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RELATIVITY

by

A. W. CONWAY, D.Sc., F.R.S.

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PREFACE

THE four chapters which follow are four lectures delivered before the Edinburgh Mathematical Colloquium on the subject of Relativity. As many of the audience had their chief interests in other branches of mathematical science, it was necessary to start *ab initio.* The best method appeared to be to treat the subject in the historical order; I have brought it down to the stage in which it was left by Minkowski.

If I have stimulated any of my audience to pursue the matter further, I shall be amply repaid for any trouble that I have taken. I wish to express my thanks to the Edinburgh Mathematical Society for the honour that they have conferred on me in inviting me to give these lectures.

ARTHUR W. CONWAY.

CONTENTS

CHAPTER I

EINSTEIN'S DEDUCTION OF FUNDAMENTAL RELATIONS

The subject of Relative Motion is one which must have presented itself at a very early stage to investigators in Applied Mathematics, but we may regard it as being placed on a definite dynamical basis by Newton. In fact, his first Law of Motion defines the absence of force in a manner which is independent of any uniform velocity of the frame of reference. When the wave theory of light became firmly established, and when a luminiferous aether was postulated as the medium, universal in extent, which carried such waves, attention was again directed to relative motion. The questions at once arose: Are we to regard the aether as being fixed? Does matter, *e.g.* the Earth, disturb the aether in its passage through it? The attempts to answer such questions gave rise to numerous investigations, experimental and theoretical, on the one hand to determine the relative motion of matter and aether, and on the other to give an adequate explanation in mathematical terms of the results of such experiments. As an introduction I shall briefly describe two such experimental researches which will bring us at once to the root of the difficulty. For those who wish to pursue the historical introduction to this subject, I can refer to Professor Whittaker's "History of the Aether," a book the value of which can be appreciated by anyone who has ever wished to trace the somewhat tangled line of thought in these matters during the last century. I shall first refer to Bradley's observation of the aberration of light. Let S (Fig. 1) be a source of light and O an observer moving with a velocity represented by $OO' = v$. A spherical wave of light diverges from S in all directions. Relative to O each point P of the sphere has two velocities, one equal to the velocity of light c and in the direction

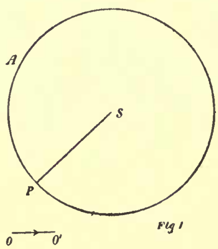

Fig 1

SP, and the other equal and opposite to OO'. The wave surface relative to the observer is thus a sphere, the radius of which is expanding at the rate c and the centre is moving at the rate v. Or again, the wave surfaces are spheres, the point S being a centre of similitude. The ray direction, or direction in which the radiant energy travels, is along the line joining S to P (Fig. 2), whilst the wave velocity is along $S'P$ where $SS'/SP=v/c$. The position of a fixed star as seen by an observer in a telescope moving with the earth will be the centre of the sphere of the system which passes through O, and the real position of the star can be calculated from the triangle $SS'O$ (Fig. 3) where $SS'/OS=v/c$ and the angle $OS'S$ is known. We thus get on the wave theory the same construction for S as on the corpuscular theory. Underlying this explanation, however, we have assumed that the aether in the neighbourhood of the earth is at rest and is undisturbed by the motion of the earth, and we are thus driven to the hypothesis of a stagnant aether, matter passing through without changing its properties. We now come to the classical researches of Michelson and Morley, which were designed to test this hypothesis.

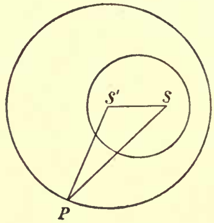

Fig 2

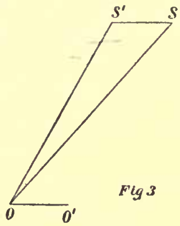

Fig 3

A ray of light coming from a source A falls on plate of glass B (Fig. 4) at an angle of 45°, and is partly transmitted to a mirror M_2, whence it is reflected back along M_2B, and finally to the observing telescope at C by reflexion at B; the other part goes over BM_1 twice in opposite directions and traverses the glass at B and reaches C. Thus the two parts of the beam are united and are in a position to produce interference bands if BM_2 is not equal to BM_1. This, however, is on the supposition that the apparatus is at rest. Suppose, however, that it has a velocity v along AB; the time taken for a light wave to travel from B to M_2 is then $l/(c-v)$, where l is the length of BM_2, and it takes a time $l/(c+v)$ to

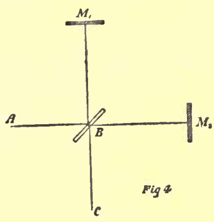

Fig 4

travel back along BM_1. Thus the whole time taken is

$$l/(c-v)+l/(c+v)=2lc/(c^2-v^2)=\frac{2l}{c}\left(1+\frac{v^2}{c^2}\right)$$

approximately if v is small compared with c.

For the reflexion from M_1, M_1 will have moved to M_1' whilst the light travelled from B to M_1' (Fig. 5). Thus if t is the time taken from B to M_1', $B\,M_1'=ct$, $M_1M_1'=vt$, and if $M_1B=l$, we have $l^2+v^2t^2=c^2t^2$, and therefore $t=l/\surd(c^2-v^2)$, and the time taken to come back to B is

$$2t=2l/\surd(c^2-v^2)=\frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right).$$

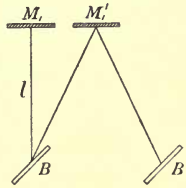

Fig 5

Comparing this with the expression above, we have a difference of time equal to lv^2/c^3, the time over BM_1 being less than BM_2 by that amount. By rotating the whole apparatus through 90°, the conditions are reversed, and the time difference of path will be of the opposite sense. In the actual experiment BM_1 was about 11 metres, and on taking v to be the velocity of the earth in its orbit, a displacement of 0·4 of an interference fringe width might be expected.

The unexpected, however, happened, and the observed displacement was certainly less than one-twentieth of this amount, and probably less than the one-fortieth.

Here we have at once a result at variance with our hypothesis that the aether is at rest. An explanation was put forward independently by Fitzgerald and Lorentz, and developed in detail by the latter. If the force of attraction between two molecules in motion is less when they are in motion at right angles to the line joining them than when they are in motion along that line, then a line such as BM_1 will measure less when it takes up the position BM_2, and we can easily see that this contraction (called the Fitzgerald-Lorentz contraction) could compensate for the difference of path in such a way that the null effect of the Michelson-Morley experiment would be adequately explained. The mathematical treatment of this theory in the hands of Lorentz and Larmor shows that all experiments on the relative motion of matter and aether must give approximately a null result.

The interpretation of these experiments received a different treatment from Einstein, whose paper in the "Annalen der

Physik," 4te Folge. 17, 1905, gave rise to most of the recent theoretical investigations which are included under the head of "Relativity." I shall now proceed to the deduction of the fundamental equations of this theory, following the above-named paper.

Let us consider a set of axes or frame of reference, which we shall term (merely for the sake of distinguishing) the *fixed* axes, and let there be another system of axes moving with reference to the former, which we shall call the *moving* axes. We shall suppose that the directions of these axes are parallel, and that the origin of the latter system moves along the x-axis of the former with a uniform velocity v. Fixing our attention for a while on the fixed system, let us suppose that at various points of space we are provided with clocks, the position of each clock being supposed fixed with reference to the axes, *i.e.* they are not in motion. We will further suppose that these clocks are synchronized, and that the synchronization is affected by the following process, which is in effect our definition of synchronization. Suppose that two points A and B are provided with clocks, and that a beam of light is flashed from A to B and reflected immediately back from B to A. The time of departure of the beam from A being t_1, and of arrival at B being t_2, whilst the final arrival at A is at the time t_3, the times t_1 and t_3 are indicated on the clock at A, and t_2 is indicated on the clock at B. Then the condition of synchronization is $t_2 - t_1 = t_3 - t_2$ or $t_1 + t_3 = 2t_2$. This of course agrees with our ordinary definition of synchronization; in fact, it states merely that the time taken for light to travel from A to B is the same as that taken to go from B to A. This definition, obvious as it appears, is the very foundation of this method of deducing the relations of Relativity.

We now lay down two fundamental assumptions.

1. The equations by which we express the sequence of natural phenomena remain unchanged when we refer them to a set of axes moving without rotation with a uniform velocity.

This is the "Principle of Relativity." We see at once that it forbids us to hope ever to be able to determine the absolute motion of our reference system. It is obviously true for the ordinary scheme of dynamics, although it should be borne in mind that a physical quantity measured in one system may not be equal to the corresponding quantity measured in the other. Thus while it is true that the change of Kinetic Energy in each system is equal to the work done by the forces, yet the number expressing the Kinetic Energy itself is different in each; in fact, the Kinetic Energy of a particle m moving with velocity w along the x-axis is $\frac{1}{2}mw^2$ when referred to the fixed system, and is equal to $\frac{1}{2}m(w-v)^2$ referred to the other system. This, as I stated, refers only to ordinary or Newtonian dynamics, which as we shall see, constitute a particular case of the more general theory which we shall develop later. This principle of Relativity is in accordance with the null effect of the Michelson-Morley and other experiments on the motion of the aether, and being thus in accordance with all known physical facts, is a valid basis for a scientific hypothesis. It will be noticed that the null effect is approximate after the theory of Lorentz and Larmor, and that conceivably by a very great increase in the accuracy of our instruments an effect other than null might be observed, but that this principle makes the null effect absolute and incapable of ever being observed. The former theory starts from certain principles and deduces the Relativity Principle as a final (and approximate result); the latter starts at this null effect and works backwards. Both theories traverse the same ground, but in opposite directions, and experimental science is at present incapable of deciding between them. The newer theory, however, appears to its admirers to be more elegant (or according to others, more artificial) in its formal presentation.

2. The second principle we make use of is that the velocity of of light which is reflected from a mirror is the same as that of light coming from a fixed source. This principle, which follows from the ordinary equations of electrodynamics, has received definite experimental proof recently in an ingenious experiment by Michelson (*Astrophysical Journal*, July 1913.)

We now proceed, having thus laid down our two principles. Supposing that in the moving system we have a system of clocks *at rest* relative to their axes, and that these clocks have been synchronized by observers in this system unconscious of their own motion. Suppose that there are two clocks in the system A and B at rest relative to the axes, and that a ray of light goes from A to B and is reflected back to A, the times being τ, τ_1, τ_2 as above. Then by the Principle of Relativity the relation $\tau_1 - \tau = \tau_2 - \tau_1$ or $\tau + \tau_2 = 2\tau_1$ must hold.

