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ELEMENTS OF APPLIED MATHEMATICS

BY

HERBERT E. COBB

PROFESSOR OF MATHEMATICS, LEWIS INSTITUTE, CHICAGO

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PREFACE

This book of problems is the result of four years' experimentation in the endeavor to make the instruction in mathematics of real service in the training of pupils for their future work. There is at the present time a widespread belief among teachers that the formal, abstract, and purely theoretical portions of algebra and geometry have been unduly emphasized. Moreover, it has been felt that mathematics is not a series of discrete subjects, each in turn to be studied and dropped without reference to the others or to the mathematical problems that arise in the shops and laboratories. Hence we have attempted to relate arithmetic, algebra, geometry, and trigonometry closely to each other, and to connect all our mathematics with the work in the shops and laboratories. This has been done largely by lists of problems based on the preceding work in mathematics and on the work in the shops and laboratories, and by simple experiments and exercises in the mathematics classrooms, where the pupil by measuring and weighing secures his own data for numerical computations and geometrical constructions.

In high schools where it is possible for the teachers to depart from traditional methods, although they must hold to a year of algebra and a year of geometry, this book of problems can be used to make a beginning in the unification of mathematics, and to make a test of work in applied problems. In the first year in algebra the problems in Chapters I–VII can be used to replace much of the abstract, formal, and lifeless material of the ordinary course. These problems afford a much-needed drill in arithmetical computation, prepare the way for geometry, and awaken the interest of the pupils in the affairs of daily life. By placing less emphasis on the formal side of geometry it is possible to make the pupil's knowledge of algebra a valuable asset in solving geometrical problems, and to give him a working knowledge of angle functions and logarithms. Chapters IX, X, and XII furnish the material for this year's work. The problems of the remaining chapters can be used in connection with the study of advanced algebra and solid geometry. They deal with various phases of real life, and in solving them the pupil finds use for all his mathematics, his physics, and his practical knowledge.

For the increasing number of intermediate industrial schools there are available at present few lists of problems of the kind brought together in this book. The methods adopted in the earlier chapters, which require the pupil to obtain his own data by measuring and weighing, are especially valuable for beginners and boys who have been out of school for several years.

The large number of problems and exercises permits the teacher to select those that are best suited to the needs of the class. In Chapters IX and XIII many of the problems contain two sets of numbers. The first set outside of the parentheses may give an integral result, while the second set may involve fractions; or the first set may give rise to a quadratic equation which can be solved by factoring, while the equation of the second set must be solved by completing the square.

Each pupil should have a triangle, protractor, pair of compasses, metric ruler, and a notebook containing plain and squared paper. Inexpensive drawing instruments can be obtained, and the pupils should be urged to use them in making rough checks of computations. They should also form the habit of making a rough estimate of the answer, and noting if the result obtained by computation is reasonable.

In the preparation of this book most of the works named in the Bibliography have been consulted. The chapter on squared paper aims to emphasize its chief uses, the representation of tables of values, and the solution of problems; and to show that the graph should be used in a common-sense way in all mathematical work.

The coöperation of the members of the department of mathematics in the Lewis Institute in the work of preparing and testing the material for this book has rendered the task less burdensome; acknowledgments are due to Assistant Professor D. Studley for the problems in Chapters XIV and XV; to Assistant Professor B. J. Thomas for aid in Chapters I, VIII, XII, and XIII; to Mr. E. H. Lay for aid in Chapters II and VI; and to Mr. A. W. Cavanaugh for aid in Chapter IX. Especial acknowledgments are due to Professor P. B. Woodworth, head of the department of physics, Lewis Institute, for his helpful coöperation with the work of the mathematics department.

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APPLIED MATHEMATICS

CHAPTER I

MEASUREMENT AND APPROXIMATE NUMBER

Exercise. Make a sketch of the whitewood block that has been given you; measure its length, breadth, and thickness in millimeters and write the dimensions on the sketch. Find the volume of the block. Have you found the exact volume? Were your measurements absolutely correct?

1. Errors. In making measurements of any kind there are always errors. We do not know whether or not the foot rule, the meter stick, or the 100-foot steel tape we are using is absolutely exact in length and graduation. Hence one source of error lies in the instruments we use. Another source of error is the inability to make correct readings. When you attempt to measure the length of a whitewood block, you will probably find that the corners are rather blunt, making it impossible to set a division of the scale exactly on the corner. Moreover, it is seldom that the end of the line you are measuring appears to coincide exactly with a division of the scale. If you are using a scale graduated to millimeters and record your measurements only to millimeters, then a length is neglected if it is less than half a millimeter, and called one millimeter if it is greater than half a millimeter.

To make a reading as correct as possible, be sure that the eye is placed directly over the division of the scale at which the reading is made. Note if the end of the scale is perfect.

2. Significant figures. A *digit* is one of the ten figures used in number expressions. A *significant figure* is a digit used to express the amount which enters the number in that particular place which the digit occupies. All figures other than zero are significant. A zero may or may not be significant. It is significant if written to show that the quantity in that place is nearer to zero than to any other digit, but a zero written merely to locate the decimal point is not significant. A zero inclosed by other digits is significant, while a final zero may or may not be significant.

For example, in the number 0.0021 the zeros are not significant. In the number .0506 the first zero is not significant, while the zero inclosed by the 5 and 6 is significant. If in a measurement a result written as 56.70 means that it is nearer 56.70 than 56.69 or 56.71, the zero is significant. In saying that a house cost about $6700, the final zeros are not significant because they merely take the place of other figures whose value we do not know or do not care to express.

3. Exact numbers. In making computations with exact numbers, multiplications and divisions are done in full, according to methods which are familiar to all students.

4. Approximate numbers. In practical calculations most of the numbers used are not exact but are approximate numbers. They are obtained by measuring, weighing, and other similar processes. Such numbers cannot be exact, for instruments are not perfect and the sense of vision does not act with absolute precision. If the length of a rectangular piece of paper were measured and found to be 614 mm., the 6 and the 1 would very likely be exact, but the 4 would be doubtful.

5. Multiplication of approximate numbers. This contracted method of multiplication gives the proper number of significant figures in the product with no waste of labor. Moreover, by omitting the doubtful figures it avoids an appearance of great accuracy in the result, which is not warranted by the data.

Exercise. Measure the length and width of a rectangular piece of paper and find its area.

Suppose the length is 614 mm. and the width is 237 mm. Let us proceed to find the area of the piece of paper, marking the doubtful figures throughout the work.

$$
\begin{array}{r}
23\mathbf{7} \\
\underline{61\mathbf{4}} \\
\mathbf{948} \\
23\mathbf{7} \\
\underline{142\mathbf{2}} \\
145\mathbf{518}
\end{array}
$$

The final three figures in the product are doubtful and may as well be replaced by zeros. Hence the area is approximately 145,000 sq. mm., or, as we sometimes say, about 145,000 sq. mm. Since many calculations are of this kind, it is a waste of time to carry out the operations in full. It is desirable to use methods which will omit the doubtful figures and retain only those which are certain.

Problem. Multiply 24.6 by 3.25.

First step	*Second step*	*Third step*
$\begin{array}{r} 24.6 \\ \underline{3.25} \\ 738 \end{array}$	$\begin{array}{r} 24.\cancel{6} \\ \underline{3.25} \\ 738 \\ 49 \end{array}$	$\begin{array}{r} 2\cancel{4}.\cancel{6} \\ \underline{3.25} \\ 738 \\ 49 \\ \underline{12} \\ 79.9 \end{array}$

First step. Start with 3 at the left in the multiplier and write the partial product as shown.

Second step. Cut off the 6 in the multiplicand and multiply by 2. Twice 6 (mentally) are 12 (1.2), which gives 1 to add. Twice 4 are 8, and 1 to add makes 9. Twice 2 are 4.

Third step. Cut off the 4 in the multiplicand and multiply by 5. 5 times 4 (mentally) are 20 (2.0), which gives 2 to add. 5 times 2 are 10, and 2 to add makes 12.

Fourth step. Add the partial products.

Fifth step. Place the decimal point by considering the number of integral figures which the product should contain. This may usually be done by making a rough estimate mentally. In this case we see that 3 times 24 are 72, and by estimating the amount to be brought up from the remaining parts we see that the product is more than 75. Hence there are two integral figures to be pointed off.

Problem. Multiply 84.6 by 4.25.

First step	*Second step*	*Third step*
$\begin{array}{r} 84.\cancel{6} \\ 4.25 \\ \hline 338 \end{array}$	$\begin{array}{r} 8\cancel{4}.\cancel{6} \\ 4.25 \\ \hline 338 \\ 17 \end{array}$	$\begin{array}{r} \cancel{8}\cancel{4}.\cancel{6} \\ 4.25 \\ \hline 338 \\ 17 \\ 4 \\ \hline 359 \end{array}$

In this case 6 is cut off before multiplying by 4 in order to keep the product to three figures. The two given numbers are doubtful in the third figure, and usually this makes the product doubtful in the third figure.

Problem. Find the product of $\pi \times 3.784 \times 460.2$.

SOLUTION.

$$\begin{array}{r} 3.\cancel{1}\cancel{4}\cancel{2} \\ 3.784 \\ \hline 9426 \\ 2199 \\ 251 \\ 12 \\ \hline 11.888 \end{array} \qquad\qquad \begin{array}{r} 1\cancel{1}.\cancel{8}\cancel{9} \\ 460.2 \\ \hline 4756 \\ 713 \\ 2 \\ \hline 5471 \end{array}$$

6. Measurements. *In making measurements to compute areas, volumes, and so on, all parts should be measured with the same relative accuracy; that is, they should all be expressed with the same number of significant figures. The calculated parts should not show more significant figures than the measured parts. Constants like π should be cut to the same number of figures as the measured parts.*

EXERCISES

1. Find the area of the printed portion of a page in your algebra.

2. Find the volume of your algebra.

3. Find the area of the top of your desk.

4. Find the area of the door.

5. Find the number of cubic feet of air in the room.

6. Find the area of one section of the blackboard.

7. Find the surface and volume of brass cylinders and prisms, and of wooden blocks.

8. Find the area of the athletic field.

9. Find the area of the ground covered by the school buildings and also the area of some of the halls and recitation rooms. Compare your results with computations made from the plans of the buildings, if they are accessible.

7. Division of approximate numbers. In dividing one approximate number by another, we shorten the work by cutting off figures in the divisor instead of adding zeros in the dividend. The principles of contracted multiplication are used in the multiplication of the divisor by the figures of the quotient. No attention is paid to the decimal point in the dividend or divisor till the quotient has been obtained. In checking multiply the quotient by the divisor. (Why ?)

Problems. 1. Divide 83.62 by 3.194.

$$\begin{array}{r|r|l}
3.\cancel{1}\cancel{9}\cancel{4} & 83.62 & 26.18 \\
\hline
 & \underline{6388} & \\
 & 1974 & \\
 & \underline{1916} & \\
 & 58 & \\
 & \underline{32} & \\
 & 26 & \\
 & \underline{25} & \\
 & 1 &
\end{array}
\qquad\qquad
\begin{array}{r}
\textit{Check} \\
2\cancel{6}.\cancel{1}\cancel{8} \\
\underline{3.194} \\
7854 \\
262 \\
235 \\
\underline{10} \\
83.61
\end{array}$$

2. Divide 41.684 by 98.247.

$$
\begin{array}{r|r|l}
\cancel{9}\cancel{8}.\cancel{2}\cancel{4}\cancel{7} & 41.684 & .42428 \\
\hline
 & 39299 & \\
\cline{2-2}
 & 2385 & \\
 & 1965 & \\
\cline{2-2}
 & 420 & \\
 & 393 & \\
\cline{2-2}
 & 27 & \\
 & 20 & \\
\cline{2-2}
 & 7 & \\
 & 7 & \\
\cline{2-2}
\end{array}
\qquad\qquad
\begin{array}{r}
\textit{Check} \\
.\cancel{4}\cancel{2}\cancel{4}\cancel{2}\cancel{8} \\
98.247 \\
\hline
38185 \\
3394 \\
85 \\
17 \\
3 \\
\hline
41.684
\end{array}
$$

The decimal point in the quotient can usually be placed quite easily by considering the number of integral figures in the divisor and dividend. In the first problem we see that 3 is contained in 83 about 26 times; in the second problem 98 is contained in 41 about .4 times.

PROBLEMS

Check the results obtained:

1. 2.142×3.152.

2. 78.14×1.314.

3. 6.718×86.42.

4. $3.142 \times .7854$.

5. $(1.4142)^2$.

6. $(1.732)^2$.

7. $(3.142)^2$.

8. $(5.164)^3$.

9. $(.6462)^3$.

10. $86.66 \div 41.37$.

11. $316.4 \div 18.74$.

12. $.916 \div .314$.

13. $\dfrac{14.16 \times 5.873}{8.614}$.

14. $3.142 \times (1.666)^2$.

15. $\dfrac{36.5 \times 192}{4.12 \times 6.33}$.

16. $\dfrac{4 \times 3.142 \times (6.023)^3}{3}$.

17. An iron bar is 9.21 in. by 2.43 in. by 1.12 in. Find its weight if 1 cu. in. of iron weighs .261 lb.

18. Find the weight of a block of oak 5.62 in. by 3.92 in. by 3.15 in. if 1 cu. in. of oak weighs .0422 lb.

19. Find the weight of an iron plate 125 in. long, 86.2 in. wide, and .562 in. thick.

20. The diameter of a piston is 16.4 in. Find its area. ($\pi = 3.14.$)

21. The radius of a circle is 12.67 in. Find its area. ($\pi = 3.142.$)

22. The diameter of a steam boiler is 56.8 in. What is its circumference?

23. The area of a rectangle is 25.37 sq. in. Find the width if the length is 11.42 in.

24. What is the length of a cylinder whose volume is 1627 cu. in. if the area of a cross section is 371.5 sq. in.?

25. A cylindrical safety-valve weight of cast iron is $15\frac{1}{4}$ in. in diameter and $3\frac{1}{8}$ in. thick. Find its weight if 1 cu. in. of cast iron weighs .261 lb.

26. A cylindrical safety-valve weight of cast iron weighs 82.5 lb. What is its diameter if it is $1\frac{1}{4}$ in. thick?

27. The diameter of a spherical safety valve of cast iron is 9.3 in. Find its weight.

28. Find the weight of a cast-iron pipe 28.5 in. long if the outer diameter is 10.9 in. and the inner diameter is 9.2 in.

29. A cylindrical water tank is 49.6 in. long and its diameter is 28.6 in. Find its volume. How many gallons will it hold?

30. A steel shaft is 68.8 in. long and its diameter is 2.58 in. Find its weight if 1 cu. in. of steel weighs .283 lb.

31. Find the weight of the water in a full cylindrical water tank 12.8 ft. in height and 6.32 ft. in diameter if 1 cu. ft. of water weighs 62.4 lb.

32. The diameter of the wheels over which a band saw runs is 3.02 ft. and the distance between the centers of the pulleys is 3.58 ft. Find the length of the saw.

33. A pulley 11.9 in. in diameter is making 185 revolutions per minute (r. p. m.). How fast is the rim traveling per minute?

34. A milling cutter 4 in. in diameter is running 150 r. p. m. What is the surface speed in feet per minute?

35. It is desired to make a 12-in. emery wheel run at a speed of 5000 ft. per minute. How many revolutions per minute must it make?

36. If we wish a milling cutter to run at a cutting speed of 266 ft. per minute, and the machine can make only 82 r. p. m., what must be the diameter of the cutter?

CHAPTER II

VERNIER AND MICROMETER CALIPERS

8. The **vernier calipers** have two jaws between which is placed the object to be measured. One jaw slides on a bar which has scales, on one side centimeters and on the other side inches.

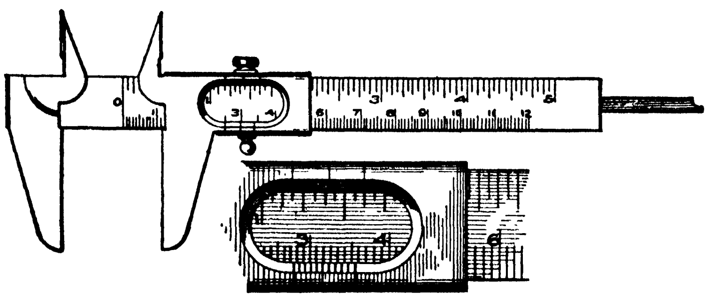

FIG. 1

The movable jaw has two small scales called *verniers*, one for each main scale.

Write the following questions and their answers in your notebook. Use the centimeter for the unit and write the results as decimal fractions.

1. (*a*) How many centimeters are marked on the main scale? (*b*) Verify by measuring with the ruler. (*c*) What is the length of the smallest division of the main scale?

2. (*a*) What is the length of the centimeter vernier? (*b*) Measure the length of the vernier with the ruler. (*c*) Verify by counting the divisions on the main scale.

3. (*a*) Into how many divisions is the vernier scale divided? (*b*) What is the length of each division?

4. Bring the jaws of the calipers together. At what point on the main scale does the first line of the vernier fall?

Make a drawing of the vernier and the scale as suggested by Fig. 2. Number the points of division as in the figure.

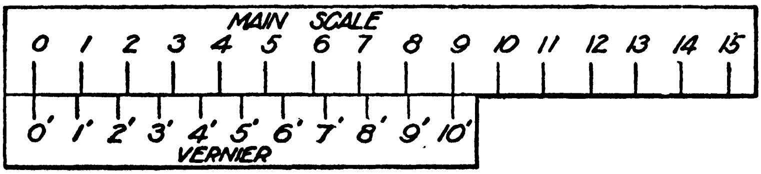

FIG. 2

Slide the vernier of the calipers along until 0′ coincides with 0.

5. (*a*) Do 1′ and 1 coincide? (*b*) What is the distance between 1′ and 1? (*c*) between 2′ and 2? (*d*) between 3′ and 3?

Now slide the vernier along until 1′ and 1 coincide.

6. (*a*) What is the distance between 0 and 0′? (*b*) between 2′ and 2?

Make 2′ and 2 coincide.

7. What is the distance between 0 and 0′ now?

8. What is the distance between 0 and 0′ when the following points coincide? (*a*) 3′ and 3; (*b*) 4′ and 4; (*c*) 7′ and 7; (*d*) 9′ and 9; (*e*) 10′ and 10.

Move the vernier until 0′ coincides with 10.

9. How far apart are the jaws? Check with the ruler.

10. When 0′ coincides with 20, how far apart are the jaws? Check.

11. When 0′ coincides with 21, how far apart are the jaws?

12. What is the distance between the jaws when the following points coincide? (*a*) 1′ and 22; (*b*) 2′ and 23; (*c*) 5′ and 26; (*d*) 8′ and 29; (*e*) 1′ and 23; (*f*) 7′ and 29; (*g*) 2′ and 26; (*h*) 3′ and 35.

9. *To measure with the vernier. Count the number of whole centimeters and millimeters to the zero line of the vernier. Then* *notice which vernier division coincides with a scale division; the number of this vernier division is the number of tenths of a millimeter.*

10. Observe carefully the following directions for making measurements. Unlock the movable jaw by means of the screw at the side. Place the object between the jaws, press these together gently but firmly with the fingers, and lock in position with the screw. Care should be taken in pressing upon the jaws as too strong a pressure may injure the instrument. If not enough pressure is applied, the reading will not be accurate.

EXERCISES

1. Place your pencil between the jaws of the calipers and measure its diameter.

2. Get a block from your instructor and measure its dimensions. Make a record of them together with the number of the block, and let the instructor check the results.

3. Get a second block. Make measurements of the length at three different places on the block and record them. Take the average of the three readings. Find dimensions in the same way. Let the instructor check the record.

4. Draw an indefinite line AB. With a point R about 1 cm. from AB as a center, and with a radius of 3 cm. draw a circle intersecting AB at C and D. Measure CD, making the measurement with the pointed ends of the jaws. Check your reading.

5. Take a sheet of squared paper and fix the vertices of a square centimeter with the point of the compasses. Measure the diagonals and take the average. Check.

6. On the same sheet of squared paper locate the vertices of a rectangle 4 cm. by 2.5 cm. Measure the diagonals and check the results.

7. Apply the sets of questions in these exercises to the inch scale and its vernier, inserting the word "inch" for "centimeter" in your record.

8. Measure the length of a block in inches and in centimeters and find out the number of centimeters in one inch.

9. On a sheet of squared paper mark out a right triangle with the legs 3 in. and 4 in. respectively. Locate the vertices with the point of the compasses and measure the hypotenuse. Show that the square of the hypotenuse is equal to the sum of the squares of the other two sides.

10. Move the zero line of the vernier opposite 1 in. on the main scale. Make the reading in centimeters. Compare the result with that obtained in Exercise 8.

11. Find the volume of a block in cubic inches and also in cubic centimeters. Check by changing the cubic inches into cubic centimeters.

12. Devise other exercises in measurement.

11. The micrometer calipers. With the micrometer calipers the object to be measured is placed between a revolving rod called the *screw*, and a fixed *stop*. The movable rod is turned by the *barrel*, which moves over a *linear scale*. The edge of the barrel is graduated into a *circular scale*.

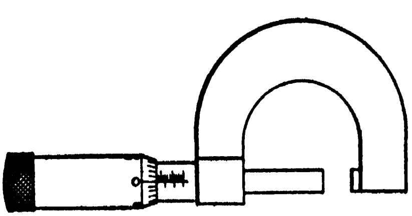

FIG. 3

12. Use of the micrometer calipers. Turn the barrel so that the screw approaches the stop and finally comes in contact with it. Now turn in the opposite direction and the screw moves away from the stop; at the same time the edge of the barrel moves over the linear scale, which shows the distance of the opening. When an object is placed in the opening between the stop and the screw, its measurement is obtained by reading the length of the linear scale exposed to view.

EXERCISES

Write the following questions and their answers in your notebook. Express your results in decimal fractions.

Turn the barrel until the entire linear scale is shown.

1. How many divisions are marked on the linear scale?

2. Determine the unit of the linear scale, whether it is a centimeter or an inch. This can be done by comparison with the English and the metric scales marked on your ruler.

3. How long in inches or centimeters is the linear scale?

4. What is the length of each division of the linear scale?

Turn the barrel until the screw comes in contact with the stop.

5. Into how many divisions is the circular scale along the edge of the barrel graduated?

6. (*a*) Does the zero line of the circular scale coincide with the line of reference of the linear scale?

(*b*) How far are they apart? Count the number of divisions of the circular scale between them. This is known as taking the zero reading.

Turn the barrel until the zero line coincides with the line of reference. From this position turn the barrel until two divisions of the linear scale have been passed over.

7. How many complete turns were made?

Bring the zero line of the barrel back to the line of reference of the linear scale. Give the barrel several complete turns and count the number of divisions passed over on the linear scale. The relation between the number of turns and the number of divisions should be carefully noted.

8. How many divisions are passed over in (*a*) two turns? (*b*) four turns? (*c*) six turns? (*d*) one turn?

9. How far in centimeters or inches does the barrel move in one complete turn?

Bring the zero line opposite the line of reference. Now move the barrel until the line 5 of the circular scale is opposite the line of reference.

10. (*a*) What part of a turn has the barrel made?

(*b*) How far in centimeters or inches did the barrel move?

(*c*) How far will the barrel move in passing over one division of the circular scale?

Turn the barrel until its edge coincides with the fifth division of the linear scale, and the zero line of the circular scale coincides with the line of reference.

11. What is the length of the opening at the end of the screw? Record the distance, and then as a rough check verify by measuring with a ruler.

With the barrel in the same position as before (at the fifth division) continue to turn so as to increase the opening at the end of the screw. Turn the barrel until the seventeenth division of the circular scale is opposite the line of reference.

12. How much is the opening at the end of the screw?

The following will illustrate: Suppose the divisions of the linear scale are $\frac{1}{40}$ (.025) of an inch, and there are 25 divisions on the circular scale. The value of one division of the circular scale will be $\frac{1}{25} \times \frac{1}{40} = .001$ in. Each division of the circular scale, therefore, measures .001 in. In Exercise 12 the distance for 5 linear divisions would be $.025 \times 5 = .125$. This added to the value of the 17 circular divisions gives $.125 + .017 = .142$ in. for the reading.

EXERCISES

Record the readings in your notebooks and give the work of the computations in full.

1. Measure the thickness of a coin. Hold the barrel lightly so that it will slip in the fingers as contact occurs. There is danger of straining the screw if it is turned up hard. Take four readings at different places on the coin and average the results.

2. Get a metal solid from your instructor and measure its dimensions. Take the average of three readings. Compute the volume. Check by using an overflow can.

3. Measure the diameter of a wire. After taking a reading release the wire and turn it about its axis through 90°; take a second reading. If the two readings do not agree, the wire is slightly flattened in section. Take several readings at different places on the wire, and the average of the readings will probably be very close to the standard diameter of the wire.

4. Find the volume of a shot. Using the specific gravity of lead, find the number of shot to the pound.

5. Find the thickness of one of the pages of your textbook. Compute its volume.

6. Devise other exercises and record them in your notebooks.

CHAPTER III

WORK AND POWER

13. Work. When a man lifts a bar of iron or pulls it along the floor, he is said to do work upon it. Evidently it takes twice as much effort to lift 50 lb. as it does to lift 25 lb., and five times as much effort to lift a box 5 ft. as it does to lift it 1 ft. That is, work depends upon two things, — distance and pressure.

Hence a *foot pound* is taken as the unit of work. It is the work done in raising 1 lb. vertically 1 ft., or it is the pressure of 1 lb. exerted over a distance of 1 ft. in any direction. If a man exerts a pressure of 25 lb. in pushing a wagon 20 ft., he has done 500 ft. lb. of work.

$$\textit{Foot pounds} = \textit{feet} \times \textit{pounds}.$$

14. Illustrations. Tie a string to a 1-lb. weight, attach a spring balance and lift it 1, 2, 3 ft. How many foot pounds of work? Lower it 1, 2, 3 ft. How much work? Pull the string horizontally over the edge of a ruler to raise the weight 1 ft. How much work? Is the amount of pressure given by the spring balance or by the pound weight? Pull the weight along the top of the desk 1 ft. How much work? Hook the spring balance under the edge of the desk and pull 2 lb. How much work? Drop the weight 1 ft. How much work ought the weight to do when it strikes the floor?

A boy weighing 60 lb. climbs up a ladder 10 ft. vertically. How much work? How much work is done when he comes down the ladder?

A boy weighing 60 lb. walks up a flight of stairs. How much work has he done when he has risen 10 ft.? Why should the answer be the same as in the preceding problem?

A stone weighing 50 lb. is on the roof of a shed 10 ft. from the ground. How much work was done to get it in that position? If it is pushed off, how much work ought it to do when it strikes the ground? Why ought the two answers to be the same number of foot pounds?

15. Power. Time is not involved in work. A man may take 4 hr. or 10 hr. to raise a ton of coal 15 ft.; in either case he has done 30,000 ft. lb. of work. But in the first case he is doing work at the rate of 125 ft. lb. per minute, while in the second case he is working at the rate of 50 ft. lb. per minute. To compare the work of men or machines, or to determine the usefulness of a machine, it is necessary to take into consideration the time required for the work.

Power is the rate of doing work. Thus if an electric crane raises a steel beam, weighing 500 lb., 80 ft. in 2 min., its rate of work is $\frac{500 \times 80}{2} = 20{,}000$ ft. lb. per minute.

The unit of power, the *horse power*, is the power required to do work at the rate of 33,000 ft. lb. per minute. If a steam crane lifts 90 T. of coal 11 ft. in 20 min., neglecting friction, the horse power of the engine is

$$\text{h. p.} = \frac{2000 \times 90 \times 11}{33{,}000 \times 20} = 3.$$

When we speak of the horse power of an engine we usually mean the indicated horse power (i. h. p.), which is calculated from the dimensions of the cylinder and the mean effective steam pressure obtained from the indicator card. The horse power actually available for work is called the brake horse power (b. h. p.), and is determined by the Prony brake or a similar device.

The horse power of a steam engine is given by the equation

$$\text{h. p.} = \frac{p \cdot l \cdot a \cdot n}{33{,}000},$$

where p = mean effective pressure in pounds per square inch,
l = length of stroke in feet,
a = area of piston in square inches,
n = number of strokes per minute, or twice the number of revolutions per minute.

PROBLEMS

In these problems no account is taken of friction and other losses.

1. If a man exerts a pressure of 56 lb. while wheeling a barrow load of earth 25 ft., find the number of foot pounds of work he does.

2. How much work is done by a steam crane in lifting a block of stone weighing 1.2 T. 30 ft.?

3. A hole is punched through an iron plate $\frac{1}{2}$ in. thick. If the punch exerts a uniform pressure of 40 T., find the work done.

4. A horse hauling a wagon exerts a constant pull of 75 lb. and travels at the rate of 4 mi. per hour. How much work will the horse do in 3 hr.? If the driver rides on the wagon, how much work does he do?

5. A man weighing 150 lb. carries 50 lb. of brick to the top of a building 40 ft. high. How much work has he done (*a*) in getting himself to the top? (*b*) in carrying the brick? How much work is done on his return trip down the ladder?

6. If a pump is raising 2000 gal. of water per hour from the bottom of a mine 400 ft. deep, how many foot pounds of work are done in 2 hr.? (A gallon of water weighs 8.3 lb.)

7. How many gallons of water would be raised per minute from a mine 600 ft. deep by an engine of 180 h. p.?

8. The plunger of a force pump is 4 in. in diameter, the length of the stroke is 3 ft., and the pressure of the water is 40 lb. per square inch. Find the work done in one stroke.

9. A well 6 ft. in diameter is dug 30 ft. deep. If the earth weighs 125 lb. per cubic foot, find the work done in raising the material.

10. A basement 20 ft. by 15 ft. is filled with water to a depth of 4 ft. How much work is done in pumping the water to the street level, 6 ft. above the basement floor? (The average distance which the water is lifted is 4 ft.)

11. A chain 40 ft. long weighing 10 lb. per foot is hanging vertically in a shaft. Construct a curve to show the work done on each foot in lifting the chain to the surface. (Assume that the first foot is lifted ½ ft., the second 1½ ft., and so on.) What is the total work done in lifting the chain?

12. How much work is done in rolling a 200-lb. barrel of flour up a plank to a platform 6 ft. high?

13. A boy who can push with a force of 40 lb. wants to roll a barrel weighing 120 lb. into a wagon 3 ft. high. How long a plank must he use? (Length of plank $\times$ 40 = 3 $\times$ 120. Why?)

14. A man can just lift a barrel weighing 200 lb. into a wagon 3½ ft. high. How much work does he do? How long a plank would he need to roll up a barrel weighing 400 lb.? 600 lb.?

15. A horse drawing a cart along a level road at the rate of 3 mi. per hour performs 42,000 ft. lb. of work in 5 min. Find the pull in pounds that the horse exerts in drawing the cart.

16. A horse attached to a capstan bar 12 ft. long exerts a pull of 120 lb. How much work is done in going around the circle 100 times?

17. How long will it take a man to pump 800 cu. ft. of water from a depth of 16 ft. if he can do 2000 ft. lb. of work per minute?

18. How much work can a 2 h. p. electric motor do in 10 min.? in 15 sec.?

19. What is the horse power of an electric crane that lifts 4 T. of coal 30 ft. per minute? If 40 per cent of the power is lost in friction and other ways, what horse power would be required?

20. Find the horse power of an engine that would pump 40 cu. ft. of water per minute from a depth of 420 ft., if 20 per cent of the power is lost.

21. A locomotive exerts a pull of 2 T. and draws a train at a speed of 20 mi. per hour. Find the horse power.

22. The weight of a train is 120 T. and the drawbar pull is 7 lb. per ton of load. Find the horse power required to keep the train running at the rate of 30 mi. per hour.

23. The drawbar pull of a locomotive pulling a passenger train at a speed of 60 mi. per hour is 5500 lb. At what horse power is the engine working?

24. What is the horse power of Niagara Falls if 700,000 T. of water pass over every minute and fall 160 ft.?

25. If a 10 h. p. pump delivers 100 gal. of water per minute, to what height can the water be pumped?

26. A derrick used in the construction of a building lifts an I-beam weighing 2 T. 50 ft. per minute. What is the horse power of the engine, if 20 per cent of the power is lost?

27. In a certain mine 400 T. of ore are raised from a depth of 1000 ft. during a day shift of 10 hr. Neglecting losses, what horse power is required to raise the ore?

28. In supplying a town with water 8,000,000 gal. are raised daily to an average height of 120 ft. What is the horse power of the engine?

29. A belt passing around two pulleys moves with a velocity of 15 ft. per second. Find the horse power transmitted if the difference in tension of the two sides of the belt is 1100 lb.

30. What is the difference in tension of the two sides of a belt that is running 3600 ft. per minute and is transmitting 280 h. p.?

31. Find the number of revolutions per minute which a driving pulley 2 ft. in diameter must make to transmit 12 h. p., if the driving force of the belt is 250 lb.

32. A belt transmits 60 h. p. to a pulley 20 in. in diameter, running at 250 r. p. m. What is the difference in pounds of the tension on the tight and slack sides?

33. In a power test of an electric motor a friction brake consisting of a strap, a weight, and a spring balance was used. The radius of the pulley was 2 in., the pull was 7 lb., and the speed of the shaft was 1800 r. p. m. What horse power did the test give?

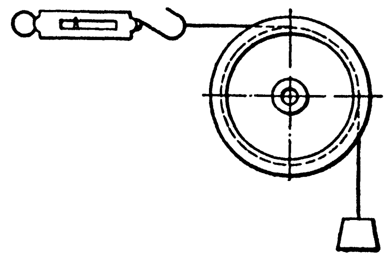

FIG. 4. FRICTION BRAKE

SOLUTION. $$\text{h. p.} = \frac{2 \times 22 \times 2 \times 1800 \times 7}{7 \times 12 \times 33{,}000} = .4.$$

34. In a power test of a small dynamo the pull was 6 lb. and the speed was 1500 r. p. m. If the radius of the driving pulley was 3 in., find the horse power.

35. In testing a motor with a Prony brake the pull was 12 lb., length of brake arm was 18 in., and the speed was 500 r. p. m. Find the horse power.

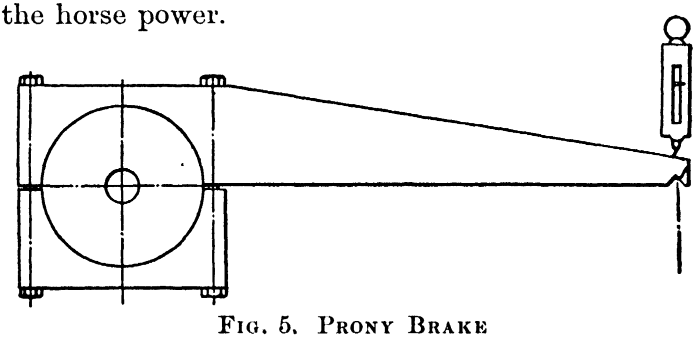

FIG. 5. PRONY BRAKE

36. In testing a Corliss engine running at 100 r. p. m. a Prony brake was used. The lever arm was 10.5 ft. and the pressure exerted at the end of the arm was 2000 lb. What was the horse power? In a second test with a pressure of 2200 lb. the speed was 90 r. p. m. Find the horse power.

37. Calculate the horse power of a steam engine from the following data: stroke, 2 ft.; diameter of cylinder, 16 in.; r. p. m., 100; mean effective pressure, 60 lb. per square inch.

38. The diameter of the cylinder of an engine is 20 in. and the length of stroke is 4 ft. Find the horse power if the engine is making 60 r. p. m. with a mean effective pressure of 60 lb. per square inch.

39. Find the horse power of a locomotive engine if the mean effective pressure is 90 lb. per square inch, each of the two cylinders is 16 in. in diameter and 24 in. long, and the driving wheels make 120 r. p. m.

40. On a side-wheel steamer the engine has a 6-ft. stroke, the shaft makes 35 r. p. m., the mean effective pressure is 30 lb. per square inch, and the diameter of the cylinder is 4 ft. Find the horse power of the engine.

41. Find the horse power of a marine engine, the diameter of the cylinder being 5 ft. 8 in., length of stroke 5 ft., r. p. m. 15, and mean effective pressure 30 lb. per square inch.

42. The diameter of the cylinder of a 514 h. p. marine engine is 5 ft., length of stroke 6 ft., r. p. m. 20. Find the mean effective pressure.

43. Find the diameter of the cylinder of a 525 h. p. steam engine: stroke, 6 ft.; r. p. m., 15; mean effective pressure, 25 lb. per square inch.

44. What diameter of cylinder will develop 10.3 h. p. with a 6-in. stroke, 300 r. p. m., and a mean effective pressure of 90 lb. per square inch?

45. The cylinder of a 55 h. p. engine is 12 in. in diameter and 28 in. long. If the mean effective pressure is 60 lb. per square inch, find the number of revolutions per minute.

16. Mechanical efficiency of machines. The useful work given out by a machine is always less than the work put into it because of the losses due to the weight of its parts, friction, and so on. If there were no losses, the efficiency would be 100 per cent.

The *efficiency* of a machine is the quotient obtained by dividing the useful work of the machine by the work put into it.

$$\textit{Efficiency} = \frac{\textit{Output}}{\textit{Input}}.$$

$$\textit{Efficiency of a steam engine} = \frac{\textit{Brake horse power}}{\textit{Indicated horse power}}.$$

In general the efficiency of a machine increases with the load up to a certain point, and then falls off. Small engines are often run at an efficiency of less than 80 per cent; large engines usually have an efficiency of 85 to 90 per cent.

PROBLEMS

1. A steam crane working at 3 h. p. raises a block of granite weighing 8 T., 50 ft. in 12 min. Find the efficiency of the crane.

SOLUTION. $\text{Output} = \dfrac{2000 \times 8 \times 50}{12}$ ft. lb. per minute.

$\text{Input} = 3 \times 33{,}000$ ft. lb. per minute.

$$\text{Efficiency} = \frac{2000 \times 8 \times 50}{12 \times 3 \times 33{,}000} = 67 \text{ per cent.}$$

2. A 6 h. p. electric crane lifts a machine weighing 15 T. at the rate of 5 ft. per minute. What is the efficiency?

3. An engine of 150 h. p. is raising 1000 gal. of water per minute from a mine 500 ft. deep. Find the efficiency of the pumping system.

4. An elevator motor of 50 h. p. raises the car and its load, 2800 lb. in all, 120 ft. in 15 sec. Find the efficiency.

5. How long will it take a 20 h. p. engine to raise 2 T. of coal from a mine 300 ft. deep, if the efficiency is 80 per cent?

6. What is the efficiency of an engine if the indicated horse power is 250 and the brake horse power is 225?

7. In lifting a weight of 256 lb. 20 ft. by means of a tackle a man hauls in 64 ft. of rope with an average pull of 110 lb. Find the efficiency of the tackle.

8. The efficiency of a set of pulleys is 75 per cent. How many pounds must be the pull, acting through 88 ft., to raise a load of 525 lb. a distance of 20 ft.?

9. A pump of 10 h. p. raises 54 cu. ft. of water per minute to a height of 80 ft. What is its efficiency?

10. A steam crane unloads coal from a vessel at the rate of 20 T. in 8 min., and lifts it a total distance of 24 ft. If the combined efficiency of the engine and crane is 70 per cent, what is the horse power of the engine?

11. Find the power required to raise 4800 gal. of water 60 ft. in 2 hr. if the efficiency of the pump is 60 per cent.

12. A centrifugal pump whose efficiency when lifting water 12 ft. is 62 per cent, is required to lift 18 cu. ft. per second to a height of 12 ft. What must be its horse power?

13. A dock 200 ft. long and 50 ft. wide is filled with water to a depth of 30 ft. It is emptied in 40 min. by a centrifugal pump which delivers the water 40 ft. above the bottom of the dock. If the combined efficiency of the engine and pump is 70 per cent, what is the horse power of the engine? (A cubic foot of sea water weighs 64 lb. The average distance which the water is lifted is 25 ft.)

14. A steam engine having a cylinder 10 in. in diameter and a stroke of 24 in. makes 80 r. p. m. and gives a brake horse power of 34 h. p. If the mean effective pressure is 50 lb. per square inch, find the efficiency.

15. In testing a Corliss engine running at 80 r. p. m. a Prony brake was used. The lever arm was 10.5 ft. and the pressure at the end of the arm was 1600 lb. The indicated horse power was 290. Find the efficiency of the engine.

16. The efficiency of a boiler is 70 per cent and of the engine 80 per cent. What is the combined efficiency?

SOLUTION. $.80 \times .70 = 56$ per cent.

17. Power is obtained from a motor. If the efficiency of the motor is 88 per cent, of the dynamo 85 per cent, and of the engine 86 per cent, what is the combined efficiency?

18. The engine which furnishes power for a centrifugal pump has an indicated horse power of 14 and an efficiency of 88 per cent. What is the efficiency of the pump if it is raising 3000 gal. of clear water 12 ft. high per minute?

19. In a test to find the efficiency of a set of pulleys the following results were obtained. Construct the efficiency curve.

Weight lifted (pounds)	5	10	15	20	25	30	35
Distance (feet) . .	1	1	1	1	1	1	1
Pull in pounds . .	3	5	6.5	8	9.5	11	12.8
Distance (feet) . .	3	3	3	3	3	3	3

20. In a test to determine the relative efficiency of centrifugal and reciprocating pumps the following results were obtained. Construct the efficiency curves.

Lift in feet	2	5	10	15	20	25	30	35	40	50	60	80	100	120	160	200	240	280
Efficiency of reciprocating pump (per cent) .			30	45	55	61	66	68	71	75	77	82	85	87	90	89	88	85
Efficiency of centrifugal pump (per cent) . . .	50	56	64	68	69	68	66	62	58	50	40							

21. In a laboratory experiment to determine the efficiency of a set of pulleys the following results were obtained. Construct the efficiency curve.

Load in grains .	40	90	140	190	240	290	340	390	440	490
Efficiency . .	13.2	26.5	36.0	43.2	49.0	53.6	57.4	60.6	63.1	65.5

22. The following results were obtained in an experiment to find the efficiency of a set of differential chain pulley blocks. Find the efficiency in each test and construct the efficiency curve.

Load in pounds .	7	21	35	49	70	98	112	126	140
Distance (feet) . .	1	1	1	1	1	1	1	1	1
Pull in pounds .	3.22	5.73	8.40	11.03	15.13	20.17	23.17	26.00	29.05
Distance (feet) . .	16	16	16	16	16	16	16	16	16

23. Find the efficiency of the following engines:

No.	Type	Pressure in lb. per sq. in.	Stroke in inches	Diameter of cylinder in inches	Revolutions per minute	Brake horse power
1	Marine . . .	37	168	110	17	4440
2	Marine . . .	25	72	70	15	441
3	Corliss . . .	90	48	30	85	1180
4	Gas engine* .	62	16	12	150	18
5	Locomotive .	80	24	17	260	504
6	High-speed .	50	16	12	246	100
7	Medium-speed	75	36	24	100	533

* Explosion every two revolutions.

CHAPTER IV

LEVERS AND BEAMS

17. Law of the lever. A rigid rod movable about a fixed point may be held in equilibrium by two or more forces. To find the relation between these forces when the lever is in a state of balance, we will make a few experiments.

EXERCISES

1. Balance a meter stick at its center; suspend on it two unequal weights so that they balance. Which weight is nearer the center? Multiply each weight by its distance from the center and compare the products. Do this with several pairs of weights. What seems to be true?

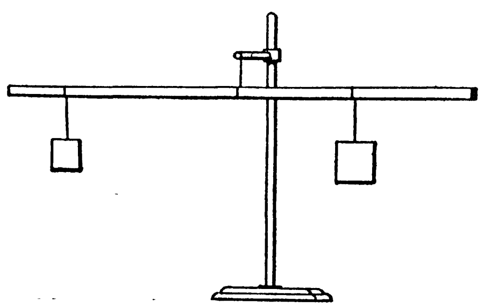

FIG. 6

2. Balance a meter stick as before, and put a 500-g. weight 12 cm. from the center; then in turn put on the following weights so that each balances the 500-g. weight. In each case record the distance from the weight to the center.

Law of the lever: $p \cdot PF = w \cdot WF$.

FIG. 7

Grams	450	400	350	300	250	200	150	120
Measured distance . .								
Computed distance . .								

Locate a point on squared paper for each weight. Units: horizontal, 1 large square = 5 cm.; vertical, 1 large square = 50 g. Draw a smooth curve through the points. Take some intermediate points on the curve and test the readings by putting the weights on the meter stick. On the same sheet of squared paper draw the curve, using the computed distances.

3. Suspend two unequal weights on one side of the center and balance them with one weight. What is the law of the lever for this case?

Definitions. The point of support is called the *fulcrum.* The product of a weight and its distance from the fulcrum is called the *leverage* of the weight. The quotient of the length of one arm of the lever divided by the length of the other is called the *mechanical advantage* of the lever.

The force which causes a lever to turn about the fulcrum may be called the *power* (p), and the body which is moved may be called the *weight* (w).

18. Three classes of levers. Levers are divided into three classes, according to the position of the power, fulcrum, and weight.

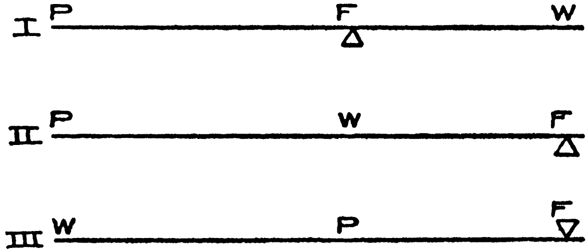

FIG. 8

Class I. When the fulcrum is between the power and the weight. Name some levers of this class.

Class II. When the weight is between the fulcrum and the power. Name some levers of this class.

Class III. When the power is between the fulcrum and the weight. Name some levers of this class.

19. Levers of the first class. With a lever of this class a large weight may be lifted by a small power; time is lost while mechanical advantage is gained.

PROBLEMS

In these problems on levers of the first class either the lever is "weightless," — that is, it is supposed to balance at the fulcrum, — or else the weight of the lever is neglected. Draw a diagram for each problem.

1. What weight 12 in. from the fulcrum will balance a 6-lb. weight 14 in. from the fulcrum?

SOLUTION. Let $w =$ the weight.

$$12\,w = 6 \times 14.$$
$$w = 7.$$

Check. $$12 \times 7 = 6 \times 14.$$
$$84 = 84.$$

2. How far from the fulcrum must a 7-lb. weight be placed to balance a 4-lb. weight 35 cm. from the fulcrum?

3. What is the weight of an object 10 in. from the fulcrum, if it balances a weight of 3 lb. 14.4 in. from the fulcrum?

4. A meter stick is balanced at the center. On one side are two weights of 10 lb. and 4 lb., 4 in. and $7\frac{1}{2}$ in. from the fulcrum respectively. How far from the fulcrum must a 7-lb. weight be placed to balance?

5. Two books weighing 250 g. and 625 g. are suspended from a meter stick to balance. The heavier book is 12 cm. from the center. How far is the other book from the center?

6. A 5-g. and a 50-g. weight are placed to balance on a meter stick suspended at its center. If the leverage is 100, how far is each weight from the center?

7. An iron casting weighing 6 lb. is broken into two pieces which balance on a meter stick when the mechanical advantage is 4. Find the weight of each piece.

8. Two boys weighing 96 and 125 lb. play at teeter. If the smaller boy is 8 ft. from the fulcrum, how far is the other boy from that point?

9. Two boys playing at teeter weigh 67 lb. and 120 lb. and are 7 ft. and 6 ft. respectively from the fulcrum. Where must a boy weighing 63 lb. sit to balance them?

10. Two bolts weighing together 392 g. balance when placed 50 cm. and 30 cm. respectively from the fulcrum. Find the weight of each.

11. A boy weighing 95 lb. has a crowbar 6 ft. long. How can he arrange things to raise a block of granite weighing 280 lb.?

12. A lever 15 ft. long balances when weights of 72 lb. and 108 lb. are hung at its ends. Find the position of the fulcrum.

Problems in which the Weight of the Lever is included

Exercise 1. Test a meter stick to see if it balances at the center. If it does not, add a small weight to make it balance.

Weigh the meter stick. It is found to weigh 162 g.

$\frac{162}{100} = 1.62$ g. per centimeter.

Attach a 200-g. weight to one end and balance as in Fig. 9.

