We see that Euclidean space admits the law of uniform translation of a rigid system relative to another. The moving system undergoes contraction in the direction of motion. This result is, apparently, the only solution of the equations (21).

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[2] A point A_i for all values of t_i may be considered as a world line in four-dimensional space-time.

CONCERNING PATHS THAT DO NOT SEPARATE A GIVEN CONTINUOUS CURVE

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In this paper it will be shown that, in space of two dimensions, every two points that do not belong to a given continuous curve may be joined by a simple continuous arc that does not disconnect that curve. First, certain auxiliary theorems will be established or, in certain cases, stated without proof.

THEOREM 1. *If M is a continuous curve there do not exist two distinct bounded complementary domains of M with the same outer*[1] *boundary.*

Theorem 1 may be proved with the aid of propositions, concerning the outer boundary of one domain with respect to another, given in footnote 4, on page 475, of my paper Concerning the Separation of Point Sets by Curves.[2]

THEOREM 2. *If a and b are distinct bounded complementary domains of the continuous curve M and the boundaries of a and b have a point P in common then P belongs either to the outer boundary of a or to the outer boundary of b.*

Theorem 2 may be proved with the aid of theorem 1.

THEOREM 3. *If J_1 and J_2 are simple closed curves enclosing the point O and each of the mutually exclusive arcs A_1B_1 and A_2B_2 has one end-point on J_1 and the other one on J_2 but no point, except its end-points, in common either with J_1 or with J_2 then the point set $J_1 + J_2 + A_1B_1 + A_2B_2$ contains a simple closed curve that encloses O and contains either A_1B_1 or A_2B_2.*

Theorem 3 may be proved with the aid of propositions established in my paper "On the Foundations of Plane Analysis Situs."[3]

THEOREM 4. *If D_1 and D_2 are distinct complementary domains of a continuous curve and B_1 and B_2 are their respective boundaries and B is the boundary of a complementary domain of the point set $D_1 + D_2 + B_1 + B_2$ then the three point sets B, B_1 and B_2 do not have more than two points in common.*

THEOREM 5. *If M is a continuous curve and K is the set of all points X such that X is common to the boundaries B_1 and B_2 of two distinct complementary domains D_1 and D_2 of M and is on the boundary of some complementary domain of the point set $D_1 + D_2 + B_1 + B_2$ then K is countable.*

Theorem 5 may be easily established with the aid of theorem 4 and the fact that no continuum has more than a countable number of complementary domains.

THEOREM 6. *If D is a bounded complementary domain of a continuous curve M and J is the outer boundary of D and I is the set of all those points of M that lie within J, then* (1) *no point of I belongs to a simple closed curve that lies in M and encloses a point of J,* (2) *if T is a maximal connected subset of I, J contains only one limit point of T.*

Part (1) of the conclusion of theorem 6 may be proved with the use of certain propositions established in my paper "On the Foundations of Plane Analysis Situs."[3] Part (2) may be proved with the help of similar considerations together with theorem 1 of R. L. Wilder's paper "Concerning Continuous Curves."[4]

THEOREM 7. *If the point O lies in a bounded complementary domain of the continuous curve M and K is the point set obtained by adding together all simple closed curves that lie in M and enclose O, then K is closed.*