To each point of this system (coordinates ξ, η, ζ) there will be a time τ, the time indicated by a synchronized clock situated there. This point has for coordinates x, y, z, referred to the fixed system, and the clock of this system denotes a time t. We have thus to discover relations between ξ, η, ζ, τ and x, y, z, t. We can see at once that the relations must be linear, otherwise the connections between infinitesimal increments of space and time would depend on the coordinates. For instance, the relation between a small triangle as observed in one system and as observed in the other must be the same wherever this triangle is situated, provided that its orientation is unaltered. In other words, we ascribe no particular properties to the origin, which may be any point of space. This property is referred to as the homogeneity of space.

We have then as the most general form of the relations the four equations

$$\begin{aligned}
\xi &= A_1 x + B_1 y + C_1 z + D_1 t \\
\eta &= A_2 x + B_2 y + C_2 z + D_2 t \\
\zeta &= A_3 x + B_3 y + C_3 z + D_3 t \\
\tau &= A\,x + B\,y + C\,z + D\,t.
\end{aligned}$$

We could add an arbitrary constant to each equation, but without loss of generality we can suppose the values of x, y, z, t, ξ, η, ζ, τ to be simultaneously zero, or more precisely, suppose that all the clocks in the fixed system are synchronized from a clock at the origin, and the same is done for clocks in the moving system, then at the instant that the "moving" origin O' and the "fixed" origin O coincide, the two clocks show the time zero. Suppose now that in the moving system a ray of light starts from the origin O' at the observed time τ, and proceeds towards a mirror placed at a *fixed* distance along the ξ-axis. It is there reflected at the time τ_1, and arrives back at the origin O' at the time τ_2. Let us see how all this appears to observers in the fixed system. They will see an origin O' moving along the axis of x, and at an invariable distance (which *they* find on measurement to be x') in front is situated a mirror. The coordinates of O' to them are $(v\,t, 0\ 0)$, and the coordinates of the mirror are $(x' + vt, 0, 0)$. A ray of light is observed to start from O' at a time t, and to overtake the mirror. It is thence reflected back to the origin which is moving forward to meet it. The reflexion thus takes place at the time

$t+x'/(V-v)$, and the reappearance of the light at O' will be at the time $t+x'/(V-v)+x'/(V+v)$. We have thus three events—the departure, the reflexion, and the arrival of the light. We know the coordinates and the times of each of them in both systems, and on inserting in the last of the above linear equations, we get

$$\tau = Avt + Dt$$
$$\tau_1 = A\{x' + v[t + x'/(c-v)]\} + D\{t + x'/(c-v)\}$$
$$\tau_2 = Av\{t + x'/(c-v) + x'/(c+v)\} + D\{t + x'/(c-v) + x'/(c+v)\}.$$

The relation of synchronism $2\tau_1 = \tau + \tau_2$ gives then

$$Ac^2 + Dv = 0.$$

Let the mirror now be placed on the η-axis, and let similar observations be made. We have, as before, the following simultaneous values for the coordinates and the times :—

	Fixed System.	Moving System.
Departure of Light	$vt,\ 0,\ 0,\ t$	$0\ \ 0\ \ 0\ \ \tau$
Reflexion	$vt + vy'/\sqrt{(c^2-v^2)},\ y',\ 0,\ t + y'/\sqrt{(c^2-v^2)}$	$0,\ \eta',\ 0\ \ \tau_1$
Arrival	$vt + 2vy'/\sqrt{(c^2-v^2)},\ 0,\ 0,\ t + 2y'/\sqrt{c^2-v^2}$	$0\ \ 0\ \ 0\ \ \tau_2$

Making use, as before, of the equation

$$\tau + \tau_2 = 2\tau_1,$$

we get $$B = 0.$$

In a similar manner we get $C = 0$, and we thus have the relation

$$\tau = D\{t - vx/c^2\}.$$

Referring again to the equations from which τ and τ_1 were determined in the moving system, the beam of light takes a time $\tau_1 - \tau$ to go from the origin O' to the mirror. Thus the coordinate ξ of the mirror must be equal to $c(\tau_1 - \tau)$.

On substituting the values of τ_1 and τ, we get

$$\xi = Dx' = D(x - vt).$$

Comparing this with our assumed linear relation If $v = 0$ then $D_1 = D$ on

$$\xi = A_1 x + B_1 y + C_1 z + D_1 t,$$ Since $x - vt = t$

we see that $A_1 = B_1 = C_1 = 0$ and $D_1 = D$. $x = t(1+v)$

$\therefore x = t$

and $\xi = Dt$

$\therefore D_1 t = Dt$

$D_1 = D$

Let us next consider the relation between y and η. Let A be a point on the η-axis, and let the distance AO' be found on measurement according to the fixed axis system to be y, and according to the moving axis to be η. The time t taken for light to go from O' to A is $y/\sqrt{(c^2-v^2)}$. The time $\tau_1-\tau$ in the moving system is $D(t-vx/c^2)$, and on substituting $x=vt$ and $t=y/\sqrt{(c^2-v^2)}$, we get

$$\eta = D\sqrt{1-\frac{v^2}{c^2}}\cdot y.$$

In a similar manner we find $\zeta = D\sqrt{1-\dfrac{v^2}{c^2}}\cdot z$.

We have thus got so far with our linear relations

$$\begin{aligned}\xi &= D\,(x-vt)\\ \eta &= D\sqrt{1-v^2/c^2}\;y\\ \zeta &= D\sqrt{1-v^2/c^2}\;z\\ \tau &= D\,(t-vx/c^2).\end{aligned}$$

The constant D, which can be a function only of v, is so far undetermined. In the first place, we may notice that if $D=\phi(v)$, then $\phi(-v)=\phi(v)$, for the relation between η and y from symmetry must be the same for v and $-v$. Now we might have considered the moving axes to be fixed and the fixed axes to be moving, and have repeated all the above transformations which would have the same form, with, however, $-v$ in place of v,

$$\begin{aligned}x &= D\,(\xi+v\tau)\\ y &= D\sqrt{1-v^2/c^2}\;\eta\\ z &= D\sqrt{1-v^2/c^2}\;\zeta\\ t &= D\,\{\tau+v\xi/c^2\}.\end{aligned}$$

On substituting these values in the equations above, we get

$$D^2\,(1-v^2/c^2)=1,$$

so that if we denote $\quad 1-v^2/c^2 = 1/\beta^2$

we have $\quad D=1/\beta$,

and we arrive at, finally, the fundamental equations of Relativity

$$\begin{aligned}\xi &= \beta\,(x-vt)\\ \eta &= y\\ \zeta &= z\\ \tau &= \beta\,(t-vx/c^2).\end{aligned}$$

Before discussing these equations, it ought to be observed that they have been obtained by the consideration of light rays which passed along one of the coordinate axes, and as a verification we ought to consider the case of a ray which passes obliquely. Let a source of light be at the moving origin O', and suppose that a beam of light starts from O' at the time t_0 and goes to a mirror which is rigidly attached to the moving axes, and whose "fixed" coordinates at the time t are x, y and z, and that the beam arrives back again at O' at the time t_1. Now if the times observed in the moving system for these events are respectively τ_0, τ, τ_1, we have, since $OO' = vt$,

$$\tau_0 = \beta\left(t_0 - \frac{v^2}{c^2}t_0\right)$$
$$\tau = \beta\,(t - vx/c^2)$$
$$\tau_1 = \beta\,(t_1 - v^2 t_1/c^2).$$

We have then to verify the equation

$$\tau - \tau_0 = \tau_1 - \tau,$$

or, to put it in a convenient form,

$$c\,(t_1 - t) - c\,(t - t_0) = \frac{v}{c}\left\{(v\,t_1 - x) - (x - v\,t_0)\right\}.$$

In the diagram (Fig. 6) let O' and A be the position of the origin and mirror at the time t_0, O'' and A' their positions at the time t, and O''' the origin at the time t_1, all as seen by an observer in the fixed system. Then $O'A' = c\,(t - t_0)$; $O''A' = c\,(t_1 - t)$; $O'O'' = v\,(t - t_0)$; $O''O''' = v\,(t_1 - t)$. So that

$$O'O'' : O''O''' = O'A : O'''A,$$

and
$$\frac{O'A' - O'''A'}{O'O'' - O'''O''} = \frac{O'A'}{O'O''} = \frac{c}{v},$$

a relation which is easily seen to be identical with that given above.

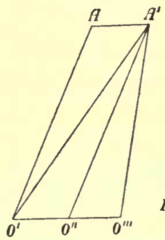

Fig 6

Returning to our fundamental equations, suppose the coordinates of any number of points P_1, P_2, etc., in the moving system, and rigidly attached to it are $(\xi_1\,\eta_1\,\zeta_1)$, $(\xi_2\,\eta_2\,\zeta_2)$, etc., and that the corresponding coordinates in the fixed system are $(x_1\,y_1\,z_1)$, $(x_2\,y_2\,z_2)$, we have then equations of the type

$$\xi_1 = \beta\,(x_1 - v\,t)$$
$$\zeta_2 = \beta\,(x_2 - v\,t)$$
$$\eta = y_1, \quad \text{etc.},$$

so that

$$(\xi_2 - \xi_1) = \beta\,(x_2 - x_1)$$
$$\eta_2 - \eta_1 = \quad y_2 - y_1$$
$$\zeta_2 - \zeta_1 = \quad x_2 - x_1, \quad \text{etc.}$$

Thus any geometrical figure in the moving system appears deformed to the observers in the fixed system. The lengths parallel to the x-axis appear to be decreased in the ratio of $1 : \beta$, whilst transverse lengths are unaltered.

Thus the ellipse $\xi^2 + \beta^2\eta^2 = \beta^2 a^2$, the eccentricity of which is v/c, becomes the circle $(x - vt)^2 + y^2 = a^2$.