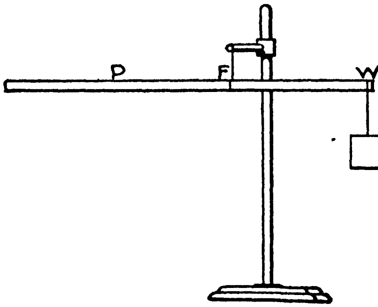

Fig. 9

The length of $FW = 22.4$ cm.
The length of $PF = 77.6$ cm.
The weight of
$FW = 22.4 \times 1.62 = 36.2$ g.
The weight of
$PF = 77.6 \times 1.62 = 125.7$ g.
161.9 g. *Check.*

The 200-g. weight and the short length of the meter stick balance the long part. Let us suppose that the weight of each part is concentrated at the center of the part, and apply the law of the lever.

$$125.7 \times \frac{77.6}{2} = 36.2 \times \frac{22.4}{2} + 200 \times 22.4.$$
$$125.7 \times 38.8 = 36.2 \times 11.2 + 4480.$$
$$4877 = 4885.$$

This checks as near as can be expected in experimental work. The measurements were made to three figures and the results differ only by one in the third place.

Hence when a uniform bar is used as a lever we may assume that the weight of each part is concentrated at the mid-point of the part.

A shorter method of solution is to consider the weight of the lever as concentrated at its center. Thus, in the preceding exercise:

$$200 \times FW = 162\ (50 - FW).$$

Solving, $$FW = 22.4.$$

Exercise 2. Make a similar test with a metal bar.

PROBLEMS

1. One end of a stick of timber weighing 10 lb. per linear foot, and 14 ft. long, is placed under a loaded wagon. If the fulcrum is 2 ft. from the end, how many pounds does the timber lift when it is horizontal?

SOLUTION. Let x = number of pounds lifted.

$$2x + 20 \times 1 = 120 \times 6.$$
$$x = 350 \text{ lb.}$$

Check. $$2 \times 350 + 20 = 120 \times 6.$$
$$720 = 720.$$

2. A lever 20 ft. long and weighing 12 lb. per linear foot is used to lift a block of granite. The fulcrum is 4 ft. from one end and a man weighing 180 lb. puts his weight on the other end. How many pounds are lifted on the stone?

3. A uniform lever 12 ft. long and weighing 36 lb. balances upon a fulcrum 4 ft. from one end when a load of x lb. is hung from that end. Find the value of x.

4. A uniform lever 10 ft. long balances about a point 1 ft. from one end when loaded at that end with 50 lb. What is the weight of the lever?

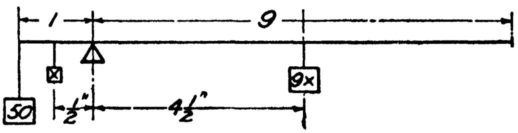

FIG. 10

SOLUTION. Let $x =$ weight of a linear foot.

$$50 \times 1 + x \times \tfrac{1}{2} = 9\,x \times 4\tfrac{1}{2}.$$
$$x = 1\tfrac{1}{4} \text{ lb.}$$
$$10\,x = 12\tfrac{1}{2} \text{ lb.}$$

Check.
$$50 \times 1 + \tfrac{1}{2} \times \tfrac{5}{4} = 9 \times \tfrac{5}{4} \times 4\tfrac{1}{2}.$$
$$50\tfrac{5}{8} = 50\tfrac{5}{8}.$$

SECOND SOLUTION. Let $x =$ weight of the lever.

$$x \times 4 = 50 \times 1.$$
$$4\,x = 50.$$
$$x = 12\tfrac{1}{2} \text{ lb.}$$

Check.
$$12\tfrac{1}{2} \times 4 = 50.$$

5. A man weighing 180 lb. stands on one end of a steel rail 30 ft. long and finds that it balances over a fulcrum at a point 2 ft. from its center. What is the weight of the rail per yard?

6. A teeter board 16 ft. long and weighing 32 lb. balances at a point 7 ft. from one end when a boy weighing 80 lb. is seated 1 ft. from this end and a second boy 1 ft. from the other end. How much does the second boy weigh?

7. A uniform lever 12 ft. long balances at a point 4 ft. from one end when 30 lb. are hung from this end and an unknown weight from the other. If the lever weighs 24 lb., find the unknown weight.

20. Levers of the second class. With a lever of the first class the weight moves in a direction opposite to that in which the power is applied. How is it with a lever of the second class?

EXERCISES

1. Place a meter stick as shown in Fig. 11, and put weights in the other pan to balance. This arrangement makes a weightless lever.

(*a*) Put 100 g. 18 in. from *F*. How many grams are required to balance it?

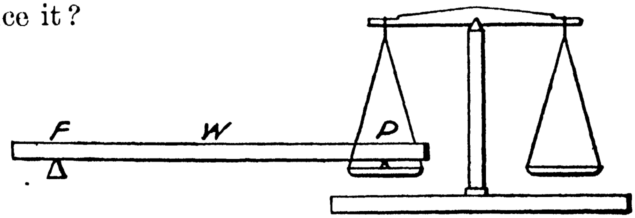

Fig. 11

Solution.
$$p \cdot PF = w \cdot WF.$$
$$36\,p = 100 \times 18.$$
$$p = 50\text{ g.}$$

Check by putting a 50-g. weight in the other pan.

(*b*) Put 100 g. 9 in. from *F* and find *p*. Check.
(*c*) Put 100 g. 27 in. from *F* and find *p*. Check.
(*d*) Put 200 g. 12 in. from *F* and find *p*. Check.

2. Lay a uniform metal bar 2 or 3 ft. long on the desk and lift one end with a spring balance. Compare the reading with the weight of the bar. Make two or three similar tests. What seems to be true? Where is the fulcrum? Where is the power? Where is the weight with reference to the fulcrum and power? Suppose the weight of the bar to be concentrated at the center and see if the law of the lever $p \cdot PF = w \cdot WF$ holds true.

3. Place a 2-lb. weight on a meter stick lying on the desk at distances of (*a*) 40 cm., (*b*) 50 cm., (*c*) 60 cm., and (*d*) 80 cm. from one end. In each case compute the pull required to lift the other end of the meter stick. Check by lifting with a spring balance.

4. Construct a graph to show the results obtained in Exercise 3. Why should it be a straight line?

PROBLEMS

1. A lever 6 ft. long has the fulcrum at one end. A weight of 120 lb. is placed on the lever 2 ft. from the fulcrum. How many pounds pressure are required at the other end to keep the lever horizontal, (*a*) neglecting the weight of the lever? (*b*) if the lever is uniform and weighs 20 lb.?

2. A man uses an 8-ft. crowbar to lift a stone weighing 800 lb. If he thrusts the lever 1 ft. under the stone, with what force must he lift to raise the stone?

3. A man is using a lever with a mechanical advantage of 6. If the load is $1\frac{1}{2}$ ft. from the fulcrum, how long is the lever?

4. A boy is wheeling a loaded wheelbarrow. The center of the total weight of 100 lb. is 2 ft. from the axle and the boy's hands are 5 ft. from the axle. What lifting force does he exert?

5. A uniform yellow-pine beam 10 ft. long weighs 38 lb. per linear foot. When it is lying horizontal a man picks up one end of the beam. How many pounds does he lift?

6. To lift a machine weighing 3000 lb. a man has a jackscrew which will lift 800 lb. and a beam 12 ft. long. If the jackscrew is placed at one end of the beam and the other end is made the fulcrum, how far from the fulcrum must he attach the machine in order to lift it?

21. Levers of the third class. In all levers of this class the power acts at a mechanical disadvantage since it must be greater than the weight. Therefore this form of the lever is used when it is desired to gain speed rather than mechanical advantage.

EXERCISES

Attach a meter stick to the base of the balance, as shown in Fig. 12, and let the meter stick rest on a triangular block placed in one pan of the balance. Put weights in the other pan to balance. This makes a weightless lever.

$$\text{Let } PF = 9 \text{ cm.}$$

1. Put 100 g. 18 cm. from F. How many grams are required to balance it?

SOLUTION.

$$p \cdot PF = w \cdot WF.$$
$$9p = 100 \times 18.$$
$$p = 200.$$

Check by placing 200 g. in the pan.

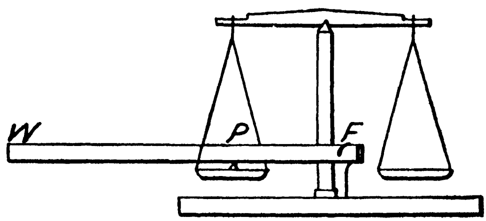

FIG. 12

2. Put 100 g. 27 cm. from F and find p. Check.

3. Put 50 g. 36 cm. from F and find p. Check.

4. Put 50 g. 45 cm. from F and find p. Check.

5. Put one end of a meter stick just under the edge of the desk. Hold the stick horizontal with a spring balance. Where are the fulcrum, weight, and power? Where may we consider the weight of each part of the meter stick to be concentrated? Weigh the meter stick and compute the pull required to hold it horizontal. Check by reading the spring balance.

6. Make the same experiment with a uniform metal bar.

PROBLEMS

1. A lever 12 ft. long has the fulcrum at one end. A pull of 80 lb. 3 ft. from the fulcrum will lift how many pounds at the other end? Neglect weight of lever.

2. The arms of a lever of the third class are 2 ft. and 6 ft. respectively. How many pounds will a pull of 60 lb. lift?

3. With a lever of the third class a pull of 65 lb. applied 6 in. from the fulcrum lifts a weight of 5 lb. at the other end of the lever. How long is the lever? Neglect its weight.

4. If the mechanical advantage of a lever is $\frac{1}{3}$, a pull of how many pounds will be required to lift 40 lb.?

5. Construct a curve to show the mechanical advantage of a lever 12 ft. long, as the power is applied 1 ft., 2 ft., 3 ft. . . . from the weight, the whole length of the lever being used.

22. Beams. The following exercises will show that a straight beam resting in a horizontal position on supports at its ends may be considered a lever of the second class.

EXERCISES

1. Test two spring balances to see if they are correct. Weigh a meter stick. Suspend it on two spring balances, as shown in Fig. 13. Read each balance. Note that each should indicate one half the weight of the meter stick. Place a 200-g. weight at the center. Read each balance.

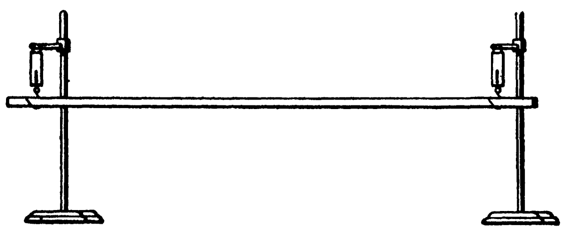

FIG. 13

2. With the meter stick as in Exercise 1, place a 200-g. weight 10, 20, 30, . . . 90 cm. from one end, and record the reading of each balance after the meter stick has been made horizontal. Construct a curve for the readings of each balance on the same sheet of squared paper.

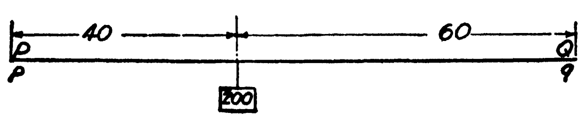

FIG. 14

To compute the reading of the balance we need only think of the beam as a lever of the second class.

Thus, when the weight is 40 cm. from one end,

$$p \times 100 = 200 \times 60,$$
$$p = 120;$$

and
$$q \times 100 = 200 \times 40,$$
$$q = 80.$$

Check. $80 + 120 = 200.$

3. Suspend a 500-g. weight 20 cm. from one end of the meter stick. Read the balances after the stick has been made horizontal. Correct for its weight. Compare with the computed readings.

4. Make similar experiments with metal bars and with two or three weights placed on the bar at the same time.

PROBLEMS

1. A man and a boy are carrying a box weighing 120 lb. on a stick 8 ft. long. If the box is 3 ft. from the man, what weight is each carrying?

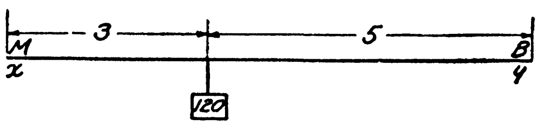

Fig. 15

Solution.

Arithmetic.

$3 + 5 = 8.$

$\frac{3}{8}$ of 120 = 45 lb., weight boy carries.

$\frac{5}{8}$ of 120 = 75 lb., weight man carries.

120. *Check.*

Algebra. Let

x = number of pounds man carries.

y = number of pounds boy carries.

$$x + y = 120.$$

$$3x = 5y.$$

Solving,

$$x = 75.$$

$$y = 45.$$

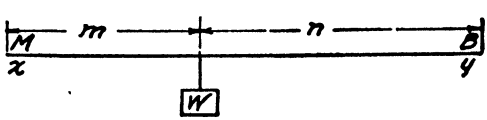

Fig. 16

A general solution.

$$x(m + n) = n \cdot W.$$

$$x = \frac{n \cdot W}{m + n}.$$

$$y(m + n) = m \cdot W.$$

$$y = \frac{m \cdot W}{m + n}.$$

2. Two men, A and B, carry a load of 400 lb. on a pole between them. The men are 15 ft. apart and the load is 7 ft. from A. How many pounds does each man carry?

3. A man and a boy are to carry 300 lb. on a pole 9 ft. long. How far from the boy must the load be placed so that he shall carry 100 lb.?

4. A beam 20 ft. long and weighing 18 lb. per linear foot rests on a support at each end. A load of 1 T. is placed 6 ft. from one end. Find the load on each support.

5. A locomotive weighing 56 T. stands on a bridge with its center of gravity 30 ft. from one end. The bridge is 80 ft. long and weighs 100 T.; it is supported by stone abutments at the ends. Find the total weight supported by each abutment.

6. A man weighing 192 lb. walks on a plank which rests on two posts 16 ft. apart. Construct curves to show the pressure on each of the posts as he walks from one to the other.

MISCELLANEOUS PROBLEMS

1. One end of a crowbar 6 ft. long is put under a rock, and a block of wood is put under the bar 4 in. from the rock. A man weighing 200 lb. puts his weight on the other end. How many pounds does he lift on the rock, and what is the pressure on the block of wood?

2. A nutcracker 6 in. long has a nut in it 1 in. from the hinge. The hand exerts a pressure of 4 lb. What is the pressure on the nut?

3. What pressure does a nut in a nutcracker withstand if it is 2.8 cm. from the hinge, and the hand exerts a pressure of 1.5 kg. 12 cm. from the hinge?

4. Two weights, P and Q, hang at the ends of a weightless lever 80 cm. long. $P = 1.2$ kg. and $Q = 3$ kg. Where is the fulcrum if the weights balance?

5. A man uses a crowbar 7 ft. long to lift a stone weighing 600 lb. If he thrusts the bar 1 ft. under the stone, with what force must he lift on the other end of the bar?

6. A safety valve is $2\frac{1}{2}$ in. in diameter and the lever is 18 in. long. The distance from the fulcrum to the center of the valve is 3 in. What weight must be hung at the end of the lever so that steam may blow off at 100 lb. per square inch, neglecting weight of valve and lever?

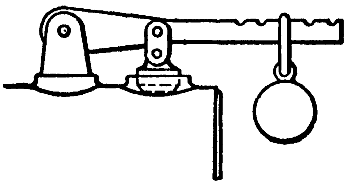

FIG. 17

7. What must be the length of the lever of a safety valve whose area is 10 sq. in., if the weight is 180 lb., steam pressure 120 lb. per square inch, and the distance from the center of the valve to the fulcrum is $3\frac{1}{2}$ in.?

8. Find the length of lever required for a safety valve 3 in. in diameter to blow off at 60 lb. per square inch, if the weight at the end of the lever is 75 lb. and the distance from the center of the valve to the fulcrum is $2\frac{1}{2}$ in.

9. In a safety valve of $3\frac{1}{2}$ in. diameter the length of the lever from fulcrum to end is 24 in., the weight is 100 lb., and the distance from fulcrum to center of valve is 3 in. Find the lowest steam pressure that will open the valve.

10. A bar 4 m. long is used by two men to carry 160 kg. If the load is 1.2 m. from one man, what weight does each carry?

11. A bar 12 ft. long and weighing 40 lb. is used by two men to carry 240 lb. How many pounds does each man carry if the load is 5 ft. from one man?

12. A man and a boy have to carry a load slung on a light pole 12 ft. long. If their carrying powers are in the ratio 8 : 5, where should the load be placed on the pole?

13. A wooden beam 15 ft. long and weighing 400 lb. carries a load of 2 T. 5 ft. from one end. Find the pressure on the support at each end of the beam.

14. A beam carrying a load of 5 T. 3 ft. from one end rests with its ends upon two supports 20 ft. apart. If the beam is uniform and weighs 2 T., calculate the pressure on each support.

15. The horizontal roadway of a bridge is 30 ft. long and its weight is 6 T. What pressure is borne by each support at the ends when a wagon weighing 2 T. is one third the way across?

16. An iron girder 20 ft. long and weighing 60 lb. per foot carries a distributed load of 1800 lb., and two concentrated loads of 1500 lb. each 6 ft. and 12 ft. respectively from one support. Calculate the pressure on each support.

17. One end of a beam 8 ft. long is set solidly in the wall, as in Fig. 18. If the beam weighs 40 lb. per linear foot, what is the bending moment at the wall?

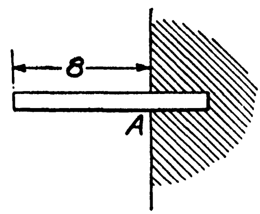

FIG. 18

SOLUTION. The bending moment at any point A is equal to the weight multiplied by its distance from A. We may assume that the weight of the beam is concentrated at its center 4 ft. from the wall. Hence the bending moment $= 320 \times 4 = 1280$ lb. ft.

18. In Fig. 18 a weight of 800 lb. is placed at the end of the beam away from the wall. What will be the total bending moment?

19. A steel beam weighing 100 lb. per linear foot projects 20 ft. from a solid wall. What is the bending moment at the wall? What weight must be placed at the outer end to make the bending moment five times as great?

20. A stiff pole 15 ft. long sticks out horizontally from a vertical wall. It would break if a weight of 30 lb. were hung at the end. How far out on the pole may a boy weighing 80 lb. go with safety?

21. A steel beam 15 ft. long projects horizontally from a vertical wall. At the end is a weight of 400 lb. Construct a curve to show the bending moments of this weight at various points on the beam from the wall to the outer end.

Suggestion. The bending moment at the wall is $400 \times 15 = 6000$ lb. ft.; 1 ft. from the wall it is $400 \times 14 = 5600$ lb. ft., and so on.

22. A beam projects horizontally 15 ft. from a vertical wall. Construct a curve to show the relation between the distance and the weight if the bending moment at the wall is kept at 1200 lb. ft.

CHAPTER V

SPECIFIC GRAVITY

23. Mass. The *mass* of a body is the quantity of matter (material) contained in it. The English unit of mass is a certain piece of platinum kept in the Exchequer Office in London. This lump of platinum is kept as a standard and is called a *pound.* The metric unit of mass is a *gram;* it is the mass of a cubic centimeter of distilled water at 4° C. (39.2° F.).

24. Weight. The *weight* of a body is the force with which the earth attracts it. The mass of a pound weight would not change if it were taken to different places on the surface of the earth, but its weight would change. A piece of brass which weighs a pound in Chicago would weigh a little more than a pound at the north pole and a little less than a pound at the equator. Why? The masses of two bodies are usually compared by comparing their weights.

25. Density. The *density* of a body is the quantity of matter in a unit volume. Thus with the foot and pound as units the density of water at 60° F. is about 62.4, since 1 cu. ft. of water at 60° F. weighs about 62.4 lb. In metric units the density of water at 4° C. is 1, since 1 ccm. of water at 4° C. weighs 1 g. The density of lead in English units is 707; that is, 1 cu. ft. of lead weighs 707 lb. In metric units the density of lead is 11.33, since 1 ccm. of lead weighs 11.33 g.

26. Specific gravity. The *specific gravity* or *relative density* of a substance is the ratio of the weight of a given volume of the substance to the weight of an equal volume of water at 4° C. (39.2° F.). Thus if a cubic inch of copper weighs .321 lb.

and a cubic inch of water weighs .0361 lb., the specific gravity of this piece of copper is $.321 \div .0361 = 8.88$. If we are told that the specific gravity of silver is 10.47, it means that a cubic foot of silver weighs 10.47 times as much as a cubic foot of water.

APPROXIMATE SPECIFIC GRAVITIES

Aluminum . .	2.67	Ice	.917	Oak, white .	.77
Brass . . .	7.82	Iron, cast . .	7.21	Pine, white .	.55
Copper . . .	8.79	Iron, wrought .	7.78	Pine, yellow .	.66
Cork . . .	.24	Lead	11.3	Silver . . .	10.47
Glass, white .	2.9	Marble . . .	2.7	Steel . . .	7.92
Granite . . .	2.6	Mercury, at 60°	13.6	Tin	7.29
Gold	19.26	Nickel . . .	8.8	Zinc . . .	7.19

Exercise. Find the specific gravity of several blocks of wood and pieces of metal.

Problem. The dimensions of a block of cast iron are $3\frac{1}{4}$ in. by $2\frac{3}{4}$ in. by 1 in., and it weighs 37.5 oz. Find its specific gravity.

$$3\tfrac{1}{4} \times 2\tfrac{3}{4} \times 1 = 8.94 \text{ cu. in.}$$
$$1 \text{ cu. in. of water} = .0361 \text{ lb.}$$
$$8.94 \text{ cu. in. of water} = .0361 \times 16 \times 8.94 \text{ oz.}$$
$$= 5.15 \text{ oz.}$$
$$\text{Sp. gr.} = \frac{\text{Weight of block of metal}}{\text{Weight of equal volume of water}}$$
$$= \frac{37.5}{5.15}$$
$$= 7.28.$$

PROBLEMS

1. What is the weight of 1 cu. in. of copper? *8.79*

SOLUTION. 1 cu in. of water = .0361 lb. .0361

Specific gravity of copper is 8.79; that is, copper is 8.79 264
times as heavy as water. 52

$\therefore$ 1 cu. in. of copper $= .0361 \times 8.79$ lb. 1

$= .317$ lb. .317

2. What is the weight of 1 cu. ft. of cast iron?

SOLUTION.

$$1 \text{ cu. ft. of water} = 62.4 \text{ lb.}$$

Specific gravity of cast iron is 7.21.

$$\therefore 1 \text{ cu. ft. of cast iron} = 62.4 \times 7.21 \text{ lb.}$$
$$= 450 \text{ lb.}$$

$$\begin{array}{r} \cancel{6}\cancel{2}.\cancel{4} \\ \underline{7.21} \\ 437 \\ 12 \\ \underline{1} \\ 450 \end{array}$$

3. Find the weight of 1 cu. in. of (*a*) aluminum; (*b*) cork; (*c*) lead; (*d*) gold; (*e*) silver; (*f*) zinc.

4. Find the weight of 1 cu. ft. of (*a*) granite; (*b*) ice; (*c*) marble; (*d*) white oak; (*e*) yellow pine.

5. What is the weight of a yellow-pine beam 20 ft. long, 8 in. wide, and 10 in. deep?

6. The ice box in a refrigerator is 24 in. by 16 in. by 10 in. How many pounds of ice will it hold?

7. A piece of copper in the form of an ordinary brick is 8 in. by 4 in. by 2 in. What is its weight? How much would a gold brick of the same size weigh?

8. A flask contains 12 cu. in. of mercury. Find the weight of the mercury.

9. Find the weight of a gallon of water.

10. What is the weight of a quart of milk if its specific gravity is 1.03?

11. How many cubic inches are there in a pound of water?

SOLUTION.

$$1 \text{ cu. in.} = .0361 \text{ lb.}$$
$$\therefore 1 \text{ lb.} = \frac{1}{.0361} \text{ cu. in.}$$
$$= 27.7 \text{ cu. in.}$$

12. An iron casting weighs 50 lb. Find its volume.

SOLUTION. Let x = number of cubic inches, in the casting.

$$.0361\,x = \text{weight of } x \text{ cu. in. of water.}$$
$$7.21 \times .0361\,x = \text{weight of } x \text{ cu. in. of cast iron.}$$
$$x = \frac{50}{7.21 \times .0361}$$
$$= 192 \text{ cu. in.}$$

13. What is the volume of 50 lb. of aluminum?

14. How many cubic feet are there in 50 lb. of cork?

15. How many cubic inches are there in a flask which just holds 6 lb. of mercury?

16. A cubic foot of bronze weighs 552 lb. What is its specific gravity?

17. Find the specific gravity of a block of limestone if a cubic foot weighs 182 lb.

18. A cubic inch of platinum weighs .776 lb. What is its specific gravity?

19. A cedar block is 5 in. by 3 in. by 2 in. and weighs 10.5 oz. Find its specific gravity.

20. .0928 cu. ft. of metal weighs 112 lb. Find its specific gravity.

21. Each edge of a cubical block of metal is 2 ft. If it weighs 4450 lb., what is its specific gravity?

22. A metal cylinder is 15.3 in. long and the radius of a cross section is 3 in. If it weighs 176.6 lb., what is its specific gravity?

23. The specific gravity of petroleum is about .8. How many gallons of petroleum can be carried in a tank car whose capacity is 45,000 lb.?

27. Advantage of the metric system. So far we have been using the English system, and we have had to remember that 1 cu. in. of water weighs .0361 lb. But in the metric system the weight of 1 ccm. of water is taken as the unit of weight and is called a gram. Thus 8 ccm. of water weighs 8 g. If a cubic centimeter of lead weighs 11.33 g., it is 11.33 times as heavy as water; hence its specific gravity is 11.33. *The weight in grams of a cubic centimeter of any substance is its specific gravity.*

Exercise. To show that 1 ccm. of water weighs 1 g.

Balance a glass graduate on the scales. Pour into it 10, 20, 30 ccm. of water, and it will be found that the weight is 10, 20, 30 g.

What is the weight of 80 ccm. of water? A dish 8 cm. by 5 cm. by 2 cm. is full of water; how many grams does the water weigh? A block of wood is 12 cm. by 10 cm. by 5 cm.; what is the weight of an equal volume of water? A brass cylinder contains 125 ccm.; what is the weight of an equal volume of water? Hence the volume of a body in cubic centimeters is equal to the weight in grams of an equal volume of water.

28. First method of finding specific gravity.

1. Weigh the solid in grams.
2. Find the volume of the solid in cubic centimeters.

$$1 \text{ ccm. of water} = 1 \text{ g.}$$

∴ the volume in cubic centimeters equals the weight of an equal volume of water.

3. $$\frac{\text{Weight in grams}}{\text{Weight of an equal volume of water}}$$
$$= \frac{\text{Weight in grams}}{\text{Volume in cubic centimeters}} = \text{Sp. gr.}$$

Exercise. Find the specific gravity of (*a*) a brass cylinder; (*b*) a brass prism; (*c*) a steel ball; (*d*) a copper wire; (*e*) an iron wire; (*f*) a pine block; (*g*) a piece of oak. Can you expect to obtain the specific gravities given in the table? Why not?

PROBLEMS

1. A block of metal 13.8 cm. by 14.2 cm. by 27.0 cm. weighs 60 kg. Find its specific gravity.

2. A cylinder is 84.3 mm. long and the radius of its base is 15.4 mm. If it weighs 157 g., what is its specific gravity?

3. A metal ball of radius 21.5 mm. weighs 292.6 g. Find its specific gravity.

4. The altitude of a cone is 42.1 mm. and the radius of the base is 14.6 mm. Find its specific gravity if it weighs 22.3 g.

5. How many times heavier is (*a*) gold than silver? (*b*) gold than aluminum? (*c*) mercury than copper? (*d*) steel than aluminum? (*e*) platinum than gold? (*f*) cork than lead?

6. The pine pattern from which an iron casting is made weighs 15 lb. About how much will the casting weigh? (The usual foundry practice is to call the ratio 16 : 1.)

29. The principle of Archimedes. This principle furnishes a convenient method of finding the specific gravity of substances.

Exercise. Weigh a brass cylinder; weigh it when suspended in water and find the difference of the weights. Lower the cylinder into an overflow can filled with water and catch the water in a beaker as it flows out. Compare the weight of this water with the difference in the weights. Do this with several pieces of metal. What seems to be true?

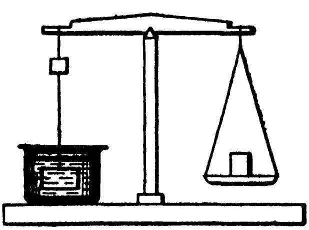

Fig. 19

Imagine a steel ball submerged in water resting on a shelf. If the shelf were taken away, the ball would sink to the bottom of the tank. Now suppose the surface of the ball contained water instead of steel, and suppose the inclosed water weighed 5 oz. If the shelf were removed, the water ball would be held in its position by the surrounding water; that is, when the steel ball is suspended in water, the water holds up 5 oz. of the total weight of the ball.

Principle of Archimedes. *Any body when suspended in water loses in weight an amount equal to the weight of its own volume of water.*

30. Second method of finding specific gravity.

1. Weigh a piece of cast iron, 156.3 g.
2. Weigh it when suspended in water, 134.3 g.
3. $156.3 - 134.3 = 22.0$ g. This is the weight of an equal volume of water.

4. Sp. gr. $= \frac{156.3}{22.0} = 7.10.$

Let W = the weight of the substance in air.
w = the weight of the substance suspended in water.

Then $\frac{W}{W - w}$ = the specific gravity of the substance.

Exercise. Find by this method the specific gravity of (*a*) brass; (*b*) copper; (*c*) cast iron; (*d*) glass; (*e*) lead; (*f*) porcelain; (*g*) an arc-light carbon.

PROBLEMS

1. How much will a brass 50-g. weight weigh in water?

SOLUTION. Let x = the weight in water.

$$\frac{50}{50 - x} = 7.82.$$

Solving, $x = 43.6$ g. Check by experiment.

2. Compute the weight in water of (*a*) 100 g. of copper; (*b*) 500 g. of zinc; (*c*) 1 kg. of silver; (*d*) 200 g. of pine; (*e*) 100 g. of cork.

3. Find the weight in water of (*a*) 1 lb. of cast iron; (*b*) 1 lb. of lead; (*c*) 5 lb. of aluminum; (*d*) 1 T. of granite; (*e*) 10 lb. of cork.

4. If a boy can lift 150 lb., how many pounds of the following substances can he lift under water: (*a*) platinum? (*b*) lead? (*c*) cast iron? (*d*) aluminum? (*e*) granite?

SOLUTION. (*a*) The problem is to find the weight in air of a mass of platinum which weighs 150 lb. in water.

Let w = the weight in air.

$$\frac{w}{w - 150} = 22 \text{ (specific gravity of platinum).}$$

Solving, $w = 157$ lb.

5. Construct a curve to show the weight in air of masses which weigh 1 lb. in water, the specific gravity varying from 1 to 20.

6. A copper cylinder weighs 80 lb. under water. How much does it weigh in air?

7. A cake of ice just floats a boy weighing 96 lb. How many cubic feet are there in it?

Suggestion. 1 cu. ft. of water weighs 62.4 lb. How much does 1 cu. ft. of ice weigh? How many pounds will 1 cu. ft. of ice float? How many cubic feet of ice are required to float 96 lb.?

8. A pine beam 1 ft. square is floating in water. If its specific gravity is .55, how long must it be to support a man weighing 180 lb.?

9. Construct a graph to show the weight in water of masses of cast iron weighing from 1 to 100 lb. in air, given that the specific gravity of cast iron is 7.2. Why should the graph be a straight line?

MISCELLANEOUS PROBLEMS

1. Find the weight of 50 ccm. of copper.

SOLUTION.

$$\begin{aligned} \text{1 ccm. of water} &= 1 \text{ g.} \\ \text{Specific gravity of copper} &= 8.79. \\ \therefore \text{ Weight of 50 ccm. of copper} &= 50 \times 8.79 \text{ g.} \\ &= 440 \text{ g.} \end{aligned}$$

2. Find the weight of (*a*) 100 ccm. of mercury; (*b*) 150 ccm. of zinc; (*c*) 300 ccm. of aluminum.

3. Find the volume of 300 g. of zinc.

SOLUTION. 1 g. of water has a volume of 1 ccm.

Specific gravity of zinc = 7.19.

∴ 7.19 g. of zinc has a volume of 1 ccm.

$$\frac{300}{7.19} = 41.7 \text{ ccm.}$$

4. Find the volume of (*a*) 50 g. of brass; (*b*) 100 g. of cork; (*c*) 100 g. of gold; (*d*) 150 g. of marble; (*e*) 1 kg. of silver.

5. The dimensions of a rectangular maple block are 8.1 cm., 5.2 cm., and 3.5 cm. If it weighs 100 g., find its specific gravity.

6. 109 ccm. of copper and 34 ccm. of zinc are melted together to form brass. Find its specific gravity.

SOLUTION. Let s = the specific gravity of the brass.

$$109 + 34 = 143 \text{ ccm., volume of the brass.}$$
$$143\,s = \text{weight of the brass.}$$
$$109 \times 8.79 = \text{weight of the copper.}$$
$$34 \times 7.19 = \text{weight of the zinc.}$$
$$143\,s = 109 \times 8.79 + 34 \times 7.19.$$

Solve for s and check.

7. 58.8 g. of copper and 25.2 g. of zinc are combined to form brass. What is its specific gravity?

SOLUTION. Let s = specific gravity of the brass.

$$58.8 + 25.2 = 84 \text{ g., weight of the brass.}$$
$$\frac{84}{s} = \text{volume of the brass.}$$
$$\frac{58.8}{8.79} = 6.69 = \text{volume of the copper.}$$
$$\frac{25.2}{7.19} = 3.50 = \text{volume of the zinc.}$$
$$\frac{84}{s} = 6.69 + 3.50.$$

Solve for s and check.

8. The specific gravity of a piece of brass weighing 123 g. is 8.22. How many grams of copper and of zinc are there in it?

SOLUTION. Let c = number of grams of copper.

z = number of grams of zinc.

$$\frac{c}{8.79} = \text{volume of the copper.}$$
$$\frac{z}{7.19} = \text{volume of the zinc.}$$
$$\frac{123}{8.22} = \text{volume of the brass.}$$
$$c + z = 123.$$
$$\frac{c}{8.79} + \frac{z}{7.19} = \frac{123}{8.22}.$$

Solve and check.

9. An alloy was formed of 79.7 ccm. of copper and 51.4 ccm. of tin. Find its specific gravity.

10. 475.2 kg. of hard gun metal was made by combining 411 kg. of copper and 64.2 kg. of tin. What was its specific gravity?

11. 336 lb. of copper and 63 lb. of zinc were combined to make brazing metal. Find its specific gravity.

Suggestion. To reduce pounds to grams multiply by 453.6. Since this factor occurs in each term of the equation, it may be divided out.

12. Nickel-aluminum consists of 20 parts of nickel and 80 parts of aluminum. Find its specific gravity.

13. What is the specific gravity of bell metal consisting of 80 per cent copper and 20 per cent tin?

14. Find the specific gravity of Tobin bronze, which consists of 58.22 per cent copper, 2.30 per cent tin, and 39.48 per cent zinc.

15. 516 g. of copper, 258 g. of nickel, and 226 g. of tin are combined to form German silver. Find its specific gravity.

16. How much copper and how much aluminum must be taken to make 200 kg. of aluminum bronze having a specific gravity of 7.69?

17. A mass of gold and quartz weighs 500 g. The specific gravity of the mass is 6.51 and of the quartz is 2.15. How many grams of gold are there in the mass?

CHAPTER VI

GEOMETRICAL CONSTRUCTIONS WITH ALGEBRAIC APPLICATIONS

NOTE. Make all drawings and constructions in a notebook. Record all the work in full, having it arranged neatly on the page. Make the constructions as accurately as possible.

31. Drawing straight lines. Keep the pencil sharp, and make the lines heavy enough to be clearly seen.

Exercise 1. Draw a line 2 in. long.

FIG. 20

To do so most accurately, draw an indefinite line AB. Then put your compasses on the scale of the ruler so that the points are 2 in. apart. With A as a center strike an arc at C. AC is the required line.

Exercise 2. Using this method, draw lines as follows: (*a*) $1\frac{7}{8}$ in.; (*b*) 1 dm.; (*c*) 1 cm.; (*d*) 83 mm.; (*e*) 3.5 cm.; (*f*) 136 mm.

32. Drawing to scale. Choose a scale that will give a good-sized figure, and below every figure record the scale used.

Exercise 3. The distance between two towns A and B is 30 mi. How could a line 6 cm. long represent that distance? Draw such a line and explain the relation that exists between the distance and the line.

Exercise 4. Draw a line 3 in. long and let it represent a distance of 36 mi. What distance is represented by 1 in.? by 2 in.? by $1\frac{1}{2}$ in.? by $2\frac{3}{8}$ in.? In this exercise the distance is said to be represented on a scale of 1 in. to 12 mi.

Exercise 5. With a scale of 1 in. to 16 ft. (1 in. = 16 ft.) draw lines to represent the distances (*a*) 8 ft.; (*b*) 12 ft.; (*c*) 24 ft.; (*d*) 36 ft.; (*e*) 18 ft.

33. Measuring straight lines. With an unmarked ruler or with the edge of your book draw a line AB. To locate the ends of the line as accurately as possible, make small marks in the paper at A and B with the point of the compasses. Care should be taken that the marks do not penetrate to the surface below. Place one point of the compasses at A and let the other fall at B. With this opening of the compasses place the points against the scale of a ruler, one point on the division marked 1 cm., and count the number of centimeters and tenths of a centimeter between the points of the compasses. On the line AB write its length as you have found it. (The end divisions of a ruler are not usually so accurate as the middle divisions; hence in making a measurement it is best not to start at the zero of the scale.)

Exercise 6. Make two crosses in your notebook and call the points of intersection M and N. Using the compasses, measure MN in inches and centimeters and record the result.

Exercise 7. Draw an indefinite line AX and mark off on it $AB = 2.8$ cm., $BC = 1.7$ cm., and $CD = 3.4$ cm. Then with your compasses measure AD. Record the length and compare it with the sum of the numbers.

Exercise 8. (*a*) Measure the lines AB, CD, and EF. Record the measurements and add them.

FIG. 21

(*b*) Draw an indefinite line AX and mark off on it AB, CD, and EF, the point C falling on B and the point E on D. Measure AF and record the result. Compare with that obtained in (*a*).

34. Angles. An *angle* is formed by two lines that meet. Thus the lines BC and BA meet at the *vertex* B, forming the angle ABC, B, or m. When three letters are used to denote an angle the letter at the vertex is read between the other two. The single small letter should be used to denote an angle when convenient.

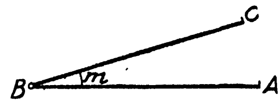

FIG. 22

The size of an angle depends on the amount of opening between the lines.

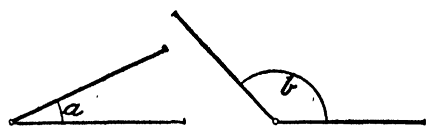

FIG. 23

A *right angle* is an angle of 90°.

An *acute angle* is less than 90°.

An *obtuse angle* is greater than 90° and less than 180°.

Thus a is an acute angle and b is an obtuse angle.

35. The protractor. To measure an angle place the protractor so that the center of the graduated circle is at the vertex of the angle and its straight side lies along one arm of the angle. Note the graduation under which the other arm of the angle passes.

Exercise 9. Take a piece of paper and fold it twice so that the creases will form four right angles at a point. Test one of the angles with the protractor.

Exercise 10. About a point construct angles of 42°, 85°, and 53°. What is the test of accuracy of construction?

Exercise 11. At each end of a line AB, 7 cm. long, construct an angle of 60° so that AB is one arm of each angle and the other arms intersect at C. Measure angle ACB, and write the number of degrees in each angle. Measure AC and BC. What is the test of accuracy of construction? Bisect angle ACB by the line CD, D being on AB. How much longer is AC than AD?

Exercise 12. Draw a large triangle. Measure each angle and write the results in the angles. What ought to be the sum?

Exercise 13. Make an angle $A = 37°$. On the horizontal arm take $AC = 6$ cm. and on the other arm take $AB = 7.5$ cm. Draw BC. Guess the number of degrees in angle ACB. Measure it.

Exercise 14. To find the distance across a lake from A to B, a surveyor selected a point C from which he could see both A and B. He measured the angle ACB, 72°, with a transit and found the distances CA and CB to be 450 ft. and 400 ft. respectively. From these measurements draw the figure to scale; measure AB and determine what distance it represents.

Exercise 15. To find the height of a building AB across a river DB measurements were made as follows: angle $ACB = 16°$, angle $ADB = 37°$, and $CD = 100$ ft. Draw to scale, and find the height of the building and the width of the river.

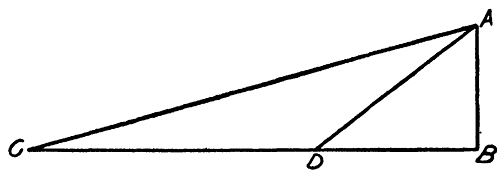

FIG. 24

Exercise 16. A man wishing to find the distance between two buoys, A and B, measured a base line CD 1500 ft. in length along the shore. At its extremities, C and D, he measured the following angles: angle $DCB = 36° 15'$, angle $BCA = 50° 45'$, angle $CDA = 43° 30'$, and angle $ADB = 72°$. Draw to scale, and find the distance between the buoys.

36. From a point in a line to draw a line at right angles (perpendicular) to it.

CONSTRUCTION. Let C be the point in AB from which the line is to be drawn. Place one point of the compasses at C and mark off on AB the equal distances CD and CE. With D and E as centers and a convenient radius describe arcs intersecting at F. Draw CF. FCB is a right angle, and CF is said to be perpendicular to AB.

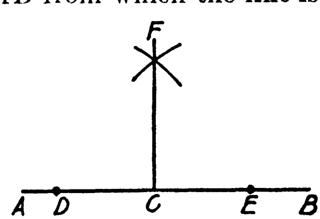

FIG. 25

Example. To construct a right triangle whose legs are 6 cm. and 8 cm. respectively.

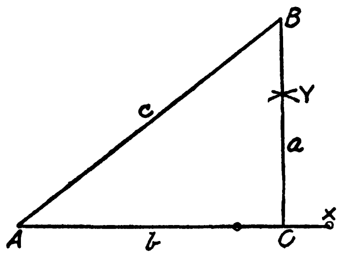

FIG. 26

CONSTRUCTION. Draw an indefinite line AX and mark off $AC = 8$ cm. At the point C construct the perpendicular CY and take $CB = 6$ cm. Draw AB, and ABC is the required triangle.

Measure $c = 9.95$ cm.

Check your construction by the formula

$$a^2 + b^2 = c^2,$$

where a and b are the legs of a right triangle and c is the hypotenuse.

$$\begin{aligned} a^2 + b^2 &= 6^2 + 8^2 \\ &= 36 + 64. \\ c^2 &= 100. \\ c^2 = 9.95^2 &= 99.0. \end{aligned}$$

$$\begin{array}{r} 9.95 \\ 9.95 \\ \hline 896 \\ 89 \\ 5 \\ \hline 99.0 \end{array}$$

Exercise 17. Construct to scale if necessary and check as in the preceding exercise, given a and b. (a) 3.5 cm. and 6.8 cm.; (b) 4.3 cm. and 9.6 cm.; (c) 84 mm. and 64 mm.; (d) 42 in. and 18 in.; (e) 28 ft. and 16 ft.; (f) 120 mi. and 200 mi.

Exercise 18. Construct a square whose side is 4 cm.

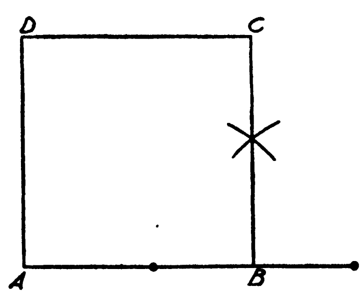

FIG. 27

CONSTRUCTION. Make $AB = 4$ cm. At B draw BX perpendicular to AB. Cut off $BC = 4$ cm. With A and C as centers and a radius of 4 cm. draw arcs intersecting at D. Draw AD and CD. $ABCD$ is the required square. Measure the diagonal and record the result on the figure. Check by applying the formula of the right triangle.

Exercise 19. Construct to scale squares whose sides are (a) 12 in.; (b) 1.8 m.; (c) 540 mm. Check by formula.

Exercise 20. Construct to scale and check, rectangles whose sides are (*a*) 78 and 48 cm.; (*b*) 32 and 54 in.; (*c*) 482 and 615 ft.

37. To construct a perpendicular to a line from a point outside the line.

Let AB be the line and C the point. With C as a center describe an arc cutting AB at D and E. With D and E as centers and a convenient radius describe arcs intersecting at F. Draw CF, the required perpendicular.

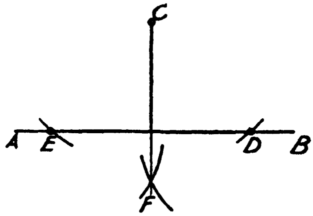

FIG. 28

Exercise 21. Construct right triangles whose legs are (*a*) 6 and 12 cm.; (*b*) 5 and 9 cm. Draw perpendiculars from the vertex of the right angle to the hypotenuse. Measure and check.

Exercise 22. Draw a large triangle and construct a perpendicular from the vertex to the base. Measure the sides of the two right triangles formed and check by the formula.

38. To construct a triangle whose sides are given.

Exercise 23. Construct a triangle whose sides are 7, 8, and 10 cm. respectively.

CONSTRUCTION. Draw a line AB 10 cm. long. With A as a center and a radius of 7 cm. describe an arc. With B as a center and a radius of 8 cm. describe an arc cutting the first arc at C. Draw AC and BC, and ABC is the required triangle.

Exercise 24. From C in the figure of Exercise 23 draw a perpendicular to AB. Measure the sides of the right triangles and check by the formula.

Exercise 25. Construct a triangle whose sides are 7.5, 8.5, and 11 cm. respectively. Draw a perpendicular from the vertex to the base and find the area of the triangle. Check by drawing a perpendicular to another side and use its length to find the area. The perpendicular from the vertex to the base is called the altitude of the triangle.

39. To bisect a given line.

Exercise 26. Bisect a given line AB.

CONSTRUCTION. With A and B as centers and a convenient radius describe arcs intersecting at C and D. Draw CD, intersecting AB at E. Then $AE = EB$. Check by measuring.

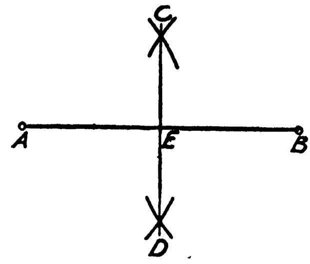

FIG. 29

Exercise 27. Draw an indefinite line AB and divide it into four equal parts, using the method of arcs. Check.

Exercise 28. Construct an equilateral triangle ABC whose sides are each 9 cm. Divide the base into four equal parts. Draw CD and CF and measure their lengths. Measure the angle ADC. Applying the formula of the right triangle, compute CD and CF.

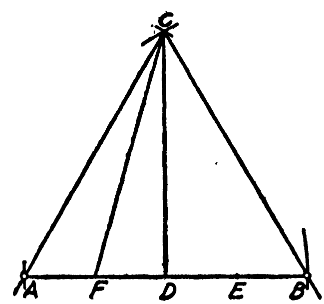

FIG. 30

40. To bisect an angle.

Exercise 29. Make an angle BAC and bisect it.

CONSTRUCTION. With A as a center and with a rather large radius mark two points D and E on AC and AB respectively. With D and E as centers and the same radius describe arcs intersecting at F. Draw AF, and angle $BAF =$ angle FAC. Check with the protractor.

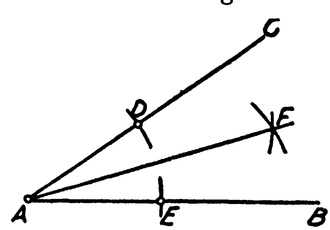

FIG. 31

Exercise 30. Draw an obtuse angle and bisect it. Check.

Exercise 31. Construct a triangle ABC with $AB = 7.6$ cm., $AC = 6.5$ cm., and angle $A = 45°$. Construct the altitude CD and measure its length. Check by computing the length of CD, using the formula of the right triangle.

41. Parallel lines. Lines that lie in the same plane and do not meet however far produced are called *parallel lines*.

Exercise 32. Construct a rectangle whose dimensions are 4.35 and 7.85 cm. respectively. Find the area to three significant figures. The opposite sides of a rectangle are parallel. Write in your notebook the sides that are parallel.

42. Parallelograms. If the opposite sides of a four-sided figure are parallel, the figure is called a *parallelogram*. $ABCD$ is a parallelogram.

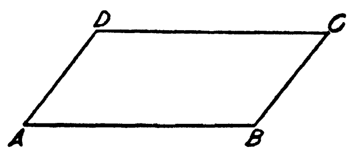

FIG. 32

Exercise 33. Construct a parallelogram with $AB = 8$ cm., $AD = 5$ cm., and angle $A = 65°$. The point C can be obtained with arcs, as in Exercise 18. Name the parallel sides. Measure all the angles.

