, as indicated below.



(1) **The regular tetrahedron.** At the center E of an equilateral triangle ABC erect a perpendicular to the plane of the triangle. On this take a point D so that $AD = AC$.

Now prove that the four triangles, ABC, ACD, ABD, BCD, are *regular* and *congruent*, and that the four trihedral angles are congruent.

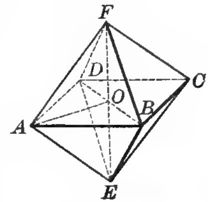
Suggestion. $AE = BE = CE$.



(2) **The regular hexahedron or cube.** At the vertices of a given square erect perpendiculars to its plane equal in length to the sides, and join their upper extremities as shown in the figure.

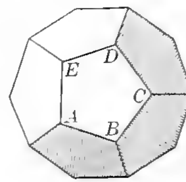
Show that six equal and congruent squares are formed and also eight congruent trihedral angles.

(3) **The regular octahedron.** Through the center O of a square $ABCD$ draw a perpendicular to the plane of the square.



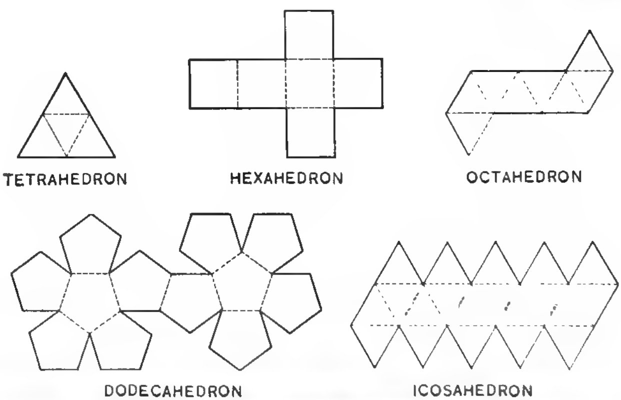
On this take points E and F such that $AF = AE = AB$. Join E and F to each of the four vertices, A, B, C, D.

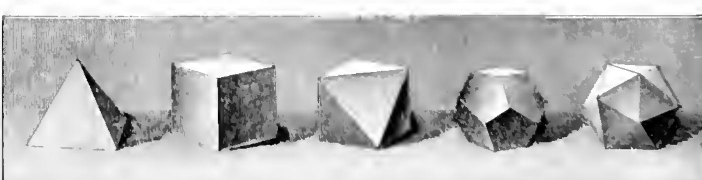
Now prove that the eight faces are congruent regular triangles, and that the six polyhedral angles are congruent.



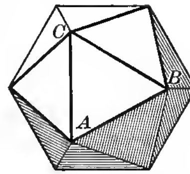
There are two other regular polyhedrons, namely:

(4) **The regular dodecahedron**, having for faces twelve regular pentagons, and





(5) **The regular icosahedron**, having for faces twenty equilateral triangles.



The geometric constructions for (4) and (5) are not so simple as for the others. However, cardboard models of all five may be made by cutting out the figures, as shown herewith, and folding them along the dotted lines. They may be held in shape by means of gum paper stuck over the joints.

626. The number of regular polyhedrons. It will now be shown that there are *not more* than these five regular polyhedrons.

There must be *at least three* faces meeting at each vertex. If these are regular triangles, there may be *three*, as in the tetrahedron, or *four*, as in the octahedron, or *five*, as in the icosahedron; *but there cannot be six*, for in that case the sum of the angles about a vertex would be 360°, and it is readily seen that this sum cannot be as great as 360°. For a proof of this fact, see § 726.

If the faces are squares, there may be *three* about a vertex, as in the cube, *but there cannot be four*, for in that case the sum of the face angles at a vertex would be 360°.

If the faces are regular pentagons, there may be *three* about a vertex, making the sum of the face angles $3 \cdot 108° = 324°$, *but there cannot be four*, for then the sum would be greater than 360°. See § 726.

Regular polygons of more than five sides cannot form the faces of a regular polyhedron, for the sum of the face angles at a vertex would in any such case be more than 360°. Show why this is so.

Hence, there cannot be more regular convex polyhedrons than those exhibited above.

627. THEOREM. *If v is the number of vertices of a convex polyhedron, e its number of edges, and f its number of faces, then $e + 2 = v + f$.*

This is called Euler's Theorem. The proof is too difficult for an elementary textbook such as this. The proofs given in the current texts are not conclusive.

628. EXERCISES.

1. Verify the above theorem by counting the number of edges, faces, and vertices in each of the regular figures given in § 625.

2. The following is a form of proof of this theorem which is often given:

Denote the number of vertices, edges, and faces of a polyhedron by V, E, and F, respectively. To prove $E + 2 = V + F$.

Proof: Taking the single face $ABCD$, the number of edges equals the number of vertices, or $E = V$. If another face be annexed, three new edges and two new vertices are added. Hence the number of edges gains one on the number of vertices, as $E = V + 1$. If still another face be added, two new edges and one new vertex are added. Hence $E = V + 2$.

With each new face that is annexed the number of edges gains one on the number of vertices, till but one face is lacking.

The last face increases neither the number of edges nor vertices. Hence, etc.

Show by putting together the faces of a cube in a certain order that the statement in italics need not be true, and hence that the proof is not conclusive.

629. THEOREM. *The sum of the face angles of any convex polyhedron is equal to four times as many right angles, less eight, as the polyhedron has vertices.*

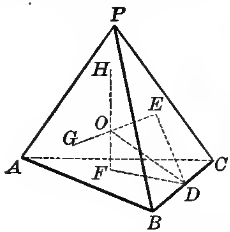
The proof depends on the preceding theorem and is not given here.

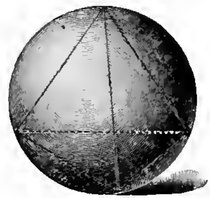
630. EXERCISE.

Verify this theorem by an examination of the regular polyhedrons.

INSCRIPTION OF REGULAR POLYHEDRONS.

631. PROBLEM. *To find a point equally distant from the four vertices of any tetrahedron.*





Given the tetrahedron *P–ABC*.

To find a point *O* equidistant from *P*, *A*, *B*, *C*.

Construction. At *D*, the middle point of *BC*, construct a plane perpendicular to *BC*.

The plane will contain the point *E*, the center of the circle circumscribed about $\triangle PBC$, and also the similar point *F* in $\triangle ABC$. (Why?)

In the plane *DEF* draw $EG \perp ED$ and $FH \perp FD$.

Then *EG* and *FH* cannot be parallel (why?), and hence meet in some point *O*.

Also $EG \perp$ to plane *PBC* and $FH \perp$ plane *ABC*. (Why?)

Proof: Now show that *O* is equidistant from *P*, *A*, *B*, *C*.

632. **Definitions.** The locus of all points in space equidistant from a given fixed point is a surface called a **sphere.** The fixed point is the **center** of the sphere, and a line-segment from the center to the surface is called a **radius.**

A polyhedron is **inscribed in a sphere** if all its vertices lie in the sphere.

The sphere is also said to be **circumscribed about the polyhedron.**

633. **EXERCISES.**

1. In the construction of § 631 show that O is the *only* point equidistant from P, A, B, and C.

2. Show that the planes perpendicular to each of the six edges of the tetrahedron at their middle points meet in the point O.

3. Does the construction of § 631 depend upon the tetrahedron being *regular?* Can a sphere be circumscribed about *any* tetrahedron?

4. Is there any limitation on the relative position of four points in order that a sphere may be passed through them?

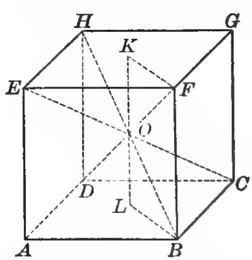
634. Four points not all lying in the same plane are said to **determine a sphere.**

Any other point, taken at random, will not, in general, lie on a sphere determined by four given points.

Hence, while any tetrahedron may be inscribed in a sphere, a polyhedron, in general, cannot.

However, any *regular* polyhedron may be inscribed in a sphere.

635. PROBLEM. *To inscribe a cube in a sphere.*

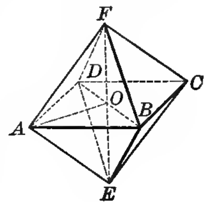


Suggestion. Show that all the diagonals meet in a common point O which is equally distant from all the vertices.

Or show that a perpendicular to one face at its center L meets the opposite face at its center K and is perpendicular to this face also, and that the middle point O of KL is the point required.

636. PROBLEM. *To inscribe an octahedron in a sphere.*

Suggestion. Making use of the construction, § 625 (3), show that O is the point equidistant from A, B, C, D, E, F. Give all the steps in full.



NOTE. The dodecahedron and icosahedron may each be inscribed in a sphere, but the proof in these cases is much more complicated.

637. EXERCISES.

1. If in the figure of § 631 the tetrahedron is *regular*, show that $OE = OF$.

2. If in Ex. 1 a sphere is described with center O and radius OE, would it touch the face PBC at any other point than E? (Why?) Would any part of the surface lie on the opposite side of ABC from O? (Why?) Is the same true of each of the other faces?

Such a sphere is said to be **inscribed in the tetrahedron** and the faces are said to be tangent to the sphere. See § 669.

3. In the figure of § 635, show that the point O is equidistant from the six faces of the cube and hence that a sphere may be inscribed.

4. In the figure of § 636 show that the point O is equidistant from the eight faces and hence that a sphere may be inscribed.

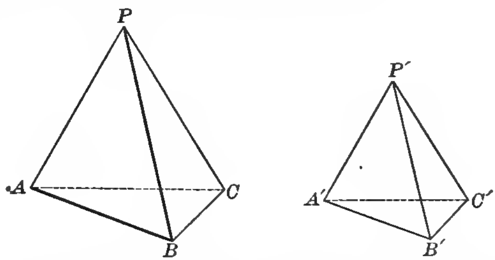
The preceding exercises show that a sphere may be inscribed in three of the regular polyhedrons, and the center in each case is the same as that of the circumscribed sphere. This is true also of the other two regular polyhedrons, but the proof is not so simple as in these cases. In the case of the tetrahedron a sphere may be inscribed *whether it is regular or not;* but if it is not regular, the center is not the same as that of the circumscribed sphere.

SIMILAR POLYHEDRONS.

638. **Definitions**. Two polyhedrons are **similar** if they have the same number of faces similar each to each and similarly placed, and have their corresponding polyhedral angles congruent.

Any two parts which are similarly placed are called **corresponding parts**, as corresponding faces, edges, vertices.

639. THEOREM. *Two tetrahedrons are similar if three faces of one are similar respectively to three faces of the other, and are similarly placed.*



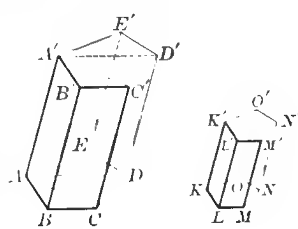
Given the tetrahedrons $P\text{–}ABC$ **and** $P'\text{–}A'B'C'$ **having** $\triangle APB \sim \triangle A'P'B'$, $\triangle APC \sim \triangle A'P'C'$, **and** $\triangle BPC \sim \triangle B'P'C'$.

To prove $P\text{–}ABC \sim P'\text{–}A'B'C'$.

Proof: (1) Show that $\triangle ABC \sim \triangle A'B'C'$.

(2) Show that trihedral $\angle$s P and P' are congruent. Likewise $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$.

Hence, by definition, the polyhedrons are similar.



640. **EXERCISES.**

1. If the two prisms in the figure are similar, name the pairs of corresponding parts. Likewise for two similar pyramids.

2. Show that a plane parallel to the base of a pyramid cuts off a pyramid similar to the given pyramid.

SUGGESTION. Use the principles of similar triangles and § 521 to show that all the requirements of the definition (§ 638) are fulfilled.

3. Does a plane parallel to the base of the prism cut off a prism similar to the given prism? Prove.

4. Show that two tetrahedrons are similar if they have a dihedral angle in one equal to a dihedral angle in the other and the including faces similar each to each and similarly placed.

5. Show that the total areas of two similar tetrahedrons are in the same ratio as the squares of any two corresponding edges.

6. Show that if each of two polyhedrons is similar to a third they are similar to each other.

641. THEOREM. *The volumes of two similar tetrahedrons are in the same ratio as the cubes of their corresponding edges.*

Given $P\text{–}ABC \sim P'\text{–}A'B'C'$, with volumes V and V'.

To prove that $$\frac{V}{V'} = \frac{\overline{PA}^3}{\overline{P'A'}^3}.$$

Proof: We have $$\frac{V}{V'} = \frac{PA \cdot PB \cdot PC}{P'A' \cdot P'B' \cdot P'C'}. \quad (\S\ 601)$$

Now use the properties of similar triangles to complete the proof. Use the figure of § 639.

642. **EXERCISES.**

1. Two similar tetrahedral mounds have a pair of corresponding dimensions 3 ft. and 4 ft. If one mound contains 40 cu. ft. of earth, how much does the other contain?

2. The edges of a tetrahedron are 3, 4, 5, 6, 7, and 10. Find the edges of a similar tetrahedron containing 64 times the volume.

3. Find what fraction of the altitude of a tetrahedron must be cut off by a plane parallel to the base, measuring from the vertex, in order that the new pyramid thus detached may have one third of the original volume.

643. Definitions. Two figures are said to have a **center of similitude** O, if for any two points A and B of the one the lines AO and BO meet the other in two points, A' and B', called **corresponding points**, such that

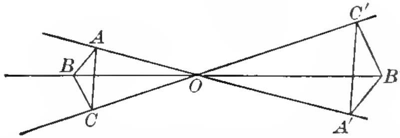
$$\frac{AO}{A'O} = \frac{BO}{B'O}.$$

See figures under §§ 189–194.

644. THEOREM. *Any two figures which have a center of similitude are similar.*

Proof: (1) *Two triangles.*

Given $$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'}.$$

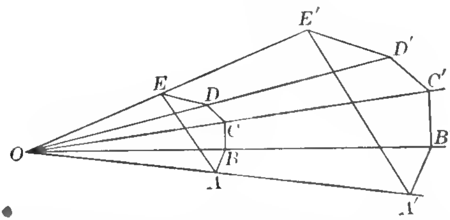


Let the student prove that $\triangle ABC \sim \triangle A'B'C'$.

In case the triangles do not lie in the same plane, use § 489 to show that the corresponding $\measuredangle$s are equal.

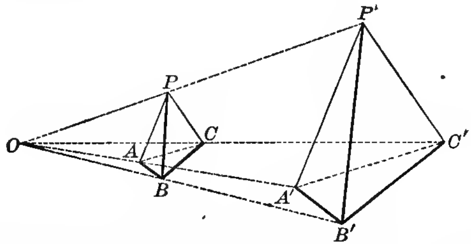
(2) *Two polygons.*

Given $$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'}, \text{ etc.}$$



Give the proof both for polygons in the same plane and not in the same plane.

(3) *Two tetrahedrons.*



With the same hypothesis as before, we must prove $\triangle PAB \sim P'A'B'$, $\triangle PBC \sim \triangle P'B'C'$, etc., and then use § 639.

(4) *Any two polyhedrons.*

(*a*) Prove corresponding polygonal faces similar to each other.

(*b*) Prove corresponding polyhedral angles equal to each other.

The last step requires not only equal *face angles* about the vertex, as in the case of the tetrahedron, but also equal *dihedral* angles. Note that two dihedral angles are equal if their faces are parallel right face to right face and left face to left face. (Why?)

(5) Consider *any two figures whatsoever* having a center of similitude.

(*a*) Take any three points A, B, C in one figure and the three corresponding points A', B', C' in the other.

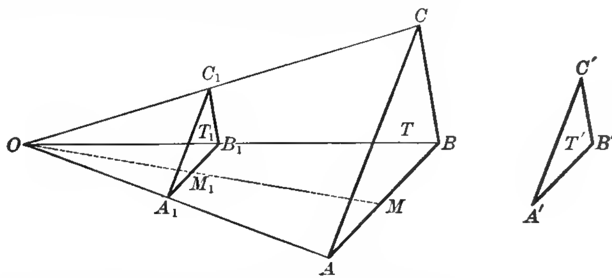
Then AB and $A'B'$, AC and $A'C'$, etc., are called **corresponding linear dimensions**, and the triangles ABC and $A'B'C'$ are **corresponding triangles.**

(*b*) It is clear that any two corresponding linear dimensions have the same ratio as any other two, and that any two corresponding triangles are similar.

In this sense the two figures are said to be **similar.**

645. Definition. The **ratio of similitude** of two similar figures is the common ratio of their corresponding linear dimensions. This ratio is the same as the **distance ratio** of corresponding points from the center of similitude.

646. THEOREM. *Two similar triangles may be so placed as to have a center of similitude.*



Given the similar triangles T and T', in which

$$\frac{A'B'}{AB}=\frac{A'C'}{AC}=\frac{B'C'}{BC}.$$

To prove that they may be placed so as to have a center of similitude.

Proof: From any point O draw OA, OB, OC.

On these rays take A_1, B_1, C_1 so that

$$\frac{OA_1}{OA}=\frac{OB_1}{OB}=\frac{OC_1}{OC}=\frac{A'B'}{AB}.$$

Now show the following:

(1) $\triangle T_1 \sim \triangle T$, and hence $\triangle T_1 \sim \triangle T'$.

(2) $\triangle T_1 \cong \triangle T'$.

For this show that $A_1B_1 = A'B'$ by means of the equations $\frac{A_1B_1}{AB}=\frac{OA_1}{OA}$ and $\frac{A'B'}{AB}=\frac{OA_1}{OA}$.