A line $A\xi + B\eta + C = 0$ transforms into $Ax + B\beta\eta + C\beta = Avt$, so that parallel lines transform into parallel lines, but the angle between two lines is usually changed. If, situated in the fixed system, we caught a passing glimpse of a teacher in the moving system proving to a class that the sum of the three angles of a triangle was equal to two right angles, we should get the impression that he was demonstrating a rather involved question in non-Euclidean geometry, and that both teacher and class were either all stouter or all thinner than similar individuals in these countries, according as the axis of x or of y is the vertical.

As regards the time, it is clear from the equation

$$\tau = \beta\,(t - v\,x/c^2)$$

that the clock at O will be behind the local time at the corresponding point of the moving system. To take a numerical example, suppose that $v/c = 4/5$, so that $\beta = 5/3$, and that at noon the clocks at O and O' mark the same time, O at that time coinciding with O'. If a person in the moving system starts a certain task at noon and works until his clock shows 1 p.m., and if at that instant he catches sight of the clock at O, which is passing by, he will find that it registers only 12.36 p.m. So that if he regulated his work according to the latter, he might easily achieve the result of getting more than twenty-four hours into the day.

Again, if we suppose an observer situated at O' looks at various clocks in the fixed system as they come opposite to him, we have, on putting in $O\,O' = x = v\,t$, $t = \beta\,\tau$, or in our example $t = 5\tau/3$.

We shall now consider some kinematical results. The component velocities of a point in the moving system w_ξ, w_η are equal to $\dfrac{d\xi}{d\tau}$ and $\dfrac{d\eta}{d\tau}$, and thus

$$w_\xi = \frac{d\xi}{d\tau} = \frac{dx - v\,dt}{dt - v\,dx/c^2}$$

$$= \frac{W_x - v}{1 - v\,W_x/c^2}$$

where W_x and W_y are the components of the velocity as observed in the fixed system. From the above we get the reciprocal relation

$$W_x = \frac{w_\xi + v}{1 + v\,w_\xi\,/\,c^2},$$

another form of which is

$$(1 + v\,w_\xi\,/\,c^2)\,(1 - v\,W_x\,/\,c^2) = (1 - v^2\,/\,c^2).$$

Any one of these formulae gives us a solution of the problem of the composition of velocities having the same direction, or of finding the relative velocity of one moving point with respect to another. Some remarks may be made with respect to these equations. In the first place, if $c = \infty$, or, what is the same thing, if the velocities W and w are very small compared with the velocity of light, these formulae become

$$w_\xi = W_x - v$$

$$W_x = v + W_\xi$$

which reproduce the ordinary equations of relative motion. Again, if there is a second moving system, moving with respect to the first moving system with a velocity v' in the same direction, and if $w_\xi{}'$ $w_\eta{}'$ are the components of velocity, we have

$$w_\xi = \frac{w_\xi{}' + v'}{1 + \dfrac{w_\xi{}'\,v'}{c^2}}.$$

But

$$W_x = \frac{w_\xi + v}{1 + \dfrac{w_\xi\,v}{c^2}}$$

$$= \frac{w_\xi{}' + (v + v')\,/\,(1 + v\,v'/c^2)}{1 + w_\xi{}'\,(v + v')\,/\,c^2\,(1 + vv'\,/\,c^2)}.$$

More generally, if we consider any number of moving axes, and if the relative velocities of the origins are v, v', v'', ..., we get

$$W_x = \frac{w_\xi + u}{1 + w_\xi u/c^2}$$

where $u = \{s_1 + s_3/c^2 + s_5/c^4 + \ldots\} / \{1 + s_2/c^2 + s_4/c^4 + \ldots\}$,

s_n meaning the sum of the products of the quantities v, v', etc., taken n at a time. It is to be noticed that the expression for u is symmetrical in the velocities v, v', v''.... The various operations are in fact commutative, that is, we might have taken these velocities in any other order v'', v, v', ...

For the velocity $w_\eta = d\eta/d\tau$ we get the expression

$$\frac{W_y}{\beta\left(1 - \frac{W_x v}{c^2}\right)}$$

or

$$W_y = \frac{w_\eta}{\beta(1 + w_\xi v/c^2)}.$$

The resultant velocity W is given by

$$W^2 = W_x^2 + W_y^2$$

$$= \frac{w^2 + v^2 + 2\,w\,v\cos\theta - v^2 w^2 \sin^2\theta / c^2}{\beta^2 \{1 + w\,v\cos\theta / c^2\}^2}$$

where θ is the angle between the velocity $w(= \sqrt{w_\xi^2 + w_\eta^2})$ and v. We can draw conclusions similar to those above for different axes all moving with the same velocity v and in the same direction.

We find, however, that if the velocities are not all in the same direction, the final result is not independent of the order in which the velocities v, v', ... are taken. To illustrate, let us consider a point P, which moves with a velocity w parallel to the η-axis of a moving system, the origin moving with a velocity v along the ξ-axis, and let us compare the resultant with the velocity of a point Q, which moves with a velocity v along the ξ-axis of a moving system which moves parallel to the η-axis with a velocity w.

We have then for P

$$W_x = v$$

$$W_y = \sqrt{1 - v^2/c^2}\, w,$$

and for Q

$$W_x' = \sqrt{1 - w^2/c^2} \, . \, v$$
$$W_y' = w.$$

Thus, though the resultant velocity has the same magnitude in each case, the directions as referred to the fixed axes are different.

As a final example, let us consider a problem of a very familiar type. Two points A and B are moving along two rectangular lines AO and BO, with velocities v and w, the distance AO being equal to a and BO equal to b. What will be their shortest distance apart, and when will this occur? We find by ordinary methods that the shortest distance is $(bv \sim aw) \, / \, \sqrt{(v^2 + w^2)}$, and that the time of reaching the shortest distance is $(av + bw) \, / \, (v^2 + w^2)$. These are the distance and time as observed by a fixed observer, but if we seek for the measurement that would be made by an observer moving with A we get different results. Thus w becomes w/β and $AO = a$ becomes βa, so that the quantities given above are to be replaced by $(bv \sim aw) \, / \, \sqrt{(v^2 + w^2 \beta^{-2})}$ and

$$\frac{av\,\beta + bw\,\beta^{-1}}{v^2 + w^2\,\beta^{-2}}.$$

The expressions for the transformation of the acceleration are more complicated, thus

$$\frac{d^2\xi}{d\tau^2} = \frac{dw_\xi}{d\tau} = \frac{dw_\xi}{\beta\,(dt - v\,dx/c^2)}$$
$$= \frac{\ddot{x}}{\beta^3\,\{1 - v\,\dot{x}/c^2\}^3}.$$
$$\frac{d^2\eta}{d\tau^2} = \frac{dw_\eta}{d\tau} = \frac{\ddot{y}}{\beta^2\,\{1 - v\,\dot{x}/z^2\}^2} + \frac{v\,\dot{y}\,\ddot{x}}{\beta^2\,\{1 - v\,\dot{x}/c^2\}^3}.$$

Supposing a wave of light diverges from a point $x_0 y_0 z_0$ at a time t_0, the equation of the spherical wave is at the time t

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2\,(t - t_0)^2.$$

If we transform this by our fundamental equations, we get

$$\beta^2\,[\xi - \xi_0 - v\,(\tau - \tau_0)]^2 + (\eta - \eta_0)^2 + (\tau - \tau_0)^2$$
$$= c^2\,\beta^2\,[\tau - \tau_0 - v\,(\xi - \xi_0)\,/\,c^2]^2,$$

or
$$(\xi - \xi_0)^2 + (\eta - \eta_0)^2 + (\tau - \tau_0)^2 = c^2\,(\tau - \tau_0)^2,$$

so that a spherical wave of light transforms into a spherical wave as it ought to do, after the Principle of Relativity. This identity

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - c^2 (t - t_0)^2$$
$$\equiv (\xi - \xi_0)^2 + (\eta - \eta_0)^2 + (\zeta - \zeta_0)^2 - c^2 (\tau - \tau_0)^2$$

has been employed conversely to deduce the equations of Relativity.

We can make some interesting deductions from this identity. If we suppose that the quantities x, y, z, t differ infinitesimally from x_0, y_0, z_0, t_0, and if we put $x - x_0 = dx$, etc., then we have the following relation between infinitesimals:—

$$dx^2 + dy^2 + dz^2 - c^2\, dt^2 = d\xi^2 + d\eta^2 + d\zeta^2 - c^2\, d\tau^2.$$

If we call $$dx^2 + dy^2 + dz^2 - c^2\, dt^2 = ds^2,$$
and $$d\xi^2 + d\eta^2 + d\zeta^2 - c^2\, d\tau^2 = d\sigma^2,$$
then we have at once the relations

$$\frac{d\xi}{d\sigma} = \beta\left(\frac{dx}{ds} - v\frac{dt}{ds}\right)$$

$$\frac{d\eta}{d\sigma} = \frac{dy}{ds}$$

$$\frac{d\zeta}{d\sigma} = \frac{dz}{ds}$$

$$\frac{d\tau}{d\sigma} = \beta\left\{\frac{dt}{ds} - v\frac{dx}{ds}c^{-2}\right\},$$

so that the differential coefficients

$$\frac{dx}{ds}, \quad \frac{dy}{ds}, \quad \frac{dz}{ds}, \quad \frac{dt}{ds}$$

are transformed by the same transformation as x, y, z, t, or, to use the algebraic term, they are *cogredient*. Obviously this is true, if in place of x, y, z, t we had any set of four *cogredient* quantities.

Hence we have as examples of cogredient quantities

$$\begin{array}{cccc} x & y & z & t \\ \dfrac{dx}{ds} & \dfrac{dy}{ds} & \dfrac{dz}{ds} & \dfrac{dt}{ds} \\ \dfrac{d^2x}{ds^2} & \dfrac{d^2y}{ds^2} & \dfrac{d^2z}{ds^2} & \dfrac{d^2t}{ds^2} \end{array}.$$

CHAPTER II

TRANSFORMATION OF ELECTROMAGNETIC EQUATIONS

We pass on now to the most remarkable application of these equations.

In free aether the electric force (X, Y, Z) and the magnetic force (α, β, γ) are related to one another by the Hertz-Heaviside form of Maxwell's equations as follows :—

$$c^{-1}\frac{\partial X}{\partial t}=\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}, \qquad -c^{-1}\frac{\partial L}{\partial t}=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}.$$

$$c^{-1}\frac{\partial Y}{\partial t}=\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}, \qquad -c^{-1}\frac{\partial M}{\partial t}=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}.$$

$$c^{-1}\frac{\partial Z}{\partial t}=\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}, \qquad -c^{-1}\frac{\partial N}{\partial t}=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}.$$

These equations involve two others.