Exercise 34. Construct a parallelogram with $AB = 9.45$ cm., $BC = 4.15$ cm., and angle $B = 115°$. From D construct DE perpendicular to AB, E being on AB. The line DE is the altitude of the parallelogram. Measure DE and find the area of the parallelogram.

43. To draw a line parallel to a given line.

Exercise 35. Construct a triangle with $AB = 8$ cm., $BC = 9$ cm., and $AC = 6$ cm. Take $CD = 4$ cm. Through D draw DE parallel to AB. (Construct the parallelogram $ADFG$.) Measure CE, or y, and record its length. The equation $\frac{4}{2} = \frac{y}{9 - y}$ will give the length of CE. Solve the equation and compare with the measured length.

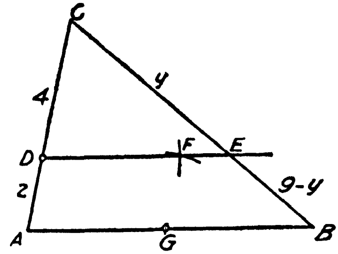

FIG. 33

Exercise 36. Construct a triangle ABC whose sides are: $AB = 7$ cm., $BC = 9$ cm., and $CA = 11$ cm. On BC take

$BD = 3$ cm., and through D draw a parallel to AB. Measure the lengths of the two parts of AC and check by an equation like that in Exercise 35.

44. To construct an angle equal to a given angle.

Exercise 37. At the point D on DE to construct an angle equal to angle A.

CONSTRUCTION. With A as a center and a rather large radius describe the arc BC cutting AX at B and AY at C. With D as a center and the same radius describe an arc FG cutting DE at F. Take

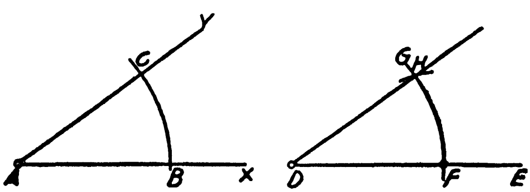

FIG. 34

off with the compasses the distance BC; then with F as a center and BC as a radius describe an arc cutting FG at H. Draw DH. Angle D is the required angle equal to A. Check with the protractor.

Exercise 38. Make angles of (*a*) 40°, (*b*) 58°, (*c*) 140°, and construct angles equal to them.

Exercise 39. Construct a triangle ABC, making $AB = 8.4$ cm., $BC = 6.8$ cm., and $AC = 7.2$ cm. Draw a line $DE = 4.2$ cm. At D make an angle EDF equal to angle BAC, and at E make an angle DEF equal to angle ABC. Produce the two lines till they meet at F. Measure the sides and angles of the triangle DEF and compare them with the corresponding parts of the triangle ABC.

Triangles which have their corresponding angles equal and their corresponding sides proportional are called *similar triangles.*

Exercise 40. The angle of elevation of a church steeple at a point 300 ft. from its base was found to be 16°. Construct a similar triangle, that is, draw to scale and find the height of the steeple.

Exercise 41. At a distance of 500 ft. the angle of elevation of the top of one of the "big trees" of California is 31°. How tall is the tree?

Exercise 42. Make some practical problems and solve them.

PROBLEMS

Record all measurements and give the work in full in your notebooks.

1. The two legs of a right triangle are 15 and 36 ft. respectively. Construct the triangle to scale, stating scale used. Measure the hypotenuse. Check by applying the formula of the right triangle.

2. Construct a rectangle 4 cm. by 7 cm. Measure the diagonal. Check.

3. A right angle may be constructed as shown in Fig. 35. ABC is an equilateral triangle. $CD = BC$. AD is drawn and BAD is a right angle. Construct a right angle DAB. On AB take $AE = 8.4$ cm., and on AD take $AF = 3.5$ cm. Measure EF. Check.

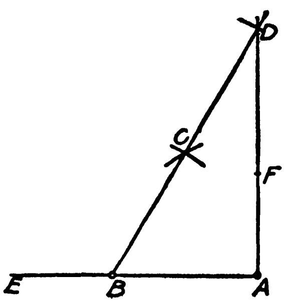

FIG. 35

4. The hypotenuse of a right triangle is 19.4 ft. and one leg is 14.2 ft. Compute the length of the other leg. Check by constructing the triangle to scale and measuring the required leg.

5. The base of a right triangle is x, the altitude is $x + 1$, and the hypotenuse is $x + 2$. Find x by applying the formula of the right triangle. Check by constructing a right triangle with the legs x and $x + 1$. Measure the hypotenuse and compare with the value of $x + 2$.

6. The following sets of expressions represent the sides of a right triangle. Solve and check as in Problem 5.

	Legs	Hypotenuse
(*a*)	x and $x + 3$	$x + 6$
(*b*)	x and $x + 7$	$x + 8$
(*c*)	x and $x - 2$	$x + 2$
(*d*)	x and $x + 4$	$x + 8$.
(*e*)	x and $x - 7$	$x + 1$
(*f*)	x and $2x - 4$	$2x - 2$
(*g*)	x and $x + 1$	$2x - 11$
(*h*)	x and $x + 5$	$2x - 5$

7. The altitude of a rectangle is 1 ft. less than the base, and the area is 20 sq. ft. Find the dimensions. Check by drawing on squared paper and counting the squares.

8. The following sets of expressions represent the sides and the area of a rectangle. Find the dimensions and check as in Problem 7.

	Sides	Area
(*a*)	x and $x - 10$	24
(*b*)	x and $x - 7$	30
(*c*)	x and $x + 12$	85
(*d*)	x and $x + 9$	90
(*e*)	x and $2x + 5$	18
(*f*)	x and $2x + 1$	36
(*g*)	x and $3x - 7$	40
(*h*)	x and $4x - 10$	24

9. Construct a right triangle ABC, denoting the base by x and the altitude by y. Complete the rectangle xy. How is the area of the rectangle found? What algebraic expression represents it in this case? What part of the rectangle is the triangle ABC? What algebraic expression represents the area of the triangle? What reason can you give for the correctness of the expression for the area of the triangle?

10. The legs of a right triangle are x and $x + 6$. Its area is 20. Find the sides of the triangle. To check, draw on squared paper a right triangle whose legs are x and $x + 6$. Find the area by counting the large squares inside the triangle.

When a part of a square looks less than a half, it is not counted; but if it looks greater than a half, it is counted as a whole square.

11. The following sets of expressions represent the legs and area of a right triangle. Find the length of the legs in each case, and check on squared paper as in Problem 10.

	Legs	Area
(*a*)	x and $x - 11$	30
(*b*)	x and $x - 12$	14
(*c*)	x and $x + 10$	28
(*d*)	x and $x - 15$	27
(*e*)	x and $2x - 7$	15
(*f*)	x and $5x - 9$	40
(*g*)	x and $3x - 1$	35
(*h*)	x and $4x - 9$	45

12. Construct a parallelogram $ABCD$. Bisect the angles A and B, and let the bisectors meet at F. Measure the angle AFB. Measure AB, BF, and FA. Apply the formula of the right triangle. Make the test in several parallelograms and state what seems to be true of the bisectors of two consecutive angles of a parallelogram.

13. Construct a triangle ABC with $CB = 8$ cm., $AB = 10.5$ cm., and $AC = 5.5$ cm. On CA take $CD = 2$ cm., and from D draw a line parallel to CB intersecting AB at E. From the formula $\frac{AD}{DC} = \frac{AE}{EB}$ find AE. Check by measuring AE.

14. In the figure of Problem 13 let $AD = x$, $DC = 3$, $AE = x + 1$, and $EB = 5$. Use the formula to find x. Find the sides AC and AB. Check by construction, taking the base any convenient length.

15. The following sets of values are the segments of the sides of a triangle formed by a line parallel to the base. Find the length of each segment and check by constructing the triangle and the parallel as in Problem 14.

	AD	DC	AE	EB
(a)	x	3	$2 - x$	$2 + x$
(b)	x	2	$x + 5$	$x - 1$
(c)	x	4	$x + 1$	$x + 7$
(d)	x	5	$4 - x$	$3 + x$
(e)	x	3	$x + 2$	$x + 5$
(f)	x	$x + 2$	$x + 4$	$2x - 5$
(g)	x	$x + 3$	$2x - 1$	x
(h)	x	$x + 4$	$3x - 2$	$x + 5$

16. The legs of a right triangle are x and y, and their sum is 15. If the area of the triangle is 27, find x and y. To check the result, construct on squared paper a right triangle whose legs are x and y. Count the large squares and compare with the given area.

17. The sum of the legs and the area of a right triangle are given by the following sets of numbers. Find x and y, and check.

	Legs	Sum of the Legs	Area
(a)	x and y	16	14
(b)	x and y	25	42
(c)	x and y	15	28
(d)	x and y	19	36

18. The difference of the legs of a right triangle and the area are given by the following sets of numbers. Find x and y, and check.

	Legs	Difference of the Legs	Area
(a)	x and y	12	32
(b)	x and y	10	48
(c)	x and y	8	64
(d)	x and y	9	35
(e)	x and y	5	102

CHAPTER VII

THE USE OF SQUARED PAPER

I. Graphical Representation of Tables of Values

45. The results of experiments and observations, statistical tables, and tabulated numerical data of all kinds can be represented by lines and curves. The graph shows at a glance relations which are not so evident in a table of values; and it also enables one to find readily values which lie between those given in the table.

Exercise. Construct a graph to represent this record of temperature, given in *The Chicago Daily News.*

3 P.M.	78	3 A.M.	73
4 P.M.	77	4 A.M.	73
5 P.M.	76	5 A.M.	72
6 P.M.	75	6 A.M.	72
7 P.M.	76	7 A.M.	72
8 P.M.	75	8 A.M.	71
9 P.M.	73	9 A.M.	71
10 P.M.	74	10 A.M.	75
11 P.M.	75	11 A.M.	74
12 midnight	73	12 noon	75
1 A.M.	73	1 P.M.	76
2 A.M.	73		

We have here two quantities, hours and degrees, so related that to a change in one there is a corresponding change in the other.

The sheets of squared paper we use have seventeen large squares each way; the side of a large square is a centimeter, and of a small square a millimeter.

The units for representing an hour and a degree should be chosen so that the picture may be of good size and still allow the whole table to be represented. Let the horizontal lines represent time and the vertical lines represent temperature.

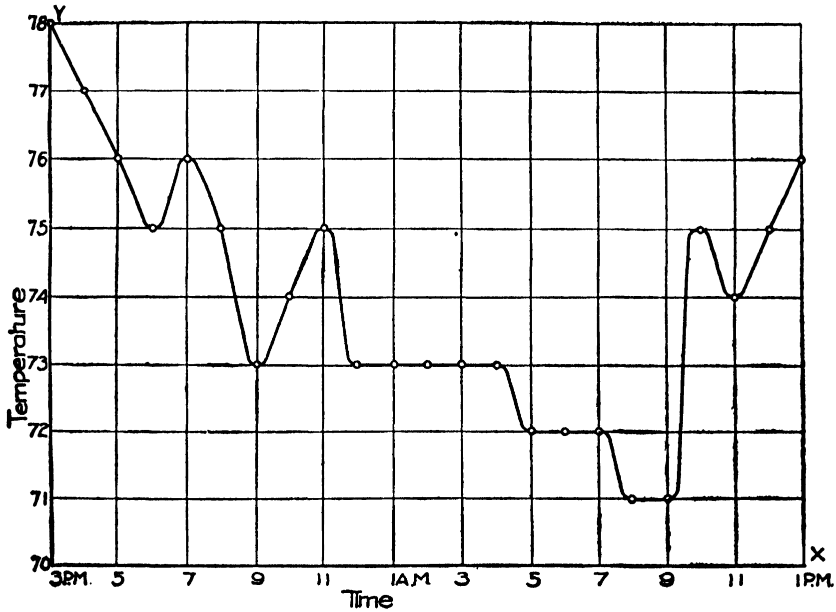

FIG. 36

The horizontal and vertical lines from which we count degrees and hours are called *axes*. We will always mark them OX and OY respectively, and call them the x-axis and the y-axis. The point O is called the *origin*.

Since the number of degrees is always greater than 70, we may call the x-axis 70° to save space. At 3 P.M. the temperature is 78°; hence on the 3 o'clock line we put a point at 78°, and so on for the other hours, as shown in Fig. 36. A smooth curve is then drawn through the points, and we have a curve which shows at a glance the change in temperature during the day.

The curve does not, of course, show the exact reading of the thermometer between the hours. However, it shows when the temperature was falling and when rising, whether the change was rapid or gradual, and in general gives a fairly correct representation of the temperature for the day.

46. Hints on the use of squared paper. All graphical work should be done in a book of squared paper where it can be referred to from time to time. Much can be learned by looking back over the curves and noting the relations between the various problems and curves. Frequently it will be found that a curve of the same shape is constructed in solving several different problems.

Each graphical solution ought to be complete in itself. The table of values or other data should be written on the sheet with the curve, or on the blank page at the left of the graph in the notebook. The axes should be lettered OX and OY, and the units written on them. It is not necessary that the units should be the same for both axes, but they should be chosen so that the whole range of values may be plotted in a figure which extends well over the sheet of squared paper.

When a curve is constructed for the sole purpose of reading off intermediate values, a large square should represent 1, 5, 10, 20, 50, 100 $\cdots$, numbers which give easy readings.

If the curve is made simply to show general changes or to solve a problem, the unit may be chosen to locate the points with the least work.

If two or more curves are constructed on the same axes, they should be numbered to correspond with the tables or data, and they can be more readily distinguished if a different kind of line is used for each curve, for example, thick and thin continuous lines, dotted lines, and so on. When convenient the various curves may be drawn in different colors; in this case the table of values should be written in the same color that is used for the curve which represents it.

EXERCISES

1. Construct a curve from the record of temperature given above with the same time unit, but let 5 mm. = 1°. From which curve can the changes be read most easily?

2. Construct several temperature curves from the weather reports in the daily papers.

3. On the same axes construct temperature curves for a day in summer and a day in winter.

4. Construct on the same axes temperature curves for several cities, e.g. Boston, Chicago, and San Francisco.

5. Place a thermometer outside the classroom window and take readings at the beginning and end of the recitation hour for two or three weeks. Construct the curve.

6. Construct several curves from tables found in newspapers, magazines, *The Daily News Almanac*, *The World Almanac*, Kent's "Mechanical Engineers' Pocket-Book," city, state, and government reports, price lists, and so on. Try to find reasons for any marked peculiarities in the curves.

7. Construct curves to show the number of hours of daylight per day for the year. (Let a heavy horizontal line near the center represent noon. From an almanac make a table of the time of sunrise and sunset on the first day of each month; locate the points and draw the two curves.) On the same sheet of squared paper make curves for different latitudes, e.g. Chicago and Dawson, Alaska, and compare the amounts of daylight.

8. A price list of the Western Electric Company gives the following price of bells. Construct the curve.

Size of gong in inches	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	5	6	7	8	10	12
Price in dollars	1.68	1.74	1.85	1.96	2.84	3.20	4.56	5.00	8.00	10.00

What is the probable price of a 9-in. gong? of an 11-in. gong?

9. The water in a glass is at a temperature of 60° F. Heat is applied to the glass, and the temperature, T, at the end of t minutes is as follows:

Minutes . .	0	5	10	15	20	25	30	35	40
Degrees . .	60	68	76	83.2	89.6	95.5	101	106	110

Construct the curve. What temperature would you expect at the end of 7 min.? of 32 min.?

10. A boat is rowed straight across a river and soundings are taken at various distances from the bank. From the table draw a section of the river bed.

Distance from bank in feet	0	5	10	15	20	25	30	35	40	45	50	52	54	56	58
Depth in feet	0	1	3	5	8	15	14	16	18	18	12	8	5	4	3

11. From the top of a cliff 1500 ft. high a bullet was shot horizontally with a velocity of 100 ft. per second. Construct a curve to show its path, if at the end of each second it has fallen the following number of feet:

Number of seconds .	0	1	2	3	4	5	6	7	8	9	10
Distance fallen . .	0	16	64	144	256	400	576	784	1024	1296	1600

Take the x-axis at the top of the sheet. On the x-axis let 1 cm. = 1 sec., or 100 ft.; on the y-axis, 1 cm. = 100 ft. In how many seconds will the bullet reach the ground if it is level? How far from the foot of the cliff will it fall? In how many seconds will it fall 600 ft.? How far will it fall in $5\frac{1}{2}$ sec.?

II. The Graph as a "Ready Reckoner"

47. Straight-line graphs. In the following exercises the graph is a straight line. Choose convenient units and let the graph extend well over the sheet of squared paper.

EXERCISES

1. Construct a graph to change inches into centimeters and centimeters into inches, given 1 in. = 2.5 cm.

CONSTRUCTION.

0 in. = 0 cm.
4 in. = 10 cm.

Locate these two points, O and P, and draw a straight line through them. Test a few points on the graph to see if the results are approximately correct. Thus at M 2 in. = 5 cm.

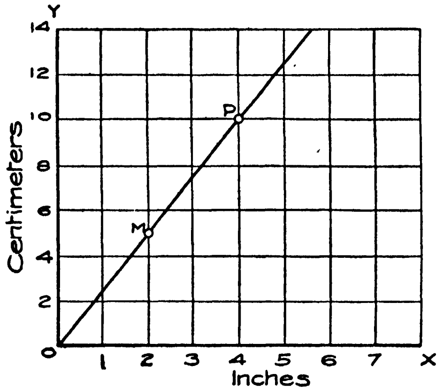

FIG. 37

2. Construct a graph to change pints to liters, given that 1 l. = 2.1 pt.

3. Construct a graph to find the circumferences of circles of diameter from 0 to 18 in., given that the circumference equals π times the diameter.

4. Construct a graph to find the velocity of a falling body, given that the velocity at any second equals 32 times the number of seconds.

5. Construct a graph to change miles per hour to feet per second, given that 30 mi. per hour equals 44 ft. per second.

6. Construct a graph to change cents to marks, given that 1 mark equals 24 cents.

7. Construct a graph to change cubic inches to gallons, given that 1 gal. equals 231 cu. in.

8. Construct on the same axes graphs to find the simple interest of $100 at 4 per cent, 5 per cent, and 6 per cent.

9. Construct a graph to find the number of amperes in a circuit of 10 ohms resistance as the voltage increases from 10 to 100 volts, given that the number of volts divided by the number of ohms equals the number of amperes.

10. The formula for the number of revolutions per minute of cutting tools in lathes is $n = \frac{3.8\,s}{d}$, where n = revolutions per minute, s = the speed in feet per minute, and d = the diameter of the rotating tool in inches. Construct a graph for a tool 6 in. in diameter, with speeds from 5 to 50 ft. per minute.

11. The resistance r of a train in pounds per ton, due to speed, is given by the formula $r = 3 + \frac{s}{6}$. Construct a graph for speeds from 5 to 60 mi. per hour.

12. The pressure of the atmosphere in pounds per square inch for readings of the barometer is given by the formula $p = .491\,b$, where p = the pressure in pounds per square inch, and b = the reading of the barometer. Construct a graph for barometer readings from 28 in. to 31 in. Use the given formula to find the pressure for the readings 28.75 in., 29.50 in., and 30 in., and compare with the pressures read from the graph.

13. Write the equations which express the relation between the two quantities in each of the preceding exercises.

Thus in Exercise 1 to change inches to centimeters we multiply the number of inches by 2.5. Therefore, representing centimeters by c and inches by i, $c = 2.5\,i$ is the equation which expresses the relation between centimeters and inches.

48. Equations expressing the relation between two quantities. In the first list of exercises the curves were constructed from tables of values determined by observation or experiment. In many cases there is no known relation between the sets of corresponding numbers. Thus in the table of temperatures the thermometer was read at intervals of one hour, and we do not know any law which will tell what the reading will be. But in the second list there is in each case a known law or relation which may be written in the form of an equation. Thus 1 in. = 2.5 cm.; hence the number of centimeters equals the number of inches multiplied by 2.5, or $c = 2.5\,i$. From this equation we can make a table of values, and from the table locate points and construct the graph. If we know that the graph is a straight line, it is necessary to determine only two points and draw a straight line through them.

All the equations in this exercise are of the first degree and all the graphs are straight lines. We may assume that when the relation between two quantities is expressed by an equation of the first degree the graph is a straight line (see sect. 52 for proof).

49. Curves. When the equation is not of the first degree the graph will be a curve which must be constructed by locating a number of points sufficient for the problem in hand.

EXERCISES

1. Construct a curve to find the area of squares whose sides are from 0 to 10 in. (Let 1 cm. horizontally = 1 in., and 1 cm. vertically = 10 sq. in.) If a = the area and s = a side of the square, what is the equation that connects the area and side? Find from the equation and from the graph the area of a square whose side is (*a*) 3.5 in.; (*b*) 7.5 in.; (*c*) 9.25 in.

2. Construct a graph to find the surface of cubes whose edges are from 0 to 10 in.

3. Construct a graph to find the area of circles of radii from 0 to 10 in., given area $= \pi r^2$.

4. Construct a graph to find the volume of cubes whose edges are from 0 to 10 in. What is the equation connecting v and e?

5. Construct a graph to find the space passed over by a falling body, given $s = 16\,t^2$, t = number of seconds.

6. The power of doing work possessed by a body in motion (kinetic energy) is given by $K = \frac{wv^2}{2\,g}$, where w = the weight

in pounds, v = the velocity of the body in feet per second, and $g = 32$. Construct a graph to show the kinetic energy of a 24-lb. shot as its velocity changes from 1600 to 600 ft. per second.

7. The volume of a gas diminishes in the same ratio as the pressure on it is increased, or pv = a constant. Given $pv = 120$, make a table of values and construct a curve to show the volume as the pressure increases from 1 lb. to 60 lb. per square inch.

8. The centrifugal force of the whole rim of a flywheel equals $\frac{wv^2}{gr}$, where w = weight of the rim in pounds, r = mean radius of the rim in feet, v = velocity of the rim in feet per second, and $g = 32.2$. Given $w = 3220$ lb. and $r = 5$ ft., construct a curve for velocities from 10 to 100 ft. per second.

9. The safe load in tons, uniformly distributed, on horizontal yellow-pine beams is $w = \frac{11\,bd^2}{15\,l}$, where b = breadth of beam in inches, d = depth of beam in inches, and l = distance between the supports in inches. Construct a curve to show the safe load on yellow-pine beams 4 in. in breadth, 12 ft. between supports, and depths from 8 to 18 in.

10. The resistance of a copper wire at 68° F. to the passage of an electric current is given by $R = \frac{10.7\,l}{d^2}$, where l = length of wire in feet and d = diameter of wire in mils (.001 in.). Construct a curve for the resistance of 1000 ft. of copper wire of diameter from 5 to 100 mils.

11. The volume of air transmitted in cubic feet per minute in pipes of various diameters is given by $Q = .327\,vd^2$, where v = velocity of flow in feet per second and d = diameter of the pipe in inches. Construct a curve to show the volume of air transmitted in pipes of diameters from 2 to 10 in. with a flow of 12 ft. per second. Without further computation construct a curve for a velocity of 24 ft. per second.

III. The Solution of Problems

50. In a graphical solution do not make a table of values unless it is necessary.

PROBLEMS

1. A travels 6 mi. per hour and B 10 mi. per hour. If B starts 2 hr. after A, when and where will they meet?

Solution. Choose units and axes as in Fig. 38. A travels 24 mi. in 4 hr. Locate this point M, and draw OA through the points O and M. B starts 2 hr. after A; hence the graph of his journey begins at C. He travels 20 mi. in 2 hr. Locate this point N and draw CB through the points C and N. P, the intersection of OA and CB, shows when and where they meet,—5 hr. after A starts and 30 mi. from the starting point.

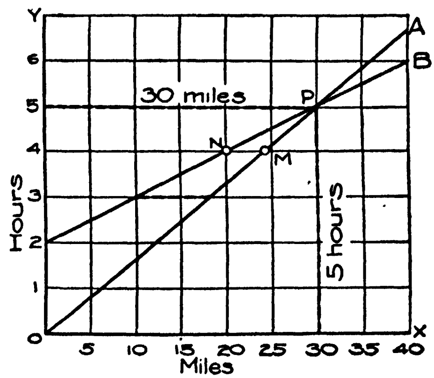

Fig. 38

The figure also shows how far they are apart at any time. Thus at the end of 3 hr. they are 8 mi. apart; this number of miles is given by the part of the 3-hr. line included between the lines OA and CB.

Solve this problem and some of the others in this list algebraically and compare the results with the graphical solution.

2. A travels 7 mi. per hour and B 5 mi. per hour. They start at the same time and travel east, A from a town M and B from a town N 15 mi. east of M. When and where will they meet?

3. Two trains start at the same time from Chicago and St. Louis respectively, 286 mi. apart; the one from Chicago travels 50 mi. per hour and the other 40 mi. per hour. When and where will they meet?

On the x-axis let a large square = 20 mi. Let St. Louis be at the lower left-hand corner, and Chicago 14.3 squares to the right.

Draw the line to represent the journey of the St. Louis train to the right, and the Chicago train to the left.

4. A cyclist starts at the rate of 300 yd. per minute, and 5 min. later another cyclist sets off after him at the rate of 500 yd. per minute. When and where will they meet? When are they 700 yd. apart?

5. A, traveling 20 mi. per day, has 80 mi. start of B, who travels 25 mi. per day. When will B overtake A?

6. A invests \$500 at 6 per cent and B invests \$1000 at 5 per cent. In how many years will A's interest differ from B's by \$300?

SOLUTION. Choose axes and units as in Fig. 39. Interest of \$500 for 10 yr. is \$300; locate point P, and draw OA through P to represent A's interest. In a similar manner draw OB to represent B's interest. Three squares vertically represent \$300. Mark off three squares on the edge of a piece of paper and with it find on what vertical line the distance between OA and OB is three squares; result, 15 yr.

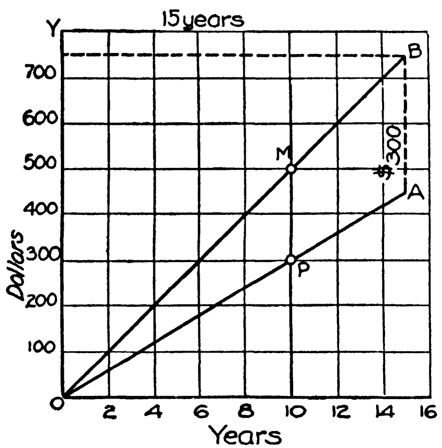

FIG. 39

7. In how many years will the interest on \$1500 at 5 per cent be \$240 greater than the interest on \$1000 at 6 per cent? When will it be \$120 greater?

8. A invests \$1000 at 5 per cent and B invests \$5000 at 4 per cent. In how many years will the amount of A's investment equal the interest of B's?

9. A invested \$2000 at $4\frac{1}{2}$ per cent, and two years later B invested \$2400 at 5 per cent. How many years elapsed before they received the same amount of interest? When was the difference of the interest \$120?

10. A man walks a certain distance and rides back in 8 hr.; he could walk both ways in 10 hr. How long would it take him to ride both ways?

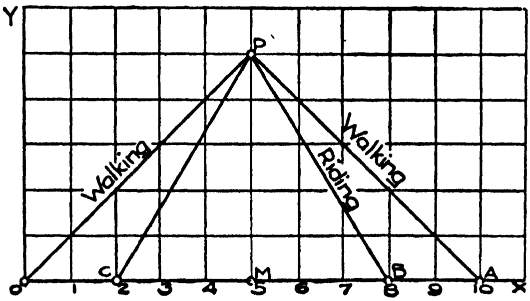

FIG. 40

SOLUTION. Let OA = 10 hr. (Fig. 40). M is the mid-point of OA. MP is any convenient length. OPA represents the journey when the man walks both ways, and OPB when he walks and rides back. It is two squares from A to B; take C two squares from O and join CP. Then CPB represents the journey when he rides both ways. $CB = 6$ hr., the time it takes him to ride both ways.

11. A man can walk to Lincoln Park in $3\frac{1}{2}$ hr. If he walks to the park and rides back in $5\frac{1}{4}$ hr., how long would it take him to ride both ways?

12. A man walks to town at the rate of 4 mi. per hour and rides back at the rate of 10 mi. per hour after remaining in town 1 hr. He was absent 8 hr. How far did he walk?

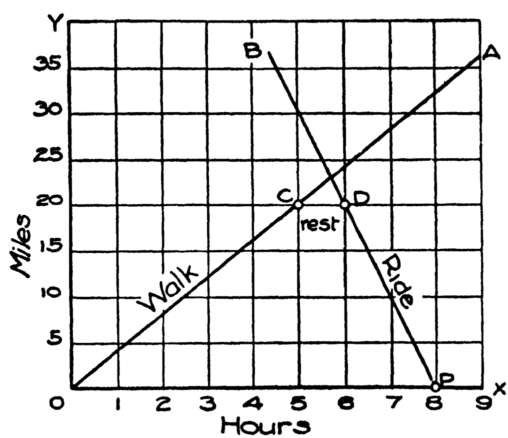

FIG. 41

SOLUTION. Choose axes and units as in Fig. 41. $OP = 8$ hr. OA is the graph of the walk and PB is the graph of the ride. If he had not remained in town, the distance of the point of intersection of the two lines from the x-axis would give the distance he walked. Since he remained in town 1 hr. we find where the horizontal distance from OA to PB equals 1 hr. This is CD on the 20-mi. line; hence the man walks 20 mi.

13. A man rides to a city at the rate of 10 mi. per hour, remains in the city 2 hr., and returns in an automobile at the rate of 15 mi. per hour. If he was absent 10 hr., how far was it to the city?

14. A boy starts out on his bicycle at the rate of 6 mi. per hour. His wheel breaks down and he walks home at the rate of $2\frac{1}{2}$ mi. per hour. How far did he ride if he reached home $8\frac{1}{2}$ hr. after starting?

Construct the graphs for the walk and ride, as in Problem 10. The intersection of the lines gives the distance.

15. A man rows at the rate of 6 mi. per hour to a town down a river and 2 mi. per hour returning. How many miles distant was the town if he was absent 12 hr. and remained in town 6 hr.?

16. If A and B can build a sidewalk in 6 and 4 da. respectively, in what time can they build it working together?

SOLUTION. Take OX in Fig. 42 any convenient length, and let $OA = 6$ da. and $XB = 4$ da. Draw XA and OB; P is the point of intersection. $PM = 2.4$ da., the required time.

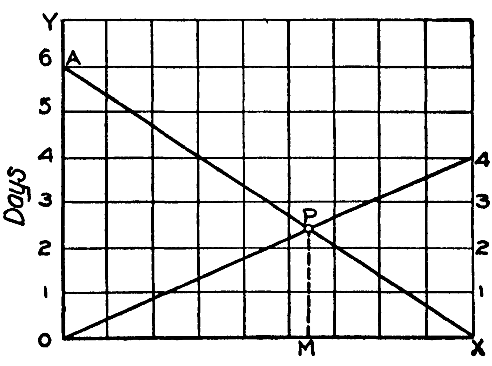

FIG. 42

17. A can do some work in 30 da. and B can do it in 20 da. How long will it take them working together?

18. If A can do some work in 12 hr. that he and B can do together in 4 hr., in what time can B do it?

As in Problem 14, draw XA for A's work. On XA take P 4 units above OX; draw OP and produce it to meet XB at B. $XB = 6$ da., the required time.

19. Two men can dig a ditch in 8 da. If one alone can dig it in 40 da., how long will it take the other man to dig it?

20. A man bought 100 lb. of brass for $13.60, paying for the copper in it 16 cents per pound and for the zinc 10 cents per pound. How many pounds of each metal are there in the brass?

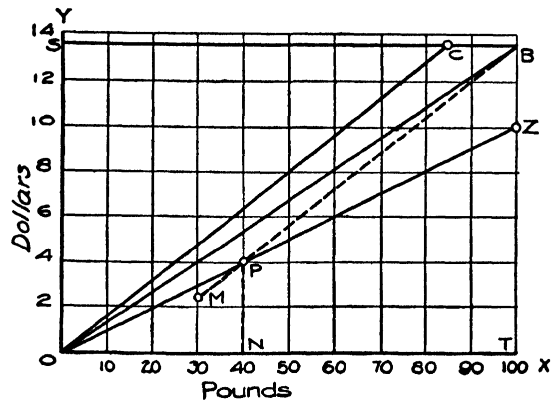

FIG. 43

SOLUTION. In working problems take the units as large as possible; they are taken small here to save space. In Fig. 43 $OS = \$13.60$. OC, OZ, and OB are the graphs for the copper, zinc, and brass respectively. Draw BM parallel to OC, intersecting OZ at P. Draw $PN \perp OX$. $ON = 40$ lb., the number of pounds of zinc; and $NT = 60$ lb., the number of pounds of copper.

Check.

$$40 + 60 = 100.$$

$$40 \times .10 + 60 \times .16 = 13.60.$$

Show that the same results are obtained by drawing BM' parallel to OZ, intersecting OC in P'. (A geometrical proof of the construction may be made by advanced students.)

21. An aluminum-zinc alloy weighing 300 lb. was sold for $60, the cost of the material. If the aluminum cost 25 cents per pound and the zinc 10 cents per pound, how many pounds of each metal were in the alloy?

22. A man bought 100 A. of land for $3250. If part of it cost him $40 an acre and part of it $15 an acre, how many acres of each kind were there?

23. A man starts off rowing at the rate of 6 mi. per hour, and half an hour later a second man sets out after him at the rate of 8 mi. per hour. (*a*) When is the first man overtaken? (*b*) How far has he rowed when overtaken? (*c*) How far apart are they when the first man has rowed 1 hr.?

24. The distance from Chicago to Milwaukee is 85 mi. An automobile leaves Chicago at 1.00 P.M. at the rate of 15 mi. per hour and another leaves Milwaukee at 1.30 P.M. at the rate of 18 mi. per hour. When and where will they meet?

25. A man walked to the top of a mountain at the rate of $2\frac{1}{3}$ mi. per hour, and down the same way at the rate of $3\frac{1}{2}$ mi. per hour. If he was out 5 hr., how far did he walk?

26. From the same place on a circular mile track two boys, A and B, start at the same moment to walk in the same direction, A 4 mi. per hour and B 3 mi. per hour. How often and at what times will they meet if they walk $1\frac{1}{2}$ hr.?

27. If the two boys in Problem 26 walk in opposite directions around the track, how often and at what times will they meet?

28. A with an old automobile travels 15 mi. an hour, and stops 5 min. at the end of each hour to make repairs. B on a new car travels 25 mi. per hour. If B starts 3 hr. after A, when and where will he overtake A?

IV. The Graphical Representation and Solution of Equations

51. Equations of the first degree. We have graphed equations which arose in concrete problems, and we will now apply the same methods to abstract equations containing the two unknowns x and y.

Exercise. Construct the graph of $x + y = 5$.

Transposing, $y = 5 - x$.

By giving values to x we have the following table:

x	8	7	6	5	4	3	2	1	0	-1	-2	-3
y	-3	-2	-1	0	1	2	3	4	5	6	7	8

For the first time in our graphical work we have to deal with negative numbers. This will cause no trouble, however, for we will simply count off the positive values of x to the right of the origin, and the negative values to the left. For positive values of y count up from the x-axis, and for negative values count down.

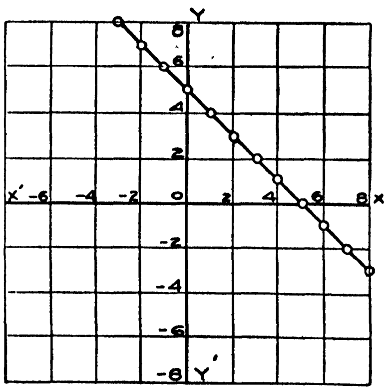

FIG. 44

Taking heavy horizontal and vertical lines near the center of the page for the x-axis and y-axis respectively, locate the points from the table and draw a line through them. The axes should always be lettered as in Fig. 44, and the units indicated on the axes or on the sides of the diagram.

It is not worth while to plot many equations of the first degree by locating points, since it will be proved in the next paragraph that such a graph is always a straight line. Hence in plotting equations of the first degree it is necessary to locate only two points. These points should be some distance apart in order that the graph may be fairly accurate.

52. Theorem. *The graph of an equation of the first degree is a straight line.*

Proof. Any equation of the first degree can be reduced to the form

$$y = mx + b \qquad (1)$$

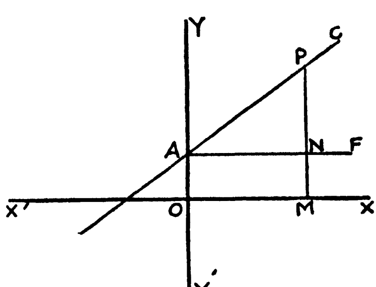

FIG. 45

by transposing, uniting, and dividing by the coefficient of y. Let P be a point on the graph of $y = mx + b$. Draw $PM \perp OX$. Then, for the point P, $OM = x$ and $PM = y$. In equation (1) put $x = 0$; then $y = b$, that is, the graph of (1) cuts the y-axis at the point $(0, b)$. Let $OA = b$. Through P and A draw the straight line AC. Through A draw AF parallel to OX, cutting PM at N.

From (1), $$m = \frac{y - b}{x}.$$

From the figure, $$y - b = PN,\qquad \text{Why?}$$

and $$x = AN.\qquad \text{Why?}$$

Therefore $$\frac{PN}{AN} = \frac{y - b}{x} = m.$$

That is, for any point P on the graph of $y = mx + b$ the ratio PN/AN is constant, since m is some fixed number. Hence, by the properties of similar triangles (what are they?), any point whose x and y satisfy equation (1) lies on the straight line AC.

EXERCISES

Plot the following equations:

1. $x + y = 6$.
2. $x - y = 6$.
3. $x + y = -6$.
4. $-x + y = 6$.
5. $2x + 3y = 6$.
6. $5x - 7y = 35$.

53. Equations of degree higher than the first. The graph of an equation of degree higher than the first is a curve, which can be drawn with sufficient accuracy by locating a number of points.

Exercise. Plot $y = x^2 - 6x + 5$.

If we wish to take the side of a large square $= 1$ on both axes, it is necessary to begin the table of values with some value of x that will bring the point on the paper. If we start with $x = 8$, then $y = 21$, and the point $(8, 21)$ is off the paper; hence we begin with $x = 7$.

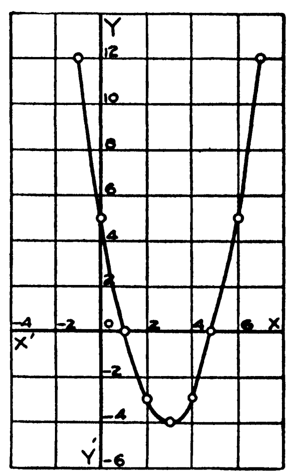

FIG. 46

x	7	6	5	4	3	2	1	0	-1
y	12	5	0	-3	-4	-3	0	5	12

Usually it is necessary to locate points close together to determine the true shape of the curve at some particular point. Thus from the given equation:

x	3.5	3.2	3.1	2.9	2.8	2.5
y	-3.75	-3.96	-3.99	-3.99	-3.96	-3.75

These additional points show that the curve is rounded at $(3, -4)$. This point is called the *turning point* of the curve.

54. The purpose of graphical representation. From this curve we may learn two things: (1) the x of the points where it intersects the x-axis, 1 and 5, are the roots of the equation $x^2 - 6x + 5 = 0$; (2) the y of the turning point, -4, gives the least value of the expression $x^2 - 6x + 5$ (see Chapter VIII).

EXERCISES

Plot these equations. In the first four find the least value of the expression and the roots of the equation when $y = 0$:

1. $y = x^2 - 4x - 5$.
2. $y = x^2 - 6x + 9$.
3. $y = x^2 - x - 6$.
4. $y = x^2 + x - 2$.
5. $x^2 + y^2 = 25$ (circle).
6. $y^2 = 8x$ (parabola).
7. $9x^2 + 25y^2 = 225$ (ellipse).
8. $4x^2 - 9y^2 = 36$ (hyperbola).

55. A short method of computing the table of values for equations of degree higher than the second. This method can be used also in checking the roots of equations.

Exercise 1. Plot $y = x^3 - 5x^2 - 2x + 24$.

Let $x = 6$. $\qquad x^3 = xx^2 = 6x^2$.

$$\therefore x^3 - 5x^2 - 2x + 24 = 6x^2 - 5x^2 - 2x + 24$$
$$= x^2 - 2x + 24.$$
$$x^2 = xx = 6x.$$
$$\therefore x^2 - 2x + 24 = 6x - 2x + 24 = 4x + 24.$$
$$4x = 4 \times 6.$$
$$\therefore 4x + 24 = 24 + 24 = 48.$$
$$\therefore y = 48 \text{ when } x = 6.$$

The coefficients only need be written and the work can be put in the following form:

$$\begin{array}{rrrr|l} 1 - 5 & - 2 & + 24 & & \underline{6} \\ & 6 & + 6 & + 24 & \\ \hline & 1 & + 4 & + 48 & \end{array}$$

After the coefficients are written we multiply the first one at the left by 6 and add the product to the second, obtaining 1. This sum is multiplied by 6 and added to the third coefficient, and so on.

If any power of x is lacking, write 0 for the coefficient of the missing term. Thus, if $y = x^4 + 3x^2 + 2x + 5$, write the coefficients $1 + 0 + 3 + 2 + 5$.

TABLE OF VALUES FOR $y = x^3 - 5x^2 - 2x + 24$

x	6	5	4	$3\frac{1}{2}$	3	2	1	0	-1	-2	-3
y	48	14	0	$-1\frac{3}{8}$	0	8	18	24	20	0	-42

Locate axes and choose convenient units, as in Fig. 47. Since $y = 0$ for $x = 4$ and $x = 3$, it is necessary to locate one or more points between $x = 4$ and $x = 3$ to get the curve fairly accurate. The roots of the equation $x^3 - 5x^2 - 2x + 24 = 0$ are seen to be -2, 3, and 4.

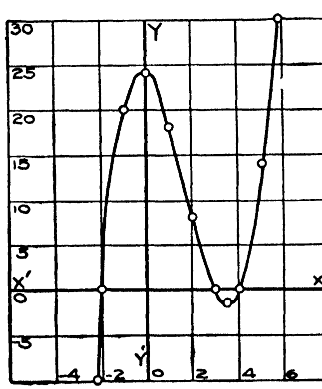

FIG. 47

Exercise 2. Plot $y = x^3 - 6x^2 - 2$.

Exercise 3. Plot

$y = x^4 + x^3 - 13x^2 - x + 12$.

TABLE OF VALUES

x	4	3.5	3	2	1	0	-1	-2	-3	-4	-4.5
y	120	44.2	0	-18	0	12	0	-30	-48	0	72

Find the table of values by the short method. The choice of units in Fig. 48 makes the curve of good form for a study of its properties. The roots of the equation $x^4 + x^3 - 13x^2 - x + 12 = 0$ are seen to be $-4, -1, 1$, and 3. How can the position of the three turning points be found?

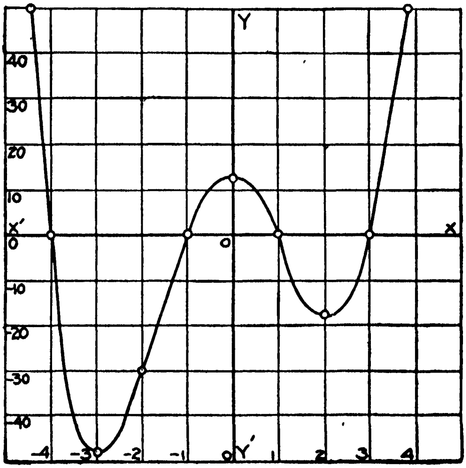

FIG. 48

56. Helpful principles in plotting curves. For equations in the form y equal an expression containing x, with no root signs and no term in the denominator containing x, the following principles are useful in plotting the curves:

1. The number of turning points cannot be greater than the degree of the equation less one. Thus an equation of the fourth degree cannot have more than three turning points.

2. A line parallel to the y-axis can cut the curve only once.

3. If the equation is of odd degree, the ends of the curve are on the opposite sides of the x-axis.

4. If the equation is of even degree, both ends of the curve are on the same side of the x-axis.

5. The number of times the curve cuts the x-axis cannot be greater than the degree of the equation.

EXERCISES

Construct curves to represent the following equations:

1. $y = x^3 + 2x^2 - x - 2$.

2. $y = x^3 + x^2 - x - 1$.

3. $y = x^3 + 3x^2 - 6x - 8$.

4. $y = x^3 - 4x^2$.

5. $y = x^4 - 10x^2 + 8$.

6. $y = x^4 - 4x^2 + 4x - 4$.

57. Solution of simultaneous equations. Equations like

$$x + y = 8 \qquad\qquad x^2 + y^2 = 25$$
$$2x + 3y = 18 \quad \text{and} \quad 3x + 4y = 25$$

can be solved by plotting the curves on the same axes and noting where they intersect. The x and the y of each point of intersection gives a pair of values which satisfies each equation. The graphical solution shows clearly how many pairs of values there are, and why a certain value of x must be taken with a certain value of y. In many cases, however, the algebraic solution can be made more quickly. But squared paper is of real service in solving equations of degree higher than the second containing one unknown.

58. Solution of equations of any degree; real roots. The principle involved in graphical solution is readily seen by looking at the curves already plotted. Suppose we wish to solve the equation $x^2 - 6x + 5 = 0$; that is, we want to find values of x which make the expression $x^2 - 6x + 5$ zero. Put $y = x^2 - 6x + 5$ and we obtain the curve in Fig. 46. At the point where the curve cuts the x-axis y is 0. Since the curve cuts the x-axis at $x = 1$ and $x = 5$, the solutions of $x^2 - 6x + 5 = 0$ are 1 and 5. Look over the curves you have plotted and determine the solutions when possible. If the roots of an equation are small whole numbers, they can easily be found by factoring the given expression. If the given expression cannot be factored, the roots can be found to as many decimal places as are needed by graphical methods.

Exercise. Solve $x^3 - 5x^2 - 2x + 20 = 0$.

Put $y = x^3 - 5x^2 - 2x + 20$ and compute the following table of values:

x	5	4.5	4	3.5	3	2.5	2	1	0	-1	-1.5	-2
y	10	.875	-4	-5.375	-4	$-.625$	4	14	20	16	8.375	-4

Time is saved by plotting the curve rather accurately where it cuts the x-axis.

Fig. 49 shows that the roots of the equation lie between 4 and 5, 2 and 3, and -1 and -2. We will find the first root to two decimal places. Since the curve seems to cut the x-axis between $x = 4.4$ and $x = 4.5$, we substitute these two values in

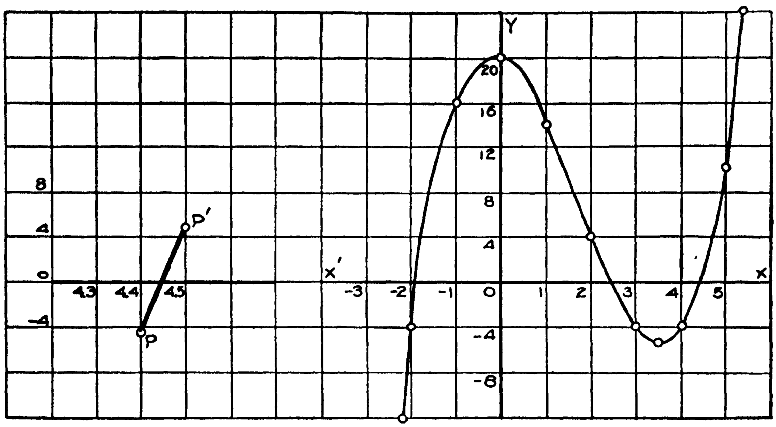

FIG. 49

the equation, obtaining for $x = 4.4$, $y = -.416$; and for $x = 4.5$, $y = .875$. The change in sign shows that the curve does cut the x-axis between these two points, and the root to two figures is 4.4.

The next thing is to draw the part of the curve between $x = 4.4$ and $x = 4.5$ to a larger scale, as in Fig. 49. The two points P and P' may be joined by a straight line which, in general, will lie close to the curve. The curve seems to cross the x-axis between $x = 4.43$ and $x = 4.44$. For $x = 4.43$, $y = -.0462$; and for $x = 4.44$, $y = .0803$. The change of sign shows that the curve does cross the x-axis between these two values of x. Hence the root to two decimal places is 4.43. In a similar manner the root could be found to any desired number of decimal places.

Find the other two roots to two decimal places.