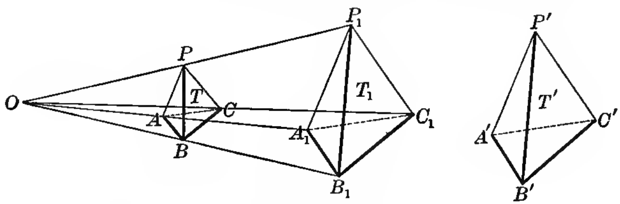
Likewise $A_1C_1 = A'C'$ and $B_1C_1 = B'C'$.

(3) Finally, $$\frac{OM_1}{OM} = \frac{OA_1}{OA},$$

where M and M_1 are any two corresponding points whatever.

Hence O is the required center of similitude.

647. THEOREM. *Two similar tetrahedrons may be so placed as to have a center of similitude.*



Given the similar tetrahedrons T and T'.

To prove that they can be placed so as to have a center of similitude.

Proof: With O as a center of similitude, construct T_1, making
$$\frac{OA}{OA_1} = \frac{OB}{OB_1} = \text{etc.} = \frac{AB}{A'B'}.$$

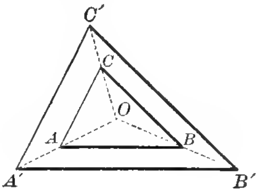
Now show as in § 646 that $T_1 \cong T'$, and hence that T' can be placed in the position T_1 so as to have with T the center of similitude O.

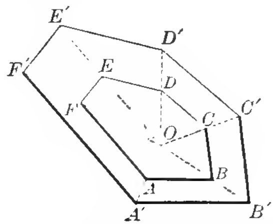
Give all the steps in detail.

648. THEOREM. *Any two similar polyhedrons may be placed so as to have a center of similitude.*

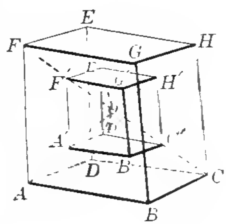
Suggestion. The argument is precisely similar to that of § 647. Give it in full.

649. In the figures for the preceding theorems the center of similitude has been taken *between* the two figures or *beyond* them both. The center may be taken equally well *within* them, as in the following illustrations :





In the case of similar convex polyhedrons with the center of similitude thus placed, the faces are the bases of pyramids whose vertices are all at the center of similitude.



If, further, the polygonal faces be divided into triangles by drawing their diagonals, these triangles become the bases of tetrahedrons, all of whose vertices are at the center of similitude.

Moreover, each inner tetrahedron is similar to its corresponding outer tetrahedron. (Why ?)

The volumes of the two similar polyhedrons are thus composed of the sums of sets of similar tetrahedrons.

650. THEOREM. *The volumes of any two similar polyhedrons have the same ratio as the cubes of their corresponding edges.*

Proof : Place the polyhedrons whose volumes are V and V' so as to have their centers of similitude within them as in the figures of § 649.

Call the volumes of the similar tetrahedrons $T_1, T_2, T_3, \cdots,$ and $T_1', T_2', T_3', \cdots,$ and let AB and $A'B'$ be two corresponding edges.

Then we have

$$\frac{\overline{AB}^3}{\overline{A'B'}^3} = \frac{T_1}{T_1'} = \frac{T_2}{T_2'} = \frac{T_3}{T_3'} = \cdots. \quad \text{(Why?)}$$

And $$\frac{T_1 + T_2 + T_3 + \cdots}{T_1' + T_2' + T_3' + \cdots} = \frac{T_1}{T_1'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}. \quad \text{(Why?)}$$

But $$T_1 + T_2 + T_3 \cdots = V \text{ and } T_1' + T_2' + T_3' \cdots = V'.$$

Hence, $$\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

651. COROLLARY. *The volumes of any two similar solids are in the same ratio as the cubes of any two corresponding linear dimensions.*

This proposition may be rendered evident by noticing that any two similar three-dimensional figures may be built up to any degree of approximation by means of pairs of similar tetrahedrons similarly placed. The proposition then holds of any two corresponding figures used in the approximations.

Note that the ratio of similitude of two similar figures may be obtained from the ratio of any pair of corresponding linear dimensions.

652. EXERCISES.

1. If two coal bins are of the same shape and one is twice as long as the other, what is the ratio of their cubical contents?

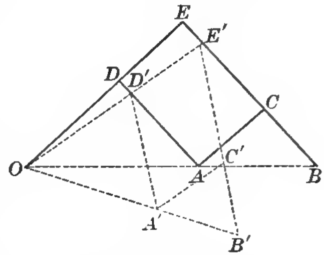
2. What is the ratio of the lengths of the two bins in the preceding example if one holds twice as much coal as the other?

3. Two water tanks are of the same shape. Find the ratio of their capacities if their ratio of similitude is $\frac{2}{5}$.

4. In the preceding what must be the ratio of similitude in order that the ratio of their capacities shall be $\frac{3}{7}$?

APPLICATIONS OF SIMILARITY.

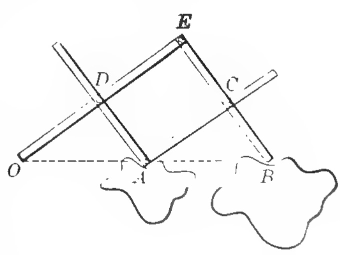
653. The theorem that any two figures which have a center of similitude are similar is the geometric basis of many mechanical contrivances for enlarging or reducing both plane and solid figures; that is, for constructing figures similar to given figures and having a given ratio of similitude with them.



The essential property of all such contrivances is that one point O is kept fixed, while two points A and B are allowed to move so that O, A, and B always remain in a straight line, and so that the ratio $\frac{OA}{OB}$ remains the same. See §§ 432–435.

In the first figure O is a fixed point. Segments OD, CB, and the sides of the parallelogram $ACED$ are of fixed length.

Prove that if B is once so taken on the line EC as to be in the line OA, the points O, A, and B will always remain collinear, and that $\frac{OA}{OB}$ remains a fixed ratio.



In the second figure is shown an ordinary **pantograph** used for copying and at the same time for reducing or enlarging maps, designs, etc. The lengths of the various segments are adjustable, as shown, thus obtaining any desired scale.

The same contrivance may be used for copying figures in *space* and at the same time reducing or enlarging them.

654. Now consider any two similar figures whatever so placed as to have a center of similitude O. We have seen that if points AB and $A'B'$ are corresponding points of the two figures, then the ratio of the **corresponding linear dimensions** AB and $A'B'$ is equal to the ratio of similitude $\frac{m}{n}$ of the two figures.

Also if A, B, C, D and A', B', C', D' are corresponding points, then $\triangle ABC$ and $A'B'C'$, and the tetrahedrons $ABCD$ and $A'B'C'D'$ are similar, and we have

$$\frac{\text{Area } ABC}{\text{Area } A'B'C'} = \frac{m^2}{n^2} \text{ and } \frac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \frac{m^3}{n^3}.$$

The points A, B, C and A', B', C' determine two planes, each of which intercepts a certain plane figure in the solid figure to which the points belong. These two plane figures we call **corresponding cross sections.**

We assume without full argument that:—

THEOREM. (1) *The ratio of the areas of any pair of corresponding cross sections or any pair of corresponding surfaces of similar figures is equal to the square of their ratio of similitude,* and

(2) *The ratio of the volumes of any two similar figures is equal to the cube of their ratio of similitude.*

Thus the ratio of the radii or of the diameters of two spheres is their ratio of similitude; likewise the ratio of the lengths or of the diameters of two shells used in gunnery, or the ratio of the heights of two men of similar build.

The fact that the ratio of the areas of corresponding surfaces of similar solids is equal to the *square* of their ratio of similitude, while the ratio of their volumes equals the *cube* of this ratio is one of the most important and far-reaching conclusions of geometry.

SUMMARY OF CHAPTER XI.

1. Describe the five regular solids according to the form and number of their faces. Why can there not be more than these five?

2. Compare the relation of the regular solids to the sphere with that of regular polygons in the plane to the circle.

3. Review the process of construction of the tetrahedron, cube, and octahedron.

4. Form all five regular polyhedrons by cardboard models as indicated in § 625.

5. Make a list of the definitions concerning similar polyhedrons.

6. Make a list of the theorems concerning similar polyhedrons.

7. Explain the relation of two figures which have a center of similitude.

8. What theorem of this chapter is referred to as of unusual importance in the problems and applications?

9. State the applications of this chapter which appeal to you as especially interesting or useful. Return to this question, after studying those which follow.

PROBLEMS AND APPLICATIONS.

1. If it is known that a steel wire of radius r will carry a certain weight w, how great a weight will a wire of the same material carry if its radius is $2r$?

SUGGESTION. The tensile strengths of wires are in the same ratio as their cross-section areas.

2. Find the ratio of the diameters of two wires of the same material if one is capable of carrying twice the load of the other; three times the load.

3. In a laboratory experiment a heavy iron ball is suspended by a steel wire. In suspending another ball of twice the diameter a wire of twice the radius of the first one is used. Is this perfectly safe if it is known that the first wire will just safely carry the ball suspended from it? Discuss fully.

4. In two schoolrooms of the same shape (similar figures) but of different size, the same *proportion* of the floor space is occupied by desks. Which contains the larger amount of air for each pupil?

5. It is decided to erect a school building exactly like another already built, except that every linear dimension is to be increased by ten per cent; that is, each room is to be ten per cent longer, wider, and higher, and so for all parts of the building. If the air in the ventilating flues flows with the same velocity in the two buildings, in which will the air in a room be entirely renewed the more quickly?

SUGGESTION. Note that the ratio of the amount of air discharged by two flues under such conditions is equal to the ratio of their cross-section areas.

6. If the shells used in guns are similar in shape, find the ratio of the total surface areas of an eight-inch and a twelve-inch shell.

7. Find the ratio of the weights of the shells in the preceding problem, weights being in the same ratio as the volumes.

8. If a man 5 ft. 9 in. tall weighs 165 lb., what should be the weight of a man 6 ft. 1 in. tall, supposing them to be similar in shape?

9. What is the diameter of a gun which fires a shell weighing twice as much as a shell fired from an eight-inch gun, supposing the shells to be similar bodies?

10. The ocean liner *Mauretania* is 790 feet in length. What must be the length of a ship having twice her tonnage, supposing the boats to be similar in shape?

11. The steamship *Lusitania* is 790 feet long, with a tonnage of 32,500, and the *Olympic* is 882 feet long, with a tonnage of 45,000. Are these vessels similar in shape? If not, which has the greater capacity in proportion to its length?

12. Supposing two trees to be similar in shape, what is the diameter of a tree whose volume is three times that of one whose diameter is 2 feet? What is the diameter if the volume is 5 times that of the given tree? What if it is n times that of the given tree?

13. Two balloons of similar shape are so related that the total surface area of one is 5 times that of the other. Find the ratio of their volumes.

See page 444 for further applications.

CHAPTER XII.

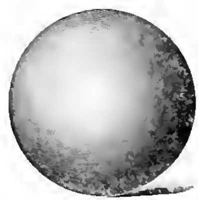
THE SPHERE.

PLANE SECTIONS OF THE SPHERE.

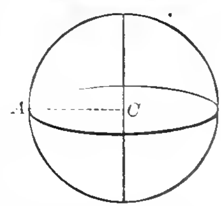
655. Definitions. A **sphere** consists of all points in space which are equally distant from a fixed point, and of these points only. The fixed point is called the center of the sphere. (See § 632.)

A sphere divides space into two parts such that any point which does not lie on the sphere lies within it or outside it.

The sphere may be developed by revolving a circle about a diameter as a fixed axis.



A line-segment joining any two points on a sphere and passing through its center is a **diameter**. A segment joining the center to any point on the sphere is a **radius**.



If the distance from a point to the center of the sphere is less than the radius, the point is **within** the sphere, and if greater than the radius it is **outside** the sphere.

A sphere may be designated by a single letter at its center, or more explicitly by naming its center and radius.

Thus the sphere C means the sphere whose center is C, and the sphere CA is the sphere whose center is C and whose radius is CA.

Two spheres are said to be **equal** if they have equal radii.

656. Theorem. *All radii of the same sphere or of equal spheres are equal. All diameters of the same sphere or of equal spheres are equal.*

These statements follow directly from the definitions.

657. EXERCISES.

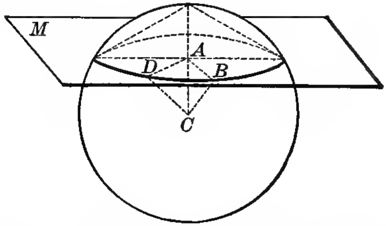
1. How does the definition of a sphere differ from that of a circle? State each in terms of a *locus*.

2. If two spheres have the same center, show that they are either equal or one lies entirely inside the other.

3. In how many points can a straight line meet a sphere?

4. Does every *line* through an interior point of a sphere meet it? In how many points?

658. Theorem. *A section of a sphere made by a plane is a circle.*



Given a sphere with center *C* cut by the plane *M*.

To prove that the points common to the sphere and the plane form a circle.

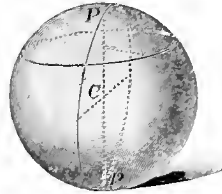
Proof: From the center C draw CA perpendicular to the plane M.

Let B and D be any two points common to the plane and the sphere. Complete the figure, and prove $AB = AD$.

Hence, any two points common to the plane and the sphere are equidistant from A.

How must this proof be modified in case the plane M passes through the center of the sphere?

659. Definitions. A circle is said to be **on a sphere** if all its points lie on the sphere.

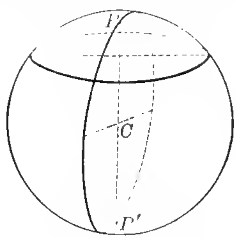


The line perpendicular to the plane of a circle at its center is called the **axis** of the circle.

The points in which the axis of a circle on a sphere meets the sphere are called the **poles of the circle**.

If the plane of a circle on a sphere passes through the center of the sphere, it is called a **great circle** of the sphere, and if not, it is called a **small circle**.

660. THEOREM. (1) *The axis of any circle on a sphere passes through the center of the sphere.*



(2) *The center of a great circle is the center of the sphere.*

(3) *All great circles are equal and bisect each other.*

(4) *Three points on a sphere determine a circle on the sphere.*

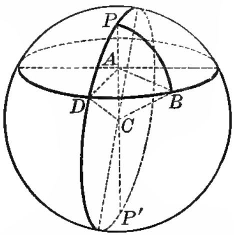
(5) *Through two given points on a sphere there is one and only one great circle unless these points are at opposite ends of a diameter.*

(6) *Every great circle bisects a sphere.*

The proofs of these statements follow easily from the definitions. Let the student give the proofs in detail.

661. **Definition.** The **distance** between two points on a sphere is the distance measured between these points along the minor arc of the great circle through them.

662. THEOREM. *All points of a circle on a sphere are equidistant from either pole of the circle.*



Given P a pole of the circle whose center is A, and let B and D be any two points on this circle.

To prove that the great circle arcs PB and PD are equal.

Suggestion. Let C be the center of the sphere.

Prove that $\angle ACB = \angle ACD$.

Hence, chord $PB =$ chord PD and $\overset{\frown}{PB} = \overset{\frown}{PD}$.

Extend the radius PC to meet the sphere in P'.

Prove that P' is also equidistant from any two points of the given circle.

663. **Definition.** The common distance from the pole of a circle to all points on it is called the **polar distance of the circle.** One fourth of a great circle is a **quadrant.**

664. COROLLARY 1. *The polar distance of a great circle is a quadrant.*

665. COROLLARY 2. *If a point P is at a quadrant's distance from each of two points not at the extremities of the same diameter, it is the pole of the great circle through these points.*

666. It follows from the preceding theorem that, if a *spherical* blackboard is at hand, circles may be constructed on it by means of crayon and string the same as on a *plane* blackboard. Likewise, curve-legged compasses may be used.

667. EXERCISES.

1. How many small circles can be passed through two points on a sphere? How many great circles? Show why.

2. If two points are at the extremities of the same diameter of a sphere, how many great circles can be passed through these points?

3. What great circles on the earth's surface pass through both poles? If a great circle passes through one pole, must it pass through the other?

4. If P is at a quadrant's distance from each of two points A and B, and if these points are at opposite ends of the same diameter, is P the pole of any circle through A and B? Of how many circles?

5. If A and B are at opposite ends of a diameter, can a small circle be passed through them?

6. If two circles on a sphere have the same poles, prove that their planes are parallel.

7. What is the locus of all points on a sphere at a quadrant's distance from a given point?

8. What is the locus of all points on a sphere at any fixed distance from a given point on the sphere? What is the greatest such distance possible? Discuss fully.

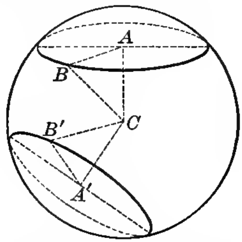
9. If two planes cutting a sphere are parallel, compare the positions of the poles of the circles thus formed.

10. Find the locus of the centers of a set of circles on a sphere formed by a set of parallel planes cutting it.

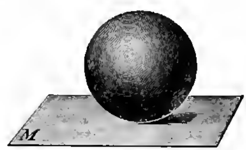
11. AB is a fixed diameter of a sphere. A plane containing AB is made to revolve about it as an axis. Find the locus of the poles of the great circles on the sphere made by this revolving plane. How are the points A and B related to this locus?

668. THEOREM. *If two planes cutting a sphere are equidistant from the center, the circles thus formed are equal; and conversely,*

If two planes cut a sphere in equal circles, the planes are equidistant from the center.