If we differentiate the first of each of those triads by x, the second by y, the third by z, and add each triad together, the right-hand sides vanish and the left-hand sides become respectively

$$\frac{\partial}{\partial t}\left\{\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right\}=0.$$

$$\frac{\partial}{\partial t}\left\{\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right\}=0,$$

from which we deduce

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0.$$

$$\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}=0.$$

Here (X, Y, Z) denote the force in dynes on an electrostic unit, and (L, M, N) the force in dynes on unit-magnet pole in the electromagnetic system.

Let us transform the second of these equations

$$c^{-1}\frac{\partial Y}{\partial t}=\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x},$$

we have

$$\frac{\partial}{\partial t} = \beta\frac{\partial}{\partial\tau} - v\beta\frac{\partial}{\partial\xi}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial\zeta}$$

$$\frac{\partial}{\partial x} = \beta\frac{\partial}{\partial\xi} - \frac{v\beta}{c^2}\frac{\partial}{\partial\tau},$$

so that the equation becomes

$$c^{-1}\left(\beta\frac{\partial Y}{\partial\tau} - v\beta\frac{\partial Y}{\partial\xi}\right) = \frac{\partial L}{\partial\zeta} - \beta\frac{\partial N}{\partial\xi} + \frac{v\beta}{c^2}\frac{\partial N}{\partial\tau}$$

or

$$c^{-2}\frac{\partial Y'}{\partial\tau} = \frac{\partial L'}{\partial\zeta} - \frac{\partial N'}{\partial\xi}$$

where

$$Y' = \beta\left(Y - \frac{v}{c}N\right)$$

$$L' = L$$

$$N' = \beta\left(N - \frac{v}{c}Y\right).$$

In the same way we can deduce the complete set of transformed equations in the form

$$c^{-1}\frac{\partial X'}{\partial\tau} = \frac{\partial N'}{\partial\eta} - \frac{\partial M'}{\partial\zeta}$$

$$c^{-1}\frac{\partial Y'}{\partial\tau} = \frac{\partial L'}{\partial\zeta} - \frac{\partial N'}{\partial\xi}$$

$$c^{-1}\frac{\partial Z'}{\partial\tau} = \frac{\partial M'}{\partial\xi} - \frac{\partial L'}{\partial\eta}$$

$$-c^{-1}\frac{\partial L'}{\partial\tau} = \frac{\partial Z'}{\partial\eta} - \frac{\partial Y'}{\partial\zeta}$$

$$-c^{-1}\frac{\partial M'}{\partial\tau} = \frac{\partial X'}{\partial\zeta} - \frac{\partial Z'}{\partial\xi}$$

$$-c^{-1}\frac{\partial N'}{\partial\tau} = \frac{\partial Y'}{\partial\xi} - \frac{\partial X'}{\partial\eta}$$

$$\frac{\partial X'}{\partial\xi} + \frac{\partial Y'}{\partial\eta} + \frac{\partial Z'}{\partial\zeta} = 0$$

$$\frac{\partial L'}{\partial\xi} + \frac{\partial M'}{\partial\eta} + \frac{\partial N'}{\partial\zeta} = 0$$

where
$$X' = X \qquad L' = L$$
$$Y' = \beta\left(Y - \frac{v}{c}N\right) \qquad M' = \beta\left(M + \frac{v}{c}Z\right)$$
$$Z' = \beta\left(Z + \frac{v}{c}M\right) \qquad N' = \beta\left(N - \frac{v}{c}Y\right).$$

We have thus the remarkable fact that the equations which express the interconnection of electric and magnetic forces remain of the same form when transferred to moving axes. This, granting the Principle of Relativity, may be looked on as a proof of these equations.

If we in the fixed system observe any point charge to have unit strength, then in the moving system the charge must be observed to have the same strength, and so we have the following statement:—Let a unit charge be situated at O' and move with it. It will experience a mechanical force of (X', Y', Z') dynes. In the fixed system we will observe a mechanical force

$$X,\ \beta\left(Y - \frac{v}{c}N\right),\ \beta\left(Z + \frac{v}{c}M\right),$$

so that we have an electric force X, βY, βZ, and an electromotive force which is β times the vector product of velocity and the magnetic force. When the ratio v/c is small, β is nearly equal to 1, and these expressions agree with those of Maxwell for the above forces.

We now pass on to the case in which, instead of free aether, we have present electric currents of strengths in electrostatic units given by the components U, V, and W. The electrodynamical equations are then modified by the introduction of certain terms. Thus,

$$c^{-1}\left(\frac{\partial X}{\partial t} + 4\pi U\right) = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$$

$$c^{-1}\left(\frac{\partial Y}{\partial t} + 4\pi V\right) = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}$$

$$c^{-1}\left(\frac{\partial Z}{\partial t} + 4\pi W\right) = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}.$$

$$-c^{-1}\frac{\partial L}{\partial t}=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}$$

$$-c^{-1}\frac{\partial M}{\partial t}=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}$$

$$-c^{-1}\frac{\partial N}{\partial t}=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}.$$

If ρ is the volume density, we have also the equation of continuity

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=-\frac{\partial \rho}{\partial t},$$

and finally

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=4\pi\rho$$

$$\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}=0.$$

On transforming as before we get equations of the type

$$c^{-1}\left(4\pi U'+\frac{\partial X'}{\partial \tau}\right)=\frac{\partial N'}{\partial \eta}-\frac{\partial M'}{\partial \zeta},$$

etc., etc.,

where X', Y', Z', L', M', N' have the same meaning as before, and

$$U'=\beta(U-v\rho)$$
$$V'=V$$
$$W'=W$$
$$\rho'=\beta(\rho-vU/c^2).$$

We notice that $U\ V\ W\ \rho$ is cogredient with $x\ y\ z\ t$. We may also notice that the velocity of a convection current at any point is given by the vector U/ρ, V/ρ, W/ρ, and from the above expressions these satisfy as they ought the laws for the composition of velocities.

If there is a distribution of electricity ρ' at relative rest in the moving system, we have $U'=V'=W'=0$ and $U=v\rho$ and $\rho'=\beta\rho(1-v^2/c^2)=\rho\beta^{-1}$; an element of volume $d\xi\ d\eta\ d\zeta$ transforms into $\beta\,dx\,dy\,dz$; so that the element of charge transforms into an equal element of charge, for

$$\rho'\,d\xi\,d\eta\,d\zeta=\rho\,dx\,dy\,dz.$$

If the charge were spread on a surface, the element of which is $d\Sigma$ and direction cosines (λ, μ, ν), and if the measurements in the fixed system are dS and (l, m, n) respectively, we have then from the equations

$$d\xi = \beta\, dx\,; \quad d\eta = dy\,; \quad d\zeta = dz$$

$$\lambda\, d\Sigma = d\eta\, d\zeta = dy\, dz = \quad l\, dS$$

$$\mu\, d\Sigma \qquad\qquad\qquad = \beta\, m\, dS$$

$$\nu\, d\Sigma \qquad\qquad\qquad = \beta\, n\, dS$$

$$d\Sigma = \beta\, dS\left(1 - \frac{v^2 l^2}{c^2}\right)^{\frac{1}{2}},$$

and so the surface density σ' and σ, on account of the equation

$$\sigma' d\Sigma = \sigma\, d\, S,$$

which expresses the invariance of a charge, give

$$\sigma' = \sigma/\beta\left(1 - \frac{v^2 l^2}{c^2}\right)^{\frac{1}{2}}.$$

We have now materials for a complete transformation of any electrical problem from one set of axes to another. We proceed to some examples.

A unit charge fixed at O' produces in the moving system components of electric force as follows :—

$$X' = \xi \,/\, (\xi^2 + \eta^2 + \zeta^2)^{\frac{3}{2}},$$

$$Y' = \eta \,/\, (\xi^2 + \eta^2 + \zeta^2)^{\frac{3}{2}},$$

$$Z' = \zeta \,/\, (\xi^2 + \eta^2 + \zeta^2)^{\frac{3}{2}},$$

whilst $L' = M' = N' = 0$.

In the transformation we have

$$\xi^2 + \eta^2 + \zeta^2 = \beta^2 (x - vt)^2 + y^2 + z^2,$$

and we have for the electric force due to a unit charge moving with a uniform velocity along the x-axis with a velocity v

$$L = L' = 0$$

$$M' = 0 \quad \text{and therefore} \quad M = -\frac{v}{c} Z$$

$$N' = 0 \quad\;\; ,, \qquad\quad ,, \qquad N = \frac{v}{c} Y$$

$$X' = X$$

$$Y' = \beta\left(Y - \frac{v}{V} N\right) = Y\beta^{-1}$$

$$Z = \qquad\qquad\qquad = Z\beta^{-1},$$

so that finally

$$X = \frac{\beta(x - vt)}{\{\beta^2(x-vt)^2 + y^2 + z^2\}^{\frac{3}{2}}}$$

$$Y = \frac{\beta y}{\{\beta^2(x-vt)^2 + y^2 + z^2\}^{\frac{3}{2}}}$$

$$Z = \frac{\beta z}{\{\beta^2(x-vt)^2 + y^2 + z^2\}^{\frac{3}{2}}}$$

$$L = 0$$

$$M = -\frac{v\beta}{c}\frac{z}{\{\beta^2(x-vt)^2 + y^2 + z\}^{\frac{3}{2}}}$$

$$N = \frac{v\beta}{c}\frac{y}{\{\beta^2(x-vt)^2 + y^2 + z^2\}^{\frac{3}{2}}}.$$

Knowing the distribution of electricity on a conductor at rest, we can by the above methods obtain the distribution of electricity on a certain moving conductor. In fact, if $U' = V' = W' = 0$, ρ' is independent of τ, and we get $U = v\rho$, $V = 0$, $W = 0$ and $\rho = \beta\rho'$; and if the equation of the conductor is $f(\xi, \eta, \zeta) = 0$, the equation of the transformed conductor is $f(\beta(x - vt), y, z) = 0$.