PROBLEMS

Find the roots of these equations to three decimal places:

1. $x^3 - 3x^2 - 2x + 5 = 0$ (root between 1 and 2).
2. $x^3 - 4x^2 - 6x + 8 = 0$ (root between 4 and 5).
3. $x^3 + 2x^2 - 4x - 43 = 0$ (positive roots).
4. $x^4 - 12x + 7 = 0$ (positive roots).
5. $x^3 - 5x^2 + 8x - 1 = 0$ (root between 0 and 1).
6. $x^3 + 2x^2 - 3x - 9 = 0$ (root between 1 and 2).
7. $x^3 - 7x + 7 = 0$ (root between -3 and -4).
8. $x^3 - 2x^2 - x + 1 = 0$ (3 roots).
9. $x^3 - 3x + 1 = 0$ (3 roots).

V. Determination of Laws from Data obtained by Observation or Experiment

59. ***Exercise.*** Find the law of a helical spring.

In the physics laboratory a helical spring was loaded with weights of 100 g., 200 g., ⋯, and the elongation for each load was recorded in the following table:

x (grams) . . .	100	200	300	400	500	600	700	800	900
y (centimeters) .	.9	3	6.4	10.4	14.5	18.6	22.6	26.8	30.9

Plot these points carefully, choosing the units to get as large a figure as possible. Stretch a fine thread along the points and it will be found that it can be placed so that most of the points will lie close to it or on it, and that they will be rather

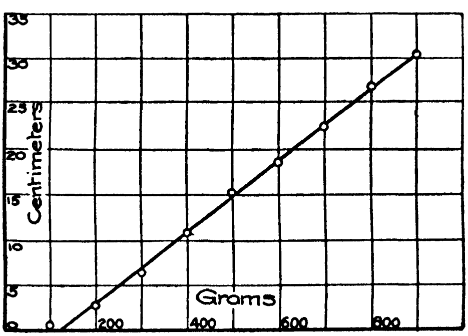

FIG. 50

evenly distributed above and below. Hence it is evident that an equation of the first degree connects the grams and centimeters. In this statement the first two loads are omitted, and no load greater than 900 g. is considered, since at that load the spring showed signs of breaking. Draw a straight line in the position of the thread.

Let us suppose that the law or equation is in the form

$$y = mx + b. \tag{1}$$

The values of m and b must be found that will best fit the data.

Take two points which lie close to the straight line and some distance apart, and substitute the x and y of these points in (1).

Taking the fourth and ninth points, we have

$$10.4 = 400\,m + b. \tag{2}$$

$$30.9 = 900\,m + b. \tag{3}$$

(3) – (2),

$$20.5 = 500\,m. \tag{4}$$

$$m = .061. \tag{5}$$

Substituting (5) in (2), $b = -6.$ (6)

Therefore $y = .041\,x - 6$ is the required equation or law.

Check. Substitute the x and y of sixth point.

$$\begin{aligned} 18.6 &= 600 \times .041 - 6 \\ &= 18.6. \end{aligned}$$

Substitute the x and y of the seventh point, we obtain $22.6 = 22.7$.

60. Straight-line laws. When the results of experimental work are plotted it frequently happens that the points lie nearly in a straight line. In such cases it is not difficult to find the law or equation by the method used in the preceding exercise. Since there are always errors in experimental work the points will not, of course, lie exactly in a straight line. If some of the points lie at a rather large distance from the straight line through several of them, it may be that the equation is not of the first degree. In the following exercises the graphs are straight lines.

EXERCISES

1. Make a helical spring by coiling a wire around a small cylinder. Arrange the spring to carry a load; take readings of the elongation for several loads and find the law of the spring.

2. Put a Fahrenheit and a Centigrade thermometer in a dish of water and take the reading of each. Vary the temperature of the water by adding hot water or ice and take several readings. Find the law connecting the readings of the two thermometers.

3. Load a thin strip of pine supported at points two feet apart and note the deflection. Vary the load and find that for loads under a certain weight the deflection is proportional to the load. For what weight does the law begin to fail?

4. Find the laws of the following helical springs:

	x (ounces)	4	5	6	7	8	9	10
1	y (inches)	5.2	5.5	5.8	6.1	6.4	6.7	7
2	y (inches)	13.2	14.0	14.8	15.6	16.4	17.2	18
3	y (inches)	3.8	5.0	6.2	7.4	8.6	9.8	10

5. l is the latent heat of steam in British thermal units (B. t. u.) at $t°$ F. Find an equation giving l in terms of t.

t	170.1	193.2	212.0	240.0	254.0
l	995.2	979.0	965.7	945.8	935.9

6. V is the volume of a certain gas in cubic centimeters at the temperature $t°$ C. If the pressure is constant, find the law connecting V and t.

t	27	33	40	55	68
V	110	112	115	120	125

7. A steel bar 107 cm. long was supported at the ends and loaded at the center with the following results. Find the equation connecting the load and deflection.

Grams	500	1000	2000	3000	4000
Deflection . .	1.18	2.35	4.72	7.15	9.42

8. In an arc-light dynamo test the voltage for the revolutions per minute was recorded. Find the laws connecting the volts and revolutions per minute.

Revolutions per minute . .	200	300	400	500	600	700
Volts	165	253	337	421	507	590

9. P is the pull in pounds required to lift a weight W by means of a differential pulley. Find the law connecting P and W.

W	50	100	150	200	250	300
P	8.0	13.4	19.0	24.4	30.1	35.6

10. When the weight W was lifted by a laboratory crane the force applied to the handle was P pounds. Find the law connecting P and W.

W	50	100	150	200	250	300	350	400
P	7.4	8.3	9.5	10.3	11.6	12.4	13.6	14.5

CHAPTER VIII

FUNCTIONALITY; MAXIMUM AND MINIMUM VALUES

61. Number scale. Real numbers are represented graphically by a straight-line scale. Zero is the dividing point between the positive and the negative field, and may be considered either positive or negative.

In going down the negative scale further and further from zero the numbers are getting smaller; that is, -10 is less than -3. The actual magnitude of a number, without regard to its sign or quality or position in the scale, is called its *absolute value*.

FIG. 51

Beginning at the extreme left and passing constantly to the right, numbers may be said to increase *continuously* from $-\infty$ through 0 to $+\infty$. Beginning at the extreme right and passing constantly to the left, numbers may be said to decrease *continuously* from $+\infty$ through 0 to $-\infty$. Beginning at any point and passing to the right gives increasing numbers, while passing to the left gives decreasing numbers.

62. Variables. A *variable* is a number which changes and passes through a series of successive values. It may pass through the whole scale of values from $-\infty$ to $+\infty$, or it may pass through a certain portion of the scale only. If the variable is confined to a portion of the number system, as from the position -15 in the scale to the position $+6$, it is said to have the *interval* -15 to $+6$.

A number is said to vary *continuously* in a given interval, a to b, if it starts with the value a and increases (or decreases) to the value b in such a way as to assume all values betwee a and b (integral, fractional, and irrational) in the order o their magnitude.

63. Inequality of numbers. One number is greater than second if a positive number must be added to the second t produce the first. Thus -3 is greater than -8, since $+5$ mus be added to -8 to obtain -3.

One number is less than a second if a positive number mus be subtracted from the second to obtain the first. Thus -1 is less than -12, since $+5$ must be subtracted from -12 t obtain -17.

The relation of inequality is usually expressed by a symbo Thus $-3 > -8$, $10 > 4$, $-17 < -12$, $2 < 7$.

64. Function of a variable. The value of an expression ir volving a variable depends upon the value of the variabl The expression is called a *function* of the variable. Thu $x^2 - 1$ is a function of x (written $f(x) = x^2 - 1$, and rea "function of x equals $x^2 - 1$"), for when x has the value $-2, -1, 0, +1, +2$ respectively, $x^2 - 1$ has the values 3, 0, -1, 0, 3.

The variable to which we may give values at will is calle the *independent variable;* but the expression or variable whic depends upon it for its value is called the *dependent variabl* or function. The volume of a cube is a function of the edg $v = f(e) = e^3$. The area of a circle is a function of the radiu $a = f(r) = \pi r^2$. The distance through which a body falls is function of the time, $s = f(t) = \frac{1}{2} gt^2$. Name the independer and dependent variables in the preceding illustrations.

Exercise. Plot the graph of the function $2x^3 - 3x^2 - 12x + 4$

Give x integral values from -3 to 4 and obtain the following table

x	-3	-2	-1	0	1	2	3	4
$2x^3 - 3x^2 - 12x + 4$	-41	0	11	4	-9	-16	5	36

You have been constructing curves by locating points from a table and drawing a smooth curve through them; you should now see that this method of plotting a function is based on the assumption that the given expression is a continuous function of x. In this case a small change in x makes a small change in the given function; hence if all values of x were taken, there would be a continuous succession of points forming a smooth curve.

In Fig. 52 imagine a perpendicular to the x-axis drawn to the curve from $x = -3$. The length of this perpendicular is the value of the function for $x = -3$. Now imagine the perpendicular to move to the right to $x = +4$, and you have a mental picture of the function varying continuously in value from -41 to $+11$, then to -16, and finally to $+36$.

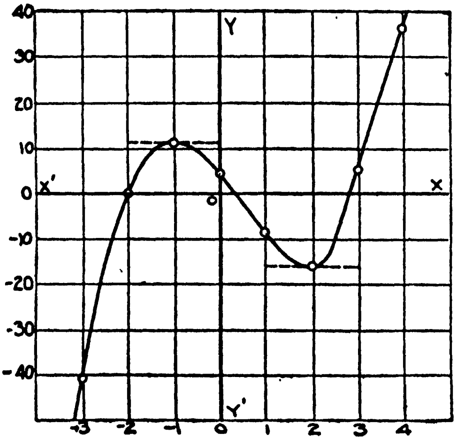

FIG. 52

For certain intervals of values of x the function is greater than zero, and for certain intervals it is less than zero. For certain definite values of x the function has the value zero. The value of the function is greater than zero in the intervals from $x = -2$ to $x = .4$ (about), and from $x = 2.9$ (about) to $x = +\infty$. The function is less than zero from $x = -\infty$ to $x = -2$, and from $x = .4$ (about) to $x = 2.9$ (about). The function has the value zero for $x = -2$ and $x = .4$ (about).

65. Maximum and minimum values. As x increases from -3 to -1, $2x^3 - 3x^2 - 12x + 4$ increases from -41 to $+11$. As x increases from -1 to $+2$, the function decreases from $+11$ to -16. As x increases from $+2$ to $+4$, the function increases from -16 to $+36$. We observe that as the variable x increases continuously, the value of the function may either increase or decrease. At any point where the function stops increasing and begins to decrease, it is said to have a *maximum value* or to be a *maximum*. In this case it occurs when $x = -1$, or when the function has the value $+11$.

When the function stops decreasing and begins to increase, it is said to have a *minimum value* or to be a *minimum*. Here it occurs when $x = 2$, or when the function has the value -16.

In other words, a function is a maximum when its value is greater than the values immediately preceding and following. In the same way a function is a minimum when its value is less than the values immediately preceding and following. The point on the curve at which there is a maximum or minimum value of the function is called a *turning point*.

66. To investigate functional variation and get an idea of regional increase and decrease, and maximum and minimum values. *Plot enough points to give the shape of the curve. The regions of increase and decrease are then readily noted. To check an apparent maximum or minimum value of the function, calculate values of the function for points close together in the immediate neighborhood and on both sides of the apparent value. That value of the function which is either greater or less than all those which immediately precede or follow is the value desired.*

PROBLEMS

1. A line 10 in. long is divided into two segments which are taken as the base and altitude of a rectangle. (*a*) Express the area of the rectangle as a function of one of the segments. (*b*) Plot this function. (*c*) Discuss the increase and decrease of area as the length of one segment changes from 0 to 10 in. (*d*) What length of segment gives a maximum area? (*e*) What is the maximum area? (*f*) Is there a minimum area?

Suggestion. Let

$$x = \text{one segment.}$$
$$10 - x = \text{other segment.}$$
$$x(10 - x) = \text{area.}$$

2. Express the sum of a variable number and its reciprocal as a function of the number. Plot the function and investigate for regional changes. What is the minimum value of the sum of a number and its reciprocal?

3. An open-top tank with a square base is to be built to contain 32 cu. ft. What should be the dimensions in order to require the smallest amount of steel plate for construction?

Suggestion. Let $x =$ a side of the base.

Then $\frac{32}{x^2} =$ depth of the tank.

$x^2 + \frac{128}{x} =$ surface of the tank.

Plot the function $x^2 + \frac{128}{x}$ and determine x for the minimum value.

4. Express the area of a variable rectangle inscribed in a circle whose radius is 4 in., as a function of the base. What are the dimensions of the rectangle of greatest possible area?

Suggestion. Make a drawing of the circle and rectangle and note how the area changes as the base of the rectangle increases from 0 to 8 in. A diagonal of the rectangle is a diameter of the circle. Why?

Let $x =$ base of the rectangle.

Then $\sqrt{64 - x^2} =$ altitude of the rectangle.

$x\sqrt{64 - x^2} =$ area of the rectangle.

Plot this function and determine the value of x that makes it a maximum.

5. Show that the largest rectangle having a perimeter of 24 in. is a square..

6. What are the dimensions of the greatest rectangle inscribed in a right triangle whose base is 12 in. and altitude 8 in.?

7. From the cube of a variable number six times the number is subtracted. What value of the variable would make this function a minimum? Discuss the functional variation in full.

8. From a variable number its logarithm is subtracted. What value of the variable number would make this difference a minimum?

9. Two towns A and B (Fig. 53) are 3 and 4 mi. respectively from the shore of a lake CD. If CD is a straight line 7 mi. long, where must a pumping station P be built to supply the towns with water with the least amount of pipe?

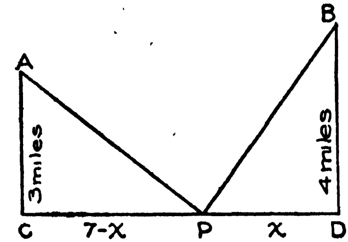

FIG. 53

10. If t represents the number of tons of coal used by a steamer on a trip, and v represents the speed of the boat per hour, the following relation holds: $t = .3 + .001\,v^3$. Other expenses are represented by one ton of coal per hour. What speed would make the cost of a 1000-mi. trip a minimum?

11. The cost of an article is 35 cents. If the number sold at different prices is given by the following table, find the selling price which would probably give the greatest profit.

Selling price in dollars . .	.50	.60	.75	.90	1.00	1.10
Number articles sold . . .	3600	3100	2640	2080	1300	700

Suggestion. First from the given table plot a curve to show the probable number sold at prices from 50 cents to $1.10. Then on the same axes with different vertical units plot the curve to show the profits at the various prices. Profit = (selling price − cost) × number sold.

To determine the turning point of the second curve somewhat closely it will be necessary to locate intermediate points; e.g. for the selling price at 80 cents and 85 cents. The number probably sold at these prices may be found from the first curve.

12. Devise other problems in maxima and minima and solve them.

CHAPTER IX

EXERCISES FOR ALGEBRAIC SOLUTION IN PLANE GEOMETRY

67. During the year given to plane geometry these exercises not only serve as a review of algebra, but they should also develop in the pupils an ability to attack successfully many geometrical problems from the algebraic side. The figures for the first exercises should be carefully drawn with ruler, compasses, and protractor, and the drawing should check the algebraic work. Later the figures may be sketched. The numbers and letters should be put on the given and required parts in the drawing, and the equations set up from the figures. Represent lines, angles, and areas by a single small letter. Check all results.

Complementary and Supplementary Angles

1. Find two complementary angles whose difference is (*a*) $20°$; (*b*) $52°$; (*c*) $5° \, 8' \, 10''$; (*d*) $x°$.

2. $x/2$ and $x/3$ ($x + 40$ and $x - 30$) are complementary angles. Find x and the angles.

3. Find the angle that is the complement of (*a*) 8 times itself; (*b*) 7 times itself; (*c*) 3 times itself; (*d*) n times itself.

4. How many degrees are there in the complementary angles which are in the ratio (*a*) $1:2$? (*b*) $4:5$? (*c*) $3.5:6.5$? (*d*) $m:n$?

5. Find the value of two supplementary angles if one is 9 (15) times as large as the other.

6. How many degrees are there in an angle that is the supplement of (*a*) 4 times itself? (*b*) 7 times itself? (*c*) $\frac{1}{3}$ of itself? (*d*) n times itself?

7. Of two supplementary adjacent angles, one lacks 7° of being 10 times as large as the other. How many degrees in each?

8. If 10° (7°) be added to one of two supplementary angles and 20° (8°) to the other, the resulting angles will be in the ratio 2 : 5 (3 : 4). Find the angles.

9. If 6° (5°) be taken from one of two supplementary angles and added to the other, the ratio of the two angles thus found is 2 : 7 (13 : 5). What are the angles?

10. To one of two supplementary angles add 11° (9°) and from the other subtract 16° (5°). The two angles thus obtained will be to each other as 3 : 4 (5 : 12). Find the angles.

11. How many degrees are there in an angle whose supplement is (*a*) 5 times its complement? (*b*) $\frac{5}{2}$ of its complement? (*c*) n times its complement?

12. Find the angle whose supplement and complement added together make 112° (208°).

13. If 3 (8) times the complement of an angle be taken from its supplement, the remainder is 10° (76°). Find the angle.

14. If 3 times an angle added to 5 times its supplement equals 20 times its complement (supplement), what is the angle?

15. The angles formed by one line meeting another are in the ratio 7 : 11 (3 : 8). How many degrees in each?

16. Construct a graph to show the complement of any angle. (Take a large square each way equal 10°. Locate a few points: $x = 10$, $y = 80$; $x = 40$, $y = 50$; $x = 90$, $y = 0$; and draw a straight line through them.) What is the equation of this line?

17. On the same sheet of squared paper construct a graph to show the supplement of any angle. What is the equation of the straight line?

18. On the same sheet of squared paper as in the last two problems draw a straight line from ($x = 0$, $y = 0$) to ($x = 80°$,

$y = 160°$). Read off a few pairs of angles given by points on this line. What is the equation of this line? On this line mark the points that answer the question, If one of two complementary (supplementary) angles is twice the other, how many degrees in each?

19. Find two complementary angles such that the sum of twice one and 3 times the other is 210°. Solve graphically.

20. Two complementary angles are in the ratio $2:3\,(7:8)$. Find the number of degrees in each. Solve graphically.

21. Three angles make up all the angular magnitude about a point. The difference of the first and second is 10° (20°), and of the second and third is 100° (2°). How many degrees in each angle?

22. The sum of four angles about a point is 360°. The second is 3 times the first, the third is 10° greater than the sum of the first and second, and the fourth is twice the first. Find the angles.

23. Of the angles formed by two intersecting lines, one is $5\,(3\frac{1}{2})$ times another. What are the angles?

Parallel Lines

24. Two parallels are cut by a transversal making one exterior angle $3\ (5\frac{3}{7})$ times the other exterior angle on the same side of the transversal. Find all the angles.

25. If two parallels are cut by a transversal making two adjacent angles differ by 20° (36° 20′), find all the angles.

26. If a transversal of two parallels makes the sum of 5 (4) times one interior angle and 2 (3) times the other interior angle on the same side of the transversal equal to 420° (625°), find all the angles.

27. The sum of one pair of alternate-interior angles formed by a transversal of two parallels is $8\,(6\frac{1}{8})$ times the sum of the other pair. Find all the angles.

TRIANGLES

28. Of the angles of a triangle the second is twice the first, and the third is 3 times the second. How many degrees in each angle?

29. Find the angles of a triangle ABC, given:

(*a*) A 3 times and B 4 times as large as C.

(*b*) A 3 times as large as C and B $\frac{1}{6}$ of C.

(*c*) A 44° and B 25° smaller than C.

(*d*) $A : B : C = 2 : 3 : 4 (3 : 5 : 10)$.

30. In a triangle ABC angle A is 6 times angle B, and angle C is $\frac{1}{3}$ of angle A. Find the three angles.

31. Find the angles of the triangle ABC when A is 43° more than $\frac{2}{3}$ of B, and B is 18° less than 4 times C.

32. The sum of the first and second angles of a triangle is twice the third angle, and the third angle added to 3 times the second equals 140° less the third angle. Find the three angles.

33. In a triangle the sum of twice the first angle, 3 times the second, and the third is 320° (400°); and the sum of the first, twice the second, and 3 times the third is 440° (310°). Find the angles.

34. In a triangle ABC, A lacks 106° of being equal to the sum of B and C, and C lacks 10° of being equal to the sum of A and B. Find the angles.

35. The vertical angle of an isosceles triangle is 68°. Find the base angles.

36. One base angle of an isosceles triangle is 25° (47°). Find the vertical angle.

37. Find the angles of an isosceles triangle if a base angle is 4 (5) times the vertical angle.

38. In an isosceles triangle the vertical angle is 36° (75°) larger than a base angle. Find the angles.

39. In an isosceles triangle 5 times a base angle added to 3 times the vertical angle equals 490° (530°). Find the angles.

40. Find the angles of an isosceles triangle in which the exterior angle at the base is 95° (140°).

41. The angle at the vertex of an isosceles triangle is $\frac{1}{3}$ ($\frac{1}{2}$) of the exterior angle at the vertex. Find the angles of the triangle.

42. A base angle of an isosceles triangle is 12 (n) times the vertical angle. Find the angles of the triangle.

43. What are the angles of an isosceles triangle in which the vertical angle is 12° more than $\frac{1}{3}$ ($\frac{1}{5}$) of the sum of the base angles?

44. Construct a graph to show the change in the vertical angle y of an isosceles triangle as a base angle x increases from 0° to 90°.

45. The vertical angle of an isosceles triangle lacks 8° (20°) of being $\frac{7}{15}$ (.9) of a right angle. Find all the angles.

46. The acute angles of a right triangle are x and $2x$ ($3y$ and $5y$). Find them.

47. The difference of the acute angles of a right triangle is 18° (37°). Find them.

48. If the acute angles of a right triangle are in the ratio (*a*) 2 : 3, (*b*) 7 : 8, (*c*) $m : n$, find the angles.

49. In a right triangle the sum of twice one acute angle and 3 times the other is 211° (192°). Find the angles.

Polygons

50. How many sides has a polygon the sum of whose interior angles is 720° (2340°)?

51. An interior angle of a regular polygon is 165° (160°). How many sides has the polygon?

52. How many sides has a polygon the sum of whose interior angles equals 2 (12) times the sum of the exterior angles?

53. How many sides has a polygon the sum of whose interior angles exceeds the sum of the exterior angles by 1080° (2700°)?

54. Construct a graph to show the sum of the angles of a polygon as the number of sides increases from 3 to 12.

55. Construct a graph to show the number of degrees in each angle of a regular polygon of n sides for values of n from 3 to 36.

56. If the number of sides of a regular polygon be increased by 2 (3), each of its interior angles is increased by 15° (10°). How many sides has the polygon?

57. By how many must the number of sides of a regular polygon of 12 (15) sides be increased in order that each interior angle may be increased 18° (6°)?

58. By how many must the number of sides of a regular polygon of 8 (20) sides be increased if each exterior angle is diminished 5° (6°)?

59. Construct a curve to show the number of degrees in an exterior angle of a regular polygon as the number of sides increases from 3 to 18.

60. The perimeter of a triangle is 176 (50.4) ft. in length and the sides are as 1 : 3 : 4 (2 : 5 : 7). Find the sides.

61. The perimeter of a triangle bears to one side the ratio 3 : 1 (15 : 4) and to another side the ratio 4 : 1 (5 : 2). What part of the perimeter is the third side?

62. The sum of the three sides, a, b, and c, of a triangle is 35 ft.; twice a is 5 ft. less than the sum of b and c, and twice c is 4 ft. more than the sum of a and b. Find each side.

63. If the perimeter and base of an isosceles triangle are in the ratio 4 : 1 (5 : 2), what part of the perimeter is one of the equal sides?

64. Find the perimeter of an isosceles triangle if it is 4 ($8\frac{1}{2}$) times the base, and one of the equal sides is 4 (55) ft. longer than the base.

65. In an isosceles right triangle the perpendicular from the vertex to the hypotenuse is 12 (30) cm. long. How long is the hypotenuse?

66. If the hypotenuse of an isosceles right triangle is 26 (8) in. long, what is the length of the perpendicular from the vertex to the hypotenuse?

PARALLELOGRAMS

67. One angle of a parallelogram is 4 (9) times its consecutive angle. Find all the angles.

68. An angle of a parallelogram is 3 ($2\frac{2}{3}$) times one of the other angles. Find all the angles.

69. Find the angles of a parallelogram if the difference of two consecutive angles is 20° (90°).

70. If two consecutive angles of a parallelogram are in the ratio 17 : 1 (4 : 5), how many degrees in each angle?

71. How many degrees in each angle of a parallelogram when an angle exceeds $\frac{1}{2}$ ($\frac{1}{3}$) of its consecutive angle by 30° (56°)?

72. The number of degrees in one angle of a parallelogram equals $\frac{1}{2}$ of the square of the number of degrees in the consecutive angle. Find all the angles.

73. Prove algebraically that if two angles x and y of a quadrilateral are supplementary, the other two angles a and b are also supplementary.

74. Find the sides of a parallelogram if one side is $\frac{2}{5}$ ($\frac{3}{4}$) of another side and the perimeter is 28 (84) cm.

75. One side of a parallelogram is 4 (5) in. longer than another side and the perimeter is 36 (58) in. Find the sides.

76. The sum of two adjacent sides of a rhomboid is $\frac{5}{3}$ ($\frac{7}{5}$) of the difference of those sides. Find the sides if the perimeter is 18.3 (82) cm.

77. One angle of a rhombus is 60°. If 5 (2) times the perimeter exceeds the square of the shorter diagonal by 19 ($13\frac{3}{4}$), find a side of the rhombus.

78. In a rhomboid two of whose sides are a and b, 3 times a exceeds twice b by 11, and the sum of twice a and 5 times b is 20. Find the perimeter.

79. In one of the triangles formed by the diagonals of a rhombus and one of the sides of the rhombus the two smaller angles are in the ratio 2 : 3 (1 : 3). Find all the angles of the rhombus.

80. The perimeter of a parallelogram is 16 (9.6), and the square of one side added to 4 (2) times an adjacent side equals 37 (8.6). Find the sides of the parallelogram.

81. In a rhombus one of whose angles is 60° the shorter diagonal is 10 in. (5 ft. 6 in.). Find the perimeter.

82. Two sides of a rectangle are x and x^2 ($3x$ and $7x$) and the perimeter is 60 (40). Find the sides.

Circles

83. The circumference of a circle is divided into three parts. Find the number of degrees in each part if the second contains 3 (6) times as many as the first part, and the third part contains 5 (7) times as many as the first part.

84. In a circle a diameter and a chord are drawn. The diameter is 4 (5) in. longer than the chord and the diameter and chord together are 18 (20) in. long. How long is each?

85. There are 100° (x°) in one of the arcs subtended by a chord. How many degrees are there in the other arc?

86. In one of the arcs subtended by a chord there are 50° (120°) more than in the other arc. How many degrees in each arc?

87. Find the side of a square inscribed in a circle whose radius is 30 (42.5) mm.

88. A triangle whose perimeter is 36 (72) mm. is inscribed in a circle. The first side is $\frac{1}{2}$ of the second and $\frac{2}{3}$ of the third. Find the three sides.

89. In a circle of radius 8 (12) in. a chord is drawn equal in length to the radius. How far is it from the center?

90. A circle containing 280 (308) sq. ft. is divided into three parts by radii. The third part equals twice the second, and the second part is 20 sq. ft. larger than the first. Find the area of each part.

91. A line 1 (3.6) ft. long intersects a circumference in two points. If the part inside the circumference is twice the length of the part outside, how long is the part which forms the chord?

92. A number of coins are placed in a row touching one another, and the length of the row is measured. 3 quarters, 2 nickels, and 5 dimes measure 204 mm.; 1 quarter, 3 nickels, and 2 dimes measure 123 mm.; and 1 quarter, 1 nickel, and 1 dime measure 63 mm. Find the diameter of each coin. Check.

93. A boy has 20 copper disks; part of them are 20 mm. in diameter and the rest are 30 mm. The sum of their diameters is 520 mm. How many of each kind has he?

94. Two diameters are drawn in a circle, making at the center one of the supplementary adjacent angles 3 times the other. How many degrees in each angle?

95. A chord 6 (4) in. long is 4 (6) in. from the center of a circle. Find the radius of the circle.

96. A chord 16 (4) in. long is at a distance of 6 (8) in. from the center of a circle. What is the length of a chord which is 3 (1) in. from the center?

97. A chord 8 (12) in. long bisects at right angles a radius. How long is the radius?

98. The radius of a circle is 5 (3) in. How far from the center is a chord 8 (4) in. long?

99. The radius of a circle is r. What is the length of a chord whose distance from the center is $\frac{1}{2}$ $(\frac{1}{3})$ r?

100. Find the length of the longest and shortest chords that can be drawn through a point 9 (6) in. from the center of a circle whose radius is 15 (8) in.

101. The sum of the longest and the shortest chords through a point 3 (8) in. from the center of a circle is 18 (64) in. Find the radius and the two chords.

102. Construct a curve to show the length of a chord in a circle of radius 8 in. as the distance of the chord from the center increases from 0 to 8 in.

103. A circle is circumscribed about a right triangle whose legs are 6 and 8 (5 and 12) in. Find the radius of the circle.

104. The legs of a right triangle inscribed in a circle are $5x$ and $12x$ (x and $3x$) and the radius of the circle is 13 (5) in. Find the sides of the triangle.

105. From the point of tangency P, a distance PA equal to twice the radius is measured off on the tangent. If the distance from A to the center of the circle is 10 (6) in., find the radius.

106. In a circle of radius 8 (5) in. two parallel chords lie on opposite sides of the center. One is twice as far from the center as the other. If the sum of the squares of the half chords is 123 (10) in., find the distance each chord is from the center.

107. The perimeter of an inscribed isosceles trapezoid is 38 (88) in. One of the parallel sides is $\frac{4}{5}$ (.7) of the other and one of the nonparallel sides is $9\frac{1}{2}$ (30) in. shorter than the longest side of the trapezoid. Find each side.

108. Two circles touch each other and their centers are 8 (a) in. apart. The diameter of one is 10 (d) in. What is the diameter of the other?

109. Two circles are tangent externally. The difference of their radii is 8 (a) in. and the distance between their centers is 12 (b) in. Find the radii.

110. The distance between the centers of two circles is 18 (a) in., which is one half the sum of their radii. Find the radii.

111. One angle of an inscribed triangle is 35° (50°) and one of its sides subtends an arc of 113° (150°). Find the other angles of the triangle.

112. The circumference of a circle is divided into three arcs in the ratio $1:2:3\,(2:3:5)$. Find the angles of the triangle formed by the chords of the arc.

113. A triangle is inscribed in a circle. The sum of the first and third angles is twice the second angle, and the difference of the first and second is 20°. How many degrees in each of the three arcs?

114. Construct a graph to show the change in an inscribed angle y, as the arc intercepted by its sides increases from 0 to 180°.

115. An isosceles triangle is inscribed in a circle. The number of degrees in the arc upon which the vertical angle stands is $8\,(3\frac{1}{5})$ times the number of degrees in a base angle of the triangle. Find the angles of the triangle.

116. Consecutive sides of an inscribed quadrilateral subtend arcs of 82°, 99°, 67°, and $x°$ respectively. Find each angle of the quadrilateral; also each of the eight angles formed by a side and a diagonal.

117. How many degrees in each angle of a quadrilateral inscribed in a circle, if the sides subtend arcs which are in the ratio $1:2:3:4\,(2:3:5:6)$?

118. A right triangle is inscribed in a circle. If one acute angle of the triangle is $\frac{2}{3}\,(\frac{3}{5})$ of the other, how many degrees in each of the three arcs?

119. $ABCD$ is an inscribed trapezoid. If the angle A is twice angle C, find each angle.

120. Two chords AB and CD intersect within a circle at P. The angle APC is 50°, arc DB is 40°, and arc AD is 160°. Find the other arcs and angles.

121. Two chords AB and CD intersect within a circle at P. Arc BD is twice arc AC, and arc CB is twice arc DA. Angle DPA is twice angle APC. Find the arcs and angles.

122. The angle y is formed by two chords AB and CD intersecting in a circle, and the two intercepted arcs AC and DB are

$90°$ and $x°$ respectively. What is the equation connecting y and x? Construct a graph to show the change in y as x increases from 0 to $90°$.

123. From a point without a circle two secants are drawn, making one of the intercepted arcs 3 (5) times the other. If the sum of the other two arcs is $200°$ ($300°$), what is the angle formed by the secants?

124. The angle y is formed by two secants intersecting without a circle. The intercepted arcs are $90°$ and $x°$ $(x < 90)$. What is the equation connecting y and x? Construct a graph to show the change in y as x increases from 0 to $90°$.

125. Two tangents drawn from an exterior point to a circle make an angle of $60°$ ($80°$). Find the two arcs. Join the points of tangency and find the other two angles in the triangle thus formed.

126. Through the ends of an arc of $45°$ ($100°$) tangents to the circle are drawn. Find the angle formed by the tangents. Find the other two angles in the triangle formed by joining the points of tangency.

127. Find the angle formed by two tangents to a circle drawn from a point at a distance from the center of the circle equal to the diameter.

128. From P, a point without a circle, two tangents PA and PB, and a secant PC are drawn. The arc AB equals $160°$ ($100°$). If the difference of the angles BPC and CPA is $10°$ ($25°$), find the angles.

129. From a point without a circle of radius 4 (8) in. a secant through the center and a tangent are drawn. If the angle formed by the secant and tangent is $30°$ ($60°$), find the distance from the point to the center of the circle, and the length of the tangent.

130. In an equilateral triangle whose sides are 40 (60) mm. a circle is inscribed. Find the radius of the circle. Find the radius of the circumscribed circle.

Ratio

131. Express the ratio of the following pairs of numbers in the simplest form:

(*a*) 168 and 252.
(*b*) 387 and 602.
(*c*) $\frac{3}{4}$ and $\frac{7}{8}$.
(*d*) $5\frac{1}{2}$ and $30\frac{1}{4}$.
(*e*) $\frac{12}{18}$ and $\frac{3}{8}$.
(*f*) .125 and 3.75.
(*g*) $6a^2x$ and $30a^4x$.
(*h*) $148x^3$ and $185x^4$.
(*i*) x^2+5x+6 and $x+3$.
(*j*) $x^2+2x-15$ and $x+5$.
(*k*) x^2+3x+2 and x^2+4x+3.
(*l*) x^2+6x+5 and $x^2+8x+15$.
(*m*) $\dfrac{x+2}{x+3}$ and $\dfrac{x^2+6x+8}{x^2+7x+12}$.

132. Squares are constructed on the lines a and b. Find the ratio of the areas:

(*a*) $a = 5$ in., $b = 10$ in.
(*b*) $a = 3\frac{1}{2}$ in., $b = 7$ in.
(*c*) $a = 4$ cm., $b = 12$ cm.
(*d*) $a = 14$ mm., $b = 35$ cm.

133. On a sheet of squared paper let the bottom line be the x-axis and the left border line be the y-axis, and the side of a square each way $= 1$. Draw a straight line through the points (0, 0) and (8, 16). Make a table of corresponding values of x and y. What is the ratio of y to x? What is the equation of the line?

134. The width y of a field is to be made $\frac{3}{5}$ of the length x. What is the equation connecting y and x? Construct a graph to show the width of the field for a length from 10 to 100 rd.

135. If the ratio of y to x is $2:3$, construct a graph to show the relation. What is the equation of the straight line?

136. If $14x - 9y = 2x - y$, find the ratio of $x:y$. Construct the graph.

137. What is the ratio of $x:y$, if $7x - 6y = 3x + 4y$?

138. If $x:y = 4:5$, find the value of the ratio $2x + y : 7x - y$. Construct the graph.

139. Find the value of the ratio $3x^2 + 2y^2 : xy + y^2$, if $x:y = 1:2$.

PROPORTION

140. Test the correctness of the following proportions:

(*a*) $\dfrac{84}{180}=\dfrac{42}{90}$.

(*b*) $\dfrac{48}{225}=\dfrac{96}{45}$.

(*c*) $\dfrac{87}{203}=\dfrac{111}{259}$.

(*d*) $\dfrac{1.25}{.26}=\dfrac{120}{24}$.

(*e*) $\dfrac{a^2+2ab+b^2}{a^2-b^2}=\dfrac{a+b}{a-b}$.

(*f*) $\dfrac{x^2+7x+10}{(x+5)^2}=\dfrac{x+2}{x+5}$.

141. Find x in the following proportions:

(*a*) $\dfrac{18}{25}=\dfrac{32}{x}$.

(*b*) $\dfrac{28}{x}=\dfrac{35}{18}$.

(*c*) $\dfrac{x}{4.8}=\dfrac{16}{.24}$.

(*d*) $\dfrac{a^2-b^2}{a+b}=\dfrac{a-b}{x}$.

(*e*) $\dfrac{x}{a^3-b^3}=\dfrac{1}{a-b}$.

(*f*) $\dfrac{a^2+10a+25}{x}=\dfrac{x}{9}$.

142. What number can be added to 7, 12, 1, and 3 (5, 19, 16, and 52) so that the resulting numbers will form a proportion?

143. Find the numbers proportional to 1, 2, 3, 4 (2, 5, 1, 3) that may be added regularly to 5, 10, 15, 40 (11, 20, 8, 14) so as to form a proportion.

144. The line joining the mid-points of the nonparallel sides of a trapezoid is 20 (42) in. long. Find the bases if one is $\frac{2}{3}$ (.4) of the other.

145. In a triangle ABC the line PQ parallel to BC divides the side AC in the ratio 3 : 4 (5 : 9). If $AB=20$ (9.8) in., find the two segments of AB.

146. The sum of the two sides of a triangle is 45 (63) in. A line parallel to the third side cuts off from the vertex segments 10 and 8 (4 and 20) in. long. Find the two sides.

147. A line 100 (6) ft. long is divided into parts in the ratio 1 : 2 : 3 : 4 (2 : 3 : 7). Find each part.

148. Three lines are in the ratio $2:3:4$ $(2:1:6)$ and their fourth proportional is 30 (24). Find the length of each line.

149. The sum of two sides of a triangle is 20 (5) in. The third side, 18 ($4\frac{1}{2}$) in. long, is a third proportional to the other two sides. Find them.

150. One side of a triangle is 2 in. longer than the first side, and the third side is 5 in. longer than the first. If one side is a mean proportional between the other two, find the three sides.

151. The three sides of a triangle are x, y, and 3. The corresponding sides of a similar triangle are 10, 20, and 15. Find x and y.

152. The sum of the three sides of a triangle, x, y, and z, is 15, and the corresponding sides of a similar triangle are $x+3$, $y+7$, and $z+5$. Find the sides of each triangle.

153. The three sides of a triangle are $3x$, $6x$, and $8x$ $(x, x+1, x+2)$, and the corresponding sides of a similar triangle are $3x^2$, $6x^2$, and $8x^2$ $(x^2, x^2+x$, and $x^2+2x)$. If the sum of the perimeters of the two triangles is 102 (75), find the sides of each triangle.

154. The sides of a triangle are 5, 8, 12 (12, 16, 20) in. Find the segments of each side made by the bisector of the opposite angle.

155. The sum of two sides of a triangle is 24 in., and the bisector of the included angle divides the third side into parts 4 and 8 in. long. Find the three sides.

156. In a triangle ABC, $AB = 12$ and $BC = 36$. From a point on AB at a distance x from A a line y is drawn to AC parallel to the base. Construct a graph to show the length of y as x increases from 0 to 12.

Right Triangles

157. The hypotenuse of a right triangle is 8 in. and one angle is 30°. Find (a) the other two sides; (b) the perpendicular from the vertex of the right angle to the hypotenuse; (c) the segments of the hypotenuse.

158. One leg of a right triangle is 2 (3) ft. longer than the other and the hypotenuse is 4 (7) ft. longer than the shorter leg. Find the three sides.

159. The legs of a right triangle are 12 and 16 (5 and 12) ft. Find (*a*) the hypotenuse; (*b*) the perpendicular from the vertex of the right angle to the hypotenuse; (*c*) the segments of the hypotenuse.

160. The perpendicular from the vertex of the right angle of a right triangle to the hypotenuse is 12 (3) in. long and the hypotenuse is 26 (6.25) in. long. Find the other two sides.

161. If the legs of a right triangle are a and b, find the perpendicular from the vertex of the right angle to the hypotenuse, and the segments of the hypotenuse.

162. One side of a right triangle is 4. Construct a curve to show the length of the hypotenuse as the other side increases from 0 to 16. (Let the bottom line be the x-axis, the left border line be the y-axis, and the side of a large square each way $= 1$. Take the side 4 on the vertical axis and locate the points of the curve with compasses. Check a few of the points by computation.)

Chords, Tangents, Secants

163. The segments of a chord made by another chord are 7 and 9 (15 and 13) in., and one segment of the latter chord is 3 (10) in. What is the other segment?

164. Two chords intersect, making the segments of one chord 2 and 12 (4 and 8) in., and one segment of the other chord 2 (14) in. longer than the other segment. Find the two chords.

165. One of two intersecting chords is 14 (17) in. long, and the product of the segments of the other chord is 45 (60). Find the segments of the first chord.

166. Two secants intersect without a circle. The external segment of one is 20 (2) in. and the internal segment is 5 (4) in. If the external segment of the other secant is 10 (3) in., find the length of the internal segment.

167. From a point without a circle two secants are drawn whose external segments are 5 and 6 (6 and 8) in. The internal segment of the former is 13 (16) in. What is the internal segment of the latter? What is the length of the tangent from the same point?

168. Two secants from a point without a circle are 24 in. and 22 in. long. If the external segment of the lesser is 5 in., what is the external segment of the greater? What is the length of the tangent from the same point?

169. A tangent and a secant are drawn to a circle from an external point. The external and internal segments of the secant are respectively 2 (3) in. and 1 (4) in. shorter than the tangent. What is the length of the tangent?

170. From a point on the tangent of a circle 6 (15) in. from the point of tangency a secant is drawn whose internal segment is 2 (3) times the external segment. Find the length of the secant.

171. A tangent intersects a secant which is drawn through the center of a circle. The length of the tangent is 4 (t) in., and the length of the external segment of the secant is 2 (s) in. Find the radius of the circle and the secant.

172. In a circle of radius 17 in. a point P is taken on the diameter 15 in. from the center. What is the product of the segments of chords through P? Denoting the segments by x and y, what is the equation that connects x and y? In this equation give values to x and make a table of values of x and y. Construct a curve to show the change of y as x increases from 2 to 32 in.

173. From a point on the circumference of a circle of 9 in. diameter a tangent 6 in. long is drawn. From the end of the tangent secants are drawn. If y is the external and x the internal segment of the secant, what is the equation connecting x and y? Construct a curve to show the length of y as x increases from 0 to 9 in. and then decreases to 0.

Area of Polygons

174. The base of a triangle is 5(3) times the altitude and the area is 90(75) sq. in. Find the base and altitude.

175. The area of a triangle is 130(42) sq. in. and the altitude is 7 in. less (5 in. more) than the base. Find these dimensions.

176. The sum of the base and altitude of a triangle is 12(23) in. and the area is 16(45) sq. in. Find the base and altitude.

177. Find the area of a right triangle whose base is 20(32) and the sum of whose hypotenuse and other side is 40(50).

178. The altitude of an equilateral triangle is 12(h) ft. Find its sides and area.

179. The altitude of a triangle is 16 in. less than the base. If the altitude is increased 3 in. and the base 12 in., the area is increased 52 sq. in. Find the base and altitude.

180. If the hypotenuse of a right triangle is 1(8) in. longer than one leg, and 8(9) in. longer than the other leg, what is the area of the triangle?

181. If the area of an equilateral triangle is $16\sqrt{3}$(60) sq. in., find the altitude and a side.

182. If a denotes the area, s a side, and h the altitude of an equilateral triangle, express each in terms of the others.

183. If a rectangle is 7(8) ft. longer than it is wide and contains 170(209) sq. ft., find its dimensions.

184. The perimeter of a rectangle is 72(132) ft. and its length is 2(5) times its width. Find its area.

185. A rectangle whose length is 8(5) ft. greater than 3(4) times its width contains 115(3750) sq. ft. Find its dimensions.

186. The area of a rectangle is 36 sq. ft. Construct a curve to show the altitude as the base increases from 1 to 36 ft.

187. The side of one square is 3(4) times as long as that of another square, and its area is 72(90) sq. yd. greater than that of the second square. What is the side of each square?

188. One side of a square is 3(6) yd. less than 2(3) times the side of a second square, and the difference in area of the squares is 45(756) sq. yd. Find the area of each square.

189. One side of a rectangle is 10(6) ft. and the other side is 2(1) ft. longer than the side of a given square. The area of the rectangle exceeds that of the square by 80(174) sq. ft. Find the side and area of the square.

190. The floor of a rectangular room contains 180(240) sq. ft., and the length of the molding around the room is 56(62) ft. Find the length and width of the room.

191. A picture including the frame is 10(9) in. longer than it is wide. The area of the frame, which is 3(6) in. wide, is 192(480) sq. in. What are the dimensions of the picture?

192. The dimensions of a picture inside the frame are 12 in. by 16 in. (5 in. by 12 in.). What is the width of the frame if its area is 288(138) sq. in.?

193. Around a square garden a path 2 ft. wide is made. If 376 sq. ft. are taken for the path, find a side of the garden.

194. Around a garden 100 ft. by 120 ft. a man wishes to make a path which shall occupy $\frac{7}{40}$ $(\frac{1}{6})$ of the area. How wide must the path be made?

195. A rectangular building having a perimeter of 140 ft. is inclosed by a fence whose distance from the building is $\frac{1}{3}$ the width of the building. If the area between the fence and building is 1800 sq. ft., find how far the fence is from the building.

196. An open-top box is made from a square piece of tin by cutting out a 5(2)-in. square from each corner and turning up the sides. How large is the original square if the box contains 180(242) cu. in.?

197. An open-top box is formed by cutting out a 1(3)-in. square from each corner of a rectangular piece of tin 2(3) times as long as it is wide, and turning up the sides. If the total surface of the box is 284(936) sq. in., find the dimensions of the piece of tin.

198. It is desired to make an open-top box from a piece of tin 30(24)(15) in. sq., by cutting out equal squares from each corner and turning up the strips. What should be the length of a side of the squares cut out to give a box of the greatest possible volume?

Suggestion. If $x =$ side of square cut out, volume of the box $= y = x(30 - 2x)^2$.

Make a table of values of y, giving x the values 1, 2, 3 $\cdots$. Locate the points and draw a smooth curve through them. The turning point of the curve will show the value of x for the greatest volume.

199. From a rectangular piece of tin 12 in. by 24 in. (16 in. by 36 in.) it is desired to make an open-top box of the largest possible volume, by cutting out equal squares from the corners and turning up the strips. What should be the length of a side of the squares?

200. The altitude of a trapezoid is 5(14) in., the area is 10(455) sq. in., and the difference of the bases is 2(11) in. Find the bases.

201. The area of a trapezoid is 90 (495) sq. ft., the line joining the mid-points of the nonparallel sides is 6 (45) ft., and the difference of the bases is 6 (12) ft. Find the bases and altitude.

202. In a trapezoid b and b' are the bases, h the altitude, and a the area. Find each in terms of the other.

203. The base of a triangle is 12 in. and the altitude increases from 0 to 20 in. Construct a graph to show the increase in area of the triangle.

204. The base and altitude of a triangle increase uniformly, and the altitude is always twice the base. Construct a curve to show the change in the area of the triangle as the base increases from 0 to 10 ft.

205. The base and altitude of a triangle are 24 in. and 9 in. respectively. What is the area of the triangle formed by a line parallel to the base and 6(8) (x) in. from the vertex?

206. In a triangle whose base is 12 in. and altitude is 16 in. a line is drawn parallel to the base and at a distance x from the vertex. If $y =$ the area of the triangle cut off from the vertex, what is the equation connecting x and y? Construct a curve to show the area of the triangle cut off as x increases from 0 to 16 in.

207. The altitude of a triangle is 2 (3) times its base. Through the mid-point of the altitude a line is drawn parallel to the base. If the area of the triangle cut off is 36 (5) sq. in., find the base and altitude of the given triangle.

208. The sum of the areas of two similar triangles is 240 (290) sq. in., and the sides of one are 2 ($2\frac{1}{2}$) times the corresponding sides of the other. Find the area of each triangle.

209. The difference of the areas of two squares is 39 (324) sq. ft., and a side of one is 3 (14) ft. longer than a side of the other. Find a side of each square.

210. The sum of the areas of two squares is 13 (221) sq. ft., and a side of one square is 1 (9) ft. shorter than a side of the other. Find a side of each square.

211. A side of one square is 5 (2) in. longer than a side of another square, and the areas of the squares are in the ratio 4 : 1 (16 : 9). What is a side of each square?