Suggestion. In the figure show that '(1) if $CA = CA'$, then $AB = A'B'$, and (2) if $AB = A'B'$, then $CA = CA'$.



669. **Definitions.** A plane which meets a sphere in only one point is **tangent to the sphere.**

Two spheres are tangent to each other if they have only one point in common.

A line is tangent to a sphere if it contains one and only one point of the sphere.

670. **EXERCISES.**

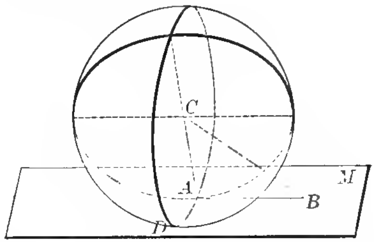
1. If a plane has more than one point in common with a sphere, must it have a circle in common with the sphere?

2. If a plane is tangent to a sphere, how many lines in the plane are tangent to the sphere?

3. Can two spheres be tangent to each other and still one be inside the other?

671. THEOREM. *A plane tangent to a sphere is perpendicular to the radius from the point of tangency; and conversely,*

A plane perpendicular to a radius at its extremity is tangent to the sphere.



Given a sphere *C* with plane *M* tangent to it at *A*.

To prove that *CA* is perpendicular to the plane *M*.

Proof: (1) It is only necessary to prove that *CA* is perpendicular to every line in *M* through *A*. (Why?)

Draw any such line *AB*.

The plane *BAC* cuts the sphere in a circle. Prove that *AB* is tangent to this circle, and hence perpendicular to *AC*.

(2) To prove the converse, note that *CA* is the shortest distance from *C* to the plane *M*. (Why?) And hence that every point of *M* except *A* is *exterior* to the sphere.

Hence, *M* is a tangent plane. (Why?)

672. **Definition.** A sphere is said to be **inscribed in a polyhedron** if every face of the polyhedron is tangent to the sphere. The polyhedron is also said to be **circumscribed about the sphere.**

673. THEOREM. *A sphere may be inscribed in any tetrahedron.*

The proof is left to the student. See § 637.

674. EXERCISES.

1. How many planes may be tangent to a sphere at one point on the sphere? How many lines?

2. Through a given point exterior to a sphere construct a line tangent to the sphere.

SUGGESTION. Let O be the center of the sphere and P the given exterior point. Pass any plane M through P and O. In the plane M construct a line through P tangent to the great circle in which M cuts the sphere.

3. How many lines tangent to a sphere can be constructed from a point outside the sphere?

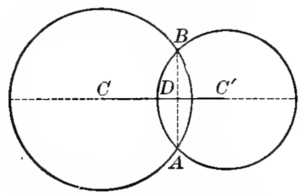
4. Through a given point exterior to a sphere construct a plane tangent to the sphere.

SUGGESTION. As in Ex. 2 construct a line through the given point tangent to the sphere. Through the point of tangency of this line pass a plane tangent to the sphere.

5. How many planes can be passed through a given exterior point tangent to the sphere?

6. How many planes tangent to a sphere can be passed through two given points A and B outside a sphere? Discuss fully if the line AB (1) meets the sphere in two points; (2) is tangent to the sphere; (3) does not meet the sphere.

675. THEOREM. *The intersection of two spheres is a circle.*



Proof: The two intersecting spheres may be developed by rotating about a fixed axis CC' two intersecting circles with centers C and C'.

Let A and B be the two points common to both circles. Then $BA \perp CC'$. (Why?)

As the figure rotates about the line CC', AB remains fixed in length and perpendicular to CC'.

Hence, B traces out a circle. See § 478,

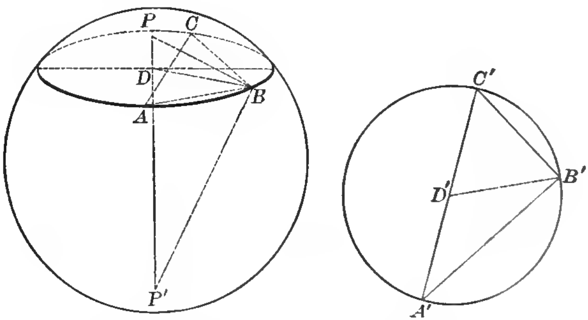
676. PROBLEM. *To find the diameter of a given material sphere.*

With any point P of the sphere as a pole, construct any circle, and on this circle select any three points A, B, C.

Using a pair of compasses, measure the straight line-segments AB, BC, CA, and construct the triangle $A'B'C'$ congruent to ABC.

Let $B'D'$ be the radius of the circle circumscribed about $A'B'C'$.

If PP' is the axis of the circle ABC on the sphere and BD the radius of this circle, then $BD = B'D'$.



Measure PB by means of the compasses.

Then PBP' is a right triangle, with BD perpendicular to its hypotenuse PP'.

PB and BD being known, we may now *compute* PD from the right triangle PBD and then compute PP' from the similar triangles PBD and $PP'D$, finding $PD : PB = PB : PP'$ or $PD \times PP' = \overline{PB}^2$. See Ex. 3, § 66.

The segment PP' may also be found by *geometric* construction; namely, draw a triangle congruent to $P'BP$.

Show how to do this when BP and BD are known.

677. EXERCISES.

1. How many points are necessary to determine a sphere? See § 632.

2. If the center of a sphere is given, how many points on the sphere are required to determine it?

3. If a plane M is tangent to a sphere at a point A, show that the plane of every great circle of the sphere through A is perpendicular to M.

4. Show that the line of centers of two intersecting spheres meets the spheres in the poles of their common circle.

5. Find the locus of the centers of all spheres tangent to a given plane at a given point.

6. Find the locus of the centers of all spheres tangent to a given line at a given point.

7. Find the locus of the centers of all spheres of given radius tangent to a fixed plane.

8. Find the locus of the centers of all spheres of given radius tangent to a fixed line.

9. Find the locus of the centers of all spheres tangent to two given intersecting planes.

10. Find the locus of the centers of all spheres tangent to all faces of a trihedral angle.

11. Show that two spheres are tangent if they meet on their line of centers. Distinguish two cases. Compare § 209.

12. State and prove the converse of the preceding proposition.

13. In plane geometry how many circles can be drawn through a given point tangent to a given line at a given point?

14. If a given sphere is tangent to a given plane M at a given point A, how many points on the sphere are required to determine it?

SUGGESTION. Suppose one point P given. Pass a plane through $P \perp$ to plane M at A. Is there only one such plane? Discuss fully.

15. Describe the set of all lines in space whose distances from the center of a sphere are all equal to the radius of the sphere.

16. Describe the set of all planes whose distances from the center of a sphere are all equal to the radius of the sphere.

TRIHEDRAL ANGLES AND SPHERICAL TRIANGLES.

678. **Definitions.** When two curves meet, they are said to **form an angle**; namely, the angle made by the tangents to the curves at their common point.

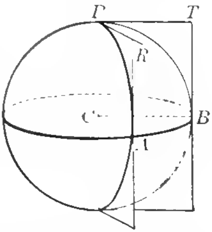
Any two planes through the center of a sphere cut out two great circles which intersect in two points and form four **spherical angles** about each of these points. Two of these angles with a common vertex are either **adjacent** or **vertical** in the same sense as the angles formed by two intersecting straight lines.

A spherical angle is **acute, right,** or **obtuse** according to the form of the angle between the tangents to its sides (arcs) at their common point.

Any two circles on a sphere which meet, whether great circles or not, form angles according to the above definition.

Only angles formed by great circles are considered in this book and the expression **spherical angle** will be understood to refer to such angles. Only angles greater than zero and not greater than two right angles are considered.

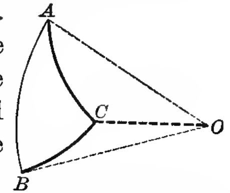
A spherical angle may be denoted by a single letter or by three letters, as in the case of a plane angle.



679. THEOREM. *A spherical angle is measured by an arc of the great circle whose pole is the vertex of the angle and which is intercepted by the sides of the angle.*

Suggestion. Show that $\overset{\frown}{AB}$ measures the dihedral angle formed by the planes PAC and PBC and that $\angle BCA = \angle TPR$.

680. Definitions. The section of a sphere made by a convex trihedral angle, whose vertex is at the center of the sphere, is called a **spherical triangle.**



The face angles of the trihedral angle are measured by the *sides* (arcs) of the spherical triangle, and its dihedral angles are equal to the *angles* of the spherical triangle.

Since each face angle of a trihedral angle is less than two right angles, it follows that each side of a spherical triangle is less than a semicircle.

681. EXERCISES.

1. Show that a spherical angle is equal to the plane angle of the dihedral angle formed by the planes of the great circles whose arcs are the sides of the spherical angle. See figure under § 679.

2. Prove that vertical spherical angles are equal.

3. Prove that the sum of the spherical angles about a point is four right angles.

4. At what angle does a meridian on the earth's surface intersect the equator?

5. Denote by P the North Pole on the earth's surface. Consider any two meridians forming an angle of one degree at P and meeting the equator in points A and B, respectively. What is the sum of the angles of the spherical triangle PAB? Compare with the sum of the angles of a plane triangle.

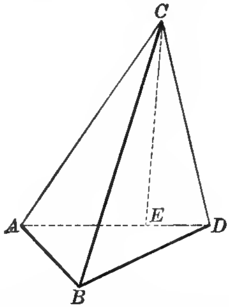
6. Is it possible to construct a spherical triangle each of whose angles is a right angle?

SUGGESTION. Consider two meridians forming a right angle at P. Such a triangle is called a trirectangular triangle.

7. In a trirectangular spherical triangle what is the length of each side in terms of degrees?

The question as to whether or not the sum of the angles of a spherical triangle is ever equal to two right angles is answered in § 712.

682. THEOREM. *The sum of two face angles of a trihedral angle is greater than the third face angle.*



Proof: Connect points A and D on two sides.

Suppose not all three face angles are equal and that $\angle ACD > \angle ACB$.

Construct CE in the face ACD, making $\angle ACE = \angle ACB$.

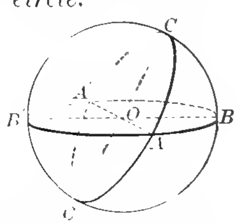
Lay off $CB = CE$ and draw AB and BD.

Now show that (1) $AB = AE$, (2) $AD < AB + BD$, (3) $ED < BD$, (4) $\angle ECD < \angle BCD$.

Hence, $\angle ACD < \angle ACB + \angle BCD$.

683. COROLLARY 1. *The sum of two sides of a spherical triangle is greater than the third side.*

684. COROLLARY 2. *The sum of the three sides of a spherical triangle is less than a great circle.*



Proof: $\overset{\frown}{ACA'} + \overset{\frown}{ABA'} =$ a great circle.

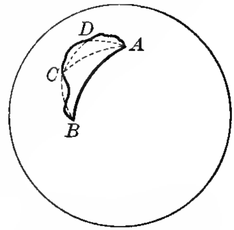
But $\overset{\frown}{AC} < \overset{\frown}{ACA'}$ and $\overset{\frown}{AB} < \overset{\frown}{ABA'}$,

and $\overset{\frown}{BC} < \overset{\frown}{BA'} + \overset{\frown}{CA'}$ by Corollary 1.

Hence, $\overset{\frown}{AC} + \overset{\frown}{CB} + \overset{\frown}{BA} <$ a great circle.

685. COROLLARY 3. *State and prove the theorems on trihedral angles corresponding to Corollaries 1 and 2.*

686. THEOREM. *The shortest distance on a sphere between two of its points is measured along the minor arc of a great circle passing through these points.*



Proof: Let A and B be any two points on a sphere, AB the minor arc of a great circle through them, and $ADCB$ any other curve on the sphere connecting A and B with points C and D on it in the order $ADCB$. Neither C nor D is on the arc AB.

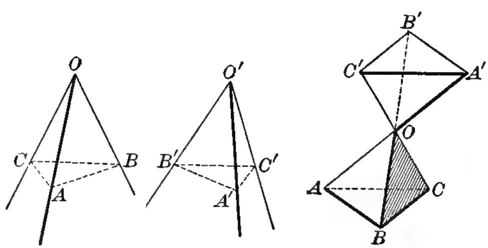
Draw the great circle arcs AD, AC, DC, and CB. Then by § 683 $\overset{\frown}{AC} + \overset{\frown}{CB} > \overset{\frown}{AB}$ and $\overset{\frown}{AD} + \overset{\frown}{DC} > \overset{\frown}{AC}$.

Hence, $\overset{\frown}{AD} + \overset{\frown}{DC} + \overset{\frown}{CB} > \overset{\frown}{AB}$.

Continuing in this manner, we obtain a succession of paths, each longer than the preceding. But by this process we get closer and closer to the length of the curve ACB.

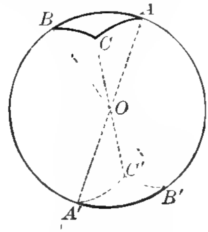
Hence, it must be greater than that of $\overset{\frown}{AB}$.

687. **Definitions.** Two trihedral angles are **symmetrical** one to the other if the face angles and the dihedral angles of one are equal respectively to the face angles and the dihedral angles of the other, but arranged in the opposite order.



Similarly, two spherical triangles are symmetrical one to the other if the sides and the angles of one are equal respectively to the sides and the angles of the other, but arranged in the opposite order.

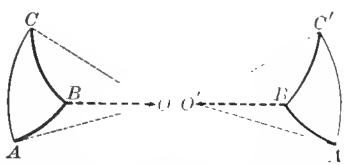
688. THEOREM. *If the radii drawn from the vertices of a spherical triangle are extended, they meet the sphere in the vertices of a triangle symmetrical to the given triangle.*



The proof is left to the student.

689. COROLLARY. *State and prove the corresponding theorem on trihedral angles.*

690. THEOREM. *Two trihedral angles having their vertices at the center of the same or of equal spheres intercept congruent spherical triangles if the trihedral angles are congruent, and symmetrical spherical triangles if they are symmetrical.*



This is an immediate consequence of §§ 680, 687.

691. COROLLARY. *If in two spherical triangles three sides of one are equal respectively to three sides of the other, and arranged in the same order, the triangles are congruent.* See § 521.

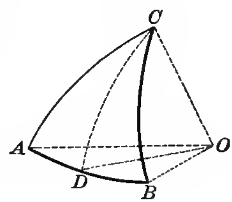
692. THEOREM. *If the face angles of one trihedral angle are equal respectively to the face angles of another, but arranged in the opposite order, the trihedral angles are symmetrical.*

Proof: By § 519 the dihedral angles of one are equal respectively to those of the other. Now verify that these parts are arranged in the opposite order.

693. COROLLARY 1. *If in two spherical triangles the three sides of one are equal to three sides of the other, but arranged in the opposite order, the triangles are symmetrical.*

694. **Definition.** A spherical triangle is isosceles if two sides are equal.

695. COROLLARY 2. *The angles opposite the equal sides of an isosceles spherical triangle are equal.*



Suggestion. Let AC and BC be the equal sides. Draw $\overset{\frown}{CD}$ to the middle point of $\overset{\frown}{AB}$.

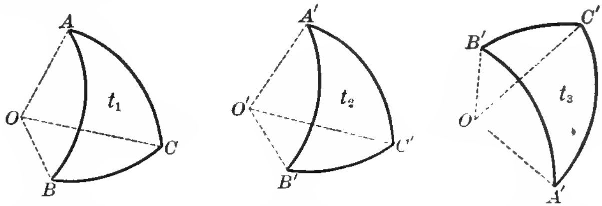
696. COROLLARY 3. *If two isosceles spherical triangles are symmetrical, they are congruent, and conversely.*

697. THEOREM. *If two trihedral angles are symmetrical to the same trihedral angle, they are congruent.*

Suggestion. Show that the corresponding parts must be arranged in the same order.

698. COROLLARY. *State and prove the corresponding theorem for spherical triangles.*

699. THEOREM. *Two spherical triangles having two sides and the included angle of one equal respectively to two sides and the included angle of the other are congruent, if the given parts are arranged in the same order, and symmetrical, if they are arranged in the opposite order.*



Proof: If the given parts are arranged in the same order, the proof may be made by superposition exactly as in § 32.

If the given parts are arranged in the opposite order, proceed as follows:

Denote the given triangles by t_1 and t_2. Construct a spherical triangle t_3 symmetrical to t_1. Then by § 698 t_2 and t_3 are congruent. Hence, if t_1 is symmetrical to t_3, it must be symmetrical to t_2.

700. COROLLARY. *State and prove the corresponding theorem for trihedral angles.*

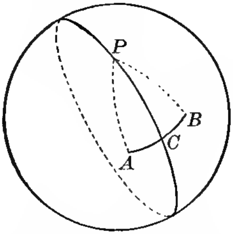
701. EXERCISES.

1. Compare fully the theorems on the congruence of plane triangles and of trihedral angles. Is there any theorem in either case for which there is no corresponding theorem in the other?

2. Compare in the same manner the theorems on the congruence of plane triangles and of spherical triangles.

3. Compare in the same manner the theorems on the congruence of trihedral angles and of spherical triangles.

702. THEOREM. *The locus of all points on a sphere equidistant from two fixed points on the sphere is a great circle bisecting at right angles the great circle arc connecting the two given points.*



Proof: Let C be the middle point of $\widehat{AB}$.

(*a*) If $\widehat{AP} = \widehat{BP}$ prove that $\triangle s$ ACP and BCP are symmetrical, and hence, $\angle ACP = \angle BCP =$ rt. $\angle$.

(*b*) If $\angle ACP = \angle BCP$, prove that $\widehat{AP} = \widehat{BP}$.

Why are steps (*a*) and (*b*) both needed?