Thus the surface density σ' on a conductor which has the form of an ellipsoid of revolution is given by

$$\sigma' = \frac{e}{4\pi ab^2\sqrt{(\xi^2/a^4 + (\eta^2 + \zeta^2)/b^4)}}$$

where the equation of the ellipsoid is

$$\frac{\xi^2}{a^2} + \frac{\eta^2 + \zeta^2}{b^2} = 1.$$

From previous equations we find that

$$\sigma = \sigma'\beta\sqrt{\left(1 - \frac{v^2\cos^2\theta}{c^2}\right)},$$

where θ is the angle which the normal to the ellipsoid in the fixed system makes with the x-axis, is the surface density on the conductor

$$\frac{\beta^2 x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1,$$

which is moving with the velocity v along the x-axis. We find, on reduction, if we put $a' = a/\beta$,

$$\sigma = \frac{e}{4\pi a' b^2\sqrt{(x^2/a'^4 + (y^2 + z^2)/b^4)}}.$$

The forces can easily in any case be obtained by making use of the potential. Thus, in the electrostatic system,

$$X' = -\frac{\partial V}{\partial \xi}$$

$$Y' = -\frac{\partial V}{\partial \eta}$$

$$Z' = -\frac{\partial V}{\partial \zeta}$$

where V, the potential, is a function only of ξ, η, ζ; we get then easily

$$X = -\beta\frac{\partial V}{\partial x} \qquad L = 0.$$

$$Y = -\beta\frac{\partial V}{\partial y} \qquad M = \frac{v\beta}{c}\frac{\partial V}{\partial z}$$

$$Z = -\beta\frac{\partial V}{\partial z} \qquad N = -\frac{v\beta}{c}\frac{\partial V}{\partial y}.$$

So that in the case of electricity at rest on a moving conductor the various forces may be obtained by differentiating a certain potential function.

As an example, the electrostatic potential of a line of length $2l$ and of charge e per unit length is given by

$$V = e\log\frac{r+r'+l}{r+r'-l}$$

where the origin O' is the middle point of the line, and

$$r^2 = (\xi - l)^2 + \eta^2 + \zeta^2$$

$$r'^2 = (\xi + l)^2 + \eta^2 + \zeta^2.$$

By means of the equations above we get the forces due to a *moving* charged line where V has now the transformed value in terms of x, y, z, and t, *i.e.*, if we put $l = \beta l'$ and $e' = \beta e$ (for the total charge $2el$ must be equal to $2e'l'$), we have

$$r^2 = \beta^2(x - vt - l')^2 + y^2 + z^2$$

$$r'^2 = \beta^2(x - vt + l')^2 + y^2 + z^2$$

and
$$V = \frac{e'}{\beta}\log\frac{r+r'+l}{r+r'-l}.$$

This then gives us the potential of a moving charged line, but it also gives the potential of a charged moving conductor, the electricity being in relative equilibrium, and the form of the conductor being a surface of the family

$$V = \text{const.}$$

This family is

$$\sqrt{\{\beta^2(x - vt - l')^2 + y^2 + z^2\}} + \sqrt{\{\beta^2(x - vt + l')^2 + y^2 + z^2\}}$$
$$= \text{constant} = 2\beta a \text{ (say)}.$$

On rationalisation we get

$$\frac{(x - vt)^2}{a^2} + \frac{y^2 + z^2}{\beta^2(a^2 - l'^2)} = 1.$$

This gives a family of spheroids, and by properly choosing l' we get a spheroid of any required eccentricity moving with a velocity v. An important case is when we take $\beta^2(a - l'^2) = a^2$, or $l' = va/c$. We have then the case of a moving sphere. Or, again, by taking $l' = 0$, we reproduce a case given above in which we get a prolate spheroid which gives the same forces as a moving point charge.

We shall now consider a case of the propagation of plane light waves. If we take

$$X = X_0 \sin\frac{2\pi}{\lambda}(lx + my + nz + ct)$$

$$Y = Y_0 \sin\frac{2\pi}{\lambda}(lx + my + nz + ct)$$

$$Z = Z_0 \sin\frac{2\pi}{\lambda}(lx + my + nz + ct)$$

$$L = L_0 \sin\frac{2\pi}{\lambda}(lx + my + nz + ct)$$

$$M = M_0 \sin\frac{2\pi}{\lambda}(lx + my + nz + ct)$$

$$N = N_0 \sin\frac{2\pi}{\lambda}(lx + my + nz + ct),$$

we have the electric and magnetic vectors for a plane-polarised beam of light coming from the direction (l, m, n), the wave length being λ, and the amplitude of the component electric and magnetic vibrations being $X_0\ Y_0\ Z_0$, $L_0\ M_0\ N_0$, etc. On insertion of these values in the electromagnetic equations we find that

$$X_0 = m\,N_0 - n\,M_0$$

$$Y_0 = n\,L_0 - l\,N_0$$

$$Z_0 = l\,M_0 - mL_0$$

equations which express the facts that the electric and magnetic vectors are in the wave front and are at right angles to one another.

Let us see how this train of waves would be measured by the moving system. We find that the argument of the circular function

$$\frac{2\pi}{\lambda}(lx + my + nz + ct)$$

becomes

$$\frac{2\pi}{\lambda}\{l\beta(\xi + v\tau) + m\eta + n\zeta + \beta c(\tau + v\xi / c^2)\},$$

or, writing this in the form

$$\frac{2\pi}{\lambda'}(l'\xi + m'\eta + n'\zeta + c\tau),$$

we get

$$l' = \frac{l + v / c}{1 + lv / c}$$

$$m' = \frac{m}{\beta(1 + lv / c)}$$

$$n' = \frac{n}{\beta(1 + lv / c)}$$

$$\lambda' = \lambda\beta(1 + lv / c).$$

This last equation gives the equation for the Döppler effect, and the first three give the effect of aberration. These latter show that since $m' / m = n' / n$, the aberration displacement is in a plane through the x-axis.

If $l' = \cos(\theta + \epsilon)$ and $l = \cos\theta$, we have

$$\cos(\theta + \epsilon) = \frac{\cos\theta + v / c}{1 + v\cos\theta / c}.$$

If v/c is small we get

$$\cos(\theta+\epsilon) = \cos\theta + (v/c)\sin^2\theta, \quad \text{or} \quad \epsilon = -v/c \sin\theta,$$

which agrees with the observations on aberration.

The relations between the values of the electric vector are

$$X' = X_0$$

$$Y_0' = \beta\left(1+\frac{lv}{c}\right) Y_0 - m\beta(v/c) X_0$$

$$Z_0' = \beta\left(1+\frac{lv}{c}\right) Z_0 - n\beta(v/c) X_0,$$

from which we could determine the change in the position of the plane of polarisation with reference to the axes.

CHAPTER III

APPLICATIONS TO RADIATION AND ELECTRON THEORY

In dealing with radiation generally we start with the electrodynamic equations. If we differentiate the third of these equations with respect to y, and the second with respect to z, and subtract the latter result from the former, we obtain

$$c^{-1}\left(\frac{\partial^2 Z}{\partial y\,\partial t}-\frac{\partial^2 Y}{\partial z\,\partial t}\right)=\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right)-\frac{\partial^2 L}{\partial x^2}-\frac{\partial^2 L}{\partial y^2}-\frac{\partial^2 L}{\partial z^2},$$

from which, on making use of the fourth equation and of the equation $\dfrac{\partial\alpha}{\partial x}+\dfrac{\partial\beta}{\partial y}+\dfrac{\partial\gamma}{\partial z}=0$, we find

$$\frac{\partial^2 L}{\partial x^2}+\frac{\partial^2 L}{\partial y^2}+\frac{\partial^2 L}{\partial z^2}=c^{-2}\frac{\partial^2 L}{\partial t^2}.$$

In the same way we find that every one of the six quantities X, Y, Z, L, M, N is annihilated by the operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-c^{-2}\frac{\partial^2}{\partial t^2}.$$

This operator plays a part in the theory of radiation similar to that of Laplace's operator $\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$ in the theory of attractions and electrostatics. In passing, we may notice that since $\dfrac{\partial}{\partial x}$, $\dfrac{\partial}{\partial y}$, $\dfrac{\partial}{\partial z}$, $\dfrac{\partial}{\partial t}$ are cogredient with $\dfrac{\partial}{\partial \xi}$, $\dfrac{\partial}{\partial \eta}$, $\dfrac{\partial}{\partial \zeta}$, $\dfrac{\partial}{\partial \tau}$ we have

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-c^{-2}\frac{\partial^2}{\partial t^2}\equiv\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}+\frac{\partial^2}{\partial \zeta^2}-c^{-2}\frac{\partial^2}{\partial \tau^2}.$$

I may also remark here that we owe to Prof. Whittaker a general solution of the equation $\partial^2 V/\partial x^2+\partial^2 V/\partial y^2+\partial^2 V/\partial z^2=c^{-2}\partial^2 V/\partial t^2$ in the form

$$V=\int_0^{\pi}\int_0^{2\pi} f(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta-ct,\ \theta,\ \phi)\,d\theta\,d\phi$$

where f is an arbitrary function.

If χ and ψ are two general solutions of the above form, Prof. Whittaker has also shown that any solution of the electromagnetic equations can be put in the form

$$c^{-1}X = \frac{\partial^2\chi}{\partial y^2} + \frac{\partial^2\chi}{\partial z^2}$$

$$c^{-1}Y = -\frac{\partial^2\chi}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial z\,\partial t}$$

$$c^{-1}Z = -\frac{\partial^2\chi}{\partial x\,\partial z} + \frac{\partial^2\psi}{\partial y\,\partial t}$$

$$L = -\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2}$$

$$M = -c^{-2}\frac{\partial^2\chi}{\partial z\,\partial t} + \frac{\partial^2\psi}{\partial x\,\partial y}$$

$$N = c^{-2}\frac{\partial^2\chi}{\partial y\,\partial t} + \frac{\partial^2\psi}{\partial x\,\partial z}.$$

If we make use of our Relativity transformation, we find the very interesting fact that these remarkable functions χ and ψ transform into themselves, and are in fact "absolute invariants" for our transformation. It will be noticed that these functions χ and ψ are symmetrical about the x-axis. We can get a more symmetrical but not more general form by introducing six functions, ϕ_1, ϕ_2, ϕ_3, ψ_1, ψ_2, ψ_3, where ϕ_1 and ψ_1 are symmetrical about the x-axis, ϕ_2 and ψ_2 symmetrical about the y-axis, ϕ_3 and ψ_3 about the z-axis. These get transformed into functions ϕ_1', ϕ_2', ϕ_3', ψ_1', ψ_2', ψ_3 according to the following scheme:—

$$\phi_1' = \phi_1 \qquad\qquad \psi_1' = \psi_1$$

$$\phi_2' = \beta(\phi_2 - v\psi_3) \qquad\qquad \psi_2' = \beta(\psi_2 + v\phi_3)$$

$$\phi_3' = \beta(\phi_3 + v\psi_2) \qquad\qquad \psi_3' = \beta(\psi_3 - v\psi_2)$$

so that we have the curious result that these functions ϕ_1, ϕ_2, ϕ_3, ψ_1, ψ_2, ψ_3 are cogredient with X, Y, Z, L, M, N.