Proof. Suppose P is a limit point of K. Then if P does not belong to K it is a limit point either of N or of N_2 where N is the sum of the set G of all simple closed curves that lie in M and enclose both O and P and N_2 is the sum of all those that enclose O but not P. I will assume[5] that P is a limit point of N. If P lies on the boundary of a complementary domain of M it must lie on the outer boundary of that domain, and if the simple closed curve[1] which constitutes this outer boundary should enclose O then P would belong to K. Suppose that P does not belong to the boundary of a complementary domain of M whose outer boundary encloses O. If there exist any complementary domains of M whose boundaries contain P, let H denote the set of all the outer boundaries of such domains, let U denote the sum of the interiors of all the curves of the set H and let T denote the point set consisting of U plus its boundary. The curve M does not[6] have more than a finite number of complementary domains of diameter greater than a given positive positive number. Hence there are not more than a finite number of curves of the set H of diameter greater than such a number. But the point set composed of any such curve plus its interior is itself a continuous curve and T is the sum of all such point sets. It follows, by a theorem of Sierpinski's,[7] that T is a continuous curve. Therefore $M + T$ is a continuous curve. But P does not belong to the boundary of any complementary domain of $M + T$. Hence there exists[8] a sequence α of simple closed curves $j_1, j_2, j_3, \ldots$, all belonging to $M + T$ and enclosing P, such that for every n the diameter of j_n is less than $1/n$, and j_n encloses j_{n+1} but does not enclose O. Suppose some curve J of the set G contains two distinct points A and B in common with a curve Q of the set H. If P does not belong to J, the point O is enclosed by the simple closed curve L consisting of a certain arc of J together with a certain interval of the arc APB of Q. But L lies wholly in M, contains P and encloses O. Hence, in this case, P belongs to K. Suppose some curve J of the set G contains only one point A in common with one curve Q_1 of the set H and only one point B in common with a certain other curve Q_2 of the set H and suppose that the points A and B are distinct. If AP denotes one of the arcs of Q_1 from A to P, it may be shown that there is one arc of Q_2 from B to P which has no point except P in common with AP. Let BP denote such an arc of Q_2. Let AB denote the sum of the arcs AP and BP. There is an arc of J which together with AB forms a simple closed curve that lies wholly in M, encloses O and contains P. Hence, in this case also, P belongs to K. Suppose now that some curve of the set G has one and only one point E in common with T, and suppose that E is distinct from P. In this case it can easily be shown that there exist other curves of G which pass through E and that indeed P is a limit point of the sum of all such curves. The point E lies on some curve J_E of the set H. Since M is a continuous curve there exists[9] a positive number d such that every point of M whose distance from P is less than d can be joined to P by a simple continuous arc which lies wholly in M and whose diameter is less than e, the distance from P to E. But there exists a curve J of the set G which passes through E and which contains a point X at a distance from P less than d. Suppose J has no point except E in common with T. Let XP denote an arc which is a subset of M and whose diameter is less than e. Let Z denote the last point that XP has in common with J and let Y denote the first point after Z that it has in common with T. Let ZY denote the interval of XP whose end-points are Z and Y. Let J_Y denote a curve of H which contains Y. There exists an arc YE which is a subset of $J_E + J_Y$. Let ZYE denote the arc obtained by adding together the two arcs ZY and YE. There exists on J an arc EWZ such that the simple closed curve formed by the two arcs EWZ and ZYE encloses O. But this curve is a subset of M and it has two points Y and E in common with T. It follows, by a result established above, that P belongs to K. Suppose finally that no curve of the set G has a point in common with the point set T. If T exists, in other words if there exists at least one simple closed curve C belonging to the set H, then there exists an integer n such that j_n contains at least one point within C and there exists a simple closed curve J of the set G such that j_n encloses at least one point of J and it may be shown that the point set $C + j_n + J$ contains a simple closed curve which encloses O and contains P and therefore P belongs to K. If, on the other hand, T does not exist then the following argument applies. For each n there exists a simple closed curve J_n which belongs to G and contains points within j_{n+1}. There exist two mutually exclusive arcs A_nB_n and C_nD_n lying on J_n and lying, except for their end-points, wholly within j_n and wholly without j_{n+1}, the points A_n and C_n lying on j_n and the points B_n and D_n lying on j_{n+1}. If A_2 is B_1 let X_2 denote A_2 and let Y_2 denote C_2. If A_2 is not B_1 let X_2 and Y_2 denote C_2 and A_2, respectively, or A_2 and C_2, respectively, according as C_2 is or is not on that arc of j_{n+1} which has B_1 and A_2 as end-points and which does not contain D_1. In any case if B_1 is identical with X_2 let B_1X_2 denote the point B_1 and if B_1 is distinct from X_2 let B_1X_2 denote that arc of j_{n+1} which has B_1 and X_2 as end-points and which does not contain D_1. Similarly, if D_1 is Y_2 let D_1Y_2 denote D_1 and if D_1 is not Y_2 let D_1Y_2 denote that arc of j_{n+1} which has D_1 and Y_2 as end-points and which does not contain B_1. Now let X_1, Y_1, Z_1 and W_1 denote the points A_1, C_1, B_1 and D_1, respectively, and let Z_2 and W_2 denote B_2 and D_2, respectively, or D_2 and B_2, respectively, according as X_2 is identical with A_2 or with C_2. Let X_2Z_2 and Y_2W_2 denote the arcs A_2B_2 and C_2D_2 or the arcs C_2D_2 and A_2B_2, respectively, according as X_2 is identical with A_2 or with C_2. This process may be continued. Thus there exist four infinite sequences $X_1Z_1, X_2Z_2, X_3Z_3, \ldots$; $Y_1W_1, Y_2W_2, Y_3W_3, \ldots$; $Z_1X_2, Z_2X_3, Z_3X_4, \ldots$; and $W_1Y_2, W_2Y_3, W_3Y_4, \ldots$ such that, for every n, (a) X_n and Y_n lie on j_n, (b) X_nZ_n and Y_nW_n lie, except for their end-points, wholly within j_n and wholly without j_{n+1} and, indeed, one of them coincides with A_nB_n and the other one with C_nD_n, (c) Z_nX_{n+1} and W_nY_{n+1} lie on j_{n+1} and have no point in common. Let a denote the point set $P + X_1Z_1 + Z_1X_2 + X_2Z_2 + Z_2X_3 + \ldots$ and let b denote the point set $P + Y_1W_1 + W_1Y_2 + Y_2W_2 + W_2Y_3 + \ldots$. It is easy to see that a and b are simple continuous arcs. These arcs belong to M and have P as a common end-point but they have no other point in common. The point set $J_1 + a + b$ contains a simple closed curve which passes through P and encloses O. Hence P belongs to K. Thus the set K contains all of its limit points.