703. EXERCISES.

1. If two face angles of a trihedral angle are equal, the opposite dihedral angles are equal.

2. If two face angles of a trihedral angle are equal, it is congruent to its symmetrical trihedral angle.

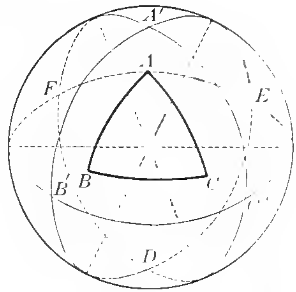
3. Show how to find a pole of the circle through three given points on the sphere.

SUGGESTION. Let the given points be A, B, C. By § 662 the pole of the circle is equidistant from A, B, and C. Connect A and B by an arc of a great circle and construct another arc of a great circle bisecting $\widehat{AB}$ perpendicularly. Similarly construct a perpendicular bisector of $\widehat{BC}$. The points in which these two arcs meet will be the poles of the circle through A, B, and C.

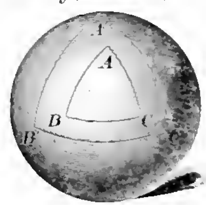
State this argument in full detail.

POLAR TRIANGLES.

704. Definition. With the vertices A, B, C of a spherical triangle as poles, construct three great circles. Each of these circles meets each of the others in two points, thus forming eight spherical triangles, as shown in the figure, namely, $A'B'C'$, $A'B'F$, $B'C'D$, $C'A'E$, $A'EF$, $B'DF$, $C'DE$, and DEF.



There is one and only one of these, namely, $A'B'C'$, such that A and A' are on the same side of $\widehat{B'C'}$, B and B' on the same side of $\widehat{A'C'}$, and C and C' on the same side of $\widehat{A'B'}$.



The triangle $A'B'C'$ as thus described is the **polar triangle** of ABC.

705. EXERCISES.

1. In the figure the parts of the great circles which are supposed to be on the front side of the figure are given in solid lines while the parts on the back side are dotted. Study the figure with care and state which triangles are entirely on the front side, which are entirely on the back side of the sphere, and which are partly on the front side and partly on the rear side of the sphere.

2. Show that the points A' and D cannot be on the same side of the circle through $B'C'$.

SUGGESTION. Can the two extremities of a diameter lie in the same hemisphere?

3. If A is a pole of the great circle through $B'C'$ and if A' is on the same side of this circle as A, show that A and A' are less than one quadrant's distance apart.

706. THEOREM. *If $A'B'C'$ is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$.*

Proof: It is required to prove (1) that A' is the pole of $\widehat{BC}$, B' the pole of $\widehat{AC}$, and C' the pole of $\widehat{AB}$, and also (2) that A and A' lie on the same side of $\widehat{BC}$, B and B' on the same side of $\widehat{AC}$, and C and C' on the same side of $\widehat{AB}$.

(1) To prove that A' is the pole of $\widehat{BC}$ we need only to show that A' is at a quadrant's distance from two points in $\widehat{BC}$. Why?

Now A' is at a quadrant's distance from B because B is the pole of $\widehat{A'C'}$. A' is also at a quadrant's distance from C because C is the pole of $\widehat{A'B'}$. Hence, A' is the pole of $\widehat{BC}$.

Similarly, B' is the pole of $\widehat{AC}$ and C' the pole of $\widehat{AB}$.

(2) To show that A and A' lie on the same side of the circle BC, we note that since A is the pole of the circle $B'C'$ and A lies on the same side of this circle with A', then A and A' are at less than a quadrant's distance. Hence, it follows that if A' is at a quadrant's distance from BC, A and A' must be on the same side of BC.

In like manner we show that B and B' lie on the same side of AC, and C and C' on the same side of AB.

707. **Definition.** If ABC and $A'B'C'$ are polar triangles, and if A is a pole of $\widehat{B'C'}$, then $\angle A$ and $\widehat{B'C'}$ are said to be **corresponding parts.**

708. EXERCISES.

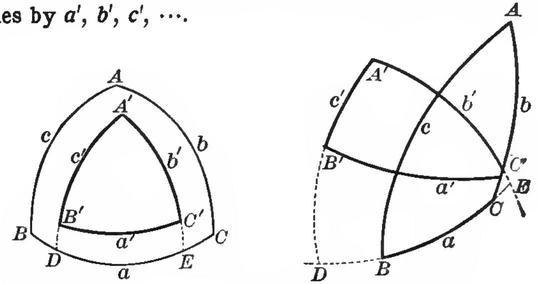
1. In the two polar triangles ABC and $A'B'C'$ name all pairs of corresponding parts.

2. Is there any spherical triangle such that its polar triangle is identical with the given triangle?

3. If one side of a spherical triangle is greater than a quadrant show that it is cut by two circles in the construction of its polar triangle.

709. THEOREM. *The sum of the measures of an angle of a spherical triangle and the corresponding arc of its polar triangle is* 180°.

Given the polar triangles ABC and $A'B'C'$. Denote the measures in degrees of the angles by A, B, C, ⋯ and of the corresponding sides by a', b', c', ⋯.



To prove that $A+a'=180°$ $A'+a=180°$
$B+b'=180°$ $B'+b=180°$
$C+c'=180°$ $C'+c=180°$

Proof: Extend (if necessary) arcs $A'B'$ and $A'C'$ till they meet the great circle BC in points D and E, respectively. Then arc DE is the measure of $\angle A'$.

Also $\widehat{BE}=90°$, and $\widehat{DC}=90°$. (Why?)

But $\widehat{BE}+\widehat{DC}=\widehat{BC}+\widehat{ED}=a+A'$.

Hence, $A'+a=180°$.

Complete the proof for the other cases.

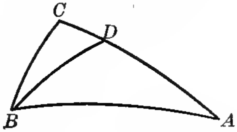
710. COROLLARY. *If two spherical triangles are congruent or symmetrical, their polar triangles are congruent or symmetrical.*

711. **EXERCISES.**

1. Does the above proof apply to the second figure?
2. If two angles of a spherical triangle are equal, it is isosceles.

SUGGESTION. Use § 709, § 695, and again § 709.

3. State and prove the theorem on trihedral angles corresponding to the preceding.



4. If two angles of a spherical triangle are unequal, the sides opposite them are unequal, the greater side being opposite the greater angle.

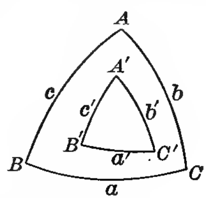
SUGGESTION. In the triangle ABC let $\angle B > \angle A$. Draw $\widehat{BD}$, making $\angle ABD = \angle A$.

Then, $\widehat{AD} = \widehat{BD}$, and $\widehat{BD} + \widehat{DC} > \widehat{BC}$.

Hence, show that $\widehat{AC} > \widehat{BC}$.

5. State and prove a theorem on trihedral angles corresponding to the preceding.

712. THEOREM. *The sum of the angles of a spherical triangle is less than six right angles and greater than two right angles.*



Given the spherical triangle *ABC*.

To prove that (1) $\angle A + \angle B + \angle C < 6$ rt. angles.

(2) $\angle A + \angle B + \angle C > 2$ rt. angles.

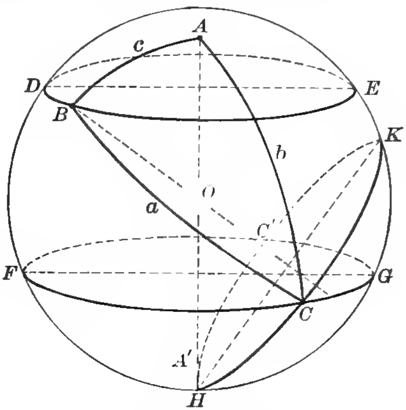
Proof: Construct the polar triangle $A'B'C'$, with sides a', b', c'.

(1) By § 709 $\angle A + \angle B + \angle C + a' + b' + c' = 6$ rt. angles. Since $a' + b' + c'$ is greater than zero, it follows that $\angle A + \angle B + \angle C < 6$ rt. angles.

(2) Using § 684, show that $\angle A + \angle B + \angle C > 2$ rt. ∠s.

713. COROLLARY. *State and prove the theorem on trihedral angles which corresponds to the preceding.*

714. PROBLEM. *On a given sphere to construct a spherical triangle when its sides are given.*



Solution. Let O be the given sphere, and a, b, c the arcs of the required triangle, and let AA' be any diameter of the sphere. With A as a pole, construct circles DBE and FCG whose polar distances from A are c and b respectively.

With B as a pole, construct a circle HCK, whose polar distance from B is a. Then construct the three great circle arcs, AB, BC, CA. ABC is the required triangle.

715. EXERCISES.

1. What restrictions if any is it necessary to impose upon the three given sides of the triangle in § 714? (§§ 683, 684.)

2. In plane geometry two congruent triangles may be constructed upon the same base and on the same side of it. Is a corresponding construction possible on the sphere?

3. If in the above construction each of two sides of the required triangle is very great, that is, nearly a semicircle, show from the construction that the third side must be very small.

4. If one side of the proposed triangle in § 714 were equal to or greater than 180°, why would that make the construction impossible?

716. PROBLEM. *To construct a spherical triangle when its three angles are given.*

Solution. Let the three given angles be A, B, C, and let a', b', c' be arcs such that $a' + \angle A = 180°$, $b' + \angle B = 180°$, $c' + \angle C = 180°$. Then the triangle whose arcs are a', b', c' will be the polar triangle of the required triangle. This latter triangle $A'B'C'$ may be constructed by the method of § 714. Then construct the polar triangle of $A'B'C'$, which will be the required triangle.

Give reasons in full for each step.

717. PROBLEM. *To construct a trihedral angle when its face angles are given.*

Solution. Construct the corresponding spherical triangle by the method of § 714.

Give the construction in full and prove each step.

718. PROBLEM. *To construct a trihedral angle when its dihedral angles are given.*

Solution. Construct the corresponding spherical triangle by the method of § 716.

Give reasons in full for each step.

719. EXERCISES.

1. If two spherical triangles having angles respectively equal are constructed on the same sphere, how are these triangles related? Prove.

2. If two trihedral angles with face angles respectively equal are constructed as in § 717, how are they related? Prove.

3. If two trihedral angles each with the same dihedral angles are constructed as in § 718, how are the trihedral angles related? Prove.

4. What restrictions if any must be placed upon the given angles A, B, C in § 716? Compare Ex. 1, § 715.

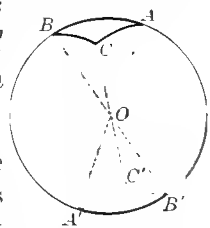
5. What restrictions if any are needed in Exs. 2 and 3?

720. THEOREM. *Two spherical triangles having two angles and the included side of one equal respectively to two angles and the included side of the other are congruent, if the given equal parts are arranged in the same order, and symmetrical, if arranged in the opposite order.*

Proof: By §§ 709 and 699 the polar triangles of the given triangles are congruent or symmetrical. Hence, by § 710, the given triangles themselves are congruent or symmetrical.

721. COROLLARY. *State and prove the theorem on trihedral angles which corresponds to the preceding.*

722. THEOREM. *Two spherical triangles having the angles of one equal respectively to the angles of the other are congruent, if the equal angles are arranged in the same order, and symmetrical, if they are arranged in the opposite order.*



Proof: By §§ 709, 691, and 693 the polar triangles of the given triangles are equal or symmetrical. Hence, by § 710, the given triangles themselves are equal or symmetrical.

723. COROLLARY. *State and prove the theorem on trihedral angles which corresponds to the preceding.*

724. Is there a theorem on plane triangles corresponding to that of § 722?

725. **EXERCISES.**

1. If the sides of a spherical triangle are 60°, 80°, 120°, find the angles of its polar triangle.

2. If the angles of a spherical triangle are 72°, 104°, 88°, find the sides of the polar triangle.

3. If a triangle is isosceles, prove that its polar triangle is isosceles.

4. If each side of a spherical triangle is a quadrant, describe its polar triangle.

5. In case each side of spherical triangle ABC is a quadrant, show that the eight triangles formed by drawing the great circles whose poles are A, B, and C are all congruent.

6. Show that for any triangle the construction of the polar triangle gives four pairs of symmetrical triangles.

7. If a triangle ABC is isosceles, show that of the eight triangles of Ex. 5 there are four pairs of congruent triangles.

8. If the triangle ABC is not isosceles, show that of the pairs of triangles proved congruent in Ex. 7 none are now congruent.

9. If the angles of a spherical triangle are 70°, 80°, and 110° respectively, find the sides of each of the eight triangles formed by the polar construction.

10. Is it possible to construct a spherical triangle whose angles are 50°, 60°, 120°?

11. Is it possible to construct a spherical triangle whose angles are 60°, 120°, 150°?

SUGGESTION. Consider the polar triangle of such triangle.

12. Consider the questions on trihedral angles corresponding to the two preceding.

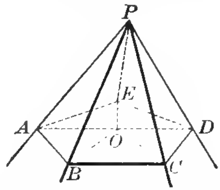
13. If the sides of a spherical triangle are 75°, 95°, and 115° respectively, find the angles of each triangle formed by the polar construction.

14. How can the theorem of § 712 be used to prove that a side of a spherical triangle cannot be as great as a semicircle?

15. If it is given that a spherical triangle is equilateral, can we infer from the theorems thus far proved that its polar triangle is equilateral?

POLYHEDRAL ANGLES AND SPHERICAL POLYGONS.

726. THEOREM. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Proof: Let $ABCDE$ be a polygonal section of the given polyhedral angle. The number of triangles thus formed having P for a vertex is equal to the number of face angles of the polyhedral angle.

Let O be any point in the base, and draw OA, OB, OC, etc.

Then $\angle PBA + \angle PBC > \angle ABC$, and $\angle PCB + \angle PCD > \angle BCD$, and so on. (Why?)

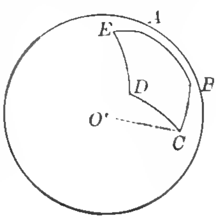
But the sum of the ∠s of the △s OAB, OBC, etc., is equal to the sum of the ∠s of the △s PAB, PBC, etc.

Hence, $\angle APB + \angle BPC + \cdots < \angle AOB + \angle BOC + \cdots$.

But the sum of the ∠s about O is four right angles.

Therefore, the sum of the face angles of the polyhedral angle is less than four right angles.

727. Definition. The section of a sphere made by a convex polyhedral angle whose vertex is the center of the sphere is called a **spherical polygon.**



Since a plane may be passed through the vertex of a polyhedral angle such that the polyhedral angle lies entirely on one side of it, it follows that a spherical polygon lies within one hemisphere.

728. Corollary. *State and prove the theorem on spherical polygons which corresponds to that of* § 726.

729. Theorem. *The sum of the angles of a spherical polygon of n sides is greater than* $2(n-2)$ *right angles and less than* $2n$ *right angles.*

Proof: Divide the polygon into $n-2$ triangles.

Hence, by § 712 the sum of the angles is greater than $2(n-2)$ right angles.

Since the polygon has n angles, and since each angle is less than two right angles, it follows that their sum is less than $2n$ right angles.

730. Corollary. *State and prove the theorem on polyhedral angles which corresponds to the preceding.*

AREAS OF SPHERICAL POLYGONS.

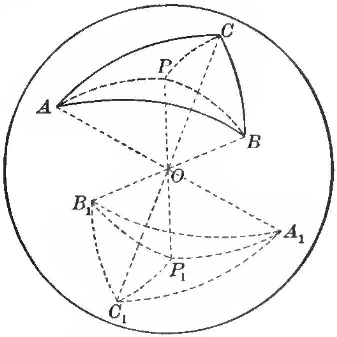
731. Definitions. A spherical polygon divides a sphere into two parts, an **exterior** and an **interior**, so that every path on the sphere passing from one to the other must cross the polygon.

We have seen (§ 727) that any spherical polygon lies within one hemisphere. The interior is that one of the two parts which lies entirely within this hemisphere.

Two spherical polygons are said to **inclose equal areas** or **to be equal** if they are congruent, or if they can be divided into polygonal surfaces which are congruent in pairs.

The terms *spherical triangle*, *spherical polygon*, are sometimes used to refer to the part of the sphere inclosed by these figures. The context will always indicate clearly in which sense they are used.

732. THEOREM. *Two symmetrical spherical triangles are equal.*



Proof: Let ABC and $A'B'C'$ be the given triangles.

Extend the radii OA, OB, OC to meet the sphere in A_1, B_1, C_1, thus forming a triangle symmetrical to $\triangle ABC$ (§ 688), and hence congruent to $\triangle A'B'C'$ (§ 698).

Let P be a pole of the circle through A, B, C. Extend PO to meet the sphere in P_1. Draw $\frown{PA}$, $\frown{PB}$, $\frown{PC}$, and $\frown{P_1A_1}$, $\frown{P_1B_1}$, and $\frown{P_1C_1}$.

Suppose that P lies within $\triangle ABC$.

Now prove that $\triangle PAB \cong \triangle P_1A_1B_1$, $\triangle PAC \cong \triangle P_1A_1C_1$, $\triangle PBC \cong P_1B_1C_1$. Note that these triangles are isosceles.

Hence, show that $\triangle ABC = \triangle A_1B_1C_1$, and therefore $\triangle ABC = \triangle A'B'C'$.

733. Definitions. A **lune** is a figure formed by two great semicircles having the same end-points. The angle between these semicircles is the **angle of the lune.**

A **birectangular spherical triangle** is one having two right angles.