When we wish to treat generally the solutions of the electromagnetic equations we begin by introducing, after the manner of Maxwell, a vector F, G, H called the vector potential. It is defined as follows—if $(L\,M\,N)$ denote the magnetic force, we have

$$L = c\left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right)$$

$$M = c\left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}\right)$$

$$N = c\left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)$$

This representation is possible on account of the fact that

$$\partial L/\partial x + \partial M/\partial y + \partial N/\partial z = 0.$$

If we introduce these values in the equations

$$c^{-1}\frac{\partial X}{\partial t} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z},\ \text{etc.},$$

we find first that

$$X + c^{-1}\frac{\partial F}{\partial t},\quad Y + c^{-1}\frac{\partial G}{\partial t},\quad Z + c^{-1}\frac{\partial H}{\partial t}$$

are differential coefficients of a function $-\phi$ (say), so that we have

$$X = -\frac{\partial \phi}{\partial x} - \frac{\partial F}{\partial t}$$

$$Y = -\frac{\partial \phi}{\partial y} - \frac{\partial G}{\partial t}$$

$$Z = -\frac{\partial \phi}{\partial z} - \frac{\partial H}{\partial t}.$$

From the equation $\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z} = 0$

and the equation $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = c^{-2}\dfrac{\partial^2 \phi}{\partial t^2}$

we find $\dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} + \dfrac{\partial H}{\partial z} + c^{-2}\dfrac{\partial \phi}{\partial t} = 0.$

The quantity ϕ is usually called the scalar potential, and we have, in terms of ϕ and the vector (F, G, H), a means of representing the field, convenient for many purposes. If we seek now for the transformed quantities ϕ', F', G', H', we have

$$X' = X = -\beta\frac{\partial \phi}{\partial \xi} + \frac{\beta v}{c^2}\frac{\partial \phi}{\partial \tau} - \beta\frac{\partial F}{\partial \tau} + v\beta\frac{\partial F}{\partial \xi}$$

and similar equations, so that

$$F' = \beta\,(F - v\,\phi\,/\,c^2)$$

$$G' = G$$

$$H' = H$$

$$\phi' = \beta\,(\phi - v\,F)\,,$$

so that the four quantities cF, cG, cH, ϕc^{-1}, are cogredient with x, y, z, t.

We have next to consider the question of dynamics. The system of Newton is known, by the accuracy of astronomical predictions and otherwise, to be true, at any rate, to a very high degree of approximation; but, on the other hand, the velocities relative to our axes of reference which we consider in ordinary dynamics, are very small compared with the speed of light. The inquiry then arises, What are the laws of dynamics when we are no longer restricted to such small velocities? We have such velocities in the famous experiments of Kaufmann. In these experiments the β-particles of radium moving with velocities almost as great as three-fourths the speed of light were subjected to transverse electric and magnetic forces, the direction of these forces being the same. The displacement due to the electric force was in the plane containing this force and the direction of the velocity; the displacement due to the magnetic force was at right angles to this plane and was proportional to the velocity. From the observations recorded, it was found that the "mass" increased as the speed increased. Various theoretical formulæ were deduced by Abraham and others which agreed well with these results. The formula of Lorentz, $m\,/\,\sqrt{(1 - v^2\,/\,c^2)}$, or $m\,\beta$, where m is the mass for slow speeds, *i.e.*, the Newtonian mass, gives, however, probably the best agreements with the observed numbers.

We shall now see what account the Relativity Principle gives of this. To begin with, the velocity changed very little in actual magnitude during the experiment, so that the motion is what is termed "quasi-stationary." In other words, suppose the particle starts with a velocity v, then if we take axes moving with velocity v, the motion of the particle relative to those axes will be slow, and therefore the laws of Newton can be applied to such a motion.

Suppose, then, that e is the charge and that the electric force is Y, and the magnetic force is M, we have then, in the moving system,

$$Y' = \beta Y; \qquad Z' = \beta v M / c.$$

We have thus, in the moving system, the equations of motion

$$m \frac{\partial^2 \eta}{\partial \tau^2} = \beta Y$$

$$m \frac{\partial^2 \zeta}{\partial \tau^2} = \beta v M / c$$

where m is the Newtonian mass. If we recall the formulae which we deduced earlier for $\partial^2 \eta / \partial \tau^2$ and $\partial^2 \xi / \partial \tau^2$, on putting $\ddot{x} = 0$ and $\dot{x} = v$, we find

$$m \beta^2 \ddot{y} = \beta Y$$

$$m \beta^2 \ddot{x} = \beta v M / c$$

$$\text{i.e.} \quad m \beta \ddot{y} = Y$$

$$m \beta \ddot{z} = v M / c,$$

so that the mass is $m\beta$, which agrees with Lorentz's formula and with experiment.

To carry this theory further, we must consider the electrical theory of inertia as applied to an electron. Suppose that we have a distribution of electricity throughout a certain volume and contained inside a certain surface, and let us term this system an electron. The volume density is ρ, and the current vector is (U, V, W). This current may be simply the convection of the electrical volume density, or it may consist partly of this and partly of a relative motion of portions of the electron. The forces acting will be assumed to be of two kinds—(1) non-electrical, (2) electrical. Of the first kind we will assume that they form a system in equilibrium amongst themselves, or, in other words, that they obey Newton's Law of equality of action and reaction or its more general expression, D'Alembert's Principle, as used in deducing the equations of motion of a rigid body.* As to (2) we assume that the expressions for them are those given by the

* An example of such a force would be a uniform hydrostatic pressure over the boundary.

theory of Maxwell, *i.e.* the mechanical forces of electrical origin per unit volume are given by

$$P = X\rho + \frac{1}{c}(VN - WM)$$

$$Q = Y\rho + \frac{1}{c}(WL - UN)$$

$$R = Z\rho + \frac{1}{c}(UM - VL).$$

We may also add the expression for the rate of working or activity of these forces

$$A = XU + YV + ZW.$$

If we express the fact that these forces are also in equilibrium amongst themselves, we get

$$\iiint P\,dx\,dy\,dz = 0$$

$$\iiint Q\,dx\,dy\,dz = 0$$

$$\iiint R\,dx\,dy\,dz = 0.$$

If we also assume that the total electrical activity is zero, we have

$$\iiint A\,dx\,dy\,dz = 0.$$

We have also the equations of the couples

$$\iiint (yR - zQ)\,dx\,dy\,dz = 0$$

$$\iiint (zP - xR)\,dx\,dy\,dz = 0$$

$$\iiint (xQ - yP)\,dx\,dy\,dz = 0.$$

These equations form the basis of the electrical theory of inertia and of the motion of electrons in the same way that D'Alembert's Principle enters into Dynamics.

To enter into this more fully would lead us too far into the Dynamics of Electrons, but a simple example may make the matter clearer. Suppose that an electron of charge e of any symmetrical shape is moving with a slow motion along the axis of x under the influence of an electric force X_0, then the total electric force is $X_0 + X_i$, where X_i arises from the motion of the electron, and so the equation

$$\iiint P\,dx\,dy\,dz = 0$$

becomes
$$\iiint X_0\,\rho\,dx\,dy\,dz + \iiint X_i\,\rho\,dx\,dy\,dz = 0$$

or
$$X_0\,e + \iiint X_i\,\rho\,dx\,dy\,dz = 0.$$

Now, on calculating X_i and finding the value of the integral, we find that the equation becomes

$$X_0\,e - mf = 0,$$

where m is a constant—the electromagnetic mass—and f is the acceleration.

The particular shape of the formula will depend on our assumptions as to the structure of the electron; but for motion, where the loss from radiation can be neglected, a convenient form which has many arguments in favour of it is the Lorentz mass-formula. This gives as the equations of motion of a particle

$$\frac{d}{dt}\,\frac{m\dot{x}}{\sqrt{(1-(\dot{x}^2+\dot{y}^2+\dot{z}^2)\,c^{-2})}} = F_x$$

$$\frac{d}{dt}\,\frac{m\dot{y}}{\sqrt{\{1-(\dot{x}^2+\dot{y}^2+\dot{z}^2)c^{-2}\}}} = F_y$$

$$\frac{d}{dt}\,\frac{m\dot{z}}{\sqrt{\{1-(\dot{x}^2+\dot{y}^2+\dot{z}^2)\,c^{-2}\}}} = F_z$$

where F_x, F_y, F_z is the mechanical force. We may notice that if this mechanical force is of electrical origin we have from the assumption of Lorentz and Larmor

$$F_x = \left[X + \frac{1}{c}(\dot{y}N - \dot{z}M)\right]e$$

where e is the charge.

If we introduce a vector P_x, P_y, P_z, such that

$$P_x = \frac{F_x}{\sqrt{\{1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, c^{-2}\}}}$$

$$P_y = \frac{F_y}{\sqrt{\{1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, c^{-2}\}}}$$

$$P_z = \frac{F_z}{\sqrt{\{1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,.\, c^{-2}\}}},$$

the equations can be put into a more symmetrical shape.