THEOREM 8. *If the point O lies in a bounded complementary domain of the continuous curve M and K is the point set obtained by adding together all simple closed curves that lie in M and enclose O, then K is the sum of a finite number of continuous curves.*

Proof. It has been shown that K is closed. Suppose it is not the sum of a finite number of continuous curves. Then, since every maximal connected subset of K contains a simple closed curve that encloses O and every one of these curves has a diameter greater than the shortest distance from O to K therefore either there is at least one maximal connected subset of K which is not a continuous curve or there exists a positive number such that infinitely many maximal connected subsets of K have diameters greater than that number. In either case there exist[10] two circles k_1 and

k_2 and a countable infinity of continua $M_1, M_2, M_3, \ldots$ such that (1) each of these continua contains at least one point of k_1 and at least one point of k_2 and is a subset of K and of the point set H which is composed of the two circles k_1 and k_2 together with all points that lie between them, (2) no two of these continua have a point in common and, indeed, no one of them is a proper connected subset of any connected point set which is common to K and H, (3) the point O does not belong to H. Let k_3 denote a circle concentric with k_1 and k_2 and lying between them. For each n, M_n has a point P_n in common with k_3 and k contains a simple closed curve J_n which encloses O and contains P_n. Since, for each n, J_n contains a point of k_3 and encloses O, and O does not lie between k_1 and k_2, therefore J_n contains at least one point of k_1 or at least one point of k_2. Let k_4 denote k_1 or k_2 according as there do or do not exist infinitely many distinct values of n such that J_n contains at least one point of k_1. There exists an infinite sequence of positive integers $n_1, n_2, n_3, \ldots$ such that, for each i, J_{n_i} contains, in common with M_{n_i}, an arc t_i which has one end-point on k_3 and the other one on k_4 and which lies, except for its end-points, wholly between k_3 and k_4. The arcs of the sequence $t_1, t_2, t_3, \ldots$ are mutually exclusive and no two of them can be joined by a connected subset of K that lies wholly between k_3 and k_4. But M is uniformly regular.[11] With the help of this fact it can be seen that there exist two positive integers i and j such that t_i and t_j can be joined by two mutually exclusive arcs E_1F_1 and E_2F_2 which are subsets of M and which lie wholly between k_3 and k_4, the points E_1 and E_2 belonging to t_i and the points F_1 and F_2 belonging to t_j. For each n $(n = 1,2)$ let A_n denote the last point that E_nF_n has in common with M_{n_j} and let B_n denote the first point following A_n that it has in common with M_{n_j}. Suppose that, either when $n = 1$ or when $n = 2, A_nB_n$ contains at least one point of J_{n_i} distinct from A_n. Then, for that value of n, let C_n denote the first such point in the order from A_n to B_n on the arc A_nB_n and let L_n denote the last point of J_{n_i}, other than C_n, in the order from E_n to C_n on the interval E_nC_n of the arc E_nF_n. The interval L_nC_n of the arc E_nF_n has its end-points L_n and C_n on J_{n_i} but it has no other point in common with J_{n_i}. Hence the point O is enclosed by a simple closed curve formed by the arc L_nC_n together with one of the arcs of J_{n_i} whose end-points are L_n and C_n. Therefore every point of L_nC_n belongs to K. But this is impossible since L_nC_n lies wholly between k_3 and k_4 and contains a point A_n of M_{n_i} and a point C_n which does not belong to M_{n_i}. Suppose, on the other hand, that J_{n_i} contains no point of A_1B_1 except A_1 and no point of A_2B_2 except A_2. In this case, for each n $(n = 1,2)$, the arc E_nF_n contains an interval L_nC_n which has only the point C_n in common with J_{n_i} and only the point L_n in common with J_{n_i}. By theorem 3 the point set $J_{n_i} +$