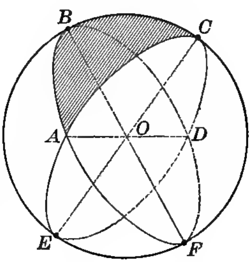
If one angle of a birectangular triangle is 1°, the triangle incloses one of 720 equal parts of the sphere. The area inclosed by such a triangle is called a **spherical degree** and is used as a unit of measure of areas inclosed by spherical polygons.

In a similar manner we define a **spherical minute** and a **spherical second.**

734. THEOREM. *The area inclosed by a lune in terms of spherical degrees is twice the angle of the lune.*

735. Definition. The number of spherical degrees by which the sum of the angles of a spherical triangle exceeds 180° is called the **spherical excess** of the triangle.

736. THEOREM. *The area of a spherical triangle in terms of spherical degrees is equal to its spherical excess.*



Proof : We are to show that area $\triangle ABC = \angle A + \angle B + \angle C - 180°$.

Consider the lunes $ACDB$, $CAEB$, $BCFA$.

We have

$\triangle ABC + \triangle BCD = ACDB = 2\angle A$.

$\triangle ABC + \triangle BAE = CAEB = 2\angle C$.

$\triangle ABC + \triangle CFA = BCFA = 2\angle B$.

Hence, adding, $3\triangle ABC +$ ⚠ BCD, BAE, $CFA = 2(\angle A + \angle B + \angle C)$.

Now ⚠ BCD and AEF are symmetrical and have equal areas.

Hence,

$2\triangle ABC +$ ⚠ ABC, AEF, BAF, $CFA = 2(\angle A + \angle B + \angle C)$.

But ⚠ ABC, AEF, BAF, CFA together constitute a hemisphere or 360 spherical degrees.

Hence, $2\triangle ABC + 360° = 2(\angle A + \angle B + \angle C)$.

Solving, $\triangle ABC = \angle A + \angle B + \angle C - 180°$.

737. Definition. The spherical excess of a spherical polygon is the sum of its angles less $(n-2)180°$, where n is the number of sides of the polygon.

738. THEOREM. *The area of a spherical polygon in terms of spherical degrees is equal to its spherical excess.*

Proof: Join one vertex of the polygon to each non-adjacent vertex, thus forming $n-2$ spherical triangles.

Now prove that the sum of the spherical excesses of these triangles is the spherical excess of the polygon and thus complete the proof.

739. **EXERCISES.**

1. What is the area in spherical degrees of a birectangular triangle one of whose angles is 54°? If one angle is 79° 30′; 106°; 14′; 36″?

2. What is the area inclosed by a lune whose angle is 45°? Note that the lune may be divided into two birectangular triangles. What is the third angle of each?

3. Between what limits is the sum of the angles of a spherical polygon of eight sides?

4. If the sum of the angles of a spherical polygon is 11 right angles, what is known about the number of its sides?

5. If the sum of the angles of a spherical polygon is 14 right angles, what is known about the number of its sides?

6. The sides of a spherical polygon are 85°, 95°, 110°. Find the area of each of the eight triangles formed by the polar construction from this triangle.

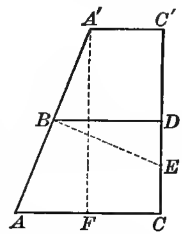
7. The area of a spherical triangle is 71 spherical degrees. One angle is 105°. Of the other two angles one is twice the other. Find all the angles of the triangle.

AREA AND VOLUME OF THE SPHERE.

740. In §§ 731–738 we discussed the areas inclosed by spherical polygons in terms of a unit directly applicable to the sphere, namely, the **spherical degree.**

We now consider the area of the sphere in terms of a plane unit of measure. It is clear that such measurement can be *approximate* only, since no plane segment however small will coincide with the spherical surface. Similarly, if the cube is used as the unit of volume, the measurement of the volume inclosed by a sphere must be *approximate*, since no set of cubes however small can be made exactly to coincide with a sphere.

The two following theorems are needed:



741. THEOREM. *The lateral area of a frustum of a right circular cone is equal to the altitude of the frustum multiplied by the length of a circle whose radius is the perpendicular distance from a point in the axis of the frustum to the middle point of an element.*

Given AA' **an element of the frustum whose middle point is** B^1, CC' **the axis of the frustum,** $BD \perp CC'$ **and** $EB \perp AA'$.

To prove that the lateral area is equal to $2\pi \cdot EB \cdot CC'$.

Proof: By § 619, the lateral area is $2\pi \cdot BD \cdot AA'$.

Hence, we must show that $EB \cdot CC' = BD \cdot AA'$.

To do this, draw $A'F \perp AC$ and show that $\triangle AFA' \sim \triangle EDB$.

742. COROLLARY. *The lateral area of a cone is equal to its altitude times the length of a circle whose radius is the perpendicular from a point in the axis to the middle point of an element.*

743. THEOREM. *Given a fixed line through the vertex of a triangle, but not crossing it. The volume swept out by the triangle as it rotates about the fixed line as an axis is equal to the numerical measure of the area generated by the side opposite the fixed vertex multiplied by one third the altitude upon that side.*

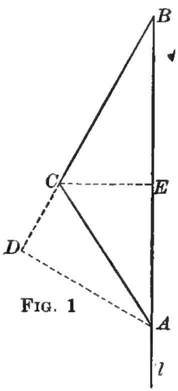


FIG. 1

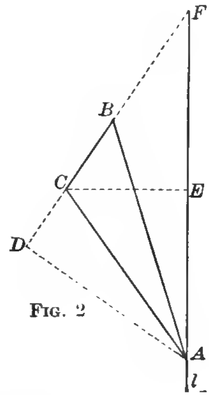


FIG. 2

Given △ *ABC* with vertex *A* in the fixed line *l*.

CASE 1. *When one side as AB lies in line l.* (Fig. 1.)

Draw $AD \perp BC$, or BC produced, and $CE \perp AB$.

Then Vol. $ABC =$ Vol. $AEC +$ Vol. BEC

$$= \frac{\pi}{3}\overline{CE}^2 \cdot AB. \quad \text{(Why?)}$$

But $\overline{CE}^2 \cdot AB = CE \cdot CE \cdot AB = CE \cdot BC \cdot AD.$ (Why?)

And by § 613, $\pi \cdot CE \cdot BC$ is the area swept out by BC.

Hence, Vol. $ABC = \frac{1}{3} AD \cdot$ (area generated by BC).

CASE 2. *When neither AB nor AC lies in line l.* (Fig. 2.)

Produce CB to meet l in F and draw AD and CE as before.

Then Vol. $ABC =$ Vol. $AFC -$ Vol. AFB.

By Case 1 Vol. $AFC = \frac{1}{3} AD \cdot$ (area generated by FC),

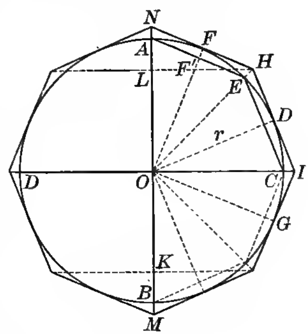
and Vol. $AFB = \frac{1}{3} AD \cdot$ (area generated by FB).

Hence, Vol. $ABC = \frac{1}{3} AD \cdot$ (area generated by $FC - FB$),

or Vol. $ABC = \frac{1}{3} AD \cdot$ (area generated by BC).

Let the student give the proof when the triangle is of other forms, *e.g.* in Case 1 when AD falls *within* the triangle, and in Case 2 when BC is parallel to l.

744. Area of the sphere. About a circle circumscribe a polygon as follows: Construct two diameters AB and CD at right angles to each other and divide each quadrant into an *even* number of parts by points, as D, E, F. At each alternate division point, beginning with the first point D, draw a tangent.



There results a regular polygon with vertices on the diameters AB and CD.

If now we construct another polygon in the same manner by dividing each quadrant into *twice* as many arcs, it will likewise have two vertices on each of the diameters AB and CD, and this may be repeated at pleasure.

Now inscribe a polygon similar to the one just circumscribed by joining the points C and E, E and A, and so on.

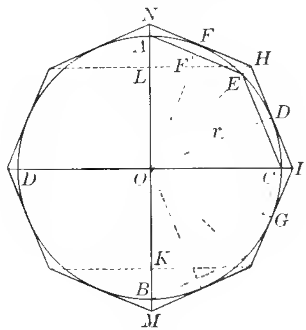
If now the whole figure is made to revolve about AB as an axis, the circle generates a sphere, and the circumscribed and inscribed polygons generate sets of circumscribed and inscribed cones and frustums of cones.

We assume the following:

745. Axiom XXIV. *A sphere has a definite area and incloses a definite volume which are less respectively than the volume and the surface of any circumscribed figure and greater than those of any inscribed convex figure.*

746. THEOREM. *The area of a sphere is $4\pi r^2$.*

Proof: Denote by r the radius of the circle which generates the sphere. That is, in the figure, $r = OF = OD = OG$.



Then by § 741 the lateral areas of the frustums whose axes are OL and OK are $2\pi r \cdot OL$ and $2\pi r \cdot OK$, and by § 742 the lateral areas of the cones whose axes are LN and KM are respectively $2\pi r \cdot LN$ and $2\pi r \cdot KM$.

Hence, the total surface of the whole circumscribed figure is

$$2\pi r \cdot MK + 2\pi r \cdot KO + 2\pi r \cdot OL + 2\pi r \cdot LN,$$

or

$$2\pi r\,(MK + KO + OL + LN) = 2\pi r \cdot MN.$$

If now a polygon of twice the number of sides is constructed, in a manner similar to the above, we obtain another circumscribed figure whose area is $2\pi r \cdot M'N'$ where M' and N' are the two vertices on the line AB.

As this process is repeated indefinitely, the vertices which lie on the line AB may be made to approach as near as we please to A and B, and hence the total surface generated approaches as near as we please to

$$A = 2\,\pi r \cdot AB = 4\,\pi r^2.$$

We now prove that A cannot *differ* from $4\,\pi r^2$.

(*a*) Suppose $A > 4\,\pi r^2$, and let $A = 4\,\pi r^2 + d$. (1)

Let k be the difference between $M'N'$ and AB.

Then the area of the circumscribed figure is

$$2\,\pi r\,(2\,r + k) = 4\,\pi r^2 + 2\,\pi r k.$$

Since k can be made as small as we please, $2\,\pi r k$ can be made smaller than d.

Hence, by Axiom XXIV, the area $> 4\,\pi r^2 + d$, which contradicts the hypothesis (1).

Therefore the area cannot be greater than $4\,\pi r^2$.

(*b*) In the inscribed figure let OF' be the apothem.

Then by § 741 and an argument like that used above, we find that the area of the inscribed figure is

$$2\,\pi \cdot AB \cdot OF' = 4\,\pi r \cdot OF'.$$

By continuing to double the number of sides, OF' may be made to approach as nearly as we please to $OF = r$.

Suppose that $A < 4\,\pi r^2$ and let $A = 4\,\pi r^2 - d'$. (2)

Let k' be the difference between r and OF'.

Then the area of the inscribed figure is

$$4\,\pi r \cdot OF' = 4\,\pi r(r - k') = 4\,\pi r^2 - 4\,\pi r k'.$$

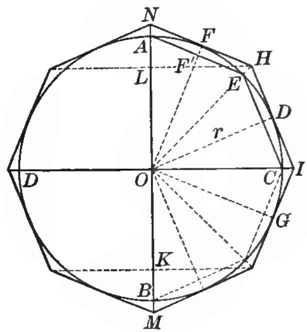
Since k' can be made as small as we please, $4\,\pi r k'$ can be made less than d'. Hence there is some inscribed figure whose area is greater than $4\,\pi r^2 - d'$. Hence by Axiom XXIV, $A > 4\,\pi r^2 - d'$, which contradicts (2).

Therefore the area cannot be less than $4\,\pi r^2$. Since the area is neither less nor greater than $4\,\pi r^2$, it is therefore equal to $4\,\pi r^2$.

747. THEOREM. *The volume of a sphere is $\frac{4}{3}\pi r^3$.*

Proof: From § 743 the volume of the figures formed by rotating the triangles OIH and OHN about AB as an axis

is $$\frac{r}{3}\text{ (area generated by } IH \text{ and } HN).$$



That is, the volume of the whole circumscribed figure is equal to the surface of the figure multiplied by $\frac{r}{3}$.

Now suppose the volume of the sphere not equal to $\frac{4}{3}\pi r^3$, but equal to $\frac{4}{3}\pi r^3 + d$.

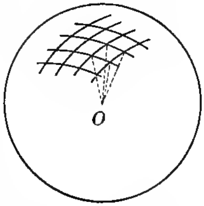
Since by the argument of § 746 we can obtain a circumscribed figure whose surface is as near as we please to $4\pi r^2$, it follows that one can be found whose volume is less than $\frac{4}{3}\pi r^3 + d$.

Hence, the volume cannot be **greater** than $\frac{4}{3}\pi r^3$.

In a similar manner, using the inscribed figures, show that the volume cannot be **less** than $\frac{4}{3}\pi r^3$.

748. The volume of a sphere by inspection. The fact that the volume of a sphere is equal to its area multiplied by one third its radius may be inferred directly from the accompanying figure.

The sphere is covered with a network of small spherical quadrilaterals. If these are taken small enough, they may be regarded as approximately *plane surfaces*.



On this supposition we have a set of pyramids with a common altitude r and the sum of their bases approximately equal to the area of the sphere.

Hence, their combined volume is $\frac{1}{3}r \times$ (area of sphere) or $\frac{1}{3}r \cdot 4\pi r^2$. That is, the volume is $\frac{4}{3}\pi r^3$.

It is clear that, by making these quadrilaterals sufficiently small, this result may be approximated as nearly as we please.

749. EXERCISES.

1. The surface of a polyhedron circumscribed about a sphere of radius 4 inches is 420 square inches. Find its volume.

2. The volume of a polyhedron circumscribed about a sphere of radius 3.5 inches is 450 cubic inches. Find its surface.

3. Given a sphere of radius 6 inches, is there any *upper* limit to the volume of its circumscribed polyhedrons? That is, can polyhedrons be circumscribed having a volume as large as we please?

4. On the same sphere is there any *lower* limit to the volume of its circumscribed polyhedrons?

5. Show that the areas of two spheres are in the same ratio as the squares of their radii or of their diameters.

6. Show that the volumes of two spheres are in the same ratio as the cubes of their radii or of their diameters.

7. Find the area of a sphere whose radius is 8 inches.

8. Find the volume of a sphere whose radius is 10 feet.

9. If the area of a sphere is 227 sq. ft., find its radius.

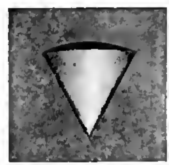
10. If the volume of a sphere is 335 cu. in., find its radius.

11. If the volumes of two spheres are 27 cu. in. and 729 cu. in., compare their radii.

750. Definition. That part of a sphere included between two parallel planes cutting it is called a **zone.** The perpendicular distance between the planes is the **altitude** of the zone.

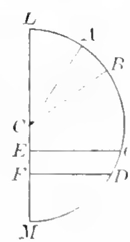
The figure formed by a zone, together with the circular plane-segments cut out by the sphere, is called a **spherical segment,** and the circular plane-segments are its **bases.**

If one of the cutting planes is tangent to the sphere, then the spherical segment and the corresponding zone are said to have but **one base.** The altitude is the perpendicular distance from the base to the tangent plane.



If one nappe of a convex conical surface has its vertex at the center of a sphere, the portion cut off by the sphere, together with the intercepted part of the spherical surface, is called a **spherical cone.**

If two spherical cones have the same axis, one lying within the other, the figure formed by their two lateral surfaces, together with the part of the sphere intercepted between them, is called a **spherical sector.**



If the two cones are right circular cones, they intercept circles on the sphere, and the zone thus included is called the **base** of the spherical sector.

If the accompanying figure be revolved about LM as an axis, then any arc, as GD or MD, generates a zone, the former with two bases, the latter with one.

The figure MDF or $FDGE$ generates a spherical segment, the former with one base, the latter with two.

The figure CAL or CBL generates a spherical cone, CL being the common axis.

The figure CBA generates a spherical sector.

751. Area of a zone. An argument precisely like that of § 746 shows that the area of a zone is

$$A = 2\pi rh$$

where h is the altitude of the zone.

That is, instead of AB, the diameter in case of the sphere, we should have the sum of the altitudes of the frustums circumscribed about the zone equal to h, the altitude of the zone.

752. Volume of a spherical cone. An argument precisely like that of § 747 shows that the volume of a spherical cone is

$$V = \frac{r}{3} \cdot A$$

where A is the area of the zone cut out of the sphere by the cone. Hence, if h is the altitude of this zone, we have

$$V = \frac{r}{3} \cdot 2\pi rh = \frac{2\pi}{3} r^2 h.$$

In like manner the **volume of a spherical sector is**

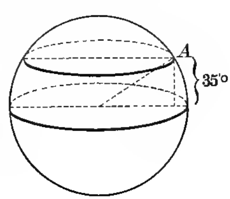
$$V = \frac{2\pi}{3} r^2 h.$$

753. **EXERCISES.**

1. The radius of a sphere is 6 and the altitude of a zone is 5. Find the area of the sphere and of the zone.

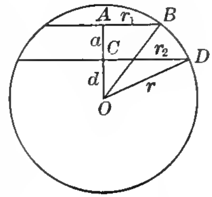
2. The area of a zone is 36π and its altitude 4. Find the radius of the sphere.

3. If the radius of a sphere is r, find the perpendicular distance from a point A to the plane of a given great circle if the distance on the sphere from A to the great circle is 35°.



4. Solve the preceding problem if the distance on the sphere from A to the given great circle is $23\frac{1}{2}$°. Also if the distance is $66\frac{1}{2}$°. (Use the tables, page 335.)