212. Construct a curve to show the area of a square as its sides increase from 0 to 13 in.

Circles and Inscribed Polygons

213. Construct a curve to show the area of a circle as its radius increases from 0 to 16 in. (Locate points for $r = 0, 2, 4, \cdots, 16$.)

214. The radius of a circle is 5 (8) (r) ft. Find a side and the area of the inscribed square.

215. What is the radius of the circle inscribed in a square whose area is 1600 (5000) (a) sq. ft.?

216. An equilateral triangle is inscribed in a circle of radius 6 (12) (r) in. Find a side, the altitude, and area of the triangle.

217. The side of an inscribed equilateral triangle is 9 (1.732) (s) in. Find the radius of the circle.

218. The sum of the side of an inscribed equilateral triangle and the radius of the circle is $5 + 5\sqrt{3}$ (10.928) (18) in. What is the length of a side and the radius?

219. The area of a regular inscribed hexagon is $24\sqrt{3}$ (17.32) (a) sq. ft. Find the radius of the circle.

220. An equilateral triangle and a regular hexagon are inscribed in a circle. Find the radius of the circle if the sum of the areas of the triangle and hexagon is $9\sqrt{3}$ ($18\sqrt{3}$) (389.7) sq. in.

221. The sum of the perimeters of two regular pentagons is 100 (225) ft., and their areas are in the ratio 1 : 9 (25 : 16). Find a side of each pentagon.

222. The difference of the perimeters of two regular octagons is 40 (80) ft., and their areas are in the ratio 1 : 4 (9 : 25). Find a side of each octagon.

223. The sum of the circumferences of two circles is 20π (176) ft., and the difference of their radii is 2 (14) ft. What are the radii?

224. The radius of one circle is 6 (1) ft. longer than the radius of another circle, and the sum of their circumferences is $113\frac{1}{7}$ (31.416) ft. Find the radii.

225. What is the radius of a circle whose area equals the area of two circles of radii (a) 3 and 4 in.? (b) 3.3 and 5.6 cm.? (c) 6.5 and 7.2 cm.? (d) r and nr?

226. What is the radius of a circle whose area equals the sum of (a) 3, (b) 6, (c) n equal circles?

227. What is the radius of a circle that is doubled in area by increasing its radius 1 (3) ft.?

228. A square and a circle have the same perimeter. Find the ratio of their areas.

229. If a square and a circle have the same area, what is the ratio of their perimeters?

230. If a circle and an equilateral triangle have the same perimeter, what is the ratio of their areas?

231. Construct on the same axes curves to show the change in area of a circle and the inscribed regular hexagon, square, and equilateral triangle, as the radius increases from 0 to 10 in.

232. The area between two concentric circles is $20\pi(286)$ sq. ft. and the difference of the radii is $2(7)$ ft. Find the radii.

233. If the area between two concentric circles is 96π (50) sq. ft., and the radius of the inner circle is $2(5)$ ft., find the radius of the larger circle.

234. In a circle of radius $12(r)$ in. it is desired to draw a concentric circle which shall bisect the area of the given circle. Find its radius.

235. The area of a circle of radius $8(r)$ in. is to be divided by a concentric circle so that the area of the ring shall be a mean proportional between the area of the given circle and of the inner circle. Find the radius.

CHAPTER X

COMMON LOGARITHMS

68. Definitions. Numbers have been reduced to powers of 10. Thus $2 = 10^{0.3010}$, $3 = 10^{0.4771}$, $125 = 10^{2.0969}$.

These exponents are called *logarithms.* The integral part of a logarithm, called the *characteristic*, can be determined easily and is not given in a table of logarithms; the decimal part, called the *mantissa*, is always taken from the table.

69. Approximate numbers. In ordinary shop practice and in much engineering work measurements are made usually to three or four figures. Thus in making a rough estimate the sides of a building lot may be measured to the nearest foot; the length of a belt may be measured to the nearest quarter of an inch; an angle may be measured to the nearest tenth of a degree. If the diameter of a pulley is measured and said to be 12.3 in., the meaning is that the diameter lies between 12.25 in. and 12.35 in., that is, the third figure is doubtful. In ordinary computations, where numbers with only three or four figures are involved, a four-place table of logarithms is used. The logarithms are not exact; they are approximate numbers in which the fourth figure is doubtful. Hence the results should not be carried beyond four figures.

70. The mantissa. To find the mantissa of the logarithm of a number from 1 to 999, e.g. 352, we look in the first column of the table at the left for the first two figures, 35, and in the column headed 2 we find the mantissa of 352, namely .5465. The mantissa of 745 is .8722.

(Let the class read the mantissas of numbers from the table till all can find the mantissa of any number quickly.)

71. The characteristic. The method of finding the characteristic is readily obtained from the following table:

$10^3 = 1000$,	$\therefore \log 1000 = 3.$	$\log 6214 = 3 + \text{a decimal.}$
$10^2 = 100$,	$\therefore \log 100 = 2.$	$\log 518 = 2 + \text{a decimal.}$
$10^1 = 10$,	$\therefore \log 10 = 1.$	$\log 83 = 1 + \text{a decimal.}$
$10^0 = 1$,	$\therefore \log 1 = 0.$	$\log 6 = 0 + \text{a decimal.}$
$10^{-1} = .1$	$\therefore \log .1 = -1.$	$\log .3 = -1 + \text{a decimal.}$
$10^{-2} = .01$,	$\therefore \log .01 = -2.$	$\log .04 = -2 + \text{a decimal.}$
$10^{-3} = .001$,	$\therefore \log .001 = -3.$	$\log .008 = -3 + \text{a decimal.}$

Since 518 lies between 100 and 1000 its logarithm lies between 2 and 3; that is, it is 2 plus a decimal.

The above table shows that the characteristic of the logarithm of an integer is one less than the number of integral figures in the number.

From the table it is also seen that the characteristic of a decimal is a negative number. Since the mantissa is always positive, it is convenient to make a little change so that the characteristic may be considered positive; this is done by adding and subtracting 10.

Thus
$$\log .2 = -1 + .3010 = 9.3010 - 10.$$
$$\log .02 = -2 + .3010 = 8.3010 - 10.$$
$$\log .002 = -3 + .3010 = 7.3010 - 10.$$

To find the characteristic of the logarithm of a decimal, begin at the decimal point and count the zeros, 9, 8, 7, $\cdots$ till the first significant figure is reached. The last count with -10 written after the mantissa is the characteristic.

72. The logarithm of a number. Since 10 is the base of our number system, 10 is taken as the base of logarithms for use in ordinary computations. This makes the work much easier, because the mantissa does not change as long as the figures in a number remain in the same order. Thus 216, 21.6, .216, and .0216 have the same mantissa.

$\log 216 = 2.3345$, i.e.	$10^{2.3345}$	$= 216.$	
Dividing both sides of the equation by 10,	10	$= 10$	
	$10^{1.3345}$	$= 21.6$	$\therefore \log 21.6 = 1.3345.$
	10	$= 10$	
	$10^{0.3345}$	$= 2.16$	$\therefore \log 2.16 = 0.3345.$
	10	$= 10$	
	$10^{9.3345-10}$	$= .216$	$\therefore \log .216 = 9.3345.-10.$
$\log 2 = 0.3010$, i.e.	$10^{0.3010}$	$= 2.$	
Multiplying both sides of the equation by 100,	10^2	$= 100$	
	$10^{2.3010}$	$= 200$	$\therefore \log 200 = 2.3010.$

Hence it is seen that moving the decimal point any number of places to the right or left is multiplying or dividing by some integral power of 10, and this affects only the characteristic.

The mantissas of numbers having one, two, or three figures are taken directly from the table. The mantissas of four-figure numbers are easily found.

Find the logarithm of 1836. The mantissa of 1836 is the same as the mantissa of 183.6, since moving the decimal point does not change the mantissa. The mantissa of 183.6 lies between the mantissas of 183 and 184; and it is .6 of the way from the mantissa of 183 to the mantissa of 184.

$$\begin{aligned} \text{Mantissa of } 184 - \text{mantissa of } 183 &= 2648 - 2625 \\ &= 23. \\ 23 \times .6 &= 14. \\ 2625 + 14 &= 2639. \\ \therefore \log 1836 &= 3.2639. \end{aligned}$$

Find log 49.23.

$$\begin{aligned} \text{Mantissa of } 493 - \text{mantissa of } 492 &= 6928 - 6920 \\ &= 8. \\ 8 \times .3 &= 2. \\ 6920 + 2 &= 6922. \\ \therefore \log 49.23 &= 1.6922. \end{aligned}$$

To find the logarithm of a number. Place the decimal point (mentally) after the third figure. Subtract the next lower mantissa from the next higher. Multiply the difference by the fourth figure of the number regarded as tenths, disregarding a fraction less than one half and calling one half or more one ; add the product to the next lower mantissa. Write the proper characteristic.

(Let the class find the logarithms of many numbers. The work should be done mentally; it can be done easily and quickly with practice.)

73. To find a number from its logarithm. Given $\log b = 1.5927$, required to find b. Looking in the table of mantissas, it is seen that 5927 lies between 5922 and 5933; the corresponding numbers are 391 and 392. Hence the number corresponding to 5927 lies between 391 and 392; that is, it is 391 plus a fraction. To find the fraction, add a zero to the difference of the given mantissa and the smaller, and divide it by the difference of the next larger and next smaller mantissas.

391	5922
391.5	5927
392	5933
	11) 50 (5

Since a difference of 11 in the mantissas makes a difference of 1 in the numbers, a difference of 5 makes a difference of $\frac{5}{11}$ in the numbers. Hence the mantissa 5927 gives the number $391\frac{5}{11} = 391.5$. But the characteristic 1 shows that there are two integral figures in the number. Therefore $b = 39.15$.

Given $\log m = 0.9145, \quad m = 8.213.$
$\log n = 8.8132 - 10, \quad n = .06504.$

To find a number from its logarithm. When the given mantissa lies between two mantissas in the table, divide the difference of these mantissas into the difference of the smaller mantissa and the given mantissa, to one decimal figure. Add this decimal *figure to the number corresponding to the smaller mantissa and place the decimal point in the position indicated by the characteristic.*

(All the work in finding a number from its logarithm should be done mentally; with practice it can be done easily and quickly.)

74. The use of logarithms in computation. Since logarithms are exponents it follows that:

I. $\log(2 \times 3) = \log 2 + \log 3.$

$$2 = 10^{0.3010}, \quad 3 = 10^{0.4771.}$$

$$2 \times 3 = 10^{0.3010} \times 10^{0.4771} = 10^{0.7781} = 6.$$

The logarithm of a product is equal to the sum of the logarithms of the factors.

II. $\log \frac{3}{2} = \log 3 - \log 2.$

$$3 \div 2 = 10^{0.4771} \div 10^{0.3010} = 10^{0.1761} = 1.5.$$

The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

III. $\log 2^3 = 3 \log 2.$

$$2^3 = (10^{0.3010})^3 = 10^{0.9030} = 8.$$

The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

IV. $\log \sqrt{3} = \log 3^{\frac{1}{2}} = \frac{1}{2} \log 3.$

$$\sqrt{3} = 3^{\frac{1}{2}} = (10^{0.4771})^{\frac{1}{2}} = 10^{0.2386} = 1.732.$$

The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.

PROBLEMS

1. Multiply 28.34 by 3.376.

$$\begin{aligned} \log 28.34 &= 1. \\ \log 3.376 &= \underline{0.\qquad} \\ \log \text{product} &= \\ \text{product} &= \end{aligned}$$

Before finding the mantissas from the table always make out an outline as above. This saves time and prevents mistakes. Keep the signs of equality and the figures exactly in columns.

SOLUTION.
$$\begin{aligned} \log 28.34 &= 1.4524 \\ \log 3.376 &= \underline{0.5284} \\ \log \text{product} &= 1.9808 \\ \text{product} &= 95.68. \end{aligned}$$

As a rough check we have $28\frac{1}{3} \times 3\frac{1}{3} = 94$.

2. Multiply 1.251 by .6453.

SOLUTION.
$$\begin{aligned} \log 1.251 &= 0.0973 \\ \log .6453 &= \underline{9.8098 - 10} \\ \log \text{product} &= 9.9071 - 10 \\ \text{product} &= .8074 \end{aligned}$$

Rough check. $.65 \times 1\frac{1}{4} = .81.$

3. Divide 31.87 by 641.2.

SOLUTION.
$$\begin{aligned} \log 31.87 &= 11.5034 - 10 \\ \log 641.2 &= \underline{2.8070} \\ \log \text{quotient} &= 8.6964 - 10 \\ \text{quotient} &= .04970. \end{aligned}$$

Rough check. $32 \div 640 = .05.$

Since the characteristic of the logarithm of the divisor is larger than the characteristic of the logarithm of the dividend, 10 is added to and subtracted from the logarithm of the dividend. Note that the quotient has four significant figures (see sect. 2). The zero must be written at the right to show that the division has been carried out to four figures.

4. Divide .8354 by .04362.

SOLUTION.
$$\begin{aligned} \log .8354 &= 9.9219 - 10 \\ \log .04362 &= \underline{8.6397 - 10} \\ \log \text{quotient} &= 1.2822 \\ \text{quotient} &= 19.15. \end{aligned}$$

Rough check. $.84 \div .044 = 19.$

5. Find $.6874^3$.

SOLUTION.
$$\begin{aligned} \log .6874 &= 9.8372 - 10 \\ & \underline{3} \\ & \,29.5116 - 30 \\ \log .6874^3 &= 9.5116 - 10 \\ .6874^3 &= .3248. \end{aligned}$$

Rough check. $.7^3 = .34.$

6. Find $\sqrt{.8231}$.

SOLUTION.
$$\begin{aligned} \log .8231 &= 9.9155 - 10 \\ &= 19.9155 - 20 \\ \tfrac{1}{2} \log .8231 &= 9.9578 - 10 \\ \sqrt{.8231} &= .9074. \end{aligned}$$

Rough check. $\sqrt{.82} = .9.$

Before dividing log .8231 by 2, 10 was added and subtracted in order that the resulting logarithm should have a -10. Similarly, in extracting the cube root of a decimal add and subtract 20.

7. 8.114×56.83.

8. $5.161 \times .0471$.

9. $86.31 \times .07832$.

10. $.0447 \times .9142$.

11. $6.320 \times 3.106 \times 8.141$.

12. $\frac{4731}{1637}$.

13. $\frac{983}{4335}$.

14. $\dfrac{2.178}{67.83}$.

15. $\dfrac{.4971}{.5382}$.

16. $(4.931)^3$.

17. $(1.237)^5$.

18. $(.8734)^3$.

19. $\sqrt{1983}$.

20. $\sqrt{.4835}$.

21. $\sqrt[3]{3.142}$.

22. $\sqrt[3]{.0687}$.

23. $\dfrac{891 \times 3.62 \times .5162}{68.14 \times 2.657}$.

24. $\dfrac{12.73 \times 9.684}{2.056 \times .8666}$.

25. $\dfrac{4 \times 3.142 \times (1.651)^3}{3}$.

26. $\dfrac{86.3 \times 4.5 \times 3.142 \times 15^2 \times 200}{33000}$.

27. Find the area of a rectangular lot 323.8 ft. long and 112.3 ft. wide.

28. The base of a triangle is 72.14 ft. and its altitude is 8.482 ft. Find its area.

29. Find the area of a square whose side is 71.18 yd.

30. The parallel sides of a trapezoid are 69.14 ft. and 38.15 ft. If the altitude is 12.83 ft., find the area.

31. Find the surface and volume of brass cylinders and prisms, wooden blocks, and so on.

32. Find the area of the blackboard in square meters.

33. Find the area of the athletic field.

34. Find the area of the ground covered by the school buildings.

35. Find the area of the block in which the school building stands.

36. Construct the logarithmic curve.

37. The area of a rectangle is 1689 sq. yd. and the length is 58.12 yd. Find its width.

38. Find the side of a square whose area is 77.83 sq. ft.

39. The volume of a cube is 2861 cu. in. Find the length of an edge.

40. What is the diameter of a piston which has an area of 172.8 sq. in.?

41. Find the diameter of a circular plate of iron of the same weight and thickness as a rectangular plate 3 ft. 4 in. by 2 ft. 8 in.

42. A steel shaft is 3.5 in. in diameter and 12 ft. 9 in. long Find its weight if 1 cu. in. of steel weighs .283 lb.

CHAPTER XI

THE SLIDE RULE

75. Use of the slide rule.* In ordinary practical work it is usual to make measurements and carry results in computations only to three or four significant figures. With the slide rule multiplications and divisions can be performed mechanically to the degree of accuracy required in this work. The slide rule is

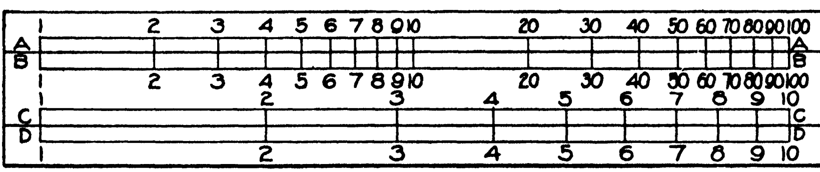

FIG. 54

widely used in technical schools and in shops and laboratories where there is a large amount of computation. It serves as a check upon the numerical solution of problems, and should be used by engineering students.

76. Description of the slide rule. The slide rule is simply a table of logarithms arranged in such a way that they can be dded and subtracted conveniently. The logarithms are not inted on the slide rule, but each number on it stands in the ition indicated by its logarithm. In Fig. 54 BC is the slide, luated on the upper and on the lower edges. These gradu- were made in the following manner: CC was divided 00 equal parts; log 2 = .301, therefore 2 was placed at

board slide rules ready for the student to cut and fit together may be f the Central Scientific Company, Chicago, at $1.10 per dozen.

the 301st graduation; log $3 = .477$, therefore 3 was placed at the 477th graduation; and so on for all the integers from 1 to 1000.

To read the numbers from 1 to 1000 we must go over the rule from left to right three times. Thus we read first 1, 2, 3, $\cdots$, 10; then beginning at 1 again and calling it 10, we read 10, 20, 30, $\cdots$, 100; then beginning at 1 again and calling it 100, we read 100, 200, 300, $\cdots$, 1000.

77. Operations with the slide rule. It is not difficult to learn to use the slide rule if at first the operations are performed with small numbers. Whenever in doubt about any operation perform it first with small numbers.

I. *Multiplication.* Multiply 3 by 2. Move the slide so as to set 1 C on 3 D; then under 2 C read the product 6 on D. Note that this is simply adding logarithms.

To find the product of two numbers, set 1 C on one of the numbers on D, and under the other number on C read the product on D.

Sometimes in multiplying we must use the 1 at the right end of scale C. Thus multiply 84 by 2. Set 1 at the right end of scale C on 84 D, under 2 C read 168 on D. We use the 1 at the left end or the right end of scale C according as it brings the second factor over scale D. In the example above, if we had set 1 at the left end of scale C on 84, then 2 C would have been off scale D.

The decimal point is placed by inspection. Thus, multiply 12.5 by 1.8. Set 1 C on 18 D, under 125 C read 225 on D. But making an approximate multiplication mentally, $12 \times 2 = 24$; hence we know that there are two integral figures in the product, giving 22.5 as the result. In all operations with the slide rule the decimal point can be placed by making an approximate mental computation.

II. *Division.* Divide 8 by 2. Set 2 C on 8 D, under 1 C read the quotient 4 on D. Note that this is simply subtracting logarithms.

To divide one number by another, set the divisor on scale C on the dividend on scale D, under 1 C read the quotient on scale D.

The decimal point is placed by inspection. Thus divide 3.44 by 16. Set 16 *C* on 344 *D*, under 1 *C* read the quotient 215 on *D*; but we see that $3 \div 16 =$ about .2; hence the quotient is .215.

III. *Combined multiplication and division.* Find the value of $\frac{24 \times 9}{6}$. Set 6 *C* on 24 *D*, under 9 *C* read the result 36 on *D*. Study this operation till the separate parts are seen clearly and understood. First the division of 24 by 6 is made by setting 6 *C* on 24 *D*, under 1 *C* we might read the quotient; but we want to multiply this quotient by 9. As 1 *C* is already on this quotient we have only to read the product 36 on scale *D* under 9 *C*.

An important problem under this case is to find the fourth term of a proportion. Thus, in the proportion $6 : 24 = 9 : x$,

$$x = \frac{24 \times 9}{6}.$$

Hence to find the fourth term of a proportion, set the first term on the second, under the third read the fourth.

IV. *Continued multiplication and division.* Here for convenience we need the runner. This is a sliding frame carrying a piece of glass which has a line on it perpendicular to the length of the rule.

1. Find the value of $3 \times 8 \times 5$.

Set 1 *C* at the right on 3 *D*, set runner on 8 *C*, set 1 *C* at the right on the runner, under 5 *C* read 12 on *D*. Hence

$$3 \times 8 \times 5 = 120.$$

2. Find the value of $\frac{54}{3 \times 6}$.

Set 3 *C* on 54 *D*, set runner on 1 *C*, set 6 *C* on runner, under 1 *C* read result 3 on *D*. Note that we have simply made two divisions.

3. Find the value of $\frac{15 \times 48}{24 \times 6}$.

Set 24 *C* on 15 *D*, set runner on 48 *C*, set 6 *C* on runner, under 1 *C* read result 5 on *D*.

4. Find the value of $\frac{8 \times 9 \times 4}{32}$.

Set 32 *C* on 8 *D*, set runner on 1 *C*, set 1 *C* at right end of slide on runner, set runner on 9 *C*, set 1 *C* on runner, under 4 *C* read result 9 on *D*.

In a similar manner any number of continued multiplications and divisions may be performed.

V. *Squares and square root.* The graduations on scale *A* at the top of the slide are arranged so that the square of every number on scale *C* stands directly above it on scale *A*. Thus above 2 is 4, above 3 is 9, and above 25 is 625. On scale *A* the distances of the numbers from 1 at the left end of the scale are proportional to the logarithms of the numbers as on scale *C*; but it is easier to learn to use scale *A* by noticing its relation to scale *C*. We read from left to right 1, 2, 3, ⋯, 10, 20, 30, ⋯, 100; then beginning at 1 again and calling it 100, we read 100, 200, 300, ⋯, 1000, 2000, 3000, ⋯, 10,000. The first 4 is either 4 or 400, that is, either the square of 2 or 20; the second 4 is either 40 or 4000, that is, either the square of 6.32 or of 63.2.

To square any number, find the number on scale C and read its square directly above it on scale A.

To extract the square root of any number, find the number on scale A and read its square root directly below it on scale C.

The upper scale is very convenient when multiplying or dividing by square roots, finding the area of circles, and so on.

1. Find the value of $8\sqrt{3}$.

Set 1 *C* at right end of scale on 3 *A*, under 8 *C* read result 13.85 on *D*.

2. Find the value of $\frac{8}{\sqrt{3}}$.

$$\frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}.$$

Set 3 *C* on 3 *A*, under 8 *C* read result 4.61 on *D*.

3. Find the value of $\frac{\sqrt{8} \times \sqrt{12}}{\sqrt{5}}$

Set 5 *B* on 8 *A*, under 12 *B* read result 4.38 on *D*.

4. Find the area of a circle whose radius is 4 ft.

Set 1 *C* on 4 *D*, above π on *B* read the area, 50.3 sq. ft., on *A*.

PROBLEMS

1. Find the value of:

1. 78×5.
2. 38.4×25.
3. 8.63×4.24.
4. $.121 \times 6.38$.
5. $\frac{12.8}{15}$.
6. $\frac{944}{16.3}$.
7. $\frac{48.8}{2.93}$.
8. $\frac{84 \times 13}{15}$.
9. $\frac{16.8 \times 4.2}{31.4}$.
10. $\frac{16\sqrt{39}}{33}$.

2. Find the area of the rectangle whose dimensions are 3.26 in. by 4.21 in.

3. The area of a rectangle is 18.6 sq. cm. and its base is 5.34 cm. Find its altitude.

4. Find the area of a circle whose radius is (*a*) 5 in.; (*b*) 1.8 in.; (*c*) 2.56 cm.; (*d*) 3.22 ft.

5. Construct a curve to show the area of circles of radius from 1 in. to 10 in.

6. Find the surfaces and volumes of brass cylinders, prisms, blocks of wood, and so on.

7. To make 865 lb. of admiralty metal, used for parts of engines on naval vessels, 752.5 lb. of copper, 43.3 lb. of zinc, and 69.2 lb. of tin were melted together. Find the per cent of each metal used.

8. 17 lb. of copper, 85 lb. of tin, 595 lb. of lead, and 153 lb. of antimony were melted together to make 850 lb. of type metal. What per cent of each metal was used?

9. If sea water contains 2.71 per cent of salt, how many tons of sea water must be taken to give 100 lb. of salt?

10. The safe load in tons, uniformly distributed, for white-oak beams is given by the formula

$$W = \frac{2\,bd^2}{3\,l},$$

where $W =$ the safe load in tons, $b =$ the breadth in inches, $d =$ the depth in inches, and $l =$ the distance between the supports in inches.

Construct a curve to show the safe load in tons for white-oak beams having a breadth of 3 in., distance between supports 13 ft., and depth from 3 in. to 15 in.

11. If $w =$ the weight of 1 lb. of any substance when suspended in water, and s its specific gravity, then

$$s = \frac{1}{1 - w}, \quad \text{or} \quad w = \frac{s - 1}{s}.$$

Construct a curve showing the weight of substances suspended in water, the specific gravity varying from .5 to 15.

CHAPTER XII

ANGLE FUNCTIONS

78. Angles. Let two lines AP and AM be coincident. Suppose AP to revolve about the point A away from AM; the amount of turn, indicated by the arrow, is called an *angle.* The amount of turn is expressed in *degrees.* A complete turn gives an angle of 360°, a half turn 180°, and a quarter turn 90°. In this chapter we will not consider angles greater than 90°.

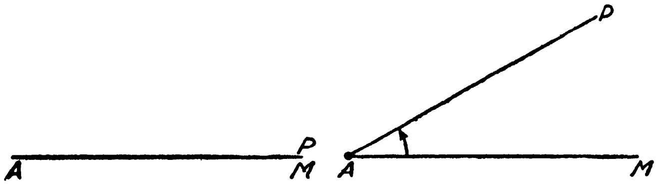

FIG. 55

The line AM which marks the beginning of the revolution is called the *initial line;* the line AP which marks the ending of the revolution is called the *terminal line* of the angle.

79. Triangle of reference. If from any point B in the terminal line of the angle a perpendicular BC is dropped to the initial line, the right triangle formed is called the *triangle of reference* for the angle. The perpendicular BC is called the *opposite side;* AC, the part of the initial line cut off by the perpendicular, is called the *adjacent side;* and AB, that part of the terminal line which belongs to the triangle of reference, is called the *hypotenuse.*

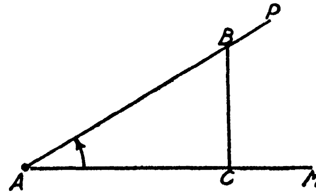

FIG. 56

80. Sine, cosine, and tangent of an angle. Given the angle A. Construct the triangle of reference, and represent the lengths of the sides by a, b, and c, set opposite the angles A, B, and C respectively.

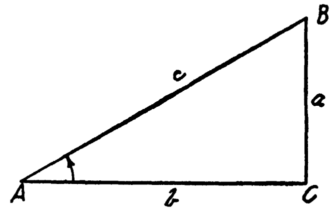

FIG. 57

$\frac{BC}{AB}=\frac{a}{c}$ $=\frac{\text{opposite side}}{\text{hypotenuse}}$ $=\sin A$ (by definition).

This ratio, called the sine of angle A, is a pure number which is usually approximate and expressed as a decimal.

$\frac{AC}{AB}=\frac{b}{c}$ $=\frac{\text{adjacent side}}{\text{hypotenuse}}$ $=\cos A$ (by definition).

This ratio, called the cosine of angle A, is a pure number which is usually approximate and expressed as a decimal.

$\frac{BC}{AC}=\frac{a}{b}$ $=\frac{\text{opposite side}}{\text{adjacent side}}$ $=\tan A$ (by definition).

This ratio, called the tangent of angle A, is a pure number which is usually approximate and expressed as a decimal.

These ratios $\sin A$, $\cos A$, and $\tan A$ are called functions of the angle A because they change in value as the angle changes. There are other functions of an angle, but as these three seem to be the more important the discussion will be limited to them.

EXERCISES

1. Make an angle A and construct the triangle of reference. Letter as before, and measure the sides a, b, and c as accurately as possible in millimeters. Use the results of the measurement to find the values of $\sin A$, $\cos A$, and $\tan A$. Carry the divisions as far as the errors in the approximation justify, and no farther.

2. Make another angle A' which differs from A. Calculate its sine, cosine, and tangent in the same manner. Compare the values of the two sines, the two cosines, and the two tangents.

If you were to continue the experiment, you would find that the ratios change in value every time the angle changes in size.

3. Make an angle and drop perpendiculars from various points on the terminal line to the initial line. Any one of the right triangles may be considered a triangle of reference for the angle. Find sin A from each triangle of reference. Compare the values. Should they all be equal? Why? Similarly for cos A and tan A.

4. In a triangle of reference ABC could $BC = 2$ in., $AB = 6$ in., and $AC = 5$ in.? Why? Could any two sides be chosen at random? Why? Could one side be chosen at random? Why?

81. Functions of 45°. Construct an angle of 45°, and in the triangle of reference make either AC or BC 1 unit long. Why is the other side 1 unit long? Why is the hypotenuse $\sqrt{2}$ units long?

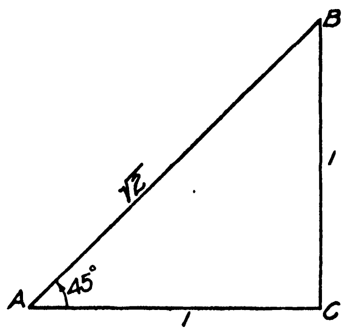

Fig. 58

$$\sin 45° = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}(1.414) = .707.$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}(1.414) = .707.$$

$$\tan 45° = \frac{1}{1} = 1.$$

This ratio is exact.

82. Functions of 30°. Construct an angle of 30°, and in the triangle of reference make the side BC opposite 30°, 1 unit long. Why is the hypotenuse AB 2 units long? Why is AC $\sqrt{3}$ units long?

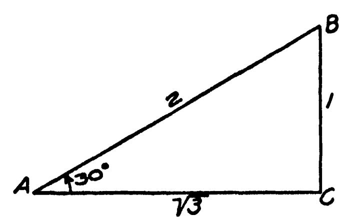

Fig. 59

$$\sin 30° = \tfrac{1}{2} = .500.$$

This ratio is exact.

$$\cos 30° = \frac{\sqrt{3}}{2} = \tfrac{1}{2}(1.732) = .866.$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3} = \tfrac{1}{3}(1.732) = .577.$$

83. Functions of 60°. Construct an angle of 60°, and in the triangle of reference make the side AC adjacent to 60°, 1 unit long. Why is the hypotenuse AB 2 units long? Why is BC $\sqrt{3}$ units long?

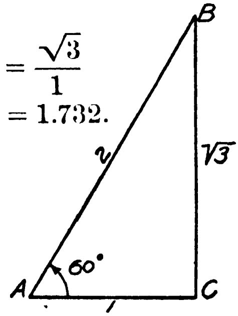

Fig. 60

$$\sin 60° = \frac{\sqrt{3}}{2} = \tfrac{1}{2}(1.732) = .866.$$

$$\cos 60° = \tfrac{1}{2} = .500.$$

This ratio is exact.

$$\tan 60° = \frac{\sqrt{3}}{1} = 1.732.$$

Show how the functions of 60° can be found from the triangle of reference for 30°.

84. Table of angle functions. The functions of angles have been calculated and tabulated. In solving problems the functions of the angle are taken from the table. The functions of a few angles, 30°, 45°, 60°, 90°, should be memorized.

PROBLEMS

1. A man standing 110 ft. from a tree on level ground finds the angle of elevation of the top of the tree to be 37° 20′. How high is the tree, and how far is the man from the top of it?

SOLUTION. Given $A = 37° 20'$.
$b = 110$ ft.

$$\frac{a}{b} = \tan A.$$
$$a = b \tan A = 110(.7627) = 83.9 \text{ ft.}$$

$$\frac{b}{c} = \cos A.$$
$$b = c \cos A.$$
$$c = \frac{b}{\cos A} = \frac{110}{.7951} = 138 \text{ ft.}$$

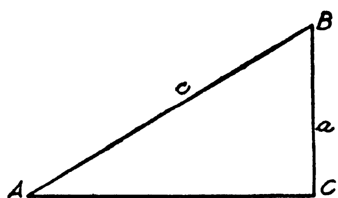

Fig. 61

Check this and all problems by constructing the triangle from the given parts. Make a good-sized drawing to scale and measure the computed parts.

2. A railroad track has a uniform slope of 5° to the horizontal. How many feet does a train rise in going a mile?

3. A ladder 24 ft. long rests against a wall. The foot of the ladder is 4 ft. 4 in. from the wall. Find the height of the top of the ladder.

4. The shadow of a tree is 38 ft. long when the angle of elevation of the sun is 42°. Find the height of the tree.

5. A ship is sailing northeast 12 mi. per hour. How fast is she sailing east?

6. A stick 8 ft. long stands vertically in a horizontal plane, and the length of the shadow is 6 ft. What is the angle of elevation of the sun?

7. What is the slope of a mountain path if it rises 118 ft. in a distance of 835 ft. along the path?

8. The top of a lighthouse is 152 ft. above sea level. If the angle of depression of a buoy is 12° 15′, how far from the lighthouse is it?

9. The chord of a circle is 4.4 in. and it subtends at the center an angle of 38°. Find the radius of the circle.

10. At a point 212 ft. from the foot of a column the angle of elevation of the top of the column is found to be 24° 28′. What is the height of the column?

11. A man 6 ft. tall stands 4 ft. 6 in. from a lamp-post. If his shadow is 17 ft. long, what is the height of the lamp-post?

12. A cable is attached to a smokestack 10 ft. below the top, and to a pile 42 ft. from the foot of the stack. If the cable makes an angle of 62° 20′ with the horizontal, find the height of the stack.

13. From the top of a lighthouse 160 ft. above sea level two vessels appear in line. If their angles of depression are 4° 20′ and 2° 45′ respectively, how many miles are they apart?

14. As the angle of elevation of the sun increases from $35° 15'$ to $64° 25'$, how many feet does the shadow of a church steeple 120 ft. high decrease?

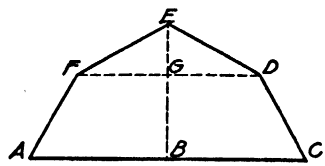

FIG. 62

15. In the gable shown in the figure angle $BAF = 60°$, angle $GFE = 30°$, $BG = 6$ ft., and $GE = 4$ ft. Find AF, FE, and AC.

16. The base AC of an isosceles trapezoid is 100 ft., and the equal sides AD and CB make angles of 60° with the base. The altitude is 40 ft. Compute the length of the upper base and the area. Draw to scale and check.

17. The pitch of a roof (angle which the rafters make with the horizontal) is 32°. If the house is 22 ft. wide, find the length of the rafters and the height of the gable.

18. A building 80 ft. long and 40 ft. wide has each side of its roof inclined 40° to the horizontal. Find the area of the roof.

19. Two towns A and B are at opposite ends of a lake. It is known that a station P is 3 mi. from A and 2 mi. from B. If the angle $PAB = 34° 30'$ and angle $PBA = 62° 40'$, find the distance between the towns.

20. Make a height or distance problem of your own and solve it.

85. Logarithmic solutions. In the preceding problems the numbers involved consist of only two or three figures; hence there would be little or no time saved in using logarithms. However, when there are several figures in the numbers, and there are three or more multiplications or divisions, logarithms should be used.

The logarithms of the angle functions are found in exactly the same way as are the logarithms of numbers. Thus, find log sin $18° 26'$.

Mantissa log sin 18° 30′ − mantissa

$$\log \sin 18^\circ 20' = 5015 - 4977$$
$$= 38.$$
$$38 \times .6 = 23.$$
$$4977 + 23 = 5000.$$
$$\therefore \log \sin 18^\circ 26' = 9.5000 - 10.$$

The sine and tangent of an angle increase as the angle increases, hence the difference for the minutes is added to the mantissa of the smaller angle taken from the table.

It is to be noted that the cosine of an angle decreases as the angle increases; hence the difference for the minutes is to be subtracted instead of added.

Thus find log cos 24° 48′.

Mantissa log cos 24° 40′ − mantissa

$$\log \cos 24^\circ 50' = 9584 - 9579$$
$$= 5.$$
$$5 \times .8 = 4.$$
$$9584 - 4 = 9580.$$
$$\therefore \log \cos 24^\circ 48' = 9.9580 - 10.$$

Given log tan x = 9.5946 − 10, find x.

21° 20′	5917
21° 2	5946
21° 30′	5954
	37)290(8

$$\therefore x = 21^\circ 28'.$$

This work should be done mentally. In class find logarithms of the functions of many angles, and the angles from the logarithms of the functions, as quickly as possible until this can be done readily.

The sine and cosine of an angle are always less than 1. Why? Hence the characteristic of the logarithm is 9 − 10, 8 − 10, and so on. The − 10 is not printed in the table, but should be written in computation.

PROBLEMS

1. In the right triangle ABC, given

$$C = 90°,$$
$$A = 28° 34',$$
$$c = 48.32 \text{ ft.}$$

Find a and b.

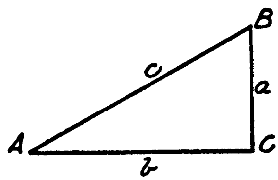

FIG. 63

SOLUTION.

$$\frac{a}{c} = \sin A. \qquad \frac{b}{c} = \cos A.$$

$$a = c \sin A. \qquad b = c \cos A.$$

$$\begin{aligned} \log c &= \\ \log \sin A &= \underline{\qquad\qquad} \\ \log a &= \\ a &= \end{aligned} \qquad \begin{aligned} \log c &= \\ \log \cos A &= \underline{\qquad\qquad} \\ \log b &= \\ b &= \end{aligned}$$

Before looking up any logarithms always make out an outline as above.

$$\frac{a}{c} = \sin A. \qquad \frac{b}{c} = \cos A.$$

$$a = c \sin A. \qquad b = c \cos A.$$

$$\begin{aligned} \log c &= 1.6841 \\ \log \sin A &= \underline{9.6796 - 10} \\ \log a &= 1.3637 \\ a &= 23.11 \text{ ft.} \end{aligned} \qquad \begin{aligned} \log c &= 1.6841 \\ \log \cos A &= \underline{9.9436 - 10} \\ \log b &= 1.6277 \\ b &= 42.43 \text{ ft.} \end{aligned}$$

Check. It may be as much work to check a problem as to solve it, but an answer is absolutely worthless unless it is known to be correct. What is the advantage of knowing how to work problems if you cannot get correct results?

$$\begin{aligned} a^2 + b^2 &= c^2 \text{ (Pythag. th.).} \\ a^2 &= c^2 - b^2 \\ &= (c - b)(c + b). \end{aligned}$$

$$\begin{aligned} c - b = 5.89 \quad \log &= 0.7701 \\ c + b = 90.75 \quad \log &= \underline{1.9579} \\ \log a^2 &= 2.7280 \\ \log a &= 1.3640 \\ a &= 23.12. \end{aligned}$$

A difference of 1 in the last figure may be expected since the logarithms are only approximate.

2. Two trees M and N are on opposite sides of a river. A line NP at right angles to MN is 432.7 ft. long and the angle NPM is $52° 27'$. What is the distance from M to N?

3. From the top of a building 156.4 ft. high the angle of depression of a street corner is $18° 46'$. Find the horizontal distance from the street corner to the building.

4. To find the height of the Auditorium tower a distance of 311.2 ft. was measured from the foot of the tower and the angle of elevation of the tower was found to be $40° 57'$. Find the height of the tower.

Solve the following right triangles, two parts being given:

5. $a = 146.8$, $b = 203.3$.
6. $b = 49.74$, $A = 53° 38'$.
7. $c = 94.53$, $B = 62° 51'$.
8. $c = 436.5$, $A = 74° 11'$.
9. $c = 110.9$, $a = 64.21$.
10. $b = 8.226$, $c = 12.15$.
11. $c = .02936$, $a = .01153$.
12. $a = .9681$, $A = 42° 17'$.

13. Find the side of an equilateral triangle inscribed in a circle of radius 52.18 in.

14. The side of an equilateral triangle inscribed in a circle is 14.26 in. Find the radius of the circle.

15. If a side of a regular pentagon is 30.24 in., find the radius of the circumscribed circle, and the apothem.

16. A regular pentagon is inscribed in a circle of radius 11.32 in. Find a side and the apothem of the pentagon.

17. The apothem of a regular polygon of 12 sides is 21.26 ft. What is the perimeter?

18. The perimeter of a regular octagonal tower is 168.4 ft. What is the area of the base of the tower?

19. A regular octagonal column is cut from a circular cylinder whose diameter is 18.32 in. Find the area of a cross section of the column.

20. A side of a regular hexagon inscribed in a circle is 28.43 ft. Find a side of a regular decagon inscribed in the same circle.

86. Area of triangles. In triangle ABC, h is the perpendicular from C to c.

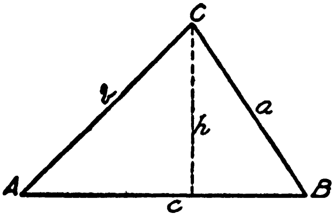

FIG. 64

$$\frac{h}{b} = \sin A.$$
$$h = b \sin A.$$

Area triangle

$$ABC = \tfrac{1}{2} \text{ base} \times \text{altitude}$$
$$= \tfrac{1}{2} ch$$
$$= \tfrac{1}{2} cb \sin A.$$

The area of a triangle equals one half the product of two sides and the sine of the included angle.

PROBLEMS

1. Find the area of a triangle ABC, given $a = 42.84$ ft., $c = 76.31$ ft., and $B = 29° 18'$.

SOLUTION.

$$\begin{aligned} 2 \text{ area} &= ac \sin B. \\ \log a &= 1.6318 \\ \log c &= 1.8826 \\ \log \sin B &= \underline{9.6896 - 10} \\ \log 2 \text{ area} &= 3.2040 \\ 2 \text{ area} &= 1600. \\ \text{area} &= 800 \text{ sq. ft.} \end{aligned}$$

Find the area of the following triangles. Check by finding the area twice, using different angles:

2. $a = 34.36$, $b = 110.5$, $c = 98.32$, $A = 17° 43'$, $C = 60° 36'$.

3. $a = 88.48$, $b = 58.59$, $c = 54.38$, $B = 40° 10'$, $C = 36° 47'$.

4. $a = 1.432$, $b = 1.583$, $c = 1.610$, $A = 53° 17'$, $B = 62° 24'$.

5. $a = 3.207$, $b = 2.367$, $c = 1.435$, $B = 42° 55'$, $C = 24° 22'$.

6. Find the area of a triangle XYZ, given $x = 184.2$ ft., $y = 381.3$ ft., and $Z = 51° 24'$.

7. The vertical angle of an isosceles triangle is $75° 18'$ and the equal sides are 16.46 ft. long. Find the area of the triangle.

8. What is the area of a parallelogram if two adjacent sides are 243.6 yd. and 315.4 yd. and the included angle is $35° 40'$?

9. Two streets make an angle of 53° 18′ with each other. The corner lot between them has a frontage of 286 ft. on one street and 324 ft. on the other. Draw to scale and find the area of the lot.

10. Two railroads cross at an angle of 21° 25′. From a point on one of them 100 rd. from the crossing how must a fence be run so that the inclosure shall contain 10 A.?

11. The survey of a field gave the following data:

$$EA = 420 \text{ ft.}$$
$$EB = 865 \text{ ft.}$$
$$EC = 875 \text{ ft.}$$
$$ED = 650 \text{ ft.}$$

$$\angle AEB = 42°.$$
$$\angle BEC = 36°.$$
$$\angle CED = 20°.$$

Draw the field to scale and find its area.

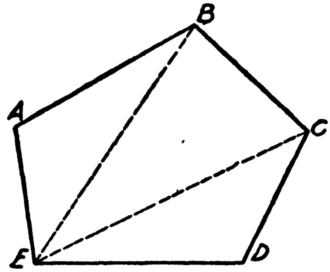

FIG. 65

12. A surveyor set his transit over the corner A of a field $ABCD$ and found the angle $DAC = 40° 12'$, and angle $CAB = 70° 54'$. AD is 52.8 rd., AC is 86.3 rd., and AB is 38.4 rd. Draw to scale and compute the area of the field.

87. Law of sines. In the triangle ABC let h be the perpendicular from the vertex C to the side c.

$$\frac{h}{b} = \sin A, \text{ and } \frac{h}{a} = \sin B.$$

By division,

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \qquad (1)$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B}. \qquad (2)$$

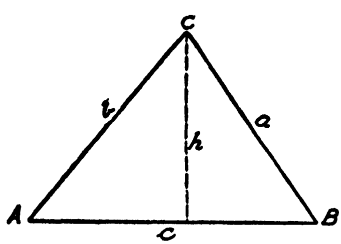

FIG. 66

What algebraic operations were used to derive (2) from (1)? What theorem in geometry could be used for this purpose?

By dropping a perpendicular from A to a we may obtain, in a similar manner,

$$\frac{c}{\sin C} = \frac{b}{\sin B}.$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Law of Sines. *In any triangle the sides are proportional to the sines of the opposite angles.*

When a side and two angles of a triangle are given we may find the other two sides by this law.

PROBLEMS

1. In a triangle ABC given $A = 36° 56'$, $B = 72° 6'$, and $a = 36.74$. Find b and c.

Solution. $C = 180° - (A + B) = 70° 58'$.

$$\frac{b}{\sin B} = \frac{a}{\sin A}. \qquad \frac{c}{\sin C} = \frac{a}{\sin A}.$$

$$b = \frac{a \sin B}{\sin A}. \qquad c = \frac{a \sin C}{\sin A}.$$

$$\begin{array}{rl} \log a = & 1.5652 \\ \log \sin B = & \underline{9.9784 - 10} \\ & 11.5436 - 10 \\ \log \sin A = & \underline{9.7788 - 10} \\ \log b = & 1.7648 \\ b = & 58.19. \end{array} \qquad \begin{array}{rl} \log a = & 1.5652 \\ \log \sin C = & \underline{9.9756 - 10} \\ & 11.5408 - 10 \\ \log \sin A = & \underline{9.7788 - 10} \\ \log c = & 1.7620 \\ c = & 57.81. \end{array}$$

Solve the following triangles and check by drawing to scale:

2.	$A = 44° 59'$,	$B = 62° 52'$,	$a = 7.942$.
3.	$A = 50° 24'$;	$C = 68° 35'$,	$b = 12.63$.
4.	$B = 72° 46'$,	$C = 41° 44'$,	$c = 203.6$.
5.	$A = 61° 18'$,	$B = 58° 32'$,	$b = 84.03$.

6. To find the distance from a point A to a point P across a river, a base line AB 1000 ft. long was measured off from A. The angles BAP and ABP were found to be $36° 18'$ and $62° 35'$ respectively. Compute the distance AP.

7. On board two ships half a mile apart it is found that the angles subtended by the other ship and a fort are $84° 16'$ and $78° 38'$ respectively. Find the distance of each ship from the fort.

8. M and N are stations on two hilltops 3684 ft. apart, and P is a station on a third hill. The angles NMP and MNP are observed to be $50° 42'$ and $63° 24'$ respectively. Find the distances MP and NP.

88. Law of cosines. In triangle ABC, h is the perpendicular from C to c.

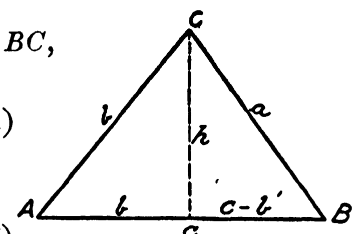

FIG. 67

In triangle on left, $b^2 = h^2 + b'^2$, (1)

and $$\frac{b'}{b} = \cos A,$$

or $$b' = b \cos A. \quad (2)$$

In triangle on right,

$$a^2 = h^2 + (c - b')^2$$
$$= h^2 + c^2 - 2b'c + b'^2.$$

Substituting (2), $$= h^2 + b'^2 + c^2 - 2bc\cos A.$$

Substituting (1), $$a^2 = b^2 + c^2 - 2bc\cos A.$$

Similarly, by dropping perpendiculars from A and B we get

$$b^2 = a^2 + c^2 - 2ac\cos B.$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

LAW OF COSINES. *In any triangle the square of any side is equal to the sum of the squares of the other two sides less twice the product of these two sides and the cosine of the included angle.*

PROBLEMS

1. Find a in the triangle ABC, given $b = 6$ in., $c = 5$ in., and $A = 29° 15'$.

SOLUTION. $$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 36 + 25 - 2 \times 6 \times 5 \times .8725$$
$$= 8.65.$$
$$a = 2.9 \text{ in.}$$

2. In triangle ABC, find A if $a = 7$, $b = 8$, $c = 9$.

SOLUTION.
$$a^2 = b^2 + c^2 - 2\,bc\cos A.$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}$$
$$= \frac{64 + 81 - 49}{2 \times 8 \times 9}$$
$$= .6667.$$
$$A = 48^\circ\, 11'.$$

3. Find B and C in the triangle in Problem 2 and check by adding the three angles.

Solve the following triangles and check by drawing to scale or otherwise:

4. $a = 10$, $b = 12$, $c = 14$.