$J_{n_j} + L_1C_1 + L_2C_2$ contains a simple closed curve enclosing O and containing either A_1C_1 or A_2C_2. Hence A_1C_1 or A_2C_2 is a subset of K. Hence P_{n_j} can be joined to P_{n_i} by a connected subset of K lying between k_1 and k_2. Thus again we have a contradiction. The truth of theorem 8 is therefore established.

THEOREM 9. *If O is a point lying in a bounded complementary domain of a continuous curve M and K is the point set obtained by adding together all simple closed curves that lie in M and enclose O, then* (1) *the boundary of every complementary domain of a maximal connected subset of K is a simple closed curve and* (2) *if T is a maximal connected subset of M-K, no maximal connected subset of K contains more than one limit point of T.*

Theorem 9 may be easily proved with the help of theorems 8 and 6.

THEOREM 10. *If A and B are distinct points not belonging to the continuous curve M, there exists a simple continuous arc from A to B which does not disconnect M.*

Proof. If A and B belong to the same complementary domain of M they can be joined by an arc lying in that domain and therefore containing no point of M. Suppose that A lies in a bounded complementary domain of M and B lies in the unbounded one.[12] Let K denote the set of points obtained by adding together all simple closed curves that lie in M and enclose A. By theorem 8, K is the sum of a finite number of continuous curves. For each point P of M-K let M_P denote the maximal connected subset of M-K which contains P and let T_P denote the set M_P together with all its limit points. Let H denote the set of all such point sets T_P for all points P of M-K. According to a theorem of W. L. Ayres,[13] if a continuous curve N is a proper subset of a continuous curve M and d is any positive number, there are not more than a finite number of maximal connected subsets of M-N of diameter greater than d. It can be seen that this proposition remains true if, instead of being a single continuous curve, N is the sum of a finite number of continuous curves. Hence there are not more than a finite number of maximal connected subsets of M-K all of diameter greater than the same positive number. It can be seen that if x and y are any two distinct maximal connected subsets of K then not more than a finite number of elements of H have one point in common with x and another point in common with y. Furthermore no maximal connected subset of M-K has limit points in each of three distinct maximal connected subsets of K. It follows, with the help of theorem 9, that the set H can be expressed as the sum of two sets H_1 and H_2, where H_1 consists of all elements h of H such that h has only one point in common with K, and H_2 is a *finite* set consisting of all those elements h of H such that h has just two points in common with K, these points belonging to different maximal connected subsets of K. Let K_1 denote the set of all points