754. PROBLEM. *To find the volume of a spherical segment.*



Solution. Let r be the radius of the sphere, and r_1 and r_2 the radii of the bases of the segment, a the altitude of the zone, and let the segment be generated by revolving the figure $ACDB$ about AO as an axis.

We have Vol. generated by $ODB = \frac{2\pi}{3} r^2 a.$ § 752

Vol. generated by $OAB = \frac{\pi}{3} r_1^2 (a+d).$ (Why?)

Vol. generated by $OCD = \frac{\pi}{3} r_2^2 d.$ (Why?)

Hence, $V = \frac{2\pi}{3} r^2 a + \frac{\pi}{3} r_1^2 (a+d) - \frac{\pi}{3} r_2^2 d$

$$= \frac{\pi}{3} [2 r^2 a + r_1^2 a + d(r_1^2 - r_2^2)]. \quad (1)$$

From $r^2 = r_2^2 + d^2$ and $r^2 = r_1^2 + (a+d)^2$

we obtain $$d = \frac{r_2^2 - r_1^2 - a^2}{2a}. \quad (2)$$

Substituting this value of d in $r^2 = r_2^2 + d^2$,

we get $$r^2 = \frac{r_2^4 + r_1^4 + a^4 - 2 r_1^2 r_2^2 + 2 a^2 r_2^2 + 2 a^2 r_1^2}{4 a^2}. \quad (3)$$

Substituting (2) and (3) in (1) and reducing, we have

$$V = \frac{\pi a}{2}(r_1^2 + r_2^2) + \frac{\pi}{6} a^3 = \frac{a}{2}(\pi r_1^2 + \pi r_2^2) + \frac{4}{3}\pi\left(\frac{a}{2}\right)^3.$$

Hence, we have the result:

THEOREM. *The volume of a spherical segment is numerically equal to the sum of the areas of its bases multiplied by half its altitude, plus the volume of a sphere whose diameter is equal to the altitude of the segment.*

SUMMARY OF CHAPTER XII.

1. Collect the theorems involving plane sections of the sphere.

2. Collect the definitions involving plane sections of the sphere.

3. State some of the principal exercises and problems connected with plane sections of the sphere.

4. Collect the definitions on trihedral angles and spherical triangles.

5. Arrange in parallel columns the pairs of theorems on trihedral angles and spherical triangles, which are proved without the use of polar triangles.

6. Collect the definitions on polar triangles.

7. Collect the theorems on polar triangles.

8. Continue the lists started in Ex. 5, adding the theorems proved by means of polar triangles.

9. Make a list of the definitions involving polyhedral angles and spherical polygons.

10. Collect the theorems involving polyhedral angles and spherical polygons.

11. Collect the theorems on the areas of spherical triangles and polygons.

12. Give the definitions and axioms pertaining to the area and volume of the sphere.

13. State all the theorems pertaining to the area and volume of the sphere.

14. Give the definitions and theorems pertaining to spherical figures, such as zones, cones, sectors, segments.

15. Collect all the mensuration formulas in this chapter.

16. Collect all the mensuration formulas of Solid Geometry.

17. Which of these formulas are illustrations of the general theorem that the surfaces of similar solids are in the same ratio as the squares of their corresponding linear dimensions? Which ones are special instances of the theorem that the volumes of similar solids are in the same ratio as the cubes of corresponding linear dimensions?

18. Describe some of the most important applications in this chapter. Return to this question after studying the following list.

PROBLEMS AND APPLICATIONS.

1. What part of the earth's surface lies in the torrid zone? What part in the temperate zones? What part in the frigid zones? The parallels $22\frac{1}{2}°$ north and south of the equator are the boundaries of the torrid zone, and the parallels $67\frac{1}{2}°$ north and south are the boundaries of the frigid zones.

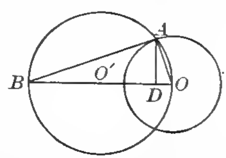
2. Find to four places of decimals the area of a sphere circumscribed about a cube whose edge is 6. No square root is to be approximated in the process, and the value of π is taken 3.1416.

3. Can the volume of the sphere in the preceding exercise be approximated without finding a square root? Find the volume.

4. Find the area of a sphere circumscribed about a rectangular parallelopiped whose sides are a, b, and c.

5. Find the volume of the sphere in the preceding example.

6. A fixed sphere with center O has its center on another sphere with center O'. Show that the area of the part of O' which lies within O is equal to the area of a great circle of the sphere O, provided the radius of the sphere O is not greater than the diameter of O'.



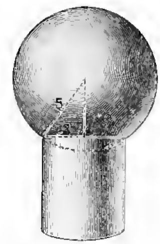
SUGGESTION. Let the figure represent a cross section through the centers of the two spheres. Connect O with A and B. Then $\overline{OA}^2 = OB \times OD$. But OD is the altitude of the zone of O', which lies within O, and OB is the diameter of the sphere O'. Hence, the area of the zone is $\pi\, BO \times OD = \pi\, \overline{OA}^2$.

7. Given a solid sphere of radius 12 inches. A cylindrical hole is bored through it so that the axis of the cylinder passes through the center of the sphere. What area of the sphere is removed if the diameter of the hole is 4 inches?

8. Find the volume removed from the sphere by the process described in the preceding exercise.

9. Through a sphere of radius r a hole of radius r_1 is bored so that the axis of the cylinder cut out passes through the center of the sphere. Find the volume of the sphere which remains.

10. A cylindrical post 6 in. in diameter is surmounted by a part of a sphere 10 in. in diameter, as shown in the figure. Find the surface and the volume of the part of the sphere used.



11. A cylindrical post 5 ft. long and 4 in. in diameter is turned so as to leave on it a part of a sphere 7 in. in diameter and having its center in the axis of the post. Find the volume of the whole post.

12. Find the volume of a spherical shell 1 inch thick if its outer diameter is 8 inches.

13. What is the diameter of a spherical shell an inch thick whose volume is half that inclosed within a sphere of the same diameter?

14. Compare the volumes and areas of a sphere and the circumscribed cylinder.

15. In a sphere of radius r inscribe a cylinder whose altitude is equal to its diameter. Compare its volume and area with those of the sphere.

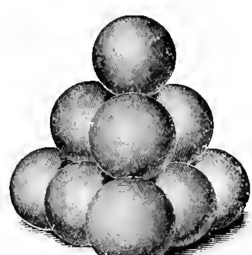
16. In a sphere inscribe a cylinder whose altitude is n times its diameter. Compare its area and volume with those of the sphere.

17. Compute the length of the diagonal of a cube in terms of its side, and also the length of the side in terms of half the diagonal.

18. Express the volume of a cube inscribed in a sphere in terms of the radius of the sphere.

19. Three spheres each of radius r are placed on a plane so that each is tangent to the other two. A fourth sphere of radius r is placed on top of them. Find the distance from the plane to the top of the upper sphere.

20. Find the vertical distance from the floor to the top of a triangular pile of spherical cannon balls, each of radius 5 inches, if there are 3 layers in the pile.



21. Solve a problem like the preceding if there are 16 layers in the pile, each shot of radius r.

22. Solve a problem like the preceding for a square pile of shot of 12 layers.

23. Solve Exercises 21 and 22 if there are n layers in each pile.

CHAPTER XIII.

VARIABLE GEOMETRIC MAGNITUDES.

GRAPHIC REPRESENTATION.

755. It is often useful to think of a geometric figure as **continuously varying** in size, or in shape, or both.

E.g. if a parallelopiped has a fixed base, say 24 square inches, but an altitude which varies continuously from 3 inches to 5 inches, then the volume varies continuously from $3 \cdot 24 = 72$ to $5 \cdot 24 = 120$ cubic inches.

We may even think of the altitude as starting at zero inches and increasing continuously, in which case the volume also starts at zero.

From this point of view many theorems may be represented **graphically**. The graph has the advantage of exhibiting the theorem at once for all its particular cases.

For a description of graphic representation, see Chapter V of the authors' High School Algebra, Elementary Course.

756. THEOREM. *The volumes of two parallelopipeds having equal bases are in the same ratio as their altitudes.*

Graphic Representation. By § 541 we have

$$Volume = base \times altitude.$$

Consider parallelopipeds each with a base whose area is A, and with altitudes h_1, h_2, h_3, and corresponding volumes V_1, V_2, V_3.

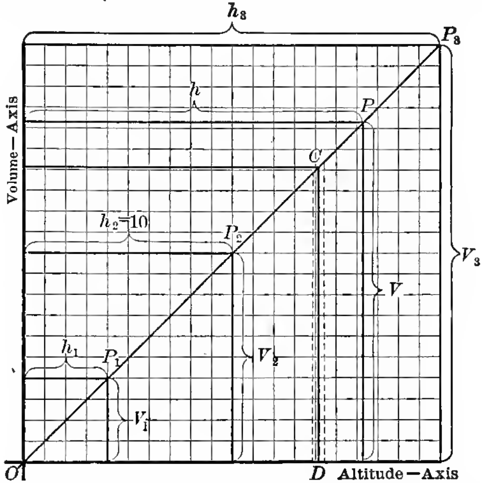
Then $$\frac{V_1}{V_2} = \frac{Ah_1}{Ah_2} = \frac{h_1}{h_2}, \quad \frac{V_2}{V_3} = \frac{Ah_2}{Ah_3} = \frac{h_2}{h_3}, \text{ etc.} \qquad (1)$$

Consider the case where $A = 10$. Let one horizontal space represent one unit of altitude, and one vertical space ten units of volume.

Thus, the point P_2 has the ordinate $V = 10$ vertical units (representing 100 units of volume), and the abscissa $h_2 = 10$ horizontal units.

Similarly, locate the points P_1 and P_3.

Using equations (1), show that O, P_1, P_2, P_3 lie in a straight line.



If we suppose that while the base of the parallelopiped remains fixed, the altitude varies continuously through all values from $h_2 = 10$ to $h_3 = 20 = 2 \times 10$, then the volume varies continuously from $10 \times 10 = 100$ to $10 \times 20 = 200$.

Using any altitude as an abscissa and the corresponding volume as an ordinate, show, as in § 368, that the point so determined lies on the line OP_2P_3.

757. The preceding theorem may also be stated:

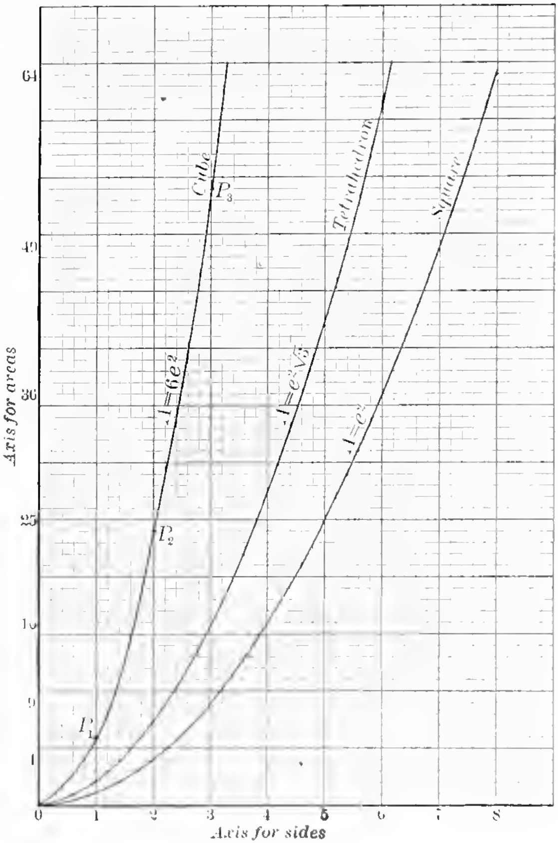
The volume of a parallelopiped with a fixed base varies directly as its altitude.

This means that if V and h are the varying volume and altitude, and V_1 and h_1 the volume and altitude at any given instant, then

$$\frac{V}{V_1} = \frac{h}{h_1} \text{ or } V = \frac{V_1}{h_1} h, \text{ or } V = kh, \text{ where } k \text{ is the } \textit{fixed ratio } \frac{V_1}{h_1}.$$

The graph representing the relation of two variables when one varies *directly* as the other is a straight line.

758. Problem. *To represent graphically the relation between the area and the edge of a cube as the edge varies continuously.*



Solution. On the horizontal axis lay off segments equal to the various values of the edge e, and on the vertical axis lay off segments equal to the corresponding areas A.

If one vertical space represents one unit of area, then the points P_1, P_2, P_3, etc., lie on the steep curve.

The student should locate many more points between those here shown, and see that a **smooth** curve can be drawn through them all.

759. The graph of the relation between two variables, one of which varies as the square of the other, is always similar to the one just given.

The lowest curve represents the relation between the side of a square and its area, and the third curve represents the relation between the edge of a regular tetrahedron and its total surface.

In each of these cases the area is said to **vary** as the square of the side of the figure. These are special cases of the theorem, § 654, that the surfaces of similar figures are in the same ratio as the squares of their corresponding near dimensions. This theorem may also be stated:

The areas of similar figures vary as the squares of their linear dimensions.

760. EXERCISES.

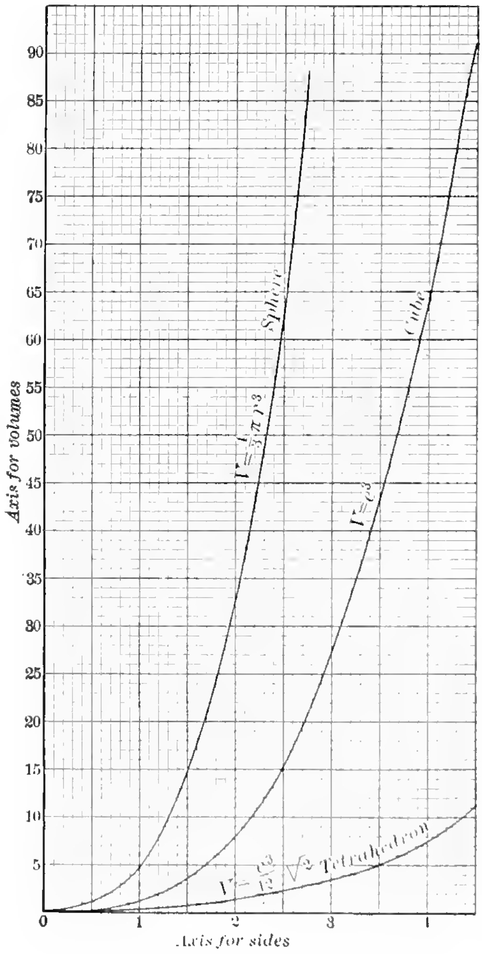
1. From the graph find approximately the area of a regular tetrahedron whose side is 2.5; one whose side is 3.7; 4.3.

2. Find approximately from the graph the edges of cubes whose total areas are 25 square units; 40 square units; 56 square units.

3. Find approximately from the graph the edges of regular tetrahedrons whose total areas are 42 square inches; 17 square inches; 55 square inches.

4. Construct a graph showing the relation between the edge and the total area of a regular octahedron.

761. PROBLEM. *To construct a graph showing the relation between the edge of a cube and its volume.*



Solution. Taking ten horizontal spaces to represent one unit of length of side of the cube and one vertical unit to represent one unit of volume, construct the middle graph shown on the page opposite.

762. The lower graph represents the relation between the length e of the edge of a regular tetrahedron and its volume V; and the upper graph represents the relation between the radius and the volume of a sphere.

In each of these cases the volume is said to vary as the cube of the given linear dimension.

These are special cases of the theorem, § 654, that the volumes of similar solids are in the same ratio as the cube of their ratio of similitude.

This theorem may now be stated:

The volumes of similar solids vary as the cubes of their linear dimensions.

763. EXERCISES.

1. From the graph read off approximately the cubes of 1.3; 2.4; 3.7.

2. From the same graph read off approximately the cube roots of 17; 46; 54; 86.

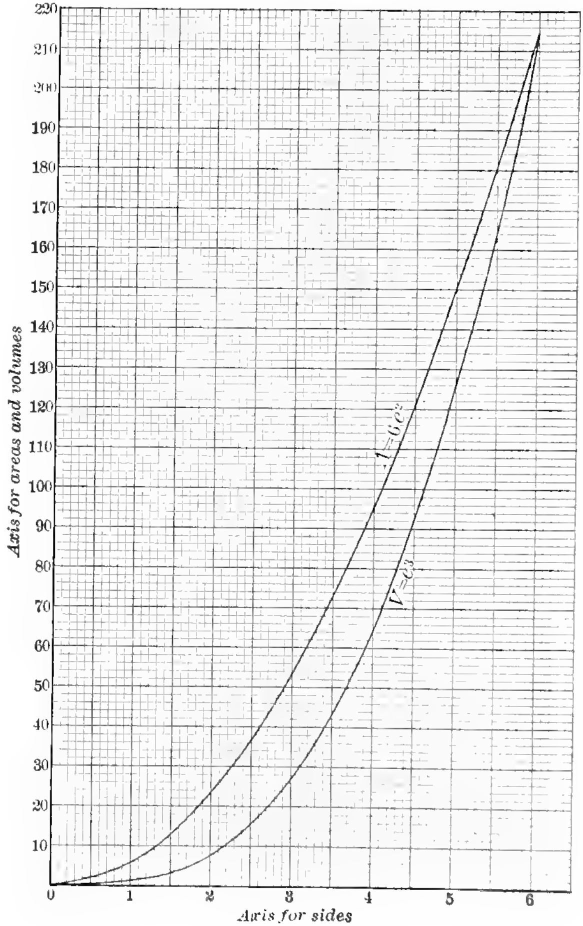
3. Find approximately by means of the graph the volume of a sphere whose radius is 1.5; 2.3; 2.7.