Writing

$$d\sigma^2 = dt^2 - c^{-2}\,(dx^2 + dy^2 + dz^2)$$

we get

$$m\,\frac{d^2 x}{d\sigma^2} = P_x$$

$$m\,\frac{d^2 y}{d\sigma^2} = P_y$$

$$m\,\frac{d^2 z}{d\sigma^2} = P_z.$$

We also have

$$m\,\frac{d^2 t}{d\sigma^2} = A$$

where
$$P_x \frac{dx}{d\sigma} + P_y \frac{dy}{d\sigma} + P_z \frac{dz}{d\sigma} - c^2 A \frac{dt}{d\sigma} = 0$$

on account of the fact that

$$\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2 - c^2\left(\frac{dt}{d\sigma}\right)^2 = -c^2,$$

and that therefore

$$\frac{dx}{d\sigma}\,\frac{d^2 x}{d\sigma^2} + \frac{dy}{d\sigma}\,\frac{d^2 y}{d\sigma^2} + \frac{dz}{d\sigma}\,\frac{d^2 z}{d\sigma^2} = c^2\,\frac{dt}{d\sigma}\,\frac{d^2 t}{d\sigma^2}\,.$$

The quantity A is $c^{-2}\dfrac{F_x\dot{x} + F_y\dot{y} + F_z\dot{z}}{\sqrt{\{1 - c^{-2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)\}}}$,

and the quantities P_x, P_y, P_z and A are cogredient with x, y, z and t.

We shall only give one example of these equations of motion.*

Suppose that a particle of mass m describes an orbit in the plane of (x, y), about a centre of force which varies inversely as the square of the distance, say $m\mu/r^2$, where μ is a constant; we have then

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1 - v^2/c^2}} = -\mu x/r^3$$

$$\frac{d}{dt}\frac{\dot{y}}{\sqrt{1 - v^2/c^2}} = -\mu y/r^3$$

where $v^2 = \dot{x}^2 + \dot{y}^2$,

or on putting $\sqrt{1 - v^2/c^2} = \beta$ and performing the differentiations

$$\beta\ddot{x} + \beta^3\dot{x}v\dot{v}/c^2 = -\mu x/r^3,$$

$$\beta\ddot{y} + \beta^3\dot{y}v\dot{v}/c^2 = -\mu y/r^3.$$

On multiplying by $\dot{x}$, $\dot{y}$ and adding we get

$$v\dot{v}\beta\{1 + \beta^2v^2/c^2\} = -\mu\dot{r}/r^2$$

or

$$\frac{v\dot{v}}{\{1 - v^2/c^2\}^{\frac{3}{2}}} = -\frac{\mu\dot{r}}{r^2}.$$

Hence we have the Energy Integral

$$\frac{1}{\sqrt{1 - v^2/c^2}} - 1 = c^{-2}\left\{\lambda + \frac{\mu}{r}\right\}$$

where λ is a constant. From which

$$1 - \frac{v^2}{c^2} = \left\{1 + c^{-2}\left(\lambda + \frac{\mu}{r}\right)\right\}^{-2}$$

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* For many other problems see Schott, *Electromagnetic Radiation.*

In the same way, by multiplying by $-y$ and x, we get the angular momentum integral

$$\beta r^2 \dot{\theta} = h \text{ (a constant)}$$

$$r^4 \dot{\theta}^2 = h^2 \left\{1 + c^{-2}\left(\lambda + \frac{\mu}{r}\right)\right\}^{-2}$$

$$\dot{r}^2 + r^2 \dot{\theta}^2 = c^2 \left\{1 - \left[1 + c^{-2}\left(\lambda + \frac{\mu}{r}\right)\right]^{-2}\right\} \text{ which leads to}$$

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} = \frac{c^2}{h^2}\left\{\left[1 + c^{-2}\left(\lambda + \frac{\mu}{r}\right)\right]^2 - 1\right\}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{h^2}\left\{2(\lambda + \mu u) + c^{-2}\{\lambda + \mu u\}^2\right\}.$$

If we put $\theta' = \theta(1 - \mu^2 / c^2 h^2)$, we get an integral of the type

$$L u = 1 + e \cos \theta', \text{ where } L \text{ is a constant.}$$

The general effect is the same as that of a force varying inversely as the cube of the distance.

CHAPTER IV

MINKOWSKI'S TRANSFORMATION

We now come to the concluding portion of our subject, namely, the form in which the preceding results have been stated by Minkowski. In the fundamental Relativity transformation let us put $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3$ instead of $x, y, z, \xi, \eta, \zeta$ respectively, and let us further put x_4 and ξ_4 instead of $i\,ct$ and $i\,c\tau$ respectively, i being the "imaginary" of algebra. We then get

$$\begin{aligned}
\xi_1 &= \beta\, x_1 + \frac{i\,v}{c}\,\beta\, x_4 \\
\xi_2 &= x_2 \\
\xi_3 &= x_3 \\
\xi_4 &= \beta\, x_4 - i\,v\frac{\beta}{c}\,x_1 .
\end{aligned}$$

Introducing the imaginary angle θ given by the equations $\cos\theta = \beta$; $-\sin\theta = i\,v\,\beta\,/\,c$, the transformation becomes

$$\begin{aligned}
\xi_1 &= x_1 \cos\theta - x_4 \sin\theta \\
\xi_2 &= x_2 \\
\xi_3 &= x_3 \\
\xi_4 &= x_1 \sin\theta + x_4 \cos\theta.
\end{aligned}$$

In order to study and interpret these equations, we shall consider first the motion of a point along a straight line; secondly, the motion of a point in a plane; and lastly, the motion of a point in space.

The rectilinear motion of a point along the x-axis is defined by two variables, x and t. It can be geometrically represented by a curve, the familiar space-time graph. In attempting to represent in the same way the relation between x_1 and x_4 considered as rectangular coordinates, we are met by the difficulty that the corresponding graph may be wholly or partially imaginary, *e.g.* as in the cases $x = ut$; $x = ut + \frac{1}{2}gt^2$. Yet the representation of such a curve helps us very much to visualize the different relations, and in fact we are accustomed to such a procedure in geometry, as, for instance, when we draw the circules and tangents through them to conics, etc.

Let the curve PQ (Fig. 7) be then supposed to represent the motion of the particle as defined by x_1 and x_4. If the motion be referred to the origin moving with velocity v, the equations

$$\xi = x_1 \cos\theta + x_4 \cos\theta\,;$$
$$\xi_4 = x_1 \sin\theta - x_4 \sin\theta$$

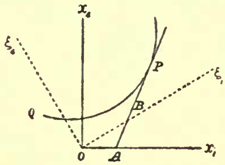

Fig 7

show that this transformation is geometrically equivalent to referring the system to new axes $O\xi_1$, $O\xi_4$, making an angle θ with the former axes respectively. This throws also a new light on the transformation of velocity

$$\frac{d\xi}{d\tau} = \frac{\dfrac{dx}{dt} - v}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}}.$$

In fact, noticing that dx_1/dx_4 is the tangent of the angle α, which AP (the tangent at P) makes with Ox_1, and that $d\xi_1/d\xi_4$ is the tangent of the angle β which AP makes with $O\xi_1$, the above relation is easily seen to be equivalent to

$$\tan\beta = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\,\tan\theta} \text{ or } \beta = \alpha - \theta.$$

The relation between the accelerations is more complicated, but it is easily put in the form

$$\frac{d^2\xi_1}{d\xi_4^2}\left[1 + \left(\frac{d\xi_1}{d\xi_4}\right)^2\right]^{-\frac{3}{2}} = \frac{d^2x_1}{dx_4^2}\left[1 + \left(\frac{dx_1}{dx_4}\right)^2\right]^{-\frac{3}{2}}.$$

We see at once that this merely asserts that the ordinary expression for curvature gives the same result no matter what rectangular axes are used.

Coming now to the motion of a point in a plane defined by the three coordinates x_1, x_2, x_4, we see that the motion can be represented by a curve in the three dimensional space of x_1, x_2, x_4, the projection of this curve on the plane of (x_1, x_2) being the actual path of the particle. If, now, the motion is referred to an origin moving with velocity v along the x-axis, we see that this is equivalent to turning the system x_1, x_2, x_4 about the axis of x_2, and if the motion is referred to system moving with velocity v' along the y-axis of this *latter* system, this is equivalent to a *subsequent* rotation about the x-axis of this latter system. The two operations thus described are not commutative. In fact, being *finite* rotations, if performed in the reverse order they would give a different result. If, however, the velocities v and v' are small compared with c, these operations *are* commutative. This fact throws a light on the conception of relative velocities in the Newtonian system. What is fixed then about the path of the particle is the curve in the (x_1, x_2, x_4) space, which remains invariable, whilst we can choose any axes to describe its properties.

Before dealing with the general motion of a particle, we shall recall some results in the transformation of rectangular axes; if we consider the scheme

$$\begin{aligned}
\xi_1 &= l_1 x_1 + l_2 x_2 + l_3 x_3 \\
\xi_2 &= m_1 x_1 + m_2 x_2 + m_3 x_3 \\
\xi_3 &= n_1 x_1 + n_2 x_2 + n_3 x_3 .
\end{aligned}$$

Then the quantities l_1, l_2, etc., satisfy various relations, such as

$$l_1^2 + l_2^2 + l_3^2 = 1\ ;\quad l_1 m_1 + l_2 m_2 + l_3 m_3 = 0\ ;\quad l_1 = m_2 n_3 - m_3 n_2, \text{ etc.}$$

A vector, that is, a directed quantity obeying the parallelogram law, can have its three components represented by distances taken along the three axes, and will thus obey the same transformation as above. We may notice that in all such cases

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = x_1^2 + x_2^2 + x_3^2.$$

Certain operators obey the same laws, and may thus be called vector operators. Thus we easily see that

$$\begin{aligned}
\frac{\partial}{\partial \xi_1} &= l_1 \frac{\partial}{\partial x_1} + l_2 \frac{\partial}{\partial x_2} + l_3 \frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial \xi_2} &= m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2} + m_3 \frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial \xi_3} &= n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} + n_3 \frac{\partial}{\partial x_3}
\end{aligned}$$

$$\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

This latter result expresses the "invariance" of Laplace's operator. *Conversely, the above transformations may be regarded as tests by means of which we can recognise whether three quantities define a vector or not.*

In the case of two vectors there are two quantities, one scalar, and the other vector, which express invariant properties independent of the axes. Thus, if $(x_1\, x_2\, x_3)$ and $(x_1'\, x_2'\, x_3')$ are two vectors, which become $(\xi_1\, \xi_2\, \xi_3)$ and $(\xi_1'\, \xi_2'\, \xi_3')$ when referred to new axes, we have at once

$$\xi_1 \xi_1' + \xi_2 \xi_2' + \xi_3 \xi_3' = x_1 x_1' + x_2 x_2' + x_3 x_3'.$$

Each side is called the *inner* product of the corresponding vectors, and is equal to the product of the magnitudes of the vectors $\sqrt{(x_1^2 + x_2^2 + x_3^2)}$ and $\sqrt{(x_1'^2 + x_2'^2 + x_3'^2)}$ into the cosine of the angle between them.