X of K such that X belongs to some point set of the set H_1, but not to any point set of the set H_2. For each point X of K_1 let N_X denote the point set obtained by adding together all point sets of the set H which contain X. Let Q_1 denote the set of all point sets N_X for all points X of K_1. With the help of the fact that there are not more than a finite number of elements of H of diameter greater than any preassigned positive number it is easy to see that every element of Q_1 is a continuous curve and that there are not more than a finite number of these curves of diameter greater than any preassigned positive number. It is also clear that no two elements of Q_1 have a point in common. Let K_2 denote the finite point set consisting of all points X of K such that X belongs to some point set of the set H_2. Let L denote the point set obtained by adding together all point sets h of the set H such that h contains a point of K_2. Let Q_2 denote the set of all maximal connected subsets of L. The set Q_2 cannot contain more elements than K_2 contains points. Hence Q_2 is a finite set. Every element of Q_2 is a continuous curve and no two of them have a point in common. No curve of the set Q_1 has a point in common with any curve of the set Q_2. Let Q denote the set of all curves q such that q belongs either to Q_1 or to Q_2. If a curve of the countable set Q separates A from B it contains[14] a simple closed curve that separates A from B and thus contains uncountably many points of K, which is impossible. Hence no element of Q separates A from B. For each element q of Q let g_q denote either q or the point set obtained by adding to q all its bounded complementary domains, according as q does or does not fail to separate the plane. Let G_1 denote the set of all point sets g_q for all elements q of Q. Let G_2 denote the set of all cut points P of M such that P lies on some simple closed curve which is a subset of M. By a theorem of G. T. Whyburn's,[15] G_2 is a countable set of points. Let G_3 denote the set of all points X of M such that X is common to the boundaries B_1 and B_2 of two distinct complementary domains D_1 and D_2 of M and is on the boundary of some complementary domain of the point set $D_1 + D_2 + B_1 + B_2$. Let G denote the set of all elements g such that g is either a point set of the set G_1 or a point of the set G_2 or of the set G_3. Let U denote the set of all continua x such that x is either an element of G or a point which does not belong to any element of G. The collection U is an upper semi-continuous[16] collection of mutually exclusive continua and every point of the plane belongs to some one of them and no one of them separates the plane. Furthermore, G is a countable subset of U. But if E is a countable set of *points*, every two points not belonging to E can[17] be joined by an arc containing no point of E. It follows[16] that there exists a simple continuous arc t with end-points at A and B and containing no point of any element of the set G. If a point P of the arc t belongs to at least one segment of t whose end-points belong to the boundary of the same complementary domain of M then

(1) there exists a segment A_PB_P which has this property and which contains every other segment that has this property with respect to this particular point P, and (2) since t contains no point of G_3, there exists *only one* domain D_P complementary to M and having both A_P and B_P on its boundary. Let W denote the set of all points P of t for which A_PB_P exists and for which A_PB_P does not belong to D_P. For each point P of W let $A_PZ_PB_P$ denote a definite arc lying wholly in D_P except for its end-points A_P and B_P and let s_P denote the segment $A_PZ_PB_P - (A_P + B_P)$. Let AB denote the arc obtained by adding M-W to the sum of all the segments s_P for all points P of W. The arc AB does not separate M. For suppose it does; then there exist two points X and Y belonging to M and such that every arc M from X to Y contains at least one point of AB. It can be shown that AB contains a point set CD, which is either a single point or an arc with C and D as end-points, such that every arc in M from X to Y contains a point of CD but such that if u is any proper subinterval of CD then there exists, in M, an arc from X to Y that contains no point of u. If CD were a point it would be a cut point of M, contrary to the fact that AB contains no point of G_1 or of G_2. Hence CD is an interval. For each positive integer n, let C_n denote a point of CD such that the diameter of the interval CC_n of CD is less than $1/n$. The curve M contains an arc t_n from X to Y that has no point in common with the interval C_nD of CD. But t_n contains at least one point of CD. Hence it contains a point of the interval CC_n. But every such point is at a distance from C less than $1/n$. Therefore C is a limit point of that maximal connected subset of $(M + CD) - CD$ that contains X. But $M + CD$ is a continuous curve. Hence[4] M contains an arc XC which has no point except C in common with CD. By a similar argument it may be shown that M contains an arc YC that has only C in common with CD and two arcs XD and YD each having only D in common with CD. The point set $XC + XD$ contains an arc CX_1D and $CY + YD$ contains an arc CY_1D. The arcs CX_1D and CY_1D have in common only their end-points C and D and together they form a simple closed curve J. The arc CD lies, except for its end-points, wholly in one of the complementary domains of J. Let I denote this domain and let E denote the other complementary domain of J. There does not exist an arc with end-points at C and D and lying, except for C and D, wholly in $(E + M) - M$. For if there were such an arc CWD then, according to the method given for constructing AB, the arc CWD would be a part of AB which is impossible since CD is a part of AB. With the help of a theorem proved by C. M. Cleveland[8] and the fact that $M + I$ is a regular continuum it follows that M contains an arc X_2Y_2 which lies wholly in E except for its end-points X_2 and Y_2 which lie, respectively, on the segment CX_1D of the arc CX_1D and the segment CX_2D of the arc CX_2D. The point set consisting of the segments CX_1D and CX_2D and the arc X_2Y_2