4. What is the radius of a sphere whose volume is 26; 52; 71; 80?

5. Construct a graph giving the relation between the volume and the length of a side of regular octahedrons.

6. By means of the graph just constructed find the volume of a regular octahedron whose side is 1.3; 2.7; 3.6.

7. From the graph constructed under Ex. 5 read off approximately the lengths of an edge of a regular octahedron whose volume is 18; also of one whose volume is 46.



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6
Axis for areas and volumes
Axis for sides
$A=6e^2$
$V=e^3$

764. The graph on the opposite page exhibits the variation of the area and the volume of the cube. We notice that for small values of e the numerical value of the area is greater than that of the volume. For $e = 6$ these values are equal, and for e greater than 6, the numerical value of the volume is greater than that of the area.

This is an instance of the general fact that if a solid figure increases in size but remains similar to its first form, then after a certain point its volume increases more rapidly than its area or the area of any cross section.

765. The fact that one variable y varies as the square or the cube of another variable x is expressed algebraically by the equations $y = kx^2$ and $y = kx^3$, respectively, where k is a constant number to be determined in any particular case.

If the value of k is known, the relations represented by these equations may be shown by a graph like those on the preceding pages.

The equation $y = kx^3$ may be plotted by multiplying the ordinate of each point of the graph $V = e^3$ (see page opposite) by the value of k.

SUMMARY OF CHAPTER XIII.

1. Give a brief summary of the graphic representation process in general and in particular, as applied to geometrical figures.

2. Make a list of the theorems which are represented graphically in this chapter.

3. Show in what respects such a representation has advantages over the algebraic representation.

4. State as formulas the important laws of variation for geometric magnitudes discussed in this chapter.

5. Make a collection of important applications given in this chapter, including those which occur in the following set.

PROBLEMS AND APPLICATIONS.

1. Assuming that the weights of schoolboys vary as the cubes of their heights, construct a graph representing the relation between their heights and weights, if a boy 5 feet 8 inches tall weighs 130 pounds.

SUGGESTION. If w represents the number of pounds in weight and h the number of feet in height, $w = kh^3$. From $w = 130$, when $h = 5\frac{2}{3}$, we have $k = .714$. For the purpose of the graph, $k = .7$ is accurate enough. Hence, we obtain the required graph by multiplying each ordinate of the graph of $V = e^3$ by .7.

2. From the graph constructed in the preceding example find the weight of a boy 5 feet tall; one 5 feet 4 inches; one 5 feet 6 inches. Compare with the weights of boys in your class.

3. If a man 6 feet tall weighs 185 pounds, construct a graph representing the weights of men of similar build and of various different heights.

4. If steamships are of the same shape, their tonnages vary as the cubes of their lengths. The *Mauretania* is 790 feet long, with a net tonnage of 32,500. Construct a graph representing the tonnage of ships of the same shape, and of various different lengths.

Other ships which at one time or another have held ocean records are: the *Deutschland*, length 686 ft. and tonnage 16,500; the *Kaiser Wilhelm der Grosse*, length 648 ft. and tonnage 14,300; the *Lucania*, length 625 ft. and tonnage 13,000 (nearly); and the *Etruria*, length 520 ft. and tonnage 8000. By means of this graph decide whether or not these boats have greater or less tonnage than the *Mauretania* as compared with their lengths.

NOTE. The following more general problems further illustrate the wide range of application of the fundamental theorem that the surfaces and the volumes of similar solids vary respectively as the squares and the cubes of their corresponding linear dimensions. These problems need not be solved by means of graphs.

5. Raindrops as they start to fall are extremely small. In the course of their descent a great many unite to form larger and larger drops. If 1000 such drops unite into one, what is the ratio of the surface of the large drop to the sum of the surfaces of the small drops?

6. Consider two machines similar in shape and of heights h_1 and h_2. Since the tensile strength of corresponding parts varies as the areas of their cross sections, it follows that the tensile strengths of corresponding parts are in the ratio $h_1^2 : h_2^2$. But the total volumes of the machines vary as the cubes of the heights; that is, the total weights are in the ratio $h_1^3 : h_2^3$. Does this fact offer any hindrance to the building of machines indefinitely large?

7. The strength of a muscle varies as its cross-section area, which in turn varies as the square of the height or length of an animal, while the weight of the animal varies as the cube of its height or length. Use these facts to explain the greater agility of small animals. For example, compare the rabbit and the elephant.

8. Assuming the velocities the same, the amounts of water flowing through pipes vary directly as their cross-section areas. How many pipes, each 4 in. in diameter, will carry as much water as one pipe 72 in. in diameter?

9. What must be the diameter of a cylindrical conduit which will carry enough water to supply ten circular intakes each eight feet in diameter?

10. A water reservoir, including its feed pipes, is replaced by another, each of whose linear dimensions is twice the corresponding dimension of the first. If the velocity of the water in the feed pipes of the new system is the same as that in the old, will it take more or less time to fill the new reservoir than it did the old? What is the ratio of the new time to the old?

11. If two engine plants are exactly similar in shape, but each linear dimension in one is three times the corresponding dimension of the other, and if the steam in the feed pipes flows with the same velocity in both, compare the speeds of the engines.

12. If two men, one 5 ft. 6 in. and the other 6 ft. 2 in. in height, are similar in structure in every respect, how much faster must the blood flow in the larger person in order that the body tissues of both shall be supplied equally well?

Suggestion. Note that the amount of tissue to be supplied varies as the cube of the height, while the cross-section area of the arteries varies as the square of the height.

CHAPTER XIV.

THEORY OF LIMITS.

GENERAL PRINCIPLES.

766. In the Plane Geometry it was found that there are segments which have no common unit of measure, that is, which are **incommensurable**, and that the ratio of the lengths of two such segments could be expressed only approximately by means of integers and ordinary fractions. Other incommensurables occurred in dealing with the length of the circle and the area inclosed by it.

In Chapters III, IV, and V we confined ourselves to an informal first treatment of these incommensurable ratios, tacitly assuming their existence and **computing them approximately**. In Chapter VII their existence was explicitly assumed, and certain theorems proved rigorously on the basis of these assumptions. In the Solid Geometry the areas and volumes of the cylinder, cone, and sphere have been treated according to this latter method.

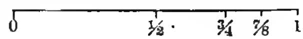
767. In returning to this subject once more we fix our attention on the **incommensurable ratios themselves**, and the method of determining them, rather than on the **process of approach** and the practical computation based on it. We have already used symbols such as $\sqrt{2}$ to represent the ratio of the lengths of incommensurable segments.

In general, the ratio between any two incommensurable geometric magnitudes of the same kind may be represented by what is called an **irrational number**; that is, a number which is *neither an integer nor a quotient of two integers.*

768. The following method for determining irrational numbers is, for simplicity, applied first to the integer 1.

Throughout this discussion the words "point on a line" and "number" will be used interchangeably.

In a straight line mark a certain point 0 (zero), and one unit to the right of it mark another point 1. Also lay off points such that their distances from 0 are $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, ⋯.



If this sequence of points is carried ever so far, it will never reach the point 1. If, however, we select a point k to the left of 1, no matter how near it, we may always go far enough along this sequence to reach points between k and 1.

769. The point 1 has two definite relations to this sequence.

(*a*) *Every point of the sequence is to the left of* 1.

(*b*) *For any fixed point* K *to the left of* 1 *there are points of the sequence between* K *and* 1.

770. We now note that 1 is the *only* point on the whole line such that both (*a*) and (*b*) are true of it. For every point to the right of 1 (*a*) is true, but (*b*) is not. For every point to the left of 1 (*b*) is true, but (*a*) is not.

It follows therefore that, while the points of the sequence merely *approach* the point 1, the sequence, *taken as a whole, serves to determine that point* just as definitely as if the numeral 1 itself were used to indicate the point.

771. The sequence 3, $2\frac{1}{2}$, $2\frac{1}{4}$, $2\frac{1}{8}$, ⋯ is a **decreasing sequence**, and the number 2 sustains relations to it similar to those described above. That is,

(*a′*) *Every point of the sequence is to the right of* 2.

(*b′*) *For every point* K *to the right of* 2 *there are points of the sequence between* K *and* 2.

772. Definitions. An endless sequence of the sort just described is called an **infinite sequence**.

Not every infinite sequence serves to single out a definite point in the manner shown above. Thus the sequence 1, 2, 3, 4, ... fails to do so, because its terms grow large beyond all bound. Such sequences are said to be **unbounded**, while the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ is **bounded**.

Again, the sequence 1, 2, 1, 2, 1, ... fails to single out a definite point. This sequence is said to be **oscillating**, since its terms increase, then decrease, then increase, etc., while $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ is **non-oscillating**.

773. The number 1 is said to be the **least upper bound** of the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$. That is, 1 is the *smallest number* beyond which the sequence does not go. 1 is also said to be the **limit of the sequence**.

Similarly, 2 is the **greatest lower bound** or the **limit** of the sequence $2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{8}, \ldots$; that is, 2 is the *greatest number* such that the sequence contains no number less than it.

774. Axiom XXV. *Every bounded increasing sequence has a least upper bound, and every bounded decreasing sequence has a greatest lower bound.*

This axiom may also be stated:

Every bounded increasing or decreasing sequence has a limit.

This axiom simply means that every such sequence singles out a definite number in the manner stated.

Thus, if we attempt to approximate the square root of 2, we obtain a sequence 1, 1.4, 1.41, 1.411, 1.4142 ..., having for its limit a *definite number* represented by $\sqrt{2}$, which corresponds to the length of the diagonal of a square whose side is unity.

775. If two sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are equal term by term, that is, if $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3, \ldots$ then they are one and the same sequence and hence by Ax. XXV they determine the same number.

The same number may be defined by two different sequences.

Thus each of the sequences, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ and $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$, has 1 as its limit.

We notice, however, that no matter what definite number we select in either of these sequences, there is a number in the other greater than it.

776. THEOREM. *Two increasing bounded sequences define the same number as their limit if neither sequence contains a number greater than every number of the other.*

Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ denote two infinite sequences, with limits A and B, such that there is no a greater than every b, and no b greater than every a.

To prove that $A = B$.

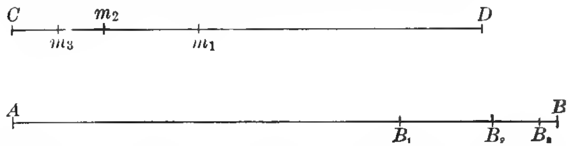
Proof: Suppose that A is not equal to B and that A is less than B. Then there must be numbers of the sequence $b_1, b_2, b_3, \cdots$ greater than A, since by § 769 there are numbers of $b_1, b_2, b_3, \cdots$ greater than any fixed number whatever which is less than B, which contradicts the hypothesis of the theorem.

In like manner we show that B is not less than A. Hence $A = B$.

777. THEOREM. *Two decreasing sequences define the same number as their limit if neither sequence contains a number less than every number of the other.*

APPLICATION OF LIMITS TO GEOMETRY.

778. PROBLEM. *On a given segment* AB *to lay off a sequence of points* B_1, B_2, B_3, *of which* B *is the limit, such that each of the segments* AB_1, AB_2, AB_3, $\cdots$ *is commensurable with a given segment* CD.



Solution. Using m_1, an exact divisor of CD, as a unit of measure, lay off on AB a segment AB_1, such that the remainder B_1B is less than m_1. Then CD and AB_1 are commensurable.

Using a unit m_2, likewise a divisor of CD, and less than B_1B, lay off AB_2 such that B_2B is less than m_2. Then $AB_2 > AB_1$, and CD and AB_2 are commensurable.

Continuing in this manner, using as units of measure segments m_3, m_4, $\cdots$, each an exact divisor of CD and each less than B_3B, B_4B, $\cdots$ respectively, we obtain a sequence of segments AB_1, AB_2, AB_3, $\cdots$, each greater than the preceding and each commensurable with CD.

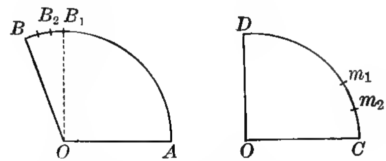
If the units m_1, m_2, m_3, $\cdots$ are so selected that they approach zero as a limit, it follows that B is the limit of the sequence B_1, B_2, B_3, $\cdots$.

If a different sequence of divisors of CD, as m_1', m_2', m_3' $\cdots$, is used, we obtain a sequence B_1', B_2', B_3' $\cdots$, likewise satisfying the conditions of the problem.

We note in regard to any two such sequences B_1, B_2, B_3, $\cdots$ and B_1', B_2', B_3', $\cdots$ determined as above, that they are both *increasing* and that each is such that *no* point of it is to the right of *every* point of the other.

Hence, by § 776 any two such sequences have the same limit B.

779. If two arcs, AB and CD, of the same circle or of equal circles are given, then, in the same manner as above, points B_1, B_2, B_3, $\cdots$ may be constructed on arc AB, forming a sequence whose limit is B, such that the arc CD is commensurable with every arc of the sequence AB_1, AB_2, AB_3 $\cdots$.



780. Definitions. If B_1, B_2, B_3, $\cdots$ is a sequence of points on the segment AB having the limit B, then the segment AB is said to **be the limit of the segments** AB_1, AB_2, AB_3, $\cdots$.

781. The **ratio of two commensurable segments** has been defined as the quotient of their numerical lengths.

The **ratio of two incommensurable segments** has not been explicitly defined, but it is now possible to do so in terms of the **limit of a sequence.**

Consider two incommensurable segments AB and CD. Let a_1, a_2, a_3, $\cdots$ be the lengths of the segments each commensurable with CD, forming a sequence whose limit is the segment AB, and let b be the length of the segment CD.

Then $\frac{a_1}{b}$, $\frac{a_2}{b}$, $\frac{a_3}{b}$, $\cdots$ is an increasing bounded sequence having a limit which we call R.

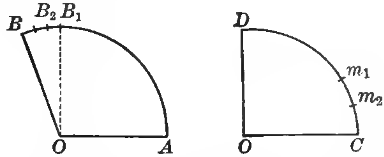
If a_1', a_2', a_3', $\cdots$ are the lengths of another sequence of segments whose limit is AB, the sequence $\frac{a_1'}{b}$, $\frac{a_2'}{b}$, $\frac{a_3'}{b}$, $\cdots$ is another increasing bounded sequence with limit R'.

By § 776 we now know that $R = R'$.

This number R is defined as **the ratio of the segments** AB and CD.

782. The application of the theory of limits to geometry consists chiefly in showing that two numbers are equal because they are the limit of the same sequence, or of sequences having the property stated in the theorem of § 776. In the following paragraphs the applications are made to the chief cases both in plane and in solid geometry.

783. THEOREM. *In the same circle or in equal circles the ratio of two central angles is the same as the ratio of their intercepted arcs.*



Proof: In case the arcs are commensurable the proof is obvious. See § 413.

If the arcs AB and CD are not commensurable, let AB_1, AB_2, AB_3, ⋯ be a sequence of arcs whose limit is AB, each arc being commensurable with the arc CD.

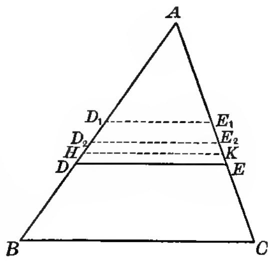
Then the sequence $\frac{AB_1}{CD}, \frac{AB_2}{CD}, \frac{AB_3}{CD}, \cdots$ has a limit R which, by definition, is the ratio of the arcs AB and CD.

Similarly the sequence $\frac{\angle AOB_1}{\angle COD}, \frac{\angle AOB_2}{\angle COD}, \frac{\angle AOB_3}{\angle COD}, \cdots$ has a limit R' which, by definition, is the ratio of $\angle AOB$ and $\angle COD$.

Since $$\frac{AB_1}{CD} = \frac{\angle AOB_1}{\angle COD}, \frac{AB_2}{CD} = \frac{\angle AOB_2}{\angle COD}, \cdots$$

these two sequences are identical and hence define the same limit. Therefore it follows that $R = R'$.

784. THEOREM. *A line parallel to the base of a triangle, and meeting the other two sides, divides them in the same ratio.*



Given the $\triangle ABC$ with $DE \parallel BC$ and cutting AB and AC.

To prove that $\frac{AD}{AB} = \frac{AE}{AC}$.

Proof: Consider the case when AD and AB are incommensurable.

Let $D_1, D_2, D_3 \cdots$ be a sequence on AB whose limit is D. Through these points draw parallels to BC, meeting AC in $E_1, E_2, E_3, \cdots$.

Then E is the limit of the sequence $E_1, E_2, E_3, \cdots$.

For suppose it is not, and that there is a point K on AE such that there is no point of $E_1, E_2, E_3, \cdots$ between K and E. Then draw a line parallel to BC through K, meeting AB in H.

But there are points of the sequence D_1, D_2, D_3, *between* H and D, and hence points of the sequence $E_1, E_2, E_3, \cdots$ *between* K and E, which shows that E is the limit of $E_1, E_2, E_3, \cdots$.

Now $\frac{AD_1}{AB} = \frac{AE_1}{AC}, \frac{AD_2}{AB} = \frac{AE_2}{AC}, \frac{AD_3}{AB} = \frac{AE_3}{AC}, \cdots$.