5. $a = 4$, $b = 5$, $c = 6$.

6. $b = 21$, $c = 19$, $A = 48^\circ 57'$.

7. $a = 14$, $b = 12$, $c = 60^\circ$.

8. Two ships leave a dock at the same time. One sails east 12 mi. per hour and the other northeast 14 mi. per hour. How far will they be apart at the end of 5 hr.?

9. From a point 5 mi. from one end of a lake and 4 mi. from the other end, the lake subtends an angle of $56^\circ\, 8'$. What is the length of the lake?

10. A and B are two stations on opposite sides of a mountain, and C is a station on top of the mountain from which A and B are visible. If $CA = 4.2$ mi. and $CB = 3.1$ mi., and angle $ACB = 88^\circ\, 12'$, find the distance from A to B, the three stations being in the same vertical plane.

89. Triangle of forces. The weight W at the end of the boom is held in position by three forces: (a) the force of gravity acting downward; (b) the tension (pull) in the tie; (c) the thrust (push) of the boom. The tension in each side of the triangle is proportional to the lengths of the sides. The tension in the mast is always taken equal to the load W; and the tension per foot is the same in each side of the triangle. Thus in Fig. 68, if $W = 2000$ lb., $AB = 10$ ft., and $BC = 16$ ft., the tension in the mast $AB = 2000$ lb. and the tension per foot $= 200$ lb

Therefore the compression in the boom $= 16 \times 200 = 3200$ lb. The tie $AC = \sqrt{10^2 + 16^2} = \sqrt{356} = 18.9$ ft., and the tension in $AC = 18.9 \times 200 = 3780$ lb.

Check.

$$2000^2 = 4{,}000{,}000$$
$$3200^2 = \underline{10{,}240{,}000}$$
$$14{,}240{,}000$$
$$3780^2 = 14{,}290{,}000$$

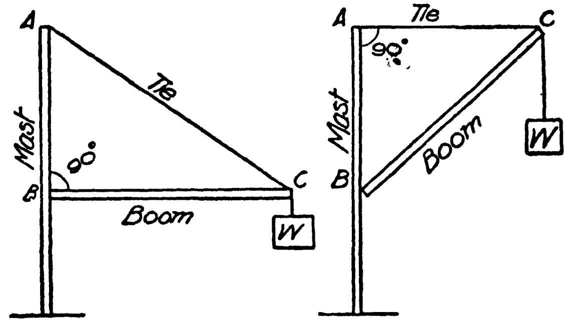

FIG. 68. A SIMPLE CRANE

Exercise. Put two screw eyes in the wall 80 cm. apart and construct a model of a crane, using a meter stick, string, and a spring balance, as shown in Fig. 69. Compute the tension for

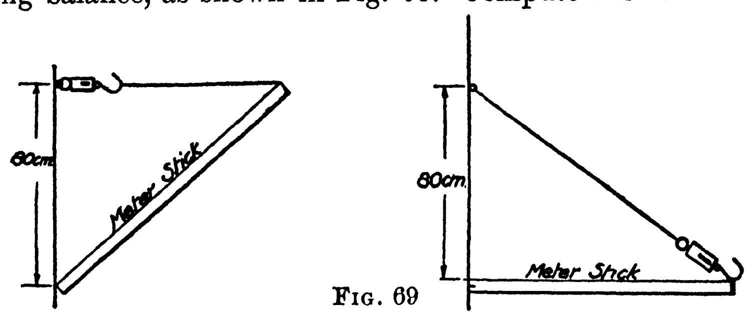

FIG. 69

different weights and check by the readings of the spring balance. After a weight has been attached the string should be shortened enough to make the string or the meter stick perpendicular to the wall in order to form a right triangle.

PROBLEMS

1. The mast of a crane is 12 ft. long and the tie 18 ft. The boom is horizontal and supports a load of 2400 lb. Find the tensions in the boom and tie.

2. The tie of a crane is horizontal. If it is 24 ft. long and the boom is 30 ft. long, find the tension in the mast, boom, and tie for a load of 4 T.

3. The tie of a crane makes an angle of 30° with the mast, and the boom is horizontal. If the boom is 20 ft. long and the load is 3000 lb., find the tension in the mast, tie, and boom.

4. The boom of a crane is 16 ft. long and makes an angle of 40° with the mast. The tie is horizontal. Find the tension in the mast, boom, and tie for a load of 2 T.

5. The boom of a crane is 20 ft. long, and when it is horizontal the tie is 30 ft. long. If the tie can stand a strain of 4200 lb., find the greatest load that can be lifted when the boom is horizontal.

6. The bracket BCD carries a load of 400 lb. at D. Find the stresses in BC, CD, and BD.

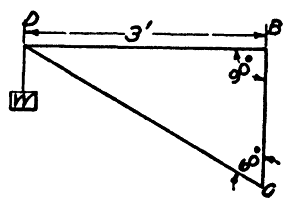

FIG. 70

7. An arc lamp weighing 20 lb. is hung on a pole, as shown in Fig. 71. Find the stresses in MP and NP.

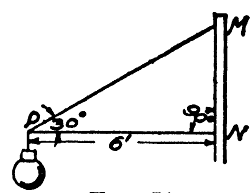

FIG. 71

8. A weight of 96 lb. is attached to a cord which is secured to the wall at a point A and is pushed out from the wall by a horizontal stick BC. If $AC = 6$ ft. and angle $BAC = 38°$, find the tension in AB and the pressure on BC.

9. A canal boat is kept 20 ft. from the towpath and the towline is 72 ft. long. If there is a pull of 144 lb. on the line, what is the effective pull?

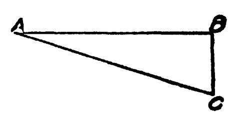

FIG. 72

SOLUTION. Let C, Fig. 72, be the position of the canal boat.

$$AB = \sqrt{72^2 - 20^2} = 69.2 \text{ ft.}$$

$\frac{144}{72} = 2$ lb., the tension per foot in AC.

$\therefore$ $69.2 \times 2 = 138.4$ lb., the effective pull.

10. The pull on the towline of a canal boat is 400 lb. and the line makes an angle of 10° with the direction of the boat. How much of the pull is effective? How much is at right angles to the direction of the boat?

11. A boat is pulled up the middle of a stream 60 ft. wide by two men on opposite sides, each pulling with a force of 100 lb. If each rope, attached to the bow of the boat, is 40 ft. long, find the effective pull on the boat.

12. Each of two horses attached to a load is pulling with a force of 200 lb. If they are pulling at an angle of 60° with each other, what is the effective pull on the load?

13. Attach two spring balances to the wall, as shown in Fig. 73, with 10 or 12 ft. of cord between them. At the center of the cord attach an 8-lb. weight. Read each balance for the tension in AC and BC.

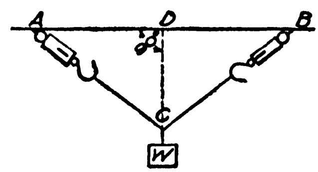

FIG. 73

Suppose $AC = 6$ ft. and $DC = 4$ ft. Compute the stress in AC.

SOLUTION. $\frac{1}{2}$ of $8 = 4$ lb., stress in DC.

$\frac{4}{4} = 1$ lb. per foot, stress in DC.

$1 \times 6 = 6$ lb., stress in AC.

Compare with result of the experiment. Make other experiments with different lengths of cord until the reason for the method of computation is understood.

14. A man weighing 180 lb. sits in the center of a hammock 12 ft. long. If the supports are 10 ft. apart, find the pull on the hammock.

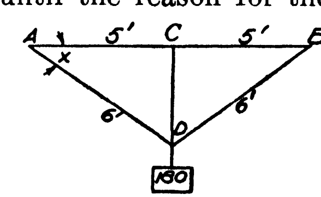

FIG. 74

SOLUTION. $CD = \sqrt{6^2 - 5^2} = 3.32$ ft.

$\frac{1}{2}$ of $180 = 90$ lb., pull in CD.

$\frac{90}{3.32} =$ pull per foot.

$\frac{90 \times 6}{3.32} = 163$ lb., pull in AD.

163 lb. = pull on hammock.

Check.

$$\cos x = \tfrac{5}{6}$$
$$= .8333.$$
$$x = 33°\ 34'.$$
$$\frac{\text{Pull in } CD}{\text{Pull in } AD} = \sin x.$$
$$\text{Pull in } CD = \sin x \times \text{pull in } AD$$
$$= .553 \times 163$$
$$= 90.2 \text{ lb.}$$
$$90.2 \times 2 = 180.4 \text{ lb., weight of the man.}$$

15. Two horses attached to a load are pulling with the same force at an angle of 60° with each other. If the combined effective pull on the load is 400 lb., how many pounds is each horse pulling?

16. Connect two light strips of wood 60 cm. long, AB and BC (Fig. 75), by a hinge at B, and put casters at A and C. Put a cord and spring balance between A and C, as shown in the figure. Hold the frame vertical, measure BD and AC, and read the balance, when $BD = 48$ cm. and $AD = 36$ cm. Attach an 8-lb. weight at B and make $AC = 72$ cm. Read the balance, and subtract the first reading to get the tension in AC due to the 8-lb. weight. Compute the tension in AC as follows:

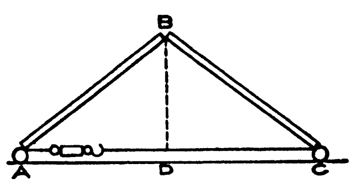

FIG. 75

$$\tfrac{4}{48} = \tfrac{1}{12} \text{ lb., tension per centimeter in } BD.$$
$$\tfrac{1}{12} \times 36 = 3 \text{ lb., tension in } AD.$$
$$\therefore \text{ tension in } AC = 3 \text{ lb.}$$

Compare with the result of the experiment. Make other experiments with different weights and distances AC, until the reason for the method of computation is understood.

17. A pair of rafters supports a weight equivalent to 800 lb. at the ridge. The pitch of the roof is 30° and the width of the building is 30 ft. Find the tension in the tie through the foot of each rafter.

18. The width of a house is 24 ft. and the rafters are 16 ft. long. If the rafters support a weight equal to 600 lb. at the ridge, find the stress in the rafters.

19. A bridge truss ABC supports a weight of 300 lb. per foot horizontally. The span is 30 ft. long. If $CD = 10$ ft., find the stresses in AC and AB. (The load at D equals one half the total load.)

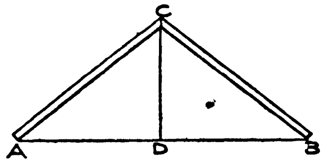

FIG. 76

20. ABC (Fig. 77) is an inverted king-post truss. $AB = 20$ ft., and the angles CAB and $ABC = 40°$. If the load at D is 4 T., find the stresses in AC and AB.

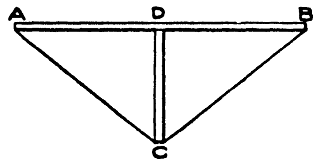

FIG. 77

CHAPTER XIII

GEOMETRICAL EXERCISES FOR ADVANCED ALGEBRA

90. A figure should be drawn for each exercise, letters or numbers put on the lines in the figure, and the equations set up from the figure. Check by drawing to scale and measuring the required parts. The first exercises involve square roots, since radicals are reviewed early. Some of the exercises should be worked out in notebooks, with emphasis placed on accuracy in drawing and neatness in arrangement.

1. Construct a graph for the squares of numbers from 0 to 13. Units: horizontal, 1 large square $= 1$; vertical, 1 large square $= 10$. What is the equation of the curve? Find $\sqrt{2}$, $\sqrt{6}$, $\sqrt{7.5}$, $\sqrt{8.25}$, $\sqrt{10}$, $\sqrt{12}$, and $\sqrt{12.5}$ to three decimal places and check by the graph.

2. Find the diagonal of a square whose side is 12 (s).

FIG. 78

SOLUTION.

$$\begin{aligned} d^2 &= 12^2 + 12^2 \text{ (Pythag. th.)} \\ &= 288 \\ d &= \sqrt{288} \\ &= \sqrt{144 \times 2} \\ &= 12\sqrt{2} \\ &= 16.97. \end{aligned}$$

$$\begin{aligned} d^2 &= s^2 + s^2 \text{ (Pythag. th.)} \\ &= 2s^2 \\ d &= s\sqrt{2}. \end{aligned}$$

3. Find the side of a square whose diagonal is 5 (d).

4. The side of an equilateral triangle is 4 (s). Find the altitude and area.

5. The altitude of an equilateral triangle is 6 (h). Find the side and area.

6. The area of an equilateral triangle is 24 (a). Find the side and altitude.

7. Find the area of a regular hexagon whose side is 3 (s).

8. Find the area of a regular hexagon whose apothem is 2 (h).

9. Find the side and apothem of a regular hexagon whose area is 36 (a).

10. A star-shaped figure is formed by constructing equilateral triangles outwardly on the four sides of a square. If the area of the entire figure is 100, find a side of the square.

11. Squares are constructed outwardly on the sides of a regular hexagon. If the area of the entire figure is 72, find a side of the hexagon.

12. From a square whose side is 12 (s) a regular octagon is formed by cutting off the corners. Find a side of the octagon.

13. The edge of a cube is 5 (e). Find a diagonal.

14. The diagonal of a cube is 8 (d). Find an edge.

15. Find the diagonal of a rectangular parallelepiped whose edges are 4, 5, and 6 (a, b, and c).

16. Find the side of an equilateral triangle whose area equals the area of a square whose diagonal is $6\sqrt{50}$.

17. Two sides of a triangle are a and b. Show that the area is $\frac{1}{4}ab$ when the included angle is 30° or 150°.

18. If two sides of a triangle are a and b, show that the area is $\frac{1}{4}\sqrt{3}\,ab$ when the included angle is 60° or 120°.

19. If two sides of a triangle are a and b, show that the area is $\frac{1}{4}\sqrt{2}\,ab$ when the included angle is 45° or 135°.

20. The sides of a triangle are 30, 60, and 80 (a, b, and c). Find the segments of each side formed by the bisector of the opposite angle.

21. The shadow cast upon level ground by a certain church steeple is 27 (37) yd. long, and at the same time the shadow of a vertical rod 5 (7) ft. high is 3 (6) ft. long. Find the height of the steeple.

22. The footpaths on the opposite sides of a street are 30 ft. apart. On one of them a bicycle rider is moving at the rate of 15 mi. per hour. If a man on the other side, walking in the opposite direction, regulates his pace so that a tree 5 ft. from his path continually hides him from the rider, at what rate does he walk?

23. One side of a triangle is divided into two equal parts and through the point of division a line is drawn parallel to the base. Into what fractional parts is the triangle divided? Similarly, when the side is divided into 3, 4, 5, $\cdots$, n equal parts?

24. Find the side of an equilateral triangle if the center of gravity is 2 (x) in. from the vertex.

25. What part of a triangle lies between the base and a line through the center of gravity parallel to the base?

26. One side of a triangle is 10 (s) in. Where must a point be taken in the given side in order that a line drawn through it, parallel to another side, will divide the triangle into two areas whose ratio is $3 : 4$ ($m : n$)?

27. The bases of a trapezoid are 16 and 10 (b_1 and b_2) and the altitude is 6 (h). Find the area of the triangle formed by producing the nonparallel sides of the trapezoid.

28. Find the side of the square inscribed in the triangle whose base is 12 (b) and altitude is 6 (h).

29. A rectangle whose length is twice its breadth is inscribed in an equilateral triangle. Find the area of the rectangle if a side of the triangle is 2.

30. Find the area of a trapezoid, given the bases 36 and 56 (b_1 and b_2) and the altitude 12 (h).

31. The bases of a trapezoid are 73 and 57 (b_1 and b_2) and each of the nonparallel sides is 17 (c). Find the area.

32. One diagonal of a trapezoid is 10 (d). The segments of the other diagonal are 6 and 9 (m and n). Find the segments of the first diagonal.

33. A trapezoid contains 480 (65) sq. ft. and its altitude is 20 (10) ft. Find the bases of the trapezoid if one of them is 4 (6) ft. longer than the other.

34. Find the area of a rectangle if its diagonal is 50 (d) ft. and the sides are in the ratio 3 : 5 ($m : n$).

35. The dimensions of a rectangle are 64 and 58 (b and h) respectively. If the length is diminished by 10 (m), how much must the breadth be increased in order to retain the same area?

36. A rectangle is 8 (h) in breadth and its diagonal is 20 (d). Upon the diagonal as a base a triangle is constructed whose area is equal to that of the rectangle. Find the altitude of the triangle.

37. The ratio of the diagonals of a rhombus is 7 : 5 ($m : n$) and their sum is 16 (k). Find the area of the rhombus.

38. The sides of a right triangle are x, $x + 7$, and $x + 8$. Find them.

39. Two telegraph poles 25 and 30 ft. high are 80 ft. apart on level ground. Find the length of the wire.

40. The chord of a circle is 8(c) and the height of the segment is 2 (h). Find the radius.

41. In a circle whose radius is 12 (r) in. a chord 4 (c) in. is drawn. Find the height of the segment.

42. Two chords 48 and 14 mm. long are on opposite sides of the center of a circle. If they are 31 mm. apart, what is the diameter of the circle?

43. Two parallel chords on the same side of the center of a circle are 48 and 14 (14 and 4) in. long. If the diameter of the circle is 50 (16) in., find the distance between the chords.

44. Find the common chord of two equal circles of radius 8 (r) in. if each circle has its center on the circumference of the other.

45. Two chords AB and CD intersect at E within a circle. If $AE = 10\,(4)$, $BE = 12\,(9)$, and $CD = 23\,(12)$, find CE and ED.

46. From a point without a circle a secant and a tangent are drawn. If the external segment of the secant is 6 (m) and the internal segment is 18 (n), find the tangent.

47. If from a point without a circle two secants are drawn whose external segments are 3 and 4 (3.2 and 5.5), and the internal segment of the latter is 17 (4.7), what is the internal segment of the former?

48. The distance between the centers of two circles whose radii are 5 and 8 (r_1 and r_2) is 26 (d). How far from the center of each circle does their common tangent intersect the line of centers? (Two solutions.)

49. The radii of two circles are 7 in. and 4 in. The distance between their centers is 12 in. Find the length of the common internal and external tangents.

50. Find the radius of a circle if the numerical measure of the area equals the measure (a) of the radius; (b) of the circumference.

51. Find the side of the largest square piece of timber that can be cut from a log 14 ft. in circumference.

52. A rectangle and a circle have equal perimeters. Find the difference of their areas if the radius of the circle is 9 (12) in. and the width of the rectangle is $\frac{3}{4}$ ($\frac{1}{2}$) its length.

53. A circle whose radius is 8 (r) has one half of its area removed by cutting a ring from the outside. What is the width of the ring?

54. Show that the ratio of the square inscribed in a semicircle to the square inscribed in the entire circle is 2 : 5.

55. Show that the ratio of the square inscribed in a semicircle to the square inscribed in a quadrant of the same circle is 8 : 5.

56. What is the ratio of the square inscribed in a quadrant of a circle to the square inscribed in the entire circle?

57. How much must be added to the circumference of a wheel whose radius is 2 (*r*) to make the radius 1 (*m*) longer?

58. If an electric cable were laid around the earth at the equator, how many feet would have to be added if the cable were raised 10 ft. above the surface of the earth?

59. A quarter-mile running track is to be laid out with straight parallel sides and semicircular ends. The track is to be 10 ft. wide, and the distance between the outer parallel edges is to be 220 ft. What must be the extreme length of the field so that a runner may cover the exact quarter of a mile by keeping in the center of the track?

60. In any triangle whose sides are a, b, and c derive a formula for the square of the side opposite an acute angle.

61. Derive a corresponding formula for the square of the side opposite an obtuse angle.

62. In a triangle whose sides are 7, 8, and 9 (4, 5, and 6) find the projections of the sides 7 and 8 (4 and 5) on 9 (6).

63. In a triangle whose sides are 10, 12, and 18 (40, 80, and 100) find the projections of the sides 18 and 10 (100 and 40) on 12 (80).

64. The sides of a triangle are 4, 5, and 7 (70, 90, and 100). Find the altitude to base 7 (100).

65. Find the area of a triangle whose sides are 8, 12, and 15 (20, 25, and 30).

66. Find the length of the common chord of two circles whose radii are 5 and 8 (10 and 17) and the distance between whose centers is 10 (21).

67. The base and altitude of a triangle are 8 and 6 (b and h) in. respectively. If the base be increased 4 (c) in., how much must the altitude be diminished in order that the area remain the same?

68. Through the vertex of a triangle whose area is 120 (100) sq. in. a line is drawn dividing it into two parts, one containing 24 (12) sq. in. more than the other. What are the segments into which the base is divided if the whole base is 20 (14) in.?

69. The base of a triangle is 6 in. and the altitude is 5 in. Find the change in area if the dimensions are (a) increased by 3 in. and 2 in. respectively; (b) diminished by 3 in. and 2 in. respectively; (c) one increased by 3 in. and the other diminished by 2 in. What is the per cent of change in each case?

70. At a distance of 60 ft. from a building the angles of elevation of the top and bottom of a tower on the building are 45° and 30° respectively. Find the height of the tower.

71. If the shadow of a tree is lengthened 60 (a) ft. as the angle of elevation of the sun changes from 45° to 30°, how high is the tree?

72. A ladder resting against a vertical wall forms an angle of 60° with the level ground. If the foot of the ladder is drawn 10 ft. farther out from the wall, the angle formed with the ground is 30°. Find the length of the ladder.

73. In a right triangle whose legs are 12 and 16 (20 and 40) find the length of the perpendicular from the vertex of the right angle to the hypotenuse, and the segments of the hypotenuse.

74. The legs of a right triangle are 9 and 12 (a and b). Find their projections on the hypotenuse.

75. The projections of the legs of a right triangle on the hypotenuse are $5\frac{2}{5}$ and $9\frac{3}{5}$ (m and n). Find the legs.

76. The sum of the three sides of a right triangle is 60 (140) in. and the hypotenuse is 26 (58) in. Find the legs and the perpendicular from the vertex of the right angle to the hypotenuse.

77. In a right triangle the perpendicular from the vertex of the right angle to the hypotenuse is $2\,(p)$ and the ratio of the segments of the hypotenuse is $4:9\,(m:n)$. Find the area of the triangle.

78. The perpendicular from the vertex of the right angle in a triangle makes the segment of the hypotenuse adjacent to the longer leg equal to the shorter leg. Find the area of the triangle when the hypotenuse is $2\,(c)$.

79. Two roads cross at right angles at A. 5 mi. from A on one road a man travels toward A at the rate of 3 mi. per hour. 6 mi. from A on the other road another man travels toward A at the rate of 6 mi. per hour. When and where will the men be 2 mi. apart?

80. Two trains run at right angles to each other, one at 30 and the other at 40 mi. per hour. The first train is 15 mi. from the crossing and is moving away from it; the second is 60 mi. from the crossing and moving toward it. When and where will the trains be 50 mi. apart?

81. How much must the length of a rectangle 16 by 12 (b by h) be increased in order to increase the diagonal $4\,(c)$?

82. The difference between the diagonal of a square and one of its sides is $2.071\,(a)$ in. Find one side and the area.

83. Find the sides of a rectangle if the perimeter is $34\,(p)$ in. and the diagonal is $13\,(d)$ in.

84. The diagonal and longer side of a rectangle are together 5 times the shorter side, and the longer side exceeds the shorter by 7. What is the area of the rectangle?

85. The perimeter of a right triangle is 24 (216) and the area is 24 (1944). Find the sides. (Solve with one, then with two, and then with three unknowns.)

86. From a square piece of tin a box is formed by cutting 6-in. squares from the corners and folding up the edges. If the volume of the box is 864 (1944) cu. in., what was the size of the original piece of tin?

87. The sum of the volumes of two cubes is 35 (2728) cu. in. and the sum of an edge of each is 5 (22) in. Find their diagonals.

88. If the edges of a rectangular box were increased by 2, 3, and 4 in. respectively, the box would become a cube and its volume would be increased by 1008 cu. in. Find the edges of the box.

89. The diagonal of a box is 125 in., the area of the lid is 4500 sq. in., and the sum of the three coterminous edges is 215 in. Find the three dimensions.

90. A rectangular piece of cloth shrinks 5 per cent in length and 2 per cent in width. The shrinkage of the perimeter is 38 in. and of the area 862.5 sq. in. Find the dimensions of the cloth.

91. If a given square be subdivided into four (n^2) equal squares and a circle inscribed in each of these squares, the sum of the areas of these circles will equal the area of the circle inscribed in the original square.

92. In a square whose side is 16 a square is inscribed by joining the mid-points of the sides in order. In this square another square is inscribed in a similar manner. This is repeated indefinitely. Find the area of the first eight inscribed squares.

93. In any triangle a triangle is inscribed by joining the mid-points of the sides. Another triangle is inscribed in this inscribed triangle in a similar manner, and so on indefinitely. How does the area of the sixth triangle compare with the area of the first?

94. An equilateral triangle is circumscribed about a circle of radius 4 (r) Find a side of the triangle.

95. A circle is inscribed in a triangle whose sides are 5, 6, and 7 (a, b, and c). Find the distances of the points of contact from the vertices of the triangles.

96. Find the radius of the circle inscribed in an isosceles trapezoid whose bases are 6 and 18 (b_1 and b_2).

97. A boy places his eyes at the surface of a smooth body of water and finds that the top of a float 1 mi. away is just visible. How far does the float project above the water?

SOLUTION. $x(8000 + x) = 1^2.$

$$x^2 + 8000\,x = 1.$$

Since x is very small compared with the diameter of the earth, we may drop x^2.

$$x = \frac{5280 \times 12}{8000} \text{ in.}$$

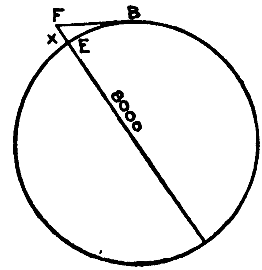

FIG. 79

98. A man 6 ft. tall standing on the seashore sees an object on the horizon. How far, in miles, is the object away from the shore?

99. From the top of a cliff 60 ft. high is barely visible the funnel of a steamer, known to be 30 ft. above the surface. How far is the steamer from the cliff?

100. The bridge of a steamer is 40 ft. above the water. How far apart are two such steamers when the bridge of one is just visible from the bridge of the other?

101. In a circle whose radius is 5 (r) a chord 8 (c) is drawn. Find the length of the chord of one half the arc.

102. Find the side of a regular polygon of twelve sides inscribed in a circle of radius 6 (r).

103. Find the side of a regular octagon inscribed in a circle of radius 8 (r).

104. The area inclosed by two concentric circles is 50 (a) sq. ft. If the radius of the inner circle is 5 (r) ft., find the radius of the outer circle.

105. Three men buy a grindstone. If the diameter is $3\,(d)$ ft., how much of the radius must each man grind off in order to obtain his share?

106. The sum of the circumferences of two circles is $56\frac{4}{7}$ ft. and the sum of their areas is $141\frac{3}{7}$ sq. ft. Find their radii. ($\pi = \frac{22}{7}$.)

107. The area of a rectangular table whose length is 5 ft. more than its breadth is equal to the area of a circular table whose radius is $3\frac{1}{2}$ ft. Find the dimensions of the table.

108. On a straight line $8\,(m)$ cm. long as a diameter describe a semicircle. On each half of the given line as diameters describe semicircles within the other semicircle. Find the radius of the circle which is tangent to the three semicircles.

109. An increase of 2 ft. in one side of an equilateral triangle enlarges the area by $4\sqrt{3}$ sq. ft. Find the side of the triangle.

110. The sides of a rectangle are 8 and $12\,(b$ and $h)$. Find the area of an equilateral triangle whose sides pass through the vertices of the rectangle.

111. The number which expresses the area of a right triangle is 1 greater than the number which expresses the length of the hypotenuse. Show that the sum of the legs of the triangle is 2 greater than the hypotenuse.

112. Find the side of the square inscribed in the common part of two circles of radius $6\,(r)$, if the center of each circle is on the circumference of the other.

113. Two parallel lines are 8 and 12 in. long respectively, and are 4 in. apart. Find the area of the two triangles formed by joining their opposite extremities.

114. How many squares may be inscribed in a triangle whose sides are 9, 12, and 15?

115. In a triangle whose sides are 3, 3, and $4\,(a, a,$ and $c)$ a line drawn across the sides 3 and $4\,(b$ and $c)$ bisects both the perimeter and the area. How far from the vertex does the line cut the sides?

CHAPTER XIV

VARIATION

91. Direct variation. If a man earns \$25 per week, the amount he earns in a given time equals \$25 multiplied by the number of weeks.

$$a = 25\,n.$$

Number of weeks	1	2	3	4	5
Amount earned	25	50	75	100	125

As the number of weeks changes the amount earned changes, but always the amount earned divided by the number of weeks equals 25.

$$\frac{a}{n} = 25.$$

We may state this fact in another way and say that the amount earned *varies directly* as the number of weeks, or $a \propto n$.

If a steel rail weighs 100 lb. per yard, the weight of the rail equals the length in yards multiplied by 100. $w = 100\,l$, or $\frac{w}{l} = 100$. Since the weight divided by the length is constant, 100, we may state this fact in the form of variation, and say that the weight *varies directly* as the length, or $w \propto l$.

Note that in direct variation an increase in one variable makes an increase in the other. The greater the length the greater the weight; the less the length the less the weight. Double the length and the weight is doubled; one fourth of the length gives one fourth the weight.

92. Definition. One number *varies directly* as another when the quotient of the first divided by the second is constant.

Exercise. On a sheet of squared paper take the lines at the bottom and left for the axes of x and y respectively, and let one square each way equal one. Draw a straight line from the lower left corner to the intersection of any two heavy lines. Make a table for the values of x, y, and $\frac{y}{x}$ for points on this line, taking $x = 1, 2, 3, \cdots, 10$. Is the quotient of y divided by x constant? Does y vary directly as x? What equation connects y and x?

PROBLEMS

1. The weight of a mass of brass varies directly as its volume. If 150 cu. in. weigh 45 lb., how many cubic inches weigh 7.5 lb.?

SOLUTION. Given $w \propto v$. (1)

By definition, $\frac{w}{v} = k$. (2)

$w = kv$. (3)

Substitute values, $45 = k \cdot 150$. (4)

Solving for k, $k = .3$. (5)

Substitute in (3), $7.5 = .3\,v$. (6)

$v = 25$ cu. in.

Arithmetical solution.

The weight of 1 cu. in. $= \frac{45}{150} = .3$ lb.

Hence it requires $\frac{7.5}{.3} = 25$ cu. in. to weigh 7.5 lb.

2. Construct a graph to show the relation between the volume and weight in Problem 1. What is the equation of the straight line? Read off some sets of values from the graph and check by the equation.

3. The weight of a mass of gold varies directly as its volume. If 60 cu. in. weighs 42 lb., find the weight of 35 cu. in.

4. Construct a graph to show the relation between the volume and weight of a mass of gold on the same axes as in Problem 2. What does the difference in the slope of the two graphs show?

5. The distance through which a body falls from rest varies as the square of the time during which it falls. If a body falls 400 ft. in 5 sec., how far will it fall in 20 sec.?

Suggestion. $d \propto t^2$. Check by arithmetical solution. $20 \div 5 = 4$. Since the distance varies as the square of the time, the body will fall $400 \times 4^2 = 6400$ ft. in 20 sec.

6. Construct a graph to show the relation between distance and time in the case of a falling body.

93. Inverse variation. A man wishes to lay out a flower bed containing 120 sq. ft. If he makes it 12 ft. long, it must be 10 ft. wide; 20 ft. long, 6 ft. wide; and so on. The greater the length the less the width. If the length is doubled, the width is halved; always $lb = 120$, or $l = \frac{120}{b}$. We say that the length *varies inversely* as the width, and write it $l \propto \frac{1}{b}$.

94. Definition. One number *varies inversely* as another when their product equals a constant.

Exercise 1. Suspend a meter stick at its center so as to balance, and attach a 500-g. weight 6 cm. from the fulcrum. Suspend on the other side a 100-g. weight to balance. How far from the fulcrum is it? Suspend other weights to balance, and make a table for the weights and distances from the fulcrum. Multiply each weight by its distance from the fulcrum. What seems to be true? If $w \cdot d = 3000$ (a constant), we may say that the distance *varies inversely* as the weight, $d \propto \frac{1}{w}$.

Exercise 2. Locate on squared paper the points from the table in Exercise 1, and draw a curve through them. Express the relation between x and y (1) as a variation; (2) as an equation.

PROBLEMS

1. The time it takes to do some work varies inversely as th number of men at work. If 6 men can do the work in 10 da., how long will it take 5 men to do it?

SOLUTION. Let t = number of days.
n = number of men.

Given $t \propto \frac{1}{n}$. (1)

By definition, $nt = k$. (2)

Substitute the given values in (2), $6 \times 10 = k$. (3)

$k = 60$. (4)

Substitute in (2), $5\,t = 60$. (5)

$t = 12$. (6)

Check by arithmetic. If 6 men can do the work in 10 da., 1 man can do it in 60 da.; and 5 men in $\frac{1}{5}$ of $60 = 12$ da.

2. The number of hours in a railway journey varies inversely as the speed. If it takes 7 hr. to go from Chicago to St. Louis at 40 mi. per hour, how long would it take at 50 mi. per hour?

3. The weight of a body varies inversely as the square of its distance from the center of the earth. If a man weighs 200 lb. on the surface of the earth (4000 mi. from the center), how much will he weigh when he is in a balloon 6 mi. from the surface?

95. Joint variation. If a carpenter saws a 2-in. plank into strips of various lengths and widths, the volume of each strip equals twice the length by the width, or $v = 2\,lb$. We may say that the volume *varies jointly* as the length and width, and write it in the form $v \propto lb$.

The number of cubic feet in a rectangular water tank 8 ft. high varies jointly as the length and width, since the number of cubic feet $= 8\,lb$.

96. Definition. One number *varies jointly* as two others when the first varies as the product of the other two.

PROBLEMS

1. The volume of a cylinder varies jointly as the altitude and the square of the radius of the base. When the altitude is 20 in. and the radius of the base is 10 in., the volume is 6284 cu. in. Find the volume when the altitude is 8 in. and the radius of the base is 6 in.

SOLUTION.

$$\begin{aligned} v &\propto hr^2. \\ v &= khr^2. \\ 6284 &= k \cdot 20 \cdot 10^2. \\ k &= 3.142. \\ v &= 3.142 \times 8 \times 6^2 \\ &= 904.9 \text{ cu. in.} \end{aligned}$$

2. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the velocity of the wind. If the pressure on 100 sq. ft. is 125 lb. when the wind is blowing 16 mi. per hour, what will be the pressure on a plate-glass window 10 by 12 ft. when the velocity of the wind is 70 mi. per hour?

97. Suggestions for the solution of problems in variation.

1. From the conditions given in the problem write the variation.
2. Change the variation to an equation.
3. Substitute the given numbers and find the value of the constant k.
4. In the equation substitute the value of k and the other numbers given in the problem.
5. Solve this equation for the required number.
6. Check.

While most of the problems in the following list should be solved by the principles of variation, some of them may be solved more easily by proportion. All results should be checked, and as far as possible the meaning of the constant should be discussed.

PROBLEMS

1. The circumference of a circle varies directly as its diameter, and when the diameter is 17.5 in. the circumference is 55.0 in. Find the circumference when the diameter is 22.7 in.

2. The velocity acquired by a falling body varies as the time of falling. If the velocity acquired in 4 sec. is 128.8 ft. per second, what velocity will be gained in 7 sec.?

3. The weight of a mass of gold varies directly as its volume. If 5 ccm. weighs 96.3 g., how many cubic centimeters will weigh 1 kg.?

4. The area of the surface of a cube varies directly as the square of its edge. What will be the edge of a cube the area of whose surface is $315\frac{3}{8}$ sq. in., if the area of the surface of a cube whose edge is $3\frac{1}{2}$ in. is $73\frac{1}{2}$ sq. in.?

5. The simple interest on a sum of money varies as the time during which it bears interest. If the interest on a certain sum is $84.20 for 5 yr., what will be the interest for $8\frac{1}{4}$ yr.?

6. The safe working load on a rope varies as the square of its girth. If the safe load on a manila rope 6 in. in girth is 1.2 T., find the girth of a rope whose safe load is 3.6 T.

7. If the friction between a wagon and the roadway varies as the total load on the wheels, and if the friction is 24 lb. when the load is 650 lb., find the friction when the load is $1\frac{1}{2}$ T.

8. The distance a body falls under the action of gravity varies as the square of the time of falling. If a body falls 403 ft. in 5 sec., in how many seconds will it fall 680 ft.?

9. The surface of a sphere varies as the square of its radius. If the surface of a sphere is 616 sq. in., by how much must its radius of 7 in. be increased in order to double its surface?

10. Given that the extension of a spring varies as the stretching force, and that a spring is stretched 10 in. by a weight of 5.2 lb., what weight will stretch the spring 7.5 in.?

11. The safe load on a rectangular beam varies jointly as the breadth and the square of the depth. If a 2 by 4 in. pine joist of given length supports safely 320 lb., what weight will a $2\frac{1}{2}$ by 10 in. beam of the same material and length safely support?

12. The weight of a disk of copper cut from a sheet of uniform thickness varies as the square of the radius. Find the weight of a circular piece of copper 12 in. in diameter, if one 7 in. in diameter weighs 4.42 oz.

13. The volume of a quantity of gas varies as the absolute temperature when the pressure is constant. If a quantity of gas occupies 3.25 cu. ft. when the temperature is 14° C., what will be its volume at 56.5° C.?

(Absolute temperature = 273° + the reading of the Centigrade thermometer.)

14. If the volume of a certain gas is 376 ccm. when the temperature is 12° C., at what temperature will the volume be 533.3 ccm., the pressure remaining the same?

15. Find the volume of a gas at − 23° C., if its volume is 200 ccm. at 27° C.

16. If the quantity of water that flows through a circular pipe varies as the square of the diameter of the pipe, and if 1.02 gal. per minute flow through a half-inch pipe, how many gallons per minute will flow through a 3-in. pipe?

17. The safe load on a wrought-iron chain varies as the square of the diameter of the section of the metal forming a link. If the safe load on a chain in which the metal is $\frac{3}{8}$ in. thick is 900 lb., what diameter of metal will be necessary in a chain that is to bear a load of 6.4 T.?

18. What is the safe load for a chain in which the diameter of a section of the metal forming a link is .9 in.?

19. The quantity of heat generated by an electric current in a given conductor for a given time varies as the square of the number of amperes. Find the amount of heat generated by a current of 25 amperes, if 224 units of heat are generated by a current of 16 amperes.

20. A current is found to generate 350 units of heat in the conductor of Problem 19. How many amperes in the current?

21. The compression of a spring under a given load varies as the cube of the mean diameter of the coils, other conditions being the same. When the diameter is 4 in. the compression is 1.64 in. What is the compression when the diameter is $6\frac{1}{2}$ in.?

22. The deflection by a given load at the middle of a beam supported at both ends varies as the cube of its length. A beam 9 ft. long is deflected .135 in. by a certain load. Find the deflection of a beam 15 ft. long by the same load.

23. The diagonal of a cube varies directly as the edge of the cube. If the diagonal of a cube is 8.66 in. when its edge is 5 in., what will be the edge of a cube whose diagonal is 13.4 in.?

24. A solid sphere of radius 3.5 in. weighs 12 lb. What is the diameter of a sphere of the same material that weighs 96 lb., given that the weight of a sphere varies as the cube of its radius?

25. The distance in miles of the offing at sea varies as the square root of the height in feet of the eye above the sea level. If the distance is 4 mi. when the height is 10 ft. 8 in., find the distance when the height is 121.5 ft.

26. According to Boyle's law the volume of a gas varies inversely as the pressure when the temperature is constant. If the volume of a gas is 600 ccm. when the pressure is 60 g. per square centimeter, find the pressure when the volume is 150 ccm.

27. If the volume of a gas is 42.5 cu. in. at a pressure of 12.6 lb. per square inch, find the pressure when the volume is 35.7 cu. in.

28. The pressure allowed in a cylindrical boiler varies inversely as its diameter. When the diameter is 42 in. the pressure allowed is 104 lb. per square inch. What pressure is allowed when the diameter is 96 in.?

29. Equal quantities of air are on opposite sides of a piston of a cylinder 16 in. long. If the piston moves 4 in. from the center, find the ratio of the pressures on the two sides of the piston.

30. The intensity of light varies inversely as the square of the distance from the source of light. If the illumination of a gas jet at a distance of 10 ft. is I, what will it be at 20 ft.? at 50 ft.?

31. A student lamp and a gas jet illuminate a screen equally when it is placed 12 ft. from the former and 20 ft. from the latter. Compare the relative intensities of the two lights.

32. How far from a lamp is a point that receives three times as much light as another point 20 ft. away?

33. How much farther from a gas jet must a book, which is 18 in. away from it, be removed in order that it may receive two thirds as much light?

34. An 8 candle power electric lamp at a distance of 6 ft. from a screen illuminates it with one half the intensity of a candle at a distance of 1 ft. 6 in. from the screen. What is the candle power of the candle?

35. In a given latitude the time of vibration of a pendulum varies as the square root of its length. If a pendulum 39.1 in. long vibrates once in a second, what is the length of a pendulum that vibrates twice in a second? three times in a second?

36. The velocity with which a liquid flows from an orifice varies as the square root of the head (depth of the liquid above the orifice). A reservoir 40 ft. high is filled with water, and when an opening is made in the side at a height of 4 ft., the water escapes with an initial velocity of 48 ft. per second. What would be the velocity if the opening were made at a height of 8 ft.?

37. The weight of a body varies inversely as the square of its distance from the center of the earth. If a body weighs 100 lb. at the earth's surface (4000 mi. from the center), what would be its weight at the summit of the highest mountain, which is $5\frac{1}{2}$ mi. high?

38. How far above the earth's surface must a body that weighs 150 lb. at the surface be removed, in order that its weight may be reduced to 96 lb.?

39. The diameter of the rivets used for a plate varies as the square root of its thickness. If $1\frac{1}{4}$-in. rivets are used for a 1-in. plate, what size of rivets is required for a $\frac{3}{4}$-in. plate? What thickness of plate can be riveted with $\frac{7}{8}$-in. rivets?

40. The volume of a gas varies inversely as the height of the mercury in a barometer, the temperature being constant. If a certain mass occupies 32 cu. in. when the barometer reads 28.8 in., what space will it occupy when the reading is 30.4 in.?

41. Compare the amounts of heat received at two points whose distances from the source of heat are in the ratio $4:3$, assuming that the intensity of heat varies inversely as the square of the distance from the source of heat.

42. If the attraction of a magnet for a piece of iron varies inversely as the square of the distance between them, and if the attraction at the distance of .1 in. is a, what will be the attraction at .2 in.? at .3 in.? at .5 in.?

43. The attractive force between two oppositely electrified balls varies inversely as the square of the distance between them. At a distance of 8 cm. the force is 3.5 g. At what distance will the force be .64 g.?

44. The compression of a spring under a given load varies inversely as the fourth power of the diameter of a cross section of the steel in the coils, other conditions being the same. If the compression is 3.5 in. when the diameter is $\frac{5}{8}$ in., what will be the compression when the diameter is $1\frac{1}{4}$ in.?

45. If 7 men in 9 weeks earn \$516.60, how many men will it take to earn \$360.80 in 4 weeks, it being given that the amount earned varies jointly as the number of men and the number of weeks?

46. The volume of a circular disk varies jointly as its thickness and the square of its radius. Two metallic disks having thicknesses 5 cm. and 3 cm., and radii 12 cm., and 20 cm. respectively, are melted and recast into a single disk 6 cm. thick. What is its radius?

47. The weight of a metal cylinder varies jointly as its length and the square of its diameter. If a cylinder 12 in. long and $4\frac{5}{8}$ in. in diameter weighs 49 lb., what is the diameter of a cylinder 20 in. long that weighs 135 lb.?

48. The volume of a cone varies jointly as its altitude and the square of the radius of its base. If the volume of a cone is 4.95 ccm. when its altitude is 2.1 cm. and its radius is 1.5 cm., find the altitude of a cone whose radius is 3 cm. and volume 33 ccm.

49. How far from a light of 9 candle power will the illumination be $2\frac{1}{2}$ times the illumination at a distance of 24 ft. from a light of 16 candle power?

50. The weight of a uniform bar of given material varies jointly as its length and the area of its cross section. If a steel bar 1 sq. in. in cross section and 1 ft. long weighs 3.3 lb., what is the weight of a T-rail 2 ft. long and $8\frac{3}{4}$ in. in cross-sectional area?

51. An ohm is the resistance offered to the flow of an electric current through a column of mercury 106 cm. long and 1 sq. mm. in cross-sectional area. What is the resistance of a column of mercury 3 m. long and 4 sq. mm. in cross-sectional area, the resistance varying directly as the length and inversely as the cross-sectional area?

52. A wire of diameter .0704 in. has a resistance of 15 ohms. Find the diameter of a wire of the same length and material whose resistance is 5.4 ohms.

53. If the resistance of 500 yd. of a certain cable is .65 ohm, what will be the resistance of 1 mi. of a cable of the same material and of one half the cross-sectional area?

54. Find the resistance of 1000 yd. of copper wire .15 in. in diameter, if the resistance of 112 yd. of copper wire .06 in. in diameter is 1 ohm. Solve also by the formula in Exercise 10, page 73, and compare the results. See Chapter XVII for definitions and explanations.

55. The resistance of a certain wire is 1.82 ohms, and the resistance of $2\frac{1}{2}$ mi. of the same wire is known to be 3.25 ohms. Find the length of the first wire.

56. The resistance of 2400 ft. of a certain copper wire of cross section 11.2 sq. mm. is 1.13 ohms. What is the resistance of 2 mi. of copper wire of cross section 6.45 sq. mm.?

57. According to Ohm's law the number of amperes flowing through an electric circuit varies directly as the number of volts of electromotive force and inversely as the number of ohms resistance. If the voltage in a certain circuit is such as to maintain a current of 10 amperes through a resistance of 40 ohms, what would be the current if the electromotive force were doubled and the resistance diminished by one third?

58. How many amperes are there in the current maintained by a dynamo whose resistance is 2.4 ohms, that of the rest of the circuit being 17.6 ohms, and the electromotive force 210 volts?

59. The resistance offered by the air to the passage of a bullet through it varies jointly as the square of its diameter and the square of its velocity. If the resistance to a bullet whose diameter is .32 in. and whose velocity is 1562.5 ft. per second is 67.5 oz., what will be the resistance to a bullet whose diameter is .5 in. and whose velocity is 1300 ft. per second?

60. From the data of Problem 59 determine the diameter of a bullet that has a resistance of 50 oz. when its velocity is 900 ft. per second.

61. If t denotes the time of revolution of a planet in its orbit about the sun, and d the mean distance of the planet from the sun, then t^2 varies as d^3. Assuming that the earth's period of revolution is 365 da. and that of Venus 225 da., find the ratio of the mean distances of these two planets from the sun.

62. The horse power that a solid steel shaft can transmit safely varies jointly as its speed in revolutions per minute and the cube of its diameter. A 4-in. solid steel shaft making 120 r. p. m. can transmit 240 h.p. How many horse power can be transmitted if the diameter of the shaft is 3 in. and its speed 100 r. p. m.?

63. The pressure of the wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. If the pressure on a square yard is $12\frac{1}{2}$ lb. when the velocity of the wind is $17\frac{1}{2}$ mi. per hour, what is the pressure on a square foot when the velocity of the wind is 45 mi. per hour?

64. The space s passed over and the time of flight t of a body projected vertically upward are connected by the relation $s = at - 16t^2$, where a is constant. If $s = 676$ ft. when $t = 6\frac{1}{2}$ sec., find s when $t = 3$ sec.