contains an arc from X_2 to Y_2 that contains no point of CD. Thus the supposition that theorem 10 is false has led to a contradiction.

[1] Moore, R. L., *Math. Zs.*, **15**, 1922 (254–260).

[2] These PROCEEDINGS, **11**, 1925 (469–476).

[3] *Trans. Amer. Math. Soc.*, **17**, 1916 (131–164).

[4] Wilder, R. L., *Fund. Math.*, **7**, 1925 (340–377).

[5] The case where P is a limit point of N_2 may be treated in a similar manner.

[6] Schoenflies, A., *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten*, Leipzig, 1908.

[7] Sierpinski, W., *Fund. Math.*, **1**, 1920 (44–60).

[8] This proposition has been proved by G. T. Whyburn with the use of a theorem proved, at my suggestion, by C. M. Cleveland and stated in *Bull. Amer. Math. Soc.*, **32**, 1926 (311).

[9] Kline, J. R., *Trans. Amer. Math. Soc.*, **21**, 1920 (p. 453, 2nd footnote); Mazurkiewicz, S., *Fund. Math.*, **1**, 1920 (166–209).

[10] Moore, R. L., *Bull. Amer. Math. Soc.*, **29**, 1923 (289–302), § 3.

[11] Hahn, H., *Wien. Ber.*, **123**, 1914 (2433–2489).

[12] Every other case may be reduced to this one by an inversion.

[13] Ayres, W. L., *Bull. Amer. Math. Soc.*, **32**, 1926 (37).

[14] See theorem 7 of my paper cited in footnote 1; also page 475 of the paper cited in footnote 2.

[15] Whyburn, G. T., *Concerning Cut Points and End-Points of Continua*, presented to the American Mathematical Society, June 12, 1926, but not yet published.

[16] Moore, R. L., *Trans. Amer. Math. Soc.*, **27**, 1925 (416–428).

[17] Kline, J. R., *Bull. Amer. Math. Soc.*, **23**, 1917 (290–291).

ON SIMPLY TRANSITIVE PRIMITIVE GROUPS

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Let G be a simply transitive primitive group of degree n. The subgroup G_1 that leaves one letter fixed is intransitive of degree $n - 1$ and is a maximal subgroup of G. Let H be an invariant subgroup of G_1 of degree $n - m(< n - 1)$. Then the largest subgroup of G in which H is invariant is G_1 and H fixes all the letters of a constituent of G_1. From a general theorem on transitive groups[1] it follows that G_1 contains exactly $m - 1$ non-invariant subgroups conjugate to H under G. If B, on the r letters $b, b_1, \ldots$ fixed by H, is a transitive constituent of G_1, there is a corresponding complete set of r conjugate subgroups $H_1, H_1', \ldots$, conjugate under G_1 and conjugate to H under G. To the transitive constituent C, on the r_1 letters $c, c_1, \ldots$ fixed by H, there corresponds the set $H_2, H_2', \ldots$ of r_1 conjugate subgroups of G_1, and so on. Since of all the m conjugates of H under G found in G_1, only H is invariant in G_1, H is a