Also the three quantities

$$x_2 x_3' - x_3 x_2', \quad x_3 x_1' - x_1 x_3', \quad x_1 x_2' - x_2 x_1'$$

will be found to satisfy the test given above for vectors. They constitute what is termed the *vector* product of the two quantities. They represent a vector the magnitude of which is the product of the magnitudes of the two vectors into the sine of the angle between their directions, and the direction is at right angles to the two vectors. It may be noticed that one of the vectors employed as above may be a vector operator, thus, if $(u_1\, u_2\, u_3)$ is a vector, the scalar quantity

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

and the vector

$$\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \quad \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_1}$$

are related to the vector $(u_1\, u_2\, u_3)$ in a manner independent of the axes.

The above remarks will prepare us for a consideration of four-dimensional space, in which we have no geometrical intuitions to guide us, but have to rely on analytical transformations. In a general orthogonal transformation in four dimensions

$$\xi_1 = l_{11} x_1 + l_{12} x_2 + l_{13} x_3 + l_{14} x_4$$

$$\xi_2 = l_{21} x_1 + l_{22} x_2 + l_{23} x_3 + l_{24} x_4,$$

$$\text{etc.},$$

where $l_{12} = l_{21}$, etc., we have

$$l_{11}^2 + l_{12}^2 + l_{13}^2 + l_{14}^2 = 1$$

$$l_{11} l_{21} + l_{12} l_{22} + l_{13} l_{23} + l_{14} l_{24} = 0, \text{ etc.}$$

and each quantity, such as l_{11}, is equal to the minor of that quantity with proper sign in the determinant formed by the l's.

We may define a "four-vector," or set of four quantities forming a vector, as a set satisfying the above transformation. We may notice that the four quantities may be vector operators, such as $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial x_3}$, $\frac{\partial}{\partial x_4}$, and that

$$\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2} + \frac{\partial^2}{\partial \xi_4^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}$$

in the same way as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2.$$

When we come to two vectors we meet as before the inner product $x_1 x_1' + x_2 x_2' + x_3 x_3' + x_4 x_4'$, which we can easily verify to be a quantity independent of the axes of reference. If, however, we try to form the vector product, we get not four but six quantities

$$x_2 x_3' - x_3 x_2', \; x_3 x_1' - x_1 x_3', \; x_1 x_2' - x_1 x_2',$$

$$x_1 x_4' - x_1' x_4, \; x_2 x_4' - x_2' x_4, \; x_3 x_4' - x_3' x_4,$$

which we may write for shortness

$$y_{23} \; y_{31} \; y_{12} \; y_{14} \; y_{24} \; y_{34} \cdot$$

We can see at once that these quantities are transformed by the transformation

$$\eta_{23} = \lambda_{11} y_{23} + \lambda_{12} y_{31} + \text{etc.},$$

where the λ's are the second minors of the l determinant.

Conversely, any six quantities transformed by what we may call the λ transformation may be called a six-vector.

A four-vector and a six-vector can be combined in two ways to form a four-vector. Thus, if (z_1, z_2, z_3, z_4) is a four-vector, we have

$$z_2 y_{34} + z_3 y_{42} + z_4 y_{23}\,,$$
$$z_3 y_{41} + z_4 y_{13} + z_1 y_{34}\,,$$
$$z_4 y_{12} + z_1 y_{24} + z_2 y_{41}\,,$$
$$z_1 y_{23} + z_2 y_{31} + z_3 y_{12}\,,$$

and

$$z_2 y_{12} + z_3 y_{13} + z_4 y_{14}\,,$$
$$z_3 y_{23} + z_4 y_{24} + z_1 y_{21}\,,$$
$$z_4 y_{34} + z_1 y_{31} + z_2 y_{32}\,,$$
$$z_1 y_{41} + z_2 y_{42} + z_3 y_{43}\,.$$

We can verify by actual substitution that these two sets of four quantities each are actually four-vectors after the definition given above. Also it may be remarked that the above statement holds true when we take, instead of (z_1, z_2, z_3, z_4), the vector operator $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial x_4)$.

After this preliminary survey of four- and six-vectors, let us return to the Relativity Transformation

$$\xi_1 = x_1 \cos\theta - x_4 \sin\theta$$
$$\xi_2 = x_2$$
$$\xi_3 = x_3$$
$$\xi_4 = x_1 \sin\theta + x_4 \cos\theta.$$

It is seen at once that the substitution is an orthogonal one, and that thus a point (x, y, z) and an associated time t correspond to a point in space of four dimensions. A new meaning of certain invariants as given above will now be at once evident. For instance, the invariance of the expression $x^2 + y^2 + z^2 - c^2t^2$ becomes in our new variables $\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, which can be interpreted as meaning that the distance of the point (x_1, x_2, x_3, x_4) from the origin remains unaltered by the orthogonal transformation. As other invariants we might mention the element of arc $\sqrt{(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)}$ and the differential operator for wave propagation $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2 + \partial^2/\partial x_4^2$. When we apply these ideas to the electromagnetic relations, a surprising symmetry becomes evident. If we denote the electric current components U, V, and W by U_1, U_2, and U_3 respectively, and the volume density of electricity ρ by $(ic)^{-1} U_4$, we have then, since it was proved that U, V, W and ρ are cogredient with x, y, z, and t, the fact that U_1, U_2, U_3, U_4 is a *four-vector*. In a similar manner, if we use the vector potential (F, G, H) and the scalar potential ϕ, and if we take quantities (F_1, F_2, F_3, F_4) defined by the equations

$$F_1 = cF, \quad F_2 = cG, \quad F_3 = cH, \quad F_4 = i\phi,$$

then (F_1, F_2, F_3, F_4) is a *four-vector*. We may notice in passing that the equation of continuity

$$\partial U / \partial x + \partial V / \partial y + \partial W / \partial z = \partial \rho / \partial t$$

becomes

$$\partial U_1 / \partial x_1 + \partial U_2 / \partial x_2 + \partial U_3 / \partial x_3 + \partial U_4 / \partial x_4 = 0.$$

From the vector operator $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$, $\partial/\partial x_4$ and the generalized potential $(F_1\, F_2\, F_3\, F_4)$ we can form a *six-vector*

$$\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \quad \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \quad \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}, \quad \frac{\partial F_4}{\partial x_1} - \frac{\partial F_1}{\partial x_4},$$

$$\frac{\partial F_4}{\partial x_2} - \frac{\partial F_2}{\partial x_4}, \quad \frac{\partial F_4}{\partial x_3} - \frac{\partial F_3}{\partial x_4}.$$

These become respectively

$$c\left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right), \quad c\left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}\right), \quad c\left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right),$$

$$i\left(\frac{\partial \phi}{\partial x} + \frac{\partial F}{\partial t}\right), \quad i\left(\frac{\partial \phi}{\partial y} + \frac{\partial G}{\partial t}\right), \quad i\left(\frac{\partial \phi}{\partial z} + \frac{\partial H}{\partial t}\right),$$

or

$$L, \quad M, \quad N, \quad -iX, \quad -iY, \quad -iZ,$$

which we may symmetrically write

$$L_{23}, \quad L_{31}, \quad L_{12}, \quad L_{14}, \quad L_{24}, \quad L_{34}.$$

Thus the magnetic and electric forces form a *six-vector*.

Using the operator $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial x_4)$ and this *six-vector*, we can form two four-vectors. The first is

$$\frac{\partial L_{34}}{\partial x_2} + \frac{\partial L_{42}}{\partial x_3} + \frac{\partial L_{23}}{\partial x_4}$$

$$\frac{\partial L_{41}}{\partial x_3} + \frac{\partial L_{13}}{\partial x_4} + \frac{\partial L_{34}}{\partial x_1}$$

$$\frac{\partial L_{12}}{\partial x_4} + \frac{\partial L_{24}}{\partial x_1} + \frac{\partial L_{41}}{\partial x_2}$$

$$\frac{\partial L_{23}}{\partial x_1} + \frac{\partial L_{31}}{\partial x_2} + \frac{\partial L_{12}}{\partial x_3}.$$

If we equate these four components to zero, we get in the earlier notation

$$-c^{-1}\frac{\partial L}{\partial t}=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}$$

$$-c^{-1}\frac{\partial M}{\partial t}=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial y}$$

$$-c^{-1}\frac{\partial N}{\partial t}=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}$$

$$\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}-\frac{\partial N}{\partial z}=0.$$

In the same way, from the other four-vector which can be formed, we can derive the other four fundamental electrodynamical equations.

As a last example, consider the four-vector

$$c^{-1}(U_2L_{12}+U_3L_{13}+U_4L_{14})$$

$$c^{-1}(U_3L_{23}+U_4L_{24}+U_1L_{21})$$

$$c^{-1}(U_4L_{34}+U_1L_{31}+U_2L_{32})$$

$$c^{-1}(U_1L_{41}+U_2L_{42}+U_3L_{43}),$$

these become on substitution

$$Xe+c^{-1}(VN-WM)$$

$$Ye+c^{-1}(WL-UN)$$

$$Ze+c^{-1}(UM-VL)$$

$$i\,c^{-1}(XU+YV+ZW).$$

The first three of these represent the mechanical force on an electric charge, and the fourth is $i\,c^{-1}$ multiplied by the activity.

For further examples, reference must be made to the work of Minkowski above referred to. What we have said is, however, sufficient to indicate the point of view of this theory. A point in ordinary space and a definite time is represented as a point in four-dimensional space. A moving particle is represented by a fixed curve in this space. The question of absolute rest in ordinary space ceases now to have any meaning. For, in the four-dimensional isotropic space, one set of axes is as good as another for describing its properties. The various electrodynamical relations take their position in a manner which reveals a symmetry which was by no means apparent in the unsymmetrical equations founded on our experimental knowledge. The whole scheme, in one aspect, is merely an analytical development of the Einstein Relativity. Both would fall together if any experimental fact appeared which would upset one, and can it not be said that the probability of both being true is increased by this elegant symmetry?

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