Hence, the two sequences,

$$\frac{AD_1}{AB}, \frac{AD_2}{AB}, \frac{AD_3}{AB}, \cdots \text{ and } \frac{AE_1}{AC}, \frac{AE_2}{AC}, \frac{AE_3}{AC}, \cdots,$$

are identical and define the same limit;

that is, $$\frac{AD}{AB} = \frac{AE}{AC}.$$

785. Definitions. If $a_1, a_2, a_3, \cdots$ is a sequence with limit a, then $ka_1, ka_2, ka_3, \cdots$ is a sequence with limit ka.

If $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ are two sequences with limits a and b respectively, then ab is the limit of the sequence $a_1b_1, a_2b_2, a_3b_3, \cdots$.

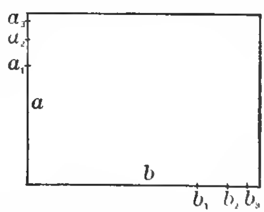
Similarly if $a_1, a_2, a_3, \cdots, b_1, b_2, b_3, \cdots$ and $c_1, c_2, c_3, \cdots$ are sequences with limits a, b, c, then abc is the limit of the sequence $a_1b_1c_1, a_2b_2c_2, a_3b_3c_3, \cdots$.

For a complete treatment it would be necessary to show that these definitions of multiplication of irrational numbers are consistent with the rest of arithmetic and also that these new sequences are such as to determine definite limits. That, however, is beyond the scope of this book.

If the sides of a rectangle are incommensurable with the unit segment, we **define its area** as follows:

Let $a_1, a_2, a_3, \cdots$ be a sequence of rational numbers whose limit is the altitude a, and let $b_1, b_2, b_3, \cdots$ be a sequence whose limit is the base b.

Then the **area of the rectangle is the limit of the sequence** $a_1b_1, a_2b_2, a_3b_3, \cdots$.

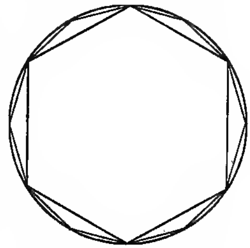


But by definition the limit of $a_1b_1, a_2b_2, a_3b_3, \cdots$ is the product ab. Hence we have —

786. Theorem. *The area of a rectangle is the product of its base and altitude.*

787. Definition. In a circle inscribe a sequence $P_1, P_2, P_3, \cdots$ of regular polygons, each having twice the number of sides of the one preceding it.

Let the perimeters of these polygons be $p_1, p_2, p_3, \cdots$ and their areas $A_1, A_2, A_3, \cdots$. Then the length c of the circle is the limit of the sequence $p_1, p_2, p_3, \cdots$ and its area A is the limit of $A_1, A_2, A_3, \cdots$.



The sequence of polygons thus inscribed is called an **approximating** sequence of polygons.

That these sequences are *increasing* and *bounded* is obvious from the figure.

788. THEOREM. *The lengths of two circles are in the same ratio as their radii, and their areas are in the same ratio as the squares of their radii.*

Proof: Let the radii of the circles whose centers are O and O' be r and r'. Denote $\frac{r'}{r}$ by k. Then $r' = kr$.

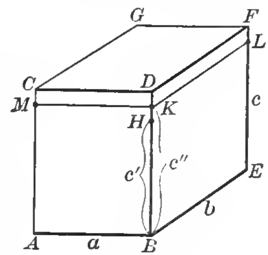
Inscribe in O an approximating sequence of polygons with perimeters $p_1, p_2, p_3, \cdots$ and areas $A_1, A_2, A_3, \cdots$. In O' inscribe a sequence of similar polygons. By §§ 347, 348, the perimeters of the latter are $kp_1, kp_2, kp_3, \cdots$ and their areas are $k^2A_1, k^2A_2, k^2A_3, \cdots$.

By § 785, if the limit of $p_1, p_2, p_3, \cdots$ is c and the limit of $A_1, A_2, A_3, \cdots$ is A, then the limits of $kp_1, kp_2, kp_3, \cdots$, and $k^2A_1, k^2A_2, k^2A_3, \cdots$ are kc and k^2A, respectively.

That is, the ratio of the lengths of the circles is $\frac{kc}{c} = k = \frac{r'}{r}$, and the ratio of their areas is $\frac{k^2A}{A} = k^2 = \frac{r'^2}{r^2}$.

If regular circumscribed polygons are used, we obtain *decreasing* sequences both for the perimeters and the areas. That the limits of these sequences are identical with those obtained above is a direct consequence. See § 417.

789. Definition. If the three concurrent edges a, b, c of a rectangular parallelopiped are incommensurable with the unit segment, the volume inclosed is defined as follows:



Let $a_1, a_2, a_3, \cdots$ be a sequence of rational numbers whose limit is the side a. Let $b_1, b_2, b_3, \cdots$ and $c_1, c_2, c_3, \cdots$, be similar sequences whose limits are respectively the dimensions b and c.

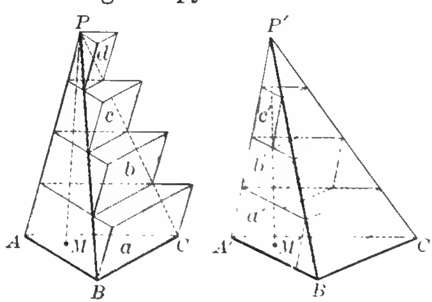
Then the **volume is the limit of the sequences** $a_1b_1c_1, a_2b_2c_2, a_3b_3c_3, \cdots$.

But the limit of this sequence is by definition the product of the limits of the three sequences $a_1, a_2, a_3, \cdots$, $b_1, b_2, b_3, \cdots$, $c_1, c_2, c_3, \cdots$, or abc.

Hence, we have —

790. THEOREM. *The volume inclosed by a rectangular parallelopiped is equal to the product of its three concurrent edges.*

791. Definition. In a triangular pyramid inscribe and circumscribe prisms as shown in the figure. See § 591 for description of the construction.



Denote the sum of the volumes of the inscribed prisms by V_1.

Using as altitudes one half the altitudes of the first set of prisms, inscribe a second set the sum of whose volumes is V_2. Continuing in this manner, we obtain a sequence of sets of prisms with volumes $V_1, V_2, V_3, \cdots$. The limit V of this sequence we **define as the volume of the pyramid.**

If circumscribed prisms are used, we get a *decreasing* sequence of volumes V_1', V_2', V_3', $\cdots$ with limit V'. That these two limits are identical follows from the fact that the sum of the volumes of the circumscribed prisms exceeds that of the inscribed prisms by exactly the volume of the lowest circumscribed prism (see § 594), and this may be made as small as we please.

792. **EXERCISE.**

1. Prove that the definition of § 789 gives the same volume for the rectangular parallelopiped, no matter what sequences of segments are used, provided their limits are the segments a, b, c.

A proof that the definition of § 791 gives the same volume for the triangular pyramid no matter into how many equal parts the altitude is first divided is rather too complicated to be attempted here. It may be proved by showing that if the altitude is divided into n and also into m equal parts, then if $n > m$, the set of $n - 1$ prisms will be greater than the set of $m - 1$ prisms.

793. THEOREM. *Two pyramids with the same altitudes and equal bases inclose equal volumes.*

Proof: Divide the two equal altitudes into the same number of equal parts and construct inscribed prisms as in § 591.

Since corresponding sets of prisms have equal volumes, it follows that the volumes of the pyramids are the limits of the same sequence, and hence identical.

794. **Convex curves.** Without specifically defining a **convex closed curve** (see § 552), we assume that in such a curve it is possible to inscribe a sequence of polygons P_1, P_2, P_3, $\cdots$, having perimeters p_1, p_2, p_3, $\cdots$ and areas A_1, A_2, A_3, $\cdots$ with limits p and A respectively, and to circumscribe a sequence of polygons P_1', P_2', P_3', $\cdots$, having perimeters p_1', p_2', p_3', $\cdots$ and areas A_1', A_2', A_3', $\cdots$ with limits p' and A' respectively, such that $p = p'$ and $A = A'$.

These limits p and A we now define **as the perimeter and the area respectively of the curve.**

795. Definition. Given any cylinder with a convex right cross section and an element e. In this cross section inscribe a sequence $P_1, P_2, P_3, \cdots$ of polygons, as in § 787, with perimeters $p_1, p_2, p_3, \cdots$ and areas $A_1, A_2, A_3, \cdots$, thus defining the perimeter p and the area A of the cross section.

Consider a set of prisms inscribed in this cylinder, of which $P_1, P_2, P_3, \cdots$ are right cross sections.

Then the areas and the volumes of these prisms are respectively $p_1e, p_2e, p_3e, \cdots$ and $A_1e, A_2e, A_3e, \cdots$.

The **lateral area and the volume of the cylinder are now defined as the limits of these sequences.** But these limits are by § 785 equal to pe and Ae respectively.

Hence, we have —

796. THEOREM. *The lateral area of a cylinder is the product of an element, and the perimeter of a right section and its volume is the product of an element and the area of a right section.*

797. EXERCISES.

1. Give examples other than those given in the text of infinite sequences which do not determine definite numbers.

2. Give two distinct sequences which determine the number 2. Show that the theorem of § 776 applies and proves that these sequences determine the same number.

3. Give two decreasing sequences each of which determines the number 3. Apply § 777 to show that these sequences determine the same number.

4. State fully the relation between a bounded increasing sequence and the number determined by it. State also the relations between a bounded decreasing sequence and the number determined by it.

5. State fully what is meant by "a limit of a sequence" both for increasing and decreasing sequences.

6. Given two incommensurable segments AB and CD. Lay off on the line AB a decreasing sequence of segments, each of which is commensurable with CD, such that the limit of the sequence is the segment AB.

7. If $a_1, a_2, a_3, \cdots$ is an increasing sequence defining the number 4, prove that $3\,a_1, 3\,a_2, 3\,a_3, \cdots$ defines the number $3 \times 4 = 12$.

Note that in case the sequence $a_1, a_2, a_3, \cdots$ defines an irrational number, we should not be able to prove the corresponding proposition without first defining what is meant by the product of a rational and an irrational number. But such definition would in that case be the proposition itself. See § 785.

8. If $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ are increasing sequences defining the numbers 3 and 5, show that the sequence $a_1b_1, a_2b_2, a_3b_3, \cdots$ defines the number 15.

In case these sequences defined irrational numbers, could a corresponding proposition be proved?

9. In the same manner as in § 781 define the ratio of two incommensurable arcs.

10. Show as above that the ratio so obtained is independent of the sequence of units of measurement used, so long as the limit of this sequence is zero.

11. Treat the ratio of two incommensurable angles in a manner similar to the treatment of arcs in the two preceding exercises.

12. Define the lateral area of a cylinder by means of circumscribed prisms, and show that this definition leads to the same result as that given in § 795.

13. Prove as above that the volume of any convex cylinder is equal to the product of its altitude and area of its base.

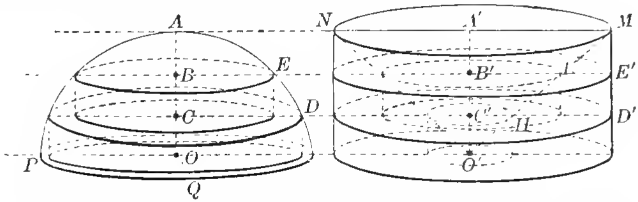
14. Prove that the lateral area of a right circular cone is equal to half the product of the slant height and the perimeter of its base.

15. Prove that the volume of any convex cone is equal to one third the product of its altitude and the area of its base.

Suggestion. The treatment required in the last three exercises is a very close paraphrase of the definitions and proof given in § 789. Observe that we cannot begin to make a proof until we have *defined* the subject matter of the theorem. That is, we must first define the areas and volumes in question.

VOLUME OF THE SPHERE.

798. Through two points P and Q of a sphere pass a great circle forming a hemisphere with center O.



Divide OA, the radius perpendicular to the plane of the great circle, into the equal parts OC, CB, and BA. Through C and B pass planes parallel to the plane of POQ, meeting the circle in points D and E, respectively.

Construct right circular cylinders with axes OC and CB and radii CD and BE. Denote by V_1 the sum of the volumes of these cylinders.

Now divide the radius OA into six equal parts and construct five cylinders in the same manner as above. Let the sum of these volumes be V_2.

Continuing in this manner, each time dividing OA into twice as many equal parts as in the preceding, we obtain a sequence of sets of cylinders and a corresponding sequence V_1, V_2, V_3, $\cdots$ of volume.

We now define **the volume inclosed by the hemisphere as the limit of the sequence** V_1, V_2, V_3, $\cdots$.

799. Construct a right circular cylinder with its base in the plane of POQ and altitude $O'A' = OA$.

Denote by F the figure formed by the lower base of the cylinder, its lateral surface and the lateral surface of the cone whose base is the upper base of the cylinder and whose vertex is at O'.

Draw segments $O'M$ and $O'N$. Let the planes through C and B cut $O'A'$ in C' and B' and $O'M$ in H and K.

Now form the cylinder $O'C'H$ whose axis is $O'C'$ and whose radius is $C'H$. Likewise form $C'B'K$.

Let V_1' denote the sum of the volumes of $O'C'D'$ and $C'B'E'$ minus the sum of the volumes $O'C'H$ and $C'B'K$.

In a similar manner, using the planes which divide OA and hence $O'A'$ into six equal parts, we form another set of five cylinders, the sum of whose volumes minus that of the smaller inside cylinders we denote by V_2'.

Continuing in this manner, we obtain a sequence of volumes V_1', V_2', V_3', $\cdots$ whose limit V' we **define as the volume of the given figure F.**

We now prove that $V_1 = V_1'$, $V_2 = V_2'$, $\cdots$.

Denote OA by r, and note that $O'B' = B'K$.

(1) Vol. $C'B'E'$ – Vol. $C'B'K = \pi C'B'(\overline{B'E'}^2 - \overline{B'K}^2) = \pi C'B'(r^2 - \overline{O'B'}^2)$.

(2) Vol. $CBE = \pi CB \cdot \overline{BE}^2 = \pi CB(r^2 - \overline{OB}^2)$, since $\overline{BE}^2 = r^2 - \overline{OB}^2$.

But $OB = O'B'$ and $CB = C'B'$.

Hence, Vol. CBE = Vol. $C'B'E'$ – Vol. $C'B'K$.

Similarly we show that

$$\text{Vol. } OCD = \text{Vol. } O'C'D' - \text{Vol. } O'C'H.$$

Hence, $V_1 = V_1'$. In like manner $V_2 = V_2'$, $V_3 = V_3'$, $\cdots$.

Hence, $V = V'$, since they are the limits of the same sequences.

But the volume of the cylinder $O'A'M$ is πr^3 and of the cone whose volume was subtracted, $\frac{\pi}{3} r^3$. That is, the volume of F is $\frac{2\pi}{3} r^3$, and hence that of the hemisphere is $\frac{2\pi}{3} r^3$.

Hence, we have —

800. THEOREM. *The volume of the sphere is* $\frac{4}{3}\pi r^3$.

801. Note that the above proof consists essentially in showing that the area of the circle BE is equal to that of the ring between the circles $B'E'$ and $B'K$, and that the area of the circle CD is equal to that of the ring between $C'D'$ and $C'H$ and so on.

Indeed this theorem and also that of § 594 are special cases of what is known as

Cavalieri's Theorem. *If two solid figures are regarded as resting on the same plane b, and if in every plane parallel to b the sections of the two figures have equal areas, the figures have equal volumes.*

The proof of this more general theorem is more difficult than any thus far given, inasmuch as it involves sequences which *oscillate*: that is, which are neither constantly increasing nor constantly decreasing.

THE AREA OF THE SPHERE.

802. About a sphere of radius r construct a sequence of circumscribed polyhedrons such that the largest face in each polyhedron becomes as small as we please when we proceed along the sequence. Let $s_1, s_2, s_3, \cdots$ be the total surfaces of these polyhedrons. This forms a decreasing sequence with limit S which we define as the **surface of the sphere.**

The volumes of these polyhedrons will be $\frac{r}{3}s_1, \frac{r}{3}s_2, \frac{r}{3}s_3, \cdots$.

Then the volume V of the sphere is defined as the limit of this sequence of volumes.

Hence, by § 785, $V = \frac{r}{3}S$. But by § 700, $V = \frac{4}{3}\pi r^3$.

Then $$S = \frac{3}{r} \cdot \frac{4}{3}\pi r^3 = 4\pi r^2.$$

Hence, we have —

803. THEOREM. *The area of the sphere is* $4\pi r^2$.

This argument is incomplete in that two distinct definitions have been given of the volume of the sphere, namely, as the limit of an increasing sequence of volumes of inscribed cylinders and also as the limit of a decreasing sequence of volume of circumscribed polyhedrons. But it has not been proved that these two definitions are equivalent; that is, that they lead to the same formula for the volume of the sphere.

The treatment of the area and volume of the sphere in most of the current textbooks on geometry is open to a similar objection. Usually two distinct definitions are used for the area of the sphere, namely, as a limit of the areas of circumscribed frustums of cones and also as the limit of the surfaces of circumscribed polyhedrons.

The argument used in §§ 740–748 is not subject to any such objection.

SUMMARY OF CHAPTER XIV.

1. Explain the separate stages of treatment of incommensurables and of limits as given in this book, including both the Plane and Solid Geometry.

2. Give an outline of the introduction to this final treatment by means of sequences.

3. Make a list of the definitions, principles, and theorems upon which the treatment of sequences is based.

4. Explain fully the way in which the principles of sequences are applied to geometric theorems. Illustrate by line-segments and arcs.

5. Make a list of the theorems which are proved here by the theory of limits, using the sequence process.

6. State in some detail how this process is used in case of the area and volume of the sphere.

7. Give all the mensuration formulas proved in this chapter.

8. Review the entire list of mensuration formulas for both Plane and Solid Geometry.

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