CHAPTER XV

EXERCISES IN SOLID GEOMETRY

98. Use short methods of multiplication and division and keep the results to a reasonable number of significant figures.

I. Numerical Exercises

1. A line 8 ft. long makes with a plane an angle of 45°. Find the length of the projection of the line upon the plane.

2. What will be the length of the projection of the line in the preceding exercise, if it makes an angle of 30° with the plane?

3. Prove that, if a line is inclined to a plane at an angle of 60°, its projection upon the plane is equal to half the line.

4. In a swimming tank the water is $5\frac{1}{2}$ ft. deep and the ceiling is 11 ft. above the water; a pole 22 ft. long rests obliquely on the bottom of the tank and touches the ceiling. How much of the pole is above the water?

5. From a point P 6 in. from a plane a perpendicular PQ is drawn to the plane; with Q as a center and a radius of $4\frac{1}{2}$ in. a circle is described in the plane; at any point R of this circle a tangent RT 10 in. long is drawn. Find the distance from P to T.

6. With a 12-ft. pole marked in feet how can you determine the foot of the perpendicular let fall to the floor from the ceiling of a room 9 ft. high?

7. If a point is 20 cm. from each of the vertices of a right triangle whose legs are 12 cm. and 16 cm. respectively, find the distance from the point to the plane of the triangle.

8. Determine the relation between (a) the edge and the diagonal of a face of a cube; (b) the edge and a diagonal of a cube.

9. The sum of the squares of the three edges of a rectangular parallelepiped is 2166 and the three edges are to each other as 1 : 2 : 3. Find the edges.

10. The dimensions of a rectangular bin are 4 ft., $4\frac{1}{2}$ ft., and 10 ft., and it is desired to treble its capacity. How can this be done if only one dimension is changed? two dimensions? all three dimensions?

11. Make a geometrical application of the equation $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

12. How much will it cost, at 40 cents per cubic yard, to dig an open ditch 80 rd. long, 6 ft. wide at the top, $2\frac{1}{4}$ ft. wide at the bottom, and 3 ft. deep?

13. How many square feet of lead will be required to line a rectangular cistern 10 ft. long, 7 ft. wide, and $4\frac{1}{2}$ ft. deep? What will be the weight of the lead if it is $\frac{1}{16}$ in. thick and a cubic inch weighs .411 lb.?

14. What is the weight of the water received upon an acre of ground during a storm in which rain falls to the depth of an inch?

15. Allowing 30 cu. ft. of air per minute for each person in this classroom, how much air must be driven into the room and how many times must the air be changed during the recitation period to insure good ventilation?

16. The cross section of a trough 12 ft. long is an equilateral triangle. When 20 gal. of water are poured into the trough, whose edges are in the same horizontal plane, how deep will the water be?

17. A room is 10 ft. high and its length is one half greater than its width. If the area of the ceiling and walls is 816 sq. ft., find the other two dimensions.

18. A block of ice $1\frac{1}{2}$ ft. by 2 ft. by 3 ft. is placed in a box 4 ft. long and 2 ft. wide. What will be the depth of water in the box after the ice melts, the specific gravity of ice being .917?

19. How large a cubical reservoir will be required to hold the water that falls on the roof of a house covering 548 sq. ft. of ground, during a shower in which $\frac{3}{8}$ of an inch of rain falls?

20. How many square yards of canvas will be required to make a tent 10 ft. by 16 ft., if the sides are 6 ft. high and the roof has $\frac{1}{2}$ pitch?

21. An oblique prism whose altitude is h has for its base a rhombus whose diagonals are k and l. Find its volume.

22. Two rectangular parallelepipeds are to each other as $5 : 18$. The dimensions of the first are 5, $13\frac{1}{2}$, and 18. Find the dimensions of the other, if they are to each other as $1 : 2 : 3$.

23. The base of a prism whose altitude is 15 cm. is a quadrilateral whose sides are 10 cm., 18 cm., 12 cm., and 16 cm., the last two forming a right angle. Find its volume.

24. A prism has for its base a triangle whose sides are to each other as $5 : 12 : 13$. If its altitude is 4 m. and its volume is 4.8 cu. m., find the sides of the base.

25. The Great Pyramid is 762 ft. square at the base and 484 ft. high. Compute its volume and its lateral area.

26. The lateral area of a regular square pyramid of wood is 144 sq. in., and one side of the base is 8 in. Find its weight, if its specific gravity is .53.

27. Determine the volume of a pyramid, one of whose lateral faces is an equilateral triangle on a side of 18 in., and whose third lateral edge is perpendicular to the other two and is 24 in. long.

28. A section of a pyramid parallel to the base contains 96 sq. ft., and its distance from the base whose area is 120 sq. ft. is 4 ft. Find the altitude of the pyramid.

29. The lateral area of a regular triangular pyramid is 64 sq. ft. and one side of the base is 8 ft. Find the altitude.

30. If a section of a pyramid parallel to the base is so taken that its area is $\frac{1}{9}$ that of the base, what part of the pyramid is that portion above the section?

31. If the sides of the base of a pyramid are 4, 6, 7, and 9, and the solid is cut by a plane parallel to the base so that the section is $\frac{1}{16}$ of the base, what will be the lengths of the sides of the section?

32. A granite obelisk in the form of a frustum of a regular quadrangular pyramid, surmounted by a pyramid of slant height 15 in., has each side of one base 1 ft. 4 in. and each side of the other base 2 ft. 3 in., and the slant height is 12 ft. If the specific gravity of the granite is 2.6, find the weight of the obelisk.

33. What will be the expense of polishing the faces of the obelisk in the preceding exercise at 50 cents per square foot?

34. What is the capacity in gallons of a reservoir 12 ft. in depth and 300 ft. long by 160 ft. wide at the top, the slope of the walls being 3 : 2?

35. A granite monument in the form of a prismoid is 16 ft. high and the dimensions of its ends are 42 in. by 28 in. and 18 in. by 12 in. respectively. What is its weight if the specific gravity of the granite is 2.7? See Kent's "Mechanical Engineers' Pocket-Book" for the definition of a prismoid and for the prismoid formula.

36. A milldam of earth with plane sloping sides and rectangular bases is 80 m. by 6 m. at the top and 66 m. by 18 m. at the bottom. If its height is 5.4 m., find its cubic contents.

37. How many cubic yards of earth will it be necessary to remove in making a cut for a railroad, which must be 14 ft. deep, 24 ft. wide, 240 ft. long at the bottom, and 170 ft. long at the top, the slope of the sides being 7 : 10?

38. Apply the prismoid formula to the regular octahedron whose edge is e.

39. The volume of a wedge whose base is 7.5 cm. by 12 cm., and whose height is 3.5 cm., is 142 ccm. Find the length of its edge, regarding the solid as a prismoid.

40. Find the weight of a steel wedge whose base measures 3 in. by 7 in., the edge 5 in., and the height 6 in., if a cubic inch of steel weighs .283 lb.

41. If a cubic foot of steel weighs 490 lb., what is the weight of a hollow steel beam 10 in. square at one end, 7 in. at the other end, and 18 ft. long, the metal being $\frac{3}{4}$ in. thick?

42. Find the cost of painting the lateral surface of an octagonal tower whose slant height is 40 ft., if the short diameter of the lower base is 12 ft. and of the upper base 3 ft., at 24 cents per square yard.

43. At what distance from the vertex of any pyramid must a lateral edge 12 ft. long be cut by planes parallel to the base, in order that the areas of the sections formed may be to each other as 2 : 3 : 5?

44. What must be the height of a prism of iron equal in weight to the sum of three other prisms of iron of the same shape, the height of the latter being 2 in., 3 in., and 4 in. respectively?

45. A block of granite weighs 2 T. and its width is 3 ft. What is the width of a block of granite of the same shape whose weight is 6 T.?

46. A block of wood of specific gravity .675 weighs 72.4 lb., and a block of steel of specific gravity 7.84 and of the same shape weighs 13.14 lb. Find the ratio of their corresponding dimensions.

47. Of two bodies of the same form, one weighs 2 lb. and its specific gravity is .24, while the other weighs 56 lb. and its specific gravity is 2.32. If one dimension of the first body is 60 cm., what is the corresponding dimension of the second?

48. An irregular mass of iron, specific gravity 7.2, weighs $42\frac{1}{2}$ lb. What is the weight of a mass of gold of the same form, specific gravity 19.3, if two corresponding lines of the two masses have the ratio 2 : 3?

49. How much tin will be required to make an open cylindrical vessel of altitude 65 cm. which shall contain 160.2 l., taking no account of seams?

50. What is the amount in cubic feet of evaporation daily from a circular fishpond 6 rd. in diameter, if the loss in depth is .04 in.?

51. How many board feet of lumber 16 in. wide can be made from a round log 20 in. in diameter and 16 ft. long?

52. The areas of two sections of a cylinder of revolution 4 ft. high, which are parallel to the axis and to each other, are 6 sq. ft. and $4\frac{2}{3}$ sq. ft. respectively. If the sections are 2 in. apart, what is the volume of the cylinder?

53. If a cubic foot of copper is drawn into a wire $\frac{1}{16}$ in. in diameter, what will be its length?

54. An irregular mass weighing 21.07 kg. is dropped into a cylindrical vessel 42 cm. in diameter, partially filled with water. If the water rises 80 cm., find the volume and specific gravity of the body.

55. How many cubic yards of stone will be required for a semicircular culvert under a railroad bank 112 ft. wide, the throat of the culvert being 6 ft. high and the walls 2 ft. thick?

56. A hollow cylindrical iron column is 14 ft. 4 in. long, 6 in. in diameter, and 1 in. thick. What is its weight if the specific gravity of iron is 7.2?

57. A steel shaft is reduced in diameter in a lathe from 5 in. to 4.5 in. Find to the nearest hundredth what part of its weight is lost.

58. In what time will a 1-in. circular pipe in which a flow of water of 1 ft. per second is maintained, fill a rectangular cistern of dimensions $3\frac{1}{2}$ ft. by 4 ft. by $7\frac{1}{3}$ ft.?

59. The plunger of a certain single-acting pump is 10 in. in diameter, has a 10-in. stroke, and makes 15 strokes per minute. How many gallons of water pass through it in 12 hr.?

60. A Holley pump has an hourly capacity of 145,800 gal. of water. If the plunger has a 40-in. stroke and makes 18 strokes per minute, what is its diameter?

61. When a pump is required to furnish 2,800,000 gal. of water in 24 hr., how many strokes per minute must the plunger make if its diameter is 30 in. and its stroke is 40 in.?

The following rule is sometimes used to calculate the horse power of a steam boiler. To the heating surface afforded by the flues is to be added two thirds of the lateral surface of the boiler, and two thirds of one flue sheet diminished by the ends of the flues. In general practice 12 sq. ft. of heating surface are considered to afford 1 h. p.

62. Compute the horse power of a steam boiler whose length is 16 ft. and diameter 6 ft., if there are 136 flues, each 16 ft. long and 3 in. in interior diameter.

63. What must be the length of the flues of a steam boiler of diameter 2 ft., containing 34 2-in. flues, in order that it may afford 12 h. p.?

64. A conical heap of grain 4 ft. high covers a space 12 ft. in diameter on the floor. How large must be a cubical bin to hold it?

65. How many square yards of canvas are required to make a conical tent 10 ft. high, such that a man 6 ft. tall may stand without stooping anywhere within 4 ft. of the center?

66. A conical vessel whose angle is 60° is filled to the depth of 8 in. with water, and when a solid cube of wood is submerged in it, the water rises 1 in. Find the edge of the cube.

67. How much ground is covered by a conical tent 9 ft. in height, which contains 162 sq. ft. of canvas?

68. A square whose side is 4 cm. revolves around one of its diagonals. Find the volume generated.

69. A rectangle 6 in. by 8 in. revolves around one of its diagonals. Determine the volume and the area of the surface generated.

70. If a sector of 120° is cut out of a circular piece of canvas 28 ft. in diameter, what are the dimensions of the conical tent that can be made out of the remainder?

71. A hollow iron cone is 4 in. long and 4 in. in diameter, and the metal is $\frac{1}{2}$ in. thick. Find its weight if a cubic inch of iron weighs .261 lb.

72. The altitudes of a cylinder and an equivalent cone are to each other as 16 : 27. Find the ratio of their other dimensions.

73. At 15 cents per square foot, what will be the cost of cementing the walls and bottom of a cistern in the form of an inverted frustum of a cone of revolution whose depth is 7 ft. and diameters 6 ft. and 3 ft., the lid $1\frac{1}{2}$ ft. square not being cemented?

74. A cone whose slant height is 16 cm. is to be divided into three parts in the ratio of 1 : 2 : 3. At what distance, measuring from the vertex, must the slant height be cut by planes parallel to the base?

75. In a sphere of radius 5 ft., what is the area of the circle whose plane is 4 ft. from the center?

76. In a sphere of radius 6 ft. how far from the center is the plane of a circle whose area is $50\frac{2}{7}$ sq. ft.?

77. What is the length of an arc of 120° in the circumference of a circle whose plane is $4\frac{1}{2}$ ft. from the center of a sphere of radius 5 ft.?

78. On a sphere of radius 6 in. what is the polar distance of a small circle whose latitude is 60°? What is the radius of the circle?

79. How many degrees in each angle of an equilateral spherical triangle whose area is $\frac{5}{16}$ of that of the sphere?

80. If a birectangular triangle is $\frac{1}{18}$ of the surface of its sphere, what is the third angle of the triangle?

81. If the diameter of the moon is 2162 mi., find its surface in square miles and its volume in cubic miles.

82. The diameter of the earth is 7918 mi. and that of the planet Mercury 3030 mi. If the density of the latter is 2.23 times that of the former, show that the mass of Mercury is nearly $\frac{1}{8}$ that of the earth.

83. If the mean diameters of the earth and the moon are 7918 and 2162 mi. respectively, show that the ratio of their surfaces is 27 : 2 nearly.

84. What is the diameter of a sphere of which a wedge of $11° 15'$ contains 359.3 cu. dm.?

85. How many bullets 1 in. in diameter can be made of 3 ft. of lead pipe $1\frac{1}{2}$ in. in exterior diameter and $\frac{1}{8}$ in. thick?

86. A steel ball 6 in. in diameter is dropped into a cylindrical vessel 8 in. in diameter, which is filled within 2 in. of the top with water. How much water will overflow?

87. If 400 lead balls each $\frac{1}{2}$ in. in diameter are melted and run into a disk $\frac{1}{16}$ in. thick, what will be the radius of the disk?

88. How many bullets of caliber .32 (.32 in. in diameter) can be made from a bar of lead $2\frac{1}{4}$ in. by 4 in. by 6 in.?

89. A marble $\frac{5}{8}$ in. in diameter is dropped into a conical glass whose diameter is 2 in. and depth 3 in., and is just covered by the water that it contains. What was the depth of the water at first?

90. What is the altitude of that zone of a sphere which equals a trirectangular triangle in area?

91. Find the surface of the zone of a sphere of radius 8 in. cut off by a plane 3 in. from the center of the sphere.

92. What is the altitude of a zone of 120 sq. in. surface, if the radius of the sphere is 10 in.?

93. How far from the surface of a sphere must a lamp be placed in order that one sixth of the surface may be illuminated?

94. Show that the portion of the earth's surface that is visible to an aëronaut at a height h above the surface is $\frac{2\pi r^2 h}{r+h}$, r being the radius of the earth. When h is small it may be dropped in the denominator, giving the approximate area $2\pi rh$.

95. On a globe of radius 7 cm. it is desired to mark off a zone whose area shall be 6.16 sq. cm. What opening of the compasses shall be used?

96. On a globe of radius 9 in. a small circle is described with an opening of the compasses of 6 in. Find the length of the circumference.

97. The altitude and radius of the base of a right cone are 12 and 9 in. respectively. Find the radius of the circle of tangency of the inscribed sphere.

98. How does the specific gravity of a spherical body compare with that of a liquid in which it floats, with one half its surface above the surface of the liquid? one third? when it is just submerged?

99. If a sphere of oak 6 in. in diameter floats in water with .3 of its surface above the surface of the water, what is the specific gravity of the oak?

100. What portion of the surface of a ball of iron of diameter 1 in. and specific gravity 7.2 will remain visible when it is dropped into a dish of mercury whose specific gravity is 13.6?

II. Graphical Exercises

99. A few of these exercises should be worked out carefully in the notebook.

1. Construct a graph to show the change in the volume of a cube as its edge increases from 0 to 12 in. What is the equation of the graph?

2. On the same axes as in Exercise 1 show graphically the change in the surface of the cube. How do the graphs show (*a*) when the surface equals the volume numerically? (*b*) when a cube has a greater surface than volume? Write on each graph its equation.

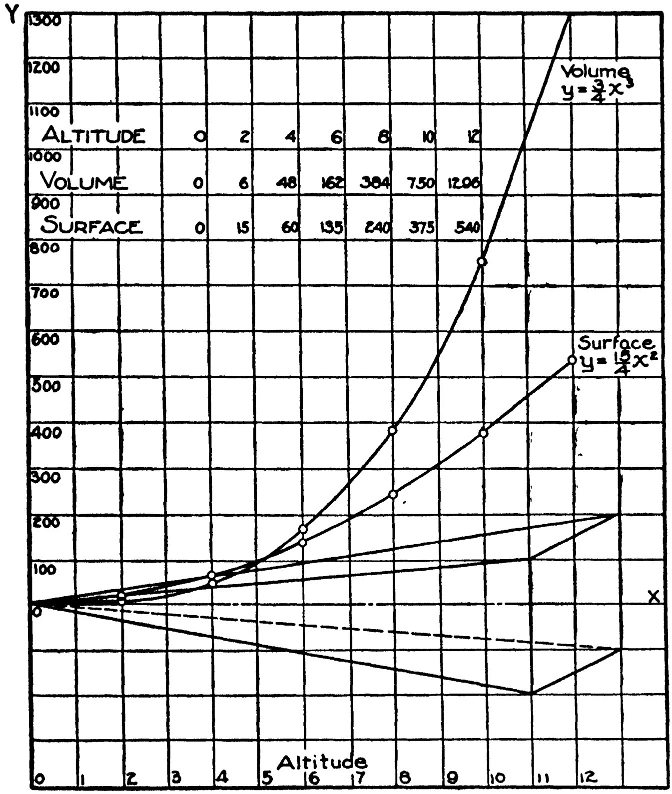

FIG. 80

3. The altitude of a regular square pyramid is 12 ft. and each side of the base is 18 ft. Show graphically (*a*) the volume, (*b*) the lateral surface of the pyramids cut off from the vertex by planes parallel to the base. Find the ratio of the surface of any of the pyramids to its volume, and use the result to check the table of values.

4. The altitude of a right cone is 12 ft. and the radius of the base is 9 ft. Show graphically (*a*) the volume, (*b*) the lateral surface of the cones cut off from the vertex by planes parallel to the base.

5. On the same axes construct graphs to show the change (*a*) in volume, (*b*) in lateral surface of a right cylinder the radius of whose base is 6 in., as its altitude increases from 0 to 15 in.

6. Represent graphically the change in the area of a section parallel to the base of a regular triangular pyramid, the side of whose base is 8 cm. and whose altitude is 12 cm.

7. On the same axes represent graphically the change (*a*) in volume, (*b*) in surface of a sphere as the radius increases from 0 to 10 in.

8. The volume of a pyramid is 60 cu. in. Construct a graph to show the relation between the base and altitude as the altitude increases from 0 to 180 in.

9. The volume of a cylinder is 440 cu. in. Construct a curve to show the relation between the radius of the base and the altitude, as the radius increases from 0 to 10 in.

10. From each corner of a square piece of tin 12 in. on a side a smaller square is cut, the remainder of the sheet being bent so as to form a rectangular open box. Determine the side of each small square in order that the capacity of the box may be as great as possible.

11. If the sheet of tin in the preceding exercise had been rectangular, 20 in. by 12 in., what then would have been the size of each small square?

12. A bin with a square base and open at the top is to be constructed to contain 400 cu. ft. of grain. What must be its dimensions to require the least amount of material?

13. A closed cylindrical oil tank is required to hold 100 bbl., each of 42 gal. What dimensions will necessitate the least steel plate in the making?

14. An open rectangular tank whose length is to be twice its width is to hold 200 gal. of water. What dimensions will require the least amount of lining for the tank?

15. The strength of a rectangular beam is proportional to the product of its breadth and the square of its depth. What are the dimensions of the strongest beam that can be cut from a round log 2 ft. in diameter?

16. If the slant height of a right cone is 12 ft., what must be the radius of its base in order that its volume may be as great as possible?

17. Determine the right cylinder of greatest lateral surface that can be inscribed in a cone of revolution whose altitude is 14 in. and radius of base 8 in.

18. Find the dimensions of the smallest cone of revolution that can be circumscribed about a cylinder whose altitude and radius are respectively 9 dm. and 3 dm.

19. The stiffness of a rectangular beam varies as the product of its breadth and the cube of its depth. Find the dimensions of the stiffest beam that can be sawed from a log 20 in. in diameter.

20. Determine the dimensions of the largest right cone that can be inscribed in a sphere of radius 5 in.

21. Find what radius of the base of a conical tent of 375 cu. ft. capacity will require the least amount of canvas in the making. Also find the relation between the altitude and the radius.

22. Find the radius of the right cylinder of greatest lateral surface that can be inscribed in a sphere whose diameter is 12 in.

23. Find the relation between the radius of the base and the altitude of a right cone whose convex surface contains 264 sq. ft., in order that the volume may be as great as possible.

24. Determine the altitude of the least cone of revolution that can be circumscribed about a sphere of radius 2 dm.

25. What must be the altitude of the cone of revolution of least lateral surface that can be circumscribed about a sphere whose radius is 4 in.?

III. Algebraic Problems

100. Make a sketch for each problem. Put the given dimensions on the figure and set up the equations from the sketch.

1. What are the other two dimensions of a rectangular parallelepiped whose length is 8 in., if its volume is 160 cu. in. and its total surface is 184 sq. in.?

2. If the three face diagonals of a rectangular solid are respectively 6, 7, and 9 cm., what must be the dimensions of the solid?

3. One dimension of a rectangular parallelepiped is 6 in., one diagonal is 12 in., and the area of one of the wholly unknown faces is 44 sq. in. What are the other two dimensions?

4. The sum of the three dimensions of a rectangular solid is 12 and the diagonal of the solid is $5\sqrt{2}$. Find its total surface.

5. The sum of a diagonal and an edge of a cube is 6. Find an edge of the cube.

6. The area of one face of a rectangular solid is 10 sq. cm., that of another is 15 sq. cm., and the total area is 100 sq. cm. Find the dimensions.

7. What are the dimensions of a rectangular solid whose entire surface is 392 sq. in., if its top contains 96 sq. in. and one end 40 sq. in.?

8. Given the diagonal of a cube equal to k. Find the volume of the cube and its surface.

9. Given the volume v and the altitude h of a regular hexagonal prism. Find s, the length of one side of the base.

10. The sides of the base of a triangular prism are as $3 : 4 : 5$, and its volume is 432 cu. ft. If the altitude is 4 ft., find the sides of the base.

11. What must be the altitude of a pyramid in order that its total area may be equal to the sum of the areas of two similar pyramids whose altitudes are respectively 6 and 4 in.?

12. What is the altitude of a pyramid whose base contains 98 sq. in., if a section parallel to the base and 4 in. from the vertex contains 32 sq. in.?

13. The volume of a pyramid with a rectangular base is 76.8 cu. in., one side of the base is 9.6 in., and the altitude exceeds the other side of the base by 2 in. Find the altitude and the other side of the base.

14. If a square pyramid has each basal edge equal to e and each lateral edge equal to e_1, show that the volume will be $v = \frac{e^2}{6}\sqrt{2(2e_1^2 - e^2)}$.

15. Given v, the volume, and s, one side of the square base of a regular quadrangular pyramid, find the lateral surface.

16. Derive an expression for the volume of a regular tetrahedron in terms of its edge e.

17. An iron plate 8 in. long and $2\frac{1}{2}$ in. thick has squared ends but uniformly and equally beveled sides, and contains 122 cu. in. If the difference of the widths of the two flat faces is 2.8 in., find those widths.

18. The lateral area of a frustum of a regular quadrangular pyramid is 281.2 sq. in., the slant height is 15.2 in., and a side of the lower base exceeds a side of the upper base by 3.75 in. Find a side of each base.

19. What must be the diameter of a cylindrical gas holder which is to hold 6,000,000 feet of gas, if its height is to be $\frac{5}{8}$ of its diameter?

20. The sum of the numerical measures of the volume and lateral area of a cylinder of revolution is 231. If the altitude is 14, what is the diameter?

21. Write the formula that gives t, the total surface of a cylinder of revolution, in terms of h, the altitude, and r, the radius of the base, and solve it for h and r.

In case of a cylinder of revolution:

22. Given t and r, find h and v.

23. Given v and r, find h and t.

24. Given v and h, find r and l (the lateral area).

25. Given l and v, find h, r, and t.

26. Given l and h, find r, v, and t.

27. Given t and v, find r, h, and l.

Suggestion. Find r by trial from $2\pi r^3 - tr + 2v = 0$ for any given numerical values of t and v (see sect. 58); then find h from $v = \pi r^2 h$, and then l from $l = 2\pi rh$.

28. How far from the axis of a cylinder of revolution whose height is h ft. and diameter d ft. must a plane parallel to the axis be passed, in order to make a section of area k sq. ft.?

29. If the total surface of a cone of revolution is 21π and the slant height is 4, find the radius and the volume of the cone.

30. The sum of the altitude and the radius of the base of a cone of revolution is 11 and their product is 10. What is the volume of the cone?

31. The lateral area of a right cone is $9\sqrt{10}\,\pi$, and its altitude is equal to 3 times the radius of its base. Find its volume.

32. Find the slant height and the radius of the base of a cone of revolution whose total surface is 462 sq. in., and the sum of the slant height and the radius is 21 in.

33. The lateral surface of a right cone whose slant height is 5 exceeds the base by $12\frac{4}{7}$. Find the radius of the base.

34. What is the radius of the upper base of a frustum of a right cone, if its volume is $.516\,\pi$ cu. dm., its altitude 1.2 dm., and the radius of its lower base .8 dm.?

35. The lateral area of a frustum of a cone of revolution is $77\,\pi$, the slant height is 7, and the altitude is $2\sqrt{6}$. Find the radii of the bases.

36. What is the volume of a frustum of a right cone the sum of the radii of whose bases is 11 and their product 28, the altitude being 7?

37. Find the radii of the bases of a frustum of a right cone, given the lateral area as $1068\frac{4}{7}$ sq. ft., the slant height as 17 ft., and the altitude as 15 ft.

38. The volumes of two spheres are to each other as $8 : 125$, and the sum of their radii is 12 in. Find the radii.

39. The product of the radii of two spheres is 22.5 and the ratio of their surfaces is $25 : 64$. What are the radii?

40. If the surface of a sphere is equal to the sum of the surfaces of two spheres whose radii are 2 in. and 4 in. respectively, how does its volume compare with the sum of their volumes?

41. What is the radius of a sphere of which a zone of 24 sq. in. is illuminated by a lamp placed 18 in. from its surface?

42. What relation must the radius of a given sphere bear to the radii of two other spheres if its surface is a mean proportional between their surfaces?

43. Compare the expression for the volume of a sphere with that for its surface, and determine how long the radius must be in order that the volume may be numerically greater than the surface.

44. In a sphere of radius 8 the radius of one small circle is a mean proportional between the radius of the sphere and the radius of another small circle, and the sum of the radii of the two small circles is 10. Find the radii of the small circles.

45. Derive an expression in one variable for the volume of a right cone inscribed in a sphere of radius r.

46. Find an expression in terms of the altitude for the total surface of a cylinder of revolution inscribed in a sphere of radius r.

47. What is the expression for the volume of a right cylinder inscribed in a right cone, altitude h, radius of base r, in terms of the radius of the cylinder?

48. Find an expression in one variable for the total surface of a right cone circumscribed about a given right cylinder.

49. What expression in one variable denotes the volume of a right cone circumscribed about a given sphere?

50. Derive the expression in one variable for the lateral surface of the cone in Exercise 49.

51. Find an expression in one variable for the volume of a right cone circumscribed about a given right cylinder.

Problems 45–51 furnish good exercises in maxima and minima by giving numerical values to the dimensions of the constant solids. Since some of the expressions are rather complicated the work of computing the table of values may be divided among the members of the class, each one computing the value of the function for a single value of the variable.

CHAPTER XVI

HEAT

101. Thermometers. Though the Fahrenheit scale is in general use in everyday life and in ordinary engineering work, the Centigrade scale is used in laboratories and all scientific work to such an extent that one should become acquainted with it. Fahrenheit (Danzig, Germany) devised his scale about 1726. He thought that the lowest possible degree of cold was obtained by mixing salt and ice; hence he took as zero the position of the mercury when placed in such a mixture. It is not known why he marked the boiling point of water 212°. The Centigrade scale was proposed by Anders Celsius (Upsala, Sweden) about 1741.

In the Fahrenheit thermometer the boiling point of water at sea level is taken at 212° and the freezing point of water at 32°. In the Centigrade thermometer the boiling point is taken at 100° and the freezing point at 0°. Hence 180° on the Fahrenheit scale equals 100° on the Centigrade scale.

$$180° \text{ F.} = 100° \text{ C.}$$

$$1° \text{ F.} = \tfrac{5}{9}° \text{ C.} \tag{1}$$

$$1° \text{ C.} = \tfrac{9}{5}° \text{ F.} \tag{2}$$

It should be remembered that a division on the Centigrade scale is longer than a division on the Fahrenheit scale. Hence in changing from degrees Centigrade to degrees Fahrenheit we get a greater number of degrees, and from Fahrenheit to Centigrade we get a smaller number of degrees.

Equations (1) and (2) enable us to change readily from one scale to the other.

PROBLEMS

1. Construct a graph to change a number of degrees of one scale to degrees of the other scale. Why is it necessary to locate only two points and draw a straight line through them?

2. Change (*a*) 90° F. to C.; (*b*) 200° F. to C.; (*c*) 40° C. to F.; (*d*) 80° C. to F.; (*e*) 150° F. to C.; (*f*) 112° F. to C. Check by the graph.

3. The sum of a number of degrees F. and a number of degrees C. is 121. When the degrees F. are changed to degrees C. and added to the number of degrees C. the result is 85. Find the number of degrees F. and C.

4. The sum of a number of degrees F. and a number of degrees C. is 53. If each number of degrees is changed into the other scale, the sum is 73. Find the number of degrees F. and C.

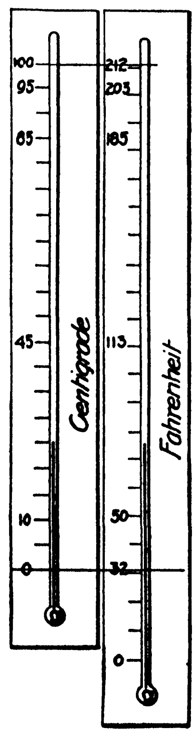

FIG. 81

102. To change thermometer readings from one scale to the other. In the above problems we were dealing with degrees not with thermometer readings. When we change thermometer readings from one scale to the other we must take account of the difference in position of the zeros on the two scales.

Thus find the C. reading when the F. reading is 80°. Looking at Fig. 81, we see that by taking 32° from 80° we get 48°, the number of degrees the F. reading is above 0° C. Then

$$48° \text{ F.} = 48° \times \tfrac{5}{9} \text{ C.} = 26.7° \text{ C.}$$

Similarly, to find the F. reading when the C. reading is 70°,

$$70° \text{ C.} = 70° \times \tfrac{9}{5} \text{ F.} = 126° \text{ F.}$$

But 126° F. takes us only to 32° F. opposite 0° C. Hence we add 32° to get the F. reading, 158°, corresponding to 70° C.

To change from F. to C. readings, subtract 32° and multiply the difference by $\frac{5}{9}$. To change from C. to F. readings, multiply by $\frac{9}{5}$ and add 32° to the product.

$$\text{C.} = \tfrac{5}{9}(\text{F.} - 32°).$$
$$\text{F.} = \tfrac{9}{5}\text{C.} + 32°.$$

103. To determine the relation of the corresponding readings of the two thermometers by experiment. Take several readings of the two thermometers on different days, or obtain the readings by putting the thermometers into water at different temperatures.

Readings obtained:

F.	32°	47°	70°	96°	118°	151°
C.	0	8	21	36	48	66

Locate these points on squared paper. Units: C., horizontal, 1 large square = 10°; F., vertical, 1 large square = 10°. On stretching a thread along these points it will be found that they lie nearly in a straight line. Draw a straight line among the points so that they are distributed evenly above and below it. This line is the graph of the equation which connects the corresponding readings. To find the equation we will suppose that it is of the form

$$\text{F.} = a\,\text{C.} + b, \tag{1}$$

where a and b are unknown numbers which must be determined. Taking the second and fifth points and substituting the readings in (1), we have

$$47 = 8a + b.$$
$$118 = 48a + b.$$

Solving these equations, we get $a = 1.77$, $b = 32.8$.

Substituting these values in (1), we get F. = 1.77° C. + 32.8°.

The readings in the experiment were not taken with sufficient exactness to give a close result (see sect. 59).

Exercise. Take several corresponding readings on the two thermometers and find the relation as above.

PROBLEMS

1. Construct a graph to change the readings of one thermometer to those of the other. Units: horizontal, 1 large square = 20° F.; vertical, 1 large square = 10° C. Take the lower left-hand corner as the origin and mark it − 40°. Show that − 40° is the same reading on both scales. Locate one other point. Why is the graph a straight line?

2. Change the reading of one thermometer to that of the other, and check by the graph:

(*a*) 78° F. to C.	(*e*) 195° F. to C.
(*b*) 18° F. to C.	(*f*) − 20° F. to C.
(*c*) 88° C. to F.	(*g*) − 30° C. to F.
(*d*) 60° C. to F.	(*h*) 0° F. to C.

3. The melting point of the following metals is given in degrees F. Change to the Centigrade scale:

Tin	442° to 446°	Copper	1929° to 1996°
Lead	608° to 618°	Cast iron	1922° to 2075°
Silver	1733° to 1873°	Steel	2372° to 2532°
Gold	1913° to 2282°	Platinum	3227°

4. The following record of temperature was taken from *The Chicago Daily News.*

3 P.M.	69	3 A.M.	67
4 P.M.	68	4 A.M.	66
5 P.M.	68	5 A.M.	65
6 P.M.	67	6 A.M.	64
7 P.M.	66	7 A.M.	65
8 P.M.	67	8 A.M.	66
9 P.M.	67	9 A.M.	67
10 P.M.	68	10 A.M.	67
11 P.M.	68	11 A.M.	68
12 midnight	68	12 noon	70
1 A.M.	69	1 P.M.	73
2 A.M.	68		

Change the readings to Centigrade by using the graph, and on the same sheet of squared paper plot a curve for each of the two sets of readings. Are the curves parallel? Why?

5. What temperature is expressed by the same number on the F. and C. scales?

6. A Fahrenheit and a Centigrade thermometer are placed in a liquid and the F. reading is found to be double the C. reading. What is the temperature of the liquid in degrees C.?

Expansion of Solids by Heat

104. Linear expansion. When a solid is subjected to changes of temperature its length changes; in general, the length increases as the temperature rises, and decreases as it falls. For ordinary temperatures the amount of change is nearly the same for each degree of increase or decrease. The following table gives results that have been secured by experiment; they are only approximate.

Linear Expansion of Solids for 1 Degree, between 0° and 212° F.

Aluminum	.00001234	Lead	.00001571
Brass, plate	.00001052	Platinum	.00000479
Copper	.00000887	Steel, cast	.00000636
Glass, white	.00000478	Steel, tempered	.00000689
Iron, wrought	.00000648	Tin	.00001163
Iron, cast	.00000556	Zinc	.00001407

The amount of expansion is seen to be very small. Thus when we say that the linear expansion of wrought iron is .00000648 we mean that the length of a wrought-iron rod 100 ft. long increases 648 millionths of a foot when the temperature of the rod rises 1 degree. However, provision must be made for this expansion in all construction work; for example, a little space is left between the ends of the rails in laying railway track, hot-water pipes have telescopic joints, and so on.

PROBLEMS

1. Find the linear expansion of copper, wrought iron, and tin for 1° C.

2. A brass wire is 200 ft. long at 0°. Find its length at 85°.

SOLUTION. $200 \times .00001052 \times 85 = .179$ ft.
$200 + .179 = 200.179$ ft. $= 200$ ft. 2.2 in.

3. A steel chain is 66 ft. long at 77°. What will be its length at 32°?

4. The iron girders of a railway bridge are 100 ft. long at a temperature of 10°. What will be the length of the girders at 90°?

5. A lead pipe is 80 ft. long at − 10°. How long will it be at 100°?

6. A brass rod is 5 m. long at 0° C. What is its length at 38° C.?

7. What is the length of a wrought-iron rod at 0° C. if it is 1.56 m. long at 100° C.?

8. What is the length of a copper wire which increases in length 1.2 in. when its temperature is raised 200°?

9. What is the area of a brass plate at 100° C. which measures 8.35 cm. by 4.16 cm. at 0° C.?

10. One brass yardstick is correct at 32° and another at 68°. What is the difference in their lengths at the same temperature?

11. A bar of metal 10 ft. long at 200° increases in length .31 in. when heated to 362°. Find the expansion of 1 ft. for 1°.

12. A plate-glass window is 10 ft. by 12 ft. How much will it change in area when its temperature changes from − 20° to 90°, if its linear expansion for 1° is .000005?

13. An iron steam pipe 200 ft. long at 190° ranges in temperature from 190° to − 4°. What must be the range of motion of an expansion joint to provide for the change in length?

14. A platinum wire and a brass wire are each 100 ft. long at 30°. How much must they be heated to make the brass wire 1 in. longer than the platinum wire?

Suggestion. Let x = the number of degrees.

$$.00001052 \times 100 \times 12x - .00000479 \times 100 \times 12 \times x = 1.$$

15. A copper bar is 10 ft. long. What must be the length of a cast-iron bar in order that the two may expand the same amount for 1°?

16. A steel tape 100 ft. long is correct at 32°. On a day when its temperature was 88° a line was measured and found to be 1 mi. long. What was the error and what was the true length of the line?

17. An iron casting shrinks about $\frac{1}{8}$ in. per linear foot when cooling down to 70°. What is the shrinkage per cubic foot?

18. The Chicago and Oak Park Elevated Railway is about 9 mi. long from Wabash Avenue to Oak Park Station. What is the difference in the length of the rails for a change in temperature from − 20° to 80°?

19. Construct a graph to show the expansion of a steel wire 100 ft. long as its temperature rises from 0° to 2000°.

20. The following table shows the change in the volume of water as its temperature rises from 0° to 17° C. Construct the graph. How does the graph show an exception to the law that the volume increases with a rise in temperature?

Temp.	Volume	Temp.	Volume	Temp.	Volume
0°	1.000000	6°	.999914	12°	1.000334
1	.999948	7	.999952	13	1.000462
2	.999911	8	1.000003	14	1.000593
3	.999889	9	1.000068	15	1.000735
4	.999883	10	1.000147	16	1.000890
5	.999891	11	1.000239	17	1.001057

MEASUREMENT OF HEAT

105. Units. When a definite quantity of heat is applied to various substances different effects are produced, depending on the nature and condition of the substance. An amount of heat may be expressed by any of its effects which can be measured; but it has been found convenient to measure heat by considering the change in temperature it produces.

Two heat units in general use are the *British thermal unit* (B. t. u.) and the *calorie*. For ordinary engineering work the unit is the *British thermal unit*, the amount of heat required to raise 1 lb. of water 1° F. A smaller unit used in laboratory investigations is the *calorie*, the amount of heat required to raise 1 g. of water 1° C. The amount of heat required to raise a quantity of water 1 degree varies with the temperature; but the variation is so small that in practical work we may neglect it and say that the same amount of heat will raise 1 lb. of water from 10° to 11° or from 211° to 212°.

106. The relation between heat and work. In sawing wood, boring iron, and so on, a part of the energy of work becomes heat. It has been found possible to determine the number of foot pounds of work required to raise the temperature of a quantity of water a certain number of degrees. The famous experiments of Joule, in England, in the years 1843–1850, showed that 772 ft. lb. of work raised the temperature of water at 60° F., 1 degree.

His apparatus consisted of a brass cylinder in which water was churned by a brass paddle wheel, which was made to revolve by a falling weight. Later experiments by other methods have given results more nearly exact, and by general consent it is now considered that 778 ft. lb. of work are required to raise the temperature of 1 lb. of water 1° F.

$$1 \text{ B. t. u.} = 778 \text{ ft. lb.}$$
$$1 \text{ ft. lb.} = .00129 \text{ B. t. u.}$$

PROBLEMS

1. How many British thermal units are required to raise the temperature of 120 lb. of water from the freezing point to the boiling point?

2. On a cold day in winter a tank 1 ft. by 2 ft. by 8 ft. was filled with water at a temperature of 100°. When the water had reached the freezing point, how much heat had been given out?

3. If 1 lb. of coal contains 13,500 B. t. u. of heat, how many pounds of coal would be required to raise the temperature of 12 cu. ft. of water 50° if there was an efficiency of 10 per cent?

4. Find the number of British thermal units required to raise the temperature of 20 lb. of lead from 70° to the melting point, 608°. (It takes only .03 as much heat to raise 1 lb. of lead 1° as it does to raise 1 lb. of water 1°.)

5. A steel ingot weighing 1 T. is red-hot (1200°). How much heat is given off as it cools to 70°? (It takes only .12 as much heat to raise 1 lb. of steel 1° as it does to raise 1 lb. of water 1°.)

6. How many foot pounds of work are required to raise the temperature of 20 lb. of water 12°?

SOLUTION. $778 \times 20 \times 12 = 186{,}720$ ft. lb.

7. The temperature of 1 cu. ft. of water was raised from 32° to 70°. How many foot pounds of work did it require?

8. Through how many feet would a weight of 1200 lb. have to fall to generate enough energy to raise the temperature of 8 lb. of water 15°?

9. Find the distance through which a weight of 2 T. could be raised by the expenditure of an amount of heat that would raise the temperature of 2 lb. of water 30°.

10. How many horse power would it take to raise the temperature of 10 cu. ft. of water from 70° to 120° in 1 hr.?

SOLUTION. $$\frac{62.4 \times 10 \times 778 \times 50}{60 \times 33{,}000} = 12.3 \text{ h.p.}$$

11. Find the number of British thermal units per minute required for an engine of the following dimensions: diameter of cylinder, 50 in.; stroke, 36 in.; revolutions per minute, 54; mean effective pressure, 28 lb. per square inch. Find also the number of pounds of coal required per hour, if 1 lb. of coal contains 13,500 B. t. u. and only 10 per cent of the heat of the coal reaches the piston.

12. How many calories are required to raise the temperature of 40 g. of water 20° C.?

13. If 126 calories of heat raised the temperature of a quantity of water 49° C., how many grams of water were there?

14. The temperature of 1 l. of water was raised from 40° C. to the boiling point. How many calories were required?

15. How many calories are there in a British thermal unit?

16. Construct a graph to change calories to British thermal units.

Specific Heat

107. ***Exercise.*** Put equal weights of water and mercury in similar dishes. Note the temperature of each. Place the dishes on an electric stove or in a dish of boiling water. After a time it will be found that there is a considerable difference in the temperatures of the mercury and water.

Since the mercury and the water have received the same amount of heat, it is evident that it takes less heat to raise the temperature of 1 lb. of mercury 1° than is required for 1 lb. of water. It is found by experiment that equal weights of different substances require different amounts of heat to raise their temperatures the same number of degrees. Thus 1 lb. of water requires 1 B. t. u. to raise its temperature 1° F.; 1 lb. of glass requires .2 B. t. u.; 1 lb. of cast iron requires .12 B. t. u.; and 1 lb. of ice requires .5 B. t. u.

108. Definition. The *specific heat* of a substance is the quotient obtained by dividing the amount of heat required to raise the temperature of a given weight of it 1° by the amount of heat required to raise the temperature of an equal amount of water 1°. (Note the similarity to specific density.)

The specific heat of substances varies a little with the temperature, but in practice it is considered to be constant.

Table of Specific Heat

Aluminum	0.21	Silver	0.06
Brass	0.09	Steel	0.12
Copper	0.09	Tin	0.06
Glass	0.20	Zinc	0.09
Iron, cast	0.12	Water	1.00
Iron, wrought	0.11	Ice	0.50
Lead	0.03	Steam	0.48
Mercury	0.03	Air	0.24

PROBLEMS

1. How many British thermal units are required to raise the temperature of 10 lb. of copper from 70° to 200°?

Solution. $200° - 70° = 130°.$

It would require 1300 B. t. u. to raise the temperature of 10 lb. of water 130°. Specific heat of copper = .09.

$$\therefore\ 1300 \times .09 = 117 \text{ B. t. u.}$$

2. How many calories are required to raise the temperature of 500 g. of lead 40° C.?

Solution. $500 \times 40 \times .03 = 600$ calories.

3. Find the amount of heat required to raise the temperature of (*a*) 20 lb. of silver from 70° to the melting point, 733°; (*b*) 30 lb. of aluminum from 70° to the melting point, 1157°; (*c*) 25 lb. of ice from 0° to 32°; (*d*) 1 kg. of mercury 80° C.

4. How many British thermal units are given off by an iron casting which weighs 50 lb., as it cools from 2000° to 70°?

5. If 1 lb. of water at 70° and 1 lb. of mercury at 70° are given the same amount of heat, how hot will the mercury become when the water is at 73°?

6. Equal weights of tin and cast iron are put into a tank of boiling water. When the tin has been heated 10°, how much has the iron been heated?

7. If 15 lb. of water at 200° and 8 lb. of water at 70° are mixed together, what is the resulting temperature?

SOLUTION. Let $t =$ the resulting temperature.

$15(200 - t) =$ number of British thermal units lost.

$8(t - 70) =$ number of British thermal units gained.

$$8(t - 70) = 15(200 - t).$$

Solving, $t = 154.8°$.

Check.
$$15(200 - 154.8) = 8(154.8 - 70).$$
$$15 \times 45.2 = 8 \times 84.8.$$
$$678 = 678.4.$$

8. 20 lb. of water at the freezing point are poured into 25 lb. of water at the boiling point. What is the temperature of the mixture?

9. A tank 2 ft. by 3 ft. by 6 ft. is two thirds full of water at 60°. If the tank is filled with water at 120°, what is the temperature of the mixture?

10. How many pounds of water at 40° must be mixed with 60 lb. of water at 100° to obtain a temperature of 80°?

SOLUTION. Let $p =$ number of pounds at 40°.

$p(80 - 40) =$ number of British thermal units gained.

$60(100 - 80) =$ number of British thermal units lost.

$$p(80 - 40) = 60(100 - 80).$$
$$p = 30 \text{ lb.}$$

Check. $30(80 - 40) = 1200$ B. t. u. gained.

$60(100 - 80) = 1200$ B. t. u. lost.

11. How many pounds of water at 180° must be mixed with 1 cu. ft. of water at 56° to obtain a temperature of 112°?

12. How many grams of water at 0° C. must be mixed with 1 kg. of water at 100° C. to obtain a temperature of 80° C.?

13. How much water at 212° and how much water at 32° must be taken to make up 36 lb. at 97°?

SOLUTION. Let $x =$ number of pounds at 212°.

$y =$ number of pounds at 32°.

$$x + y = 36. \quad (1)$$

$$x(212 - 97) = y(97 - 32). \quad (2)$$

$$115x = 65y. \quad (3)$$

$$x = \tfrac{13}{23}y. \quad (4)$$

Substitute (4) in (1), $\frac{13}{23}y + y = 36.$ (5)

$$y = 23. \quad (6)$$

Substitute (6) in (1), $x = 13.$ (7)

Check.

$$13(212 - 97) = 23(97 - 32).$$

$$13 \times 115 = 23 \times 65.$$

$$115 = 115.$$

14. How much water at 180° and at 81° must be taken to fill a tank which contains 90 lb., if it is desired to have the temperature of the mixture 125°?

15. Into a dish containing some water at 4° C. was poured some water at 75° C. How many grams of each were taken if there were in all 250 g. at a temperature of 60° C.?

16. An iron casting when red-hot (1300°) was put into a tank containing 2 cu. ft. of water at 170°. If the temperature of the water rose to 200°, what was the weight of the casting?

SOLUTION. Let $x =$ number of pounds of cast iron.

Specific heat of cast iron $= .12$.

2 cu. ft. of water $= 62.4 \times 2 = 124.8$ lb. of water.

$(1300 - 200) \times .12x =$ number of British thermal units lost by the iron.

$(200 - 170) \times 124.8 =$ number of British thermal units gained by the water.

$$(1300 - 200) \times .12x = (200 - 170) \times 124.8.$$

$$132x = 3744.$$

$$x = 28.4.$$

Check. $28.4 \times 1100 \times .12 = 3749$ B. t. u. lost.

$124.8 \times 30 = 3744$ B. t. u. gained.

17. If a mass of lead at 500° was put in a gallon of water ($8\frac{1}{3}$ lb.) and the temperature of the water rose from 77° to 80°, what was the weight of the lead?

18. An 80-lb. mass of steel at 1000° is placed in a tank containing water at 60°. If the final temperature is 70°, how many pounds of water are in the tank?

19. If 20 lb. of brass at 300° were placed in a tank containing 1 cu. ft. of water at 72°, what would be the final temperature?

20. If 500 g. of brass at 100° C. were placed in 188 g. of water at 17.5° C. and the final temperature was 33.5° C., find the specific heat of the brass.

SOLUTION. Let s = the specific heat of the brass.

$500(100 - 33.5)s$ = number of calories lost by the brass.

$188(33.5 - 17.5)$ = number of calories gained by the water.

$$500(100 - 33.5)s = 188(33.5 - 17.5).$$

$$s = \frac{188 \times 16}{500 \times 66.5}$$

$$= .0905.$$

Check. $500 \times 66.5 \times .0905 = 3010$ calories lost.

$188 \times 16 = 3008$ calories gained.

21. The following data were obtained by experiment. Find the specific heat of each metal.

No.	Water (grams)	Temperature of Water	Material	Weight used	Initial Temperature	Final Temperature
1	188	18.5°	Zinc	250 g.	100° C.	28.5° C.
2	188	11.0°	Cast iron	750 g.	100° C.	39.0° C.
3	188	16.5°	Lead	700 g.	100° C.	26.0° C.

22. 45 g. of zinc at 100° C. were immersed in 52 g. of water at 10° C. If the temperature of the water rose to 17° C., find the specific heat of the zinc, assuming that no heat was absorbed by the dish containing the water.

23. A room 20 ft. by 30 ft. by 10 ft. is to be heated from a temperature of 32° to 72°. Assuming that 1 cu. ft. of air at 32° weighs .08 lb., that the specific heat of air is .24, and that 8 per cent of the fuel is available for raising the temperature, how many pounds of hard coal (1 lb. coal = 13,500 B. t. u.) would be required?

LATENT HEAT

109. ***Exercise.*** Place a dish of melting ice on a stove. Though the melting ice and water receive heat continuously, a thermometer placed in the dish will stand at 32° F. till all the ice is melted. Then the mercury will rise till the boiling point is reached. The temperature will remain at 212° till all the water is evaporated.

110. Latent heat. This heat which goes into a substance and produces a change in form rather than an increase in temperature is called *latent* (*hidden*) *heat.*

The following table gives the approximate number of British thermal units absorbed by 1 lb. of the substance in changing from solid to liquid or liquid to solid.

LATENT HEAT OF FUSION		LATENT HEAT OF VAPORIZATION	
Bismuth	22.75	Alcohol	363 at 172° F.
Cast iron	42.5	Ether	162 at 93° F.
Ice	142.6	Mercury	117 at 580° F.
Lead	9.66	Water	965.7 at 212° F.
Silver	43.	Water	1044.4 at 100° F.
Tin	27.	Water	1091.7 at 32° F.
Zinc	54.		

PROBLEMS

1. Find the number of British thermal units required to melt the following masses of metal after they have been brought to the melting point: (*a*) 120 lb. of iron; (*b*) 24 lb. of lead; (*c*) 55 lb. of silver; (*d*) 40 lb. of tin.

SOLUTION. (*a*) $42.5 \times 120 = 5100$ B. t. u.

2. How much heat is given out by 50 lb. of molten zinc as it becomes solid?

3. How much heat is required to melt 16 lb. of tin at 70° if its melting point is 442°?

SOLUTION. Specific heat of tin = .06.

$$442° - 70° = 372°.$$

$$16 \times 372 \times .06 = 357 \text{ B.t.u. to raise to } 442°.$$

$$16 \times 27 = \underline{432} \text{ B.t.u. to melt.}$$

$$789 \text{ B.t.u., total.}$$

4. How much heat is required to melt 150 lb. of lead at 70° if its melting point is 622°?

5. 1 T. of molten iron at 2200° cooled to 70°. How much heat was given off if the melting point was 2000°?

6. A cake of ice weighing 50 lb. is at 0°. How many British thermal units are required to melt it and bring the water to the boiling point?

7. If 1 lb. of ice at 32° is put into 2 lb. of water at 80°, how much of the ice will melt?

SOLUTION. $(80° - 32°)2 = 96$ B. t. u. available to melt the ice.

$142.6 =$ number of British thermal units required to melt 1 lb. of ice at 32°.

$$\frac{96}{142.6} = .67 \text{ lb. ice melted.}$$

Check. $142.6 \times .67 = 96.$

$$\tfrac{96}{2} + 32 = 80.$$

8. How much boiling water will be required to melt 12 lb. of ice at 32°?

9. What would be the final temperature of the water if 16 lb. of ice at 32° were put into 40 lb. of boiling water?

SOLUTION. Let $t =$ the final temperature.

$142.6 \times 16 =$ number of British thermal units to melt the ice.

$40(212 - t) =$ number of British thermal units lost.

$16(t - 32) + 142.6 \times 16 =$ number of British thermal units gained.

$$40(212 - t) = 16(t - 32) + 142.6 \times 16.$$

Solve and check.

10. 5 lb. of molten lead at the melting point 610° were poured into 50 lb. of water at 70°. What is the resulting temperature?

11. 1 lb. of lead at 212° is placed on a cake of ice at 30°. How much ice will it melt?

12. How many pounds of steam at 212° will melt 20 lb. of ice at 32°?

13. How many pounds of zinc at 500° must be added to 100 lb. of water at 75° to heat it to 100°?

14. 20 lb. of ice at 0° are immersed in 200 lb. of water at 200°. What is the temperature of the water when the ice has just melted?

15. How many pounds of water at 70° would be evaporated at 212° by 1 T. of coal, assuming an efficiency of 12 per cent, and 13,500 B. t. u. per pound of coal?

16. The temperature of 1 lb. of water in a teakettle rises from 32° to 212° in ten minutes. (*a*) How long before the kettle will boil dry? (*b*) If the kettle contained 5 lb. of water, how many British thermal units would be needed to boil it dry?

SOLUTION. (*a*) $212° - 32° = 180°.$

$$\frac{180}{10} = 18 \text{ B. t. u. per minute.}$$

$$\frac{965.7}{18} = 53.7 \text{ minutes.}$$

17. If 1 lb. of ice at 0° is put on an electric stove which gives out 8 B. t. u. per minute, find the number of British thermal units and the number of minutes required (*a*) to raise the ice to 32°; (*b*) to melt the ice; (*c*) to raise the water to 212°; (*d*) to evaporate the water; (*e*) to raise the steam to 312°. Construct a graph and write the results on it. Units: horizontal, 1 large square = 10 minutes; vertical, 1 large square = 20°.

CHAPTER XVII

ELECTRICITY

111. ***Exercise.*** Into a tumbler two thirds full of water pour 2 ccm. of sulphuric acid. Stand in this solution a strip of zinc and a strip of copper each well sandpapered. Take 6 ft. of No. 20 insulated copper wire and wind about 25 turns around a large lead pencil, leaving about a foot uncoiled at each end. Cut the insulation from the ends of the wire and wrap the ends around the strips, as shown in Fig. 82. To get good connections it may be necessary to cut into the edge of the strips and wedge the wire under the pieces lifted.

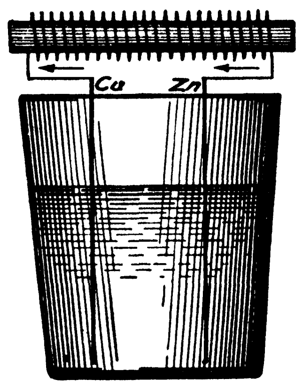

FIG. 82

Take a piece of soft wrought iron and sprinkle some iron filings on each end. Result? Place the iron within the coil, as shown in Fig. 82, and drop some iron filings on the ends. Result? Is the iron magnetized? If so, we have generated a current of electricity and magnetized the iron. (See Shepardson's "Electrical Catechism.")

112. Nature of electricity. The exact nature of electricity is not known. Some scientists think it is a condition of the ether. Others think that it is a form of energy or force. However, much is known about the laws of electricity and about methods of applying it to useful work.

113. Electromotive force. When the strips of copper and zinc were placed in the solution of sulphuric acid, the acid dissolved the zinc strip faster than it did the copper strip. We say that this caused an electrical flow from the zinc to the copper; that is, an electromotive force exists between the two strips. Whatever produces or tends to produce an electrical flow is called an *electromotive force* (e. m. f.). When the two strips are connected by the wire this action takes place continuously and there is said to be a flow of electricity from the zinc to the copper and through the wire to the zinc again. Though we cannot perceive this flow by any of our senses, we can see the effects it produces.

114. The electrical units. It is not possible nor is it necessary to give exact definitions here. However, definitions can be given which are readily understood and are sufficiently exact for practical purposes.

The volt. We may think of the electromotive force existing between the strips of zinc and copper in the cell described above, as *pressure.* It takes pressure to force a current of electricity through a wire, just as it takes pressure to drive a stream of water through a pipe. To measure this pressure we have the unit of electromotive force called a *volt* (from Volta, an Italian physicist who lived from 1745 to 1827). The pressure of a gravity or crowfoot cell is about 1.1 volts. When a wire is moved across the magnetic lines of force which exist between the poles of a magnet, an electrical flow is produced in the wire. A *volt* is the electromotive force set up in a wire that crosses magnetic lines of force at the rate of one hundred million per second. In a dynamo the armature may be thought of as a bundle of wires which cut across the lines of force of the field magnet as the armature revolves.

The ohm. The pressure (electromotive force) produces a flow of electricity which meets with resistance in the conductor. Just as the frictional resistance in a water pipe opposes the flow of water, so the electrical resistance of a conductor opposes the flow of electricity. The unit of resistance is the *ohm* (from Ohm, a German mathematician who lived from 1789 to 1854).

The ohm is nearly equal to the resistance of 1000 ft. of copper wire .1 in. in diameter. Different substances have different degrees of resistance. The resistance of metals increases slightly as the temperature rises, but the resistance of carbon (incandescent lamp filament) decreases with a rise in temperature. Thus the resistance of a 16 candle power incandescent-lamp carbon filament is about 220 ohms when hot, but it may be as high as 400 ohms when cold.

Resistance varies directly as the length and inversely as the cross section of a conductor. Thus if the resistance of 100 ft. of wire is 2 ohms, the resistance of 300 ft. of the same wire is 6 ohms; if the resistance of a wire .3 in. in diameter (cross section, 7.07 sq. in.) is 8 ohms, the resistance of a wire of the same material and length .6 in. in diameter (cross section, 28.27 sq. in.) is 2 ohms.

The ampere. The unit for measuring the rate of the electrical flow is the *ampere.* An ampere (from Ampère, a French physicist who lived from 1775 to 1836) may be defined as the current which an electromotive force of 1 volt will send through a conductor whose resistance is 1 ohm.

The number of amperes of current corresponds quite closely to the rate of flow of a stream of water. We may say that at a certain point in an electrical circuit the rate of flow is 5 amperes, just as we would say that at a certain point in a water pipe the rate of flow of the water is 10 gal. per minute.

Given an electromotive force of 1 volt, a circuit of 1 ohm resistance, and we have a current of 1 ampere.

115. Ohm's law. A very simple relation exists between the electromotive force, resistance, and current in a closed circuit.

Let $V =$ the number of volts of electromotive force,

$R =$ the number of ohms of resistance,

$A =$ the number of amperes of current,

and we have Ohm's law, $\dfrac{V}{R} = A.$

In words this law may be stated as follows: The number of volts of electromotive force divided by the number of ohms of resistance gives the number of amperes of current flowing through a circuit. This law was first formulated by Ohm in 1827.

PROBLEMS

1. How many amperes are there in a circuit of 20 ohms resistance if the dynamo generates 110 volts?

SOLUTION. $\frac{V}{R} = \frac{110}{20} = 5.5$ amperes.

2. A battery sends a current of 5 amperes through a circuit. If the electromotive force is 10 volts, find the total resistance of the circuit.

3. If a cable has a resistance of .004 ohm and a current of 20 amperes passes through it, what is the electromotive force?

4. Find the resistance of an incandescent lamp which takes a current of .5 ampere when connected to a 110-volt main.

5. If a telegraph wire has a resistance of 200 ohms, how many amperes will be sent through it by a pressure of 10 volts?

6. The wires in an electric heater will stand 8 amperes without becoming unduly heated. What must be the resistance for 110 volts?

7. A dynamo generates 110 volts. What is the total resistance of the circuit if there is a current of 40 amperes?

8. A 32 candle power lamp for a 220-volt circuit has a resistance of 330 ohms, and a 16 candle power lamp for a 110-volt circuit has a resistance of 180 ohms. Which lamp has the greater current?

9. Construct a curve to show the relation between the electromotive force and the resistance of a circuit in which the current is 20 amperes, as the resistance varies from 1 to 10 ohms.

10. If the electromotive force of a dynamo remains constantly at 120 volts, construct a curve to show the changes in the current as the resistance increases from 10 to 120 ohms.

116. Resistances in combination. In the preceding problems the resistance of the circuit was considered as a single resistance, but in practical work the circuit is made up of several parts. Thus in an electric lighting system the total resistance is made up of the resistances of the dynamo, lamps, and connecting wires. The parts of a circuit may be combined in two distinct ways.

117. Series circuits. When the different parts of a circuit are joined end to end and the whole current flows through each part, the circuit is called a *series circuit.* Let D in Fig. 83 be a dynamo maintaining an electromotive force of 110 volts measured across the terminals AB. This means that 110 volts are continuously generated and used up in forcing the current through the circuit $BCEFA$. Hence we may say that from B to A there is a *drop in voltage* of 110 volts.

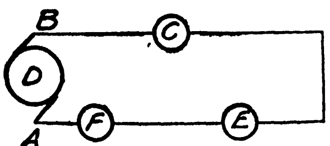

FIG. 83

Let C, E, and F be arc lights having resistances of 4.2, 4.6, and 4.8 ohms respectively, and let the resistance of the line be 4 ohms.

$$\begin{aligned}\text{Total resistance} &= 4.2 + 4.6 + 4.8 + 4\\ &= 17.6 \text{ ohms.}\end{aligned}$$

$$\text{By Ohm's law, } \frac{V}{R} = \frac{110}{17.6} = 6.25 \text{ amperes.}$$

At any point in the circuit the current is 6.25 amperes, since in a series circuit the current is constant. But there is a continual drop in the voltage along the circuit as the voltage is used up in forcing the current along its path. This drop in voltage, or drop of potential, as it is sometimes called, follows Ohm's law.

The drop in lamp $C = A \cdot R = 6.25 \times 4.2 = \ 26.2$ volts.
The drop in lamp $E = A \cdot R = 6.25 \times 4.6 = \ 28.8$ volts.
The drop in lamp $F = A \cdot R = 6.25 \times 4.8 = \ 30.0$ volts.
The drop in the line $= A \cdot R = 6.25 \times 4 \quad = \ 25.0$ volts.
Total drop $= 110.0$ volts. *Check.*

The arc lights in general use to light city streets are connected in series, and the entire current goes through each lamp.

118. Ammeter. The number of amperes of current is measured by an ammeter. It consists of a coil of wire suspended

FIG. 84

between the poles of a magnet so that it rotates through a small angle when the current passes through. The coil carries a light needle. The instrument is graduated by passing through it currents of known strength, and marking on the scale the position of the needle. The type of ammeter commonly used is cut into the circuit when a measurement is made.

119. Voltmeter. Voltage (electromotive force, drop of potential) is measured by the voltmeter. Most voltmeters are simply special forms of ammeters. The voltmeter also is graduated by experiment. It is put on circuits of known voltage and the position of the needle is marked on the scale. In using the voltmeter its terminals are connected to the ends of the parts of the circuit in which the voltage is to be measured; the

FIG. 85

reading of the voltmeter is the number of volts of electromotive force, or drop of potential. If a voltmeter is connected across the terminals of an arc light and the reading is 47 volts, it means that 47 volts are used up in running that arc light.

In Fig. 86 the ammeter A is arranged to measure the current produced by the dynamo D; and the voltmeter V is connected

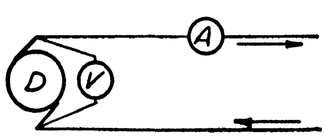

FIG. 86

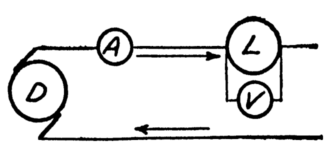

FIG. 87

to show the electromotive force between the terminals of the dynamo. Fig. 87 shows an ammeter and a voltmeter arranged to measure the current and drop in voltage in an arc lamp L.

PROBLEMS

1. Three wires having resistances of 2, 5, and 8 ohms respectively are joined end to end and a voltage of 90 volts is applied. How many amperes of current are there?

2. Two wires of resistances 6 and 8 ohms respectively are joined in series. If the current is 1.8 amperes, find the voltage.

3. Two incandescent lamps are in series and one has twice as great resistance as the other. If the voltage is 110 and the current is $\frac{1}{3}$ of an ampere, find the resistance of each lamp.

SOLUTION. Let $R =$ the resistance of one lamp.

$2R =$ the resistance of the other lamp.

$$\frac{V}{R} = \frac{110}{3R} = \frac{1}{3}.$$

$$R = 110 \text{ ohms.}$$

$$2R = 220 \text{ ohms.}$$

$$\text{Total} = 330 \text{ ohms.}$$

Check.

$$\frac{V}{R} = \frac{110}{330} = \frac{1}{3}.$$

4. Find the internal resistance of a battery which gives a current of 1.5 amperes with an electromotive force of 5 volts, if the external resistance is 1.33 ohms.

5. An iron wire and a copper wire are in series. If the voltage is 12 volts, the current 2.8 amperes, and the copper wire has a resistance of 3 ohms, find the resistance of the iron wire.

6. A circuit consists of two wires in series. An electromotive force of 30 volts gives a current of 3.2 amperes. If the length of one wire is doubled and the other is made 5 times as long, the current is .84 ampere. Find the resistance of the two wires.

7. What voltage is necessary to furnish a current of 9.6 amperes, if the circuit is made up of 2 mi. of No. 6 Brown & Sharpe gauge copper wire (resistance of 1000 ft., .395 ohm) and 10 arc lights in series, each of 4.8 ohms resistance? Find also the drop in voltage in the wire and in the lamps.

8. A dry cell is used to ring a door bell. The resistance of the wire in the bell is 1.5 ohms, of the line .5 ohm, and of the cell 1 ohm. If the electromotive force of the cell is 1.4 volts, what current will flow when the circuit is closed?

9. What is the resistance per mile of No. 20 Brown & Sharpe gauge copper wire, if the voltmeter connected to the ends of 100 ft. of the wire reads 5.13 volts and the ammeter reads 5 amperes?

10. An arc-light dynamo of 30 ohms resistance supplies a current of 6.8 amperes through 12 mi. of No. 6 Brown & Sharpe gauge copper wire to a series of 50 arc lights, each adjusted to 47 volts. Find the electromotive force of the dynamo.

Suggestion. The drop in voltage in the lamps $= 47 \times 50$. Find the drop in voltage in the dynamo and in the line by $V = R \cdot A$. The total voltage is the sum of the drop in voltage in the three parts of the circuit. Check by finding the total resistance of the circuit and dividing it into the total electromotive force; this should give 6.8 amperes.

11. In an electric lighting system there are 6 mi. of No. 6 Brown & Sharpe gauge copper wire, and 80 arc lights, each having a resistance of 4.5 ohms. The resistance of the dynamo is 3 ohms and the electromotive force is 3725 volts. Find (*a*) the total resistance; (*b*) the current; (*c*) the fall of voltage in the dynamo, line, and lamps.

12. The voltage across the mains of an electric-light circuit is 110 volts. If a voltmeter is connected across the mains in series with a resistance of 6000 ohms, it reads 70 volts. What is the resistance of the voltmeter?

SOLUTION. Since there is a drop of 70 volts in the voltmeter, there is a drop of $110 - 70 = 40$ volts in the resistance.

$$\frac{V}{R} = \frac{40}{6000} = \frac{1}{150} \text{ ampere.}$$

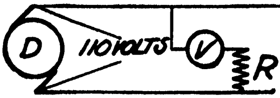

FIG. 88

Since the current is the same in all parts of the circuit, $\frac{V}{A} = \frac{70}{\frac{1}{150}} =$ 10,500 ohms, resistance of the voltmeter.

Check. $10{,}500 + 6000 = 16{,}500$ ohms.

$$\frac{V}{R} = \frac{110}{16{,}500} = \frac{1}{150} \text{ ampere.}$$

13. A coil of wire is placed in series with a voltmeter having a resistance of 18,000 ohms across 110-volt mains. If the voltmeter reading is 60 volts, find the resistance of the coil of wire.

14. A voltmeter has a resistance of 10,000 ohms. What will be the reading of the voltmeter when connected across 110-volt mains with a man having a hand-to-hand resistance of 5000 ohms?

120. Multiple circuits. When the branches of a circuit are connected so that only a part of the current flows through each of the several branches, the circuit is called a *multiple*, *parallel*, or *divided circuit*. Fig. 89 shows three incandescent lamps connected in multiple. The ordinary incandescent lamps used in houses are connected in multiple between mains from the terminals of the dynamo. The full electromotive force of the dynamo, except the drop in voltage in the wires, acts upon each lamp; but only a part of the current goes through each lamp.

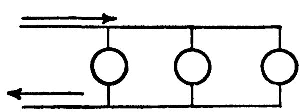

FIG. 89

121. To find the total resistance of a multiple circuit. In Fig. 90 let the drop in voltage from B to C be 12 volts, and a and b have resistances of 2 and 4 ohms respectively. The pressure (electromotive force) in each branch is 12 volts; just as in a water pipe of similar construction the pressure would be the same in each branch.

$$\frac{V}{R} = \frac{12}{2} = 6 \text{ amperes in } a.$$

$$\frac{12}{4} = 3 \text{ amperes in } b.$$

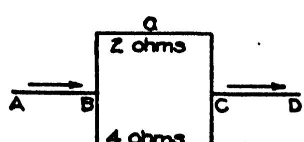

FIG. 90

Hence the total current is $6 + 3 = 9$ amperes.

The total resistance of a and b is given by

$$R = \frac{V}{A} = \frac{12}{9} = \frac{4}{3} \text{ ohms.}$$

We will now obtain a general formula for the total resistance of a multiple circuit.

Let $V =$ the drop in voltage from B to C.

$r_1 =$ resistance of a.

$r_2 =$ resistance of b.

$R =$ total resistance.

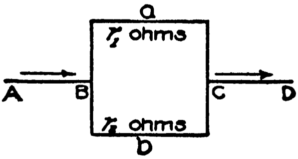

FIG. 91

$\frac{V}{r_1} =$ current in a.

$\frac{V}{r_2} =$ current in b.

$$\frac{V}{r_1} + \frac{V}{r_2} = \frac{V(r_1 + r_2)}{r_1 r_2} = \text{total current.}$$

But $\frac{V}{R} =$ total current.

$$\therefore \frac{V}{R} = \frac{V(r_1 + r_2)}{r_1 r_2},$$

or
$$R = \frac{r_1 r_2}{r_1 + r_2}. \tag{1}$$

In a multiple circuit of two branches the total resistance is the product of the resistances divided by their sum.

In a similar manner let the student work out the formulas for three and four branches, obtaining:

$$R_3 = \frac{r_1 r_2 r_3}{r_1 r_2 + r_2 r_3 + r_1 r_3}, \tag{2}$$

$$R_4 = \frac{r_1 r_2 r_3 r_4}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2}. \tag{3}$$

When two, three or four equal resistances are combined in multiple, we have from (1), total resistance $= \dfrac{r^2}{2r} = \dfrac{r}{2}$,

from (2), total resistance $= \dfrac{r^3}{3r^2} = \dfrac{r}{3}$,

from (3), total resistance $= \dfrac{r^4}{4r^3} = \dfrac{r}{4}$.

Thus when ten 16 candle power lamps of 220 ohms resistance are connected in multiple the total resistance is

$$\frac{220}{10} = 22 \text{ ohms.}$$

122. Graphical method of finding the total resistance. The total resistance of a multiple circuit can be readily determined by a graph.

EXERCISES

1. Construct a graph to find the result of combining resistances of 20 and 30 ohms in multiple.

Take OX any convenient length, and with convenient units lay off $OM = 30$ ohms, and $XN = 20$ ohms. Draw ON and XM, intersecting at A. $AB = 12$ ohms, the total resistance.

That is, $AB = \dfrac{OM \cdot XN}{OM + XN}$. (1)

Prove geometrically. The two pairs of similar triangles OBA, OXN; and XBA, XOM give two equations. Eliminating XB from these equations gives (1). (See Problem 14, p. 77.)

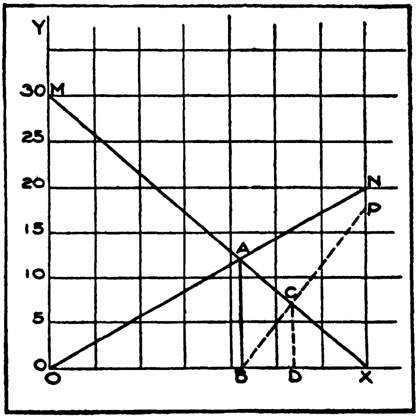

FIG. 92

A similar construction gives the total resistance of any number of resistances in multiple. Thus, given the resistances 20, 30, and 18 ohms, combine 20 and 30 ohms, as above. Then lay off $XP = 18$ ohms. Draw PB, intersecting XA at C, and CD is the total resistance.

2. What resistance must be combined with 24 ohms to obtain a total resistance of 8 ohms?

Take OX any convenient length and with a convenient unit lay off $OM = 24$ ohms. Draw MX. On MX take a point A, such that $AB = 8$ ohms. Draw OA, and extend to meet XP at N. $XN = 12$ ohms, the required resistance.

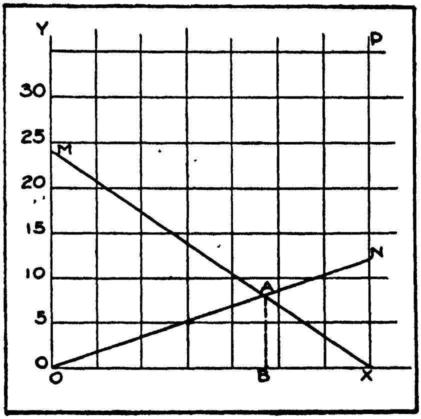

FIG. 93

The graphical method should be used to solve and check some of the following problems.

PROBLEMS

1. Three resistances of 20, 30, and 40 ohms are joined in multiple. Find the total resistance.

SOLUTION. $R = \dfrac{20 \cdot 30 \cdot 40}{20 \cdot 30 + 20 \cdot 40 + 30 \cdot 40}$
$= 9.2$ ohms.

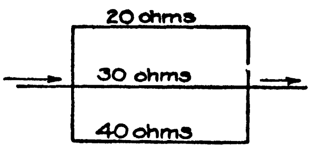

FIG. 94

2. If 110 volts be applied to the circuit in Problem 1, what is the total current and the current in each branch?

SOLUTION. $\dfrac{V}{R} = \dfrac{110}{9.2} = 12$ amperes.

$\frac{110}{20} = 5.5$ amperes.
$\frac{110}{30} = 3.7$ amperes.
$\frac{110}{40} = 2.8$ amperes.
12 amperes. *Check.*

3. Three lamps having resistances of 60, 120, and 240 ohms are connected in multiple. If they are supplied with 110 volts, find the total resistance, the total current, and the current in each lamp.

4. Two lamps of 100 and 150 ohms are put in parallel with each other, and the pair is joined in series with a lamp of 100 ohms. If the electromotive force is 200 volts, what will be the current?

5. A resistance of 10 ohms is put in parallel with an unknown resistance. If an electromotive force of 120 volts gives a current of 20 amperes, find the unknown resistance.

SOLUTION. Let $r =$ the unknown resistance.

Then $\frac{10r}{10+r} =$ the total resistance.

$$\frac{120}{20} = 6 = \text{the total resistance.}$$

$$\frac{10r}{10+r} = 6.$$

$$10r = 60 + 6r.$$

$$r = 15 \text{ ohms.}$$

Check. $\frac{10 \times 15}{10+15} = \frac{150}{25} = 6$ ohms.

6. A lamp of unknown resistance is put in parallel with a lamp of 220 ohms resistance. If a voltage of 110 volts gives a current of 1.6 amperes, what is the unknown resistance?

7. The total resistance of three wires in multiple is 1.52 ohms. If the resistance of two of the wires is 3 and 5 ohms respectively, what is the resistance of the third?

8. The total resistance of two conductors in multiple is 4.8 ohms, and the sum of the two resistances is 20; find them.

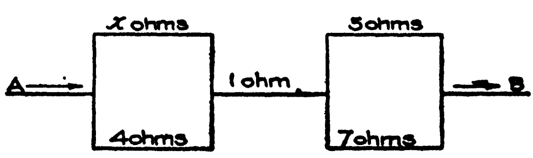

FIG. 95

9. The total resistance between *A* and *B* in Fig. 95 is 5.25 ohms. Find the resistance *x*.

10. Three resistances in parallel are in the ratio 1 : 2 : 3. If an electromotive force of 120 volts gives a current of 11 amperes, find each resistance.

11. Twenty 16 candle power 110-volt lamps are in multiple. If the resistance of each lamp is 220 ohms, what is the total resistance, and what is the current?

12. A 110-volt incandescent lighting circuit divides into three multiple circuits of 5, 8, and 10 lamps respectively. If the resistance of each lamp is 220 ohms, find (*a*) the resistance of each branch; (*b*) the total resistance; (*c*) the current; (*d*) the current in each branch.

13. Construct a curve to show the change in the resistance of a multiple circuit consisting of a number of incandescent lamps of 220 ohms, as the number of lamps increases from 1 to 20.

14. Two conductors of 12 and 18 ohms respectively are in multiple. What resistance must be placed in series with the multiple circuit to give a current of 5 amperes with an electromotive force of 110 volts?

Work and Power

123. The watt. When an electromotive force overcomes the resistance of a conductor and causes a current to flow, work is done. This is analogous to the case where work is done by the pressure of steam on the piston of an engine. The number of pounds pressure multiplied by the number of feet through which the piston is moved gives the number of foot pounds of work. Power is the rate of doing work. The unit of mechanical power is the horse power, the rate of doing work equal to 33,000 ft. lb. per minute. The unit of electrical power is the *watt* (James Watt, Scotland, 1736–1819, practically the inventor of the modern steam engine), the rate of doing work equal to 44¼ ft. lb. per minute.

Power in watts equals the number of volts multiplied by the number of amperes.

$$W = V \cdot A. \tag{1}$$

Thus if a dynamo supplies a current of 50 amperes at a voltage of 110 volts, the power delivered is $110 \times 50 = 5500$ watts.

From Ohm's law $\frac{V}{R} = A$, (1) may be written

$$W = A^2 \cdot R. \tag{2}$$

$$W = \frac{V^2}{R}. \tag{3}$$

In words: *watts equal volts multiplied by amperes;* (1)
watts equal current squared multiplied by resistance; (2)
watts equal volts squared divided by resistance. (3)

To express watts in horse power:

Since $\quad$ 1 h. p. = 33,000 ft. lb. per minute,
and $\quad$ 1 watt = $44\frac{1}{4}$ ft. lb. per minute,

$$1 \text{ h. p.} = \frac{33{,}000}{44\frac{1}{4}} \text{ watts.}$$

$$1 \text{ h. p.} = 746 \text{ watts.}$$

124. The kilowatt. For many purposes a larger unit than a watt is convenient. Hence 1000 watts, called a *kilowatt* (kw.), is sometimes taken as the unit of power.

125. The kilowatt hour. A *kilowatt hour* is a practical unit used in measuring electrical energy. It is the energy expended by 1 kw. in 1 hr. Thus 20 kw. hr. might mean 2 kw. for 10 hr., 5 kw. for 4 hr., 1 kw. for 20 hr., and so on.

$$1 \text{ kw. hr.} = 44\tfrac{1}{4} \times 1000 \times 60 \text{ ft. lb.}$$

$$1 \text{ h. p. hr.} = 33{,}000 \times 60 \text{ ft. lb.}$$

Hence

$$1 \text{ kw. hr.} = \frac{44\frac{1}{4} \times 1000 \times 60}{33{,}000 \times 60} \text{ h. p. hr.}$$

$$1 \text{ kw. hr.} = 1.34 \text{ h. p. hr.}$$

PROBLEMS

1. An arc light requires 10 amperes at 45 volts. How much power does it absorb?

SOLUTION. $W = V \cdot A = 45 \times 10 = 450$ watts.

$$\frac{450}{746} = .6 \text{ h. p.}$$

2. A 16 candle power incandescent lamp is on a 110-volt circuit and takes $\frac{1}{2}$ ampere. How many watts per candle power are required?

SOLUTION. $W = V \cdot A = 110 \cdot \frac{1}{2} = 55$ watts.

$$\frac{55}{16} = 3.5 \text{ watts per candle power.}$$

3. A dynamo has a voltage of 550 volts and is producing 40 kw. How many amperes in the current?

4. How many watts will be lost in forcing the current through the armature of a dynamo, if the resistance is .035 ohm and the current is 30 amperes?

5. A 150-kw. dynamo was supplying 273 amperes. What was the voltage of the dynamo?

6. How many horse power are required to send a current of 65 amperes through 10 mi. of No. 6 B. & S. gauge copper wire?

7. A current of 15 amperes flows through 100 ft. of iron wire whose resistance is $\frac{1}{8}$ ohm per foot. How many watts are lost in the wire?

8. With a current of 50 amperes 450 watts are absorbed in the conductor. Find the drop in voltage in the conductor.

9. A voltmeter has a resistance of 17,000 ohms. If placed in a circuit of 110 volts, how much power is required to operate it?

10. A 200-volt lamp takes $\frac{1}{3}$ ampere. How many watts are required for 30 such lamps? How many horse power are required to drive the dynamo if it has an efficiency of 90 per cent?

SOLUTION. $200 \times \frac{1}{3} \times 30 = 2000$ watts.

$$\frac{2000}{746} = 2.68 \text{ h. p. } 100 \text{ per cent efficiency.}$$

$$\frac{2.68}{.90} = 2.98 \text{ h. p. } 90 \text{ per cent efficiency.}$$

11. In a room there are thirty 16 candle power incandescent lamps, each taking .52 ampere at 110 volts; and 3 arc lights, each taking 6.8 amperes at 50 volts. How many watts and how many horse power are required to operate these lights?

12. How many incandescent lamps, each having a resistance of 220 ohms and requiring a current of .5 ampere can be run by a 10-kw. generator?

13. A 25 h. p. dynamo is running at 550 volts. How many amperes in the current? How many 16 candle power incandescent lamps can be placed on the circuit if each lamp takes 55 watts and there is a loss of 10 per cent on the line?

14. In an electric-lighting circuit there are 60 arc lights, each taking 50 volts, and 15 mi. of wire having a resistance of 2.1 ohms per mile. If the current is 9.6 amperes, how many watts are required to run the lights?

15. Find the energy in foot pounds expended per candle power in a 16 candle power incandescent lamp in 1 hr., if it takes $\frac{1}{2}$ ampere at 110 volts.

16. If a 500 candle power arc light requires 50 volts with 9.6 amperes, how many foot pounds per candle power are expended in 1 hr.? How does this compare with the result in Problem 15?

17. A 10-kw. dynamo has an efficiency of 88 per cent. How many horse power are required to drive it?

18. The efficiency of a dynamo is 85 per cent. How many horse power are required to drive it when there are 200 16 candle power lamps on the circuit, each lamp taking $\frac{1}{2}$ ampere at 110 volts?

19. How many amperes at 120 volts must be furnished a hoisting motor which is to lift 900 lb. 70 ft. per minute, if it has an efficiency of 70 per cent?

20. A motor operates a pump which in 1 hr. lifts 20,000 gal. of water (1 gal. = $8\frac{1}{3}$ lb.) 400 ft. If the combined efficiency of the pumping system is 72 per cent, what current will the motor require at 550 volts?

21. An electric street car with its load weighs 8 T.; on a level track the pull required is 20 lb. per ton. How much power is necessary at the axle to move the car 10 mi. per hour? If the motor and gearing have an efficiency of 75 per cent, how many amperes are required on a 550-volt circuit?

22. To perform a certain amount of work 30 kw. hr. are required. If the dynamo gives a current of 125 amperes at 220 volts, how long must it be used to perform this work?

SOLUTION.
$$\frac{125 \times 220}{1000} = 27.5 \text{ kw.}$$
$$\frac{30}{27.5} = 1.09 \text{ hr.}$$

23. A 5-kw. motor is used to operate a printing press 8 hr. What will be the cost of the power at 12 cents per kilowatt hour?

24. What is the cost of running a motor which requires 15 amperes at 110 volts, at 12 cents per kilowatt hour?

25. An incandescent lamp takes .6 ampere at 110 volts. If power costs 15 cents per kilowatt hour, what is the cost of operating the lamp 12 hr.?

26. An inclosed arc lamp takes 80 volts on a current of 6.6 amperes. How much does it cost to operate the lamp 12 hr. at 15 cents per kilowatt hour?

27. How many watts per candle power are required in each of the following lamps? If power costs 10 cents per kilowatt hour, how much would it cost per hour to keep each lamp at full candle power? Construct a curve to show the relation between the candle power of each lamp and the cost per candle power.

Volts	Candle power	Amperes	Ohms
110	10	.32	344
110	16	.51	216
110	20	.64	172
110	24	.76	145
110	32	1.02	108

28. From the equation $V \times A = W$ construct a series of curves on the same axes for W equal to 1, 2, 3, 4, 5 kw. Knowing the voltage and current in a circuit, by means of these curves the approximate power can be determined readily.

Heat generated by a Current

126. Heat loss in a conductor. We have seen that it takes pressure (voltage) to drive a current through a conductor, and we have computed this fall of potential. Thus if a current of 10 amperes flows through a resistance of 2 ohms, the amount of voltage required to send the current is $V = AR = 10 \times 2 = 20$ volts. We have also computed the loss of power. Thus the number of watts lost is $V \times A = 200$ watts. This power or energy which is lost in the conductor is changed into heat. We may say in the above problem that the heat loss is 200 watts per minute.

Hence to find the heat loss in a conductor we simply find the watts lost, and, if desired, change the watts into calories or British thermal units.

1 watt minute = 44.25 ft. lb. per minute.
1 B. t. u. = 778 ft. lb.

Hence 1 watt minute = .057 B. t. u. per minute.

PROBLEMS

1. Find the heat loss due to a current of 60 amperes through a resistance of 10 ohms.

SOLUTION. $W = A^2 \cdot R = 60^2 \times 10 = 36{,}000$ watts in 1 min.

2. A conductor having a resistance of 5 ohms carries a current of 18 amperes. How much heat is developed in 1 hr.?

3. How much heat is developed in a wire of 15.2 ohms resistance by a current of 8 amperes in 15 min., (*a*) in watts? (*b*) in calories? (*c*) in British thermal units?

4. A current of 36 amperes is sent over a line of 2 ohms resistance. What is the drop in voltage? What is the heat loss per hour (*a*) in watts? (*b*) in British thermal units?

5. A current of 12 amperes flows through a resistance of 3.2 ohms for 15 min., and another current of 8 amperes flows through a conductor of 2.5 ohms resistance. How long must the second current flow in order that the amount of heat generated may be the same as in the first case?

6. Construct a curve to show the heat loss in a conductor as the resistance changes from 1 to 10 ohms while the current remains constantly 5 amperes.

7. Construct a curve to show the heat loss in a conductor of 1 ohm resistance as the current varies from 10 to 20 amperes.

8. In a conductor of 10 ohms resistance the voltage increases from 10 to 1000 volts. Construct a curve to show the heat loss.

9. A Leclanché cell used to ring a doorbell has an electromotive force of 1.6 volts and the current is .75 ampere. If the wire has a resistance of .4 ohm, what per cent of the power is the heat loss in the line?

SOLUTION. $1.6 \times .75 = 1.2$ watts, total power.

$A^2 \cdot R = .75^2 \times .4 = .23$ watts, heat loss in line.

$$\frac{.23}{1.2} = 19 \text{ per cent.}$$

10. The dynamo of an arc-light system furnishes a current of 9.6 amperes at 3000 volts. The circuit is made up of 16 mi. of No. 6 B. & S. gauge copper wire. What per cent of the power is the heat loss in the line?

11. A 6-in.-plate stove requires 5.5 amperes at 110 volts. What is the cost of running it 30 min. if the current costs 6 cents per kilowatt hour?

12. Find the cost of heating a $6\frac{1}{2}$-lb. flatiron for 3 hr., if it takes 4 amperes at 110 volts, at 6 cents per kilowatt hour.

13. An electric radiator takes 13.6 amperes at 110 volts. Find the cost for 8 hr. at 6 cents per kilowatt hour.

14. In an electric heater there is a coil of iron wire 224 ft. in length having a resistance of $\frac{1}{8}$ ohm per foot. If it is connected to a 110-volt circuit, how much heat is generated?

15. It is desired to make an electric soldering iron to be heated by a coil of No. 27 German silver wire of resistance 1.25 ohms per foot. How many feet will be required to give 200 watts on a 500-volt circuit?

Wiring for Light and Power

127. The mil. In electrical calculations involving the diameter of wire, the *mil* is usually taken as the unit of length.

$$1 \text{ mil} = .001 \text{ in.}$$

A *circular mil* is a circle whose diameter is 1 mil.

$$\text{A circular mil} = \pi r^2 = \pi \times .5^2 = .7854 \text{ sq. mils.}$$

$$1 \text{ circular mil} = .7854 \text{ sq. mils.}$$

$$1 \text{ sq. mil} = 1.273 \text{ circular mils.}$$

Circles are to each other as the squares of their diameters. *Hence to find the area of a circle in circular mils, square its diameter expressed in mils.*

A *mil foot* of wire is 1 ft. long and 1 mil in diameter. In practice the resistance of 1 mil-ft. of copper wire is usually taken as 10.7 ohms.

PROBLEMS

1. The diameter of a wire is $\frac{1}{4}$ in. Find (*a*) its diameter in mils; (*b*) its cross section in circular mils.

2. How many circular mils in the cross section of a wire of diameter (*a*) $\frac{1}{2}$ in.? (*b*) .125 in.? (*c*) .06 in.?

3. Find the diameter and area in square mils of a wire whose cross section is (*a*) 10,381 circular mils; (*b*) 26,250 circular mils; (*c*) 105,590 circular mils.

4. A copper bar is 1 in. by $\frac{1}{2}$ in. Find the area of a cross section in square mils and in circular mils.

5. Find the resistance of 1000 ft. of copper wire 40 mils in diameter.

SOLUTION. The cross section $= 40^2 = 1600$ circular mils.
Resistance of 1 mil foot $= 10.7$ ohms.
Resistance of 1 ft. of wire of 40 mils diameter

$$= \frac{10.7}{1600} = .00669 \text{ ohm.}$$

Resistance of 1000 ft. of wire of 40 mils diameter

$$= .00669 \times 1000 = 6.69 \text{ ohms.}$$

6. Find the resistance of 1000 ft. of copper wire that has a diameter of (*a*) 460 mils; (*b*) 289.3 mils; (*c*) .1 in.; (*d*) 40.3 mils; (*e*) 20 mils.

7. A current of 75 amperes is sent through 1 mi. of copper wire 229.4 mils in diameter. Find the drop in voltage.

SOLUTION. Cross section $= 229.4^2 = 52{,}620$ circular mils.

$$\text{Resistance of 1 ft.} = \frac{10.7}{52{,}620} = .000203 \text{ ohm.}$$

$$\text{Resistance of 1 mile} = .000203 \times 5280 = 1.07 \text{ ohms.}$$

$$V = A \cdot R = 75 \times 1.07 = 80.3 \text{ volts.}$$

8. What is the drop in voltage in a circuit of 5 mi. of copper wire 162 mils in diameter if the current is 40 amperes?

9. With a current of 210 amperes what will be the drop in voltage in 2500 ft. of copper wire 460 mils in diameter?

10. How many circular mils are required in a power line 500 ft. long with a current of 150 amperes, if a drop of 12 volts is allowed?

SOLUTION. Let n = number of circular mils required.

$$10.7 \times 500 = 5350 \text{ ohms per circular mil for 500 ft.}$$

$$\frac{5350}{n} = \text{number of ohms resistance of } n \text{ circular mils for 500 ft.}$$

$$\frac{5350 \times 150}{n} = \text{number of volts in drop.}$$

$$\therefore \frac{5350 \times 150}{n} = 12.$$

$$n = 66{,}880 \text{ circular mils.}$$

Check. $$\frac{10.7}{66{,}880} = .00016 \text{ ohm, resistance of 1 ft. of the wire.}$$

$$.00016 \times 500 = .08 \text{ ohm, resistance of 500 ft.}$$

$$V = AR = 150 \times .08 = 12.$$

From the above solution we may obtain the following formula, which is in general use for finding the size of conductor required to carry a given load.

$$\text{No. circular mils} = \frac{21.4 \times \text{distance one way in feet} \times \text{amperes}}{\text{volts lost}}.$$

11. A motor is 300 ft. from the dynamo. How many circular mils are required for a current of 90 amperes, if a drop of 6 volts is allowed?

12. Find the number of circular mils required to deliver 10 kw. to a motor at a distance of 200 ft., with 100 volts pressure at the motor, if a drop of 5 volts is allowed in the line.

13. Find the number of circular mils required to transmit 25 kw., with a 20 per cent drop in the voltage, a distance of 10 mi., if the voltage at the load is to be (*a*) 100 volts; (*b*) 500 volts; (*c*) 1000 volts.

Suggestion. $\frac{100}{.80} = 125$ volts at the dynamo.

14. It is required to deliver 120 h. p. to a motor 2 mi. away, from a dynamo which has a voltage of 550 volts. If the line loss is to be not more than 16 per cent, find the cross section of the wire in circular mils and the number of pounds of copper required.

Dynamos and Motors

128. Construction. When a closed wire is rotated between the poles of a magnet so as to cut the lines of force, a current flows in the wire. The dynamo is constructed on this principle. The *armature* is the part of the machine in which the current is generated, and in most machines the armature revolves. The *field* is the space between the poles of the magnets in which the armature revolves. The magnets are pieces of soft iron, which are magnetized by a current from the machine itself or from a separate dynamo. This current flows in coils which are placed around the magnets. In Fig. 96 the armature is represented by a single wire revolving in the field.

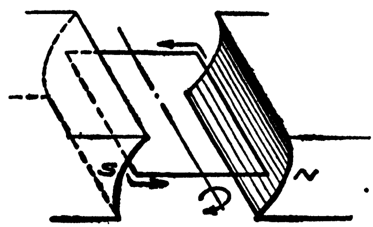

FIG. 96

129. The field coils connected in three ways. Direct current dynamos generally excite their own fields; and there are three ways of connecting the field-magnet coils.

1. *Series-wound dynamos.* The field-magnet coils are connected to the armature so that the whole current generated passes through them.

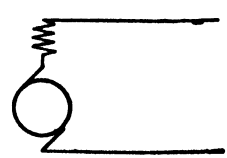
FIG. 97

2. *Shunt-wound dynamos.* The field-magnet coils are connected in multiple with the terminals of the armature; hence only a part of the current goes through them. These coils consist of many turns of comparatively fine wire.

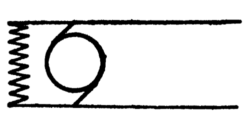
FIG. 98

3. *Compound-wound dynamos.* The field magnets are wound with two sets of coils, one in series and one in multiple with the armature.

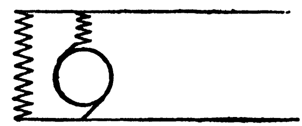

Fig. 99

Motors are also wound in these three ways.

130. Electrical efficiency of dynamos and motors. Since it requires pressure (voltage) to drive a current through the armature and field coils, there is a loss of power in a dynamo and in a motor. This loss is sometimes called the *copper loss*. Electrical efficiency takes into account only the copper loss.

$$\text{Electrical efficiency of a dynamo} = \frac{\text{Power given out}}{\text{Power generated}}.$$

$$\text{Electrical efficiency of a motor} = \frac{\text{Power left for useful work}}{\text{Power supplied to motor}}.$$

PROBLEMS

1. The output of a series-wound dynamo is 5 kw. at a voltage of 110 volts. The resistance of the armature is .06 ohm and of the field coil .072 ohm. Find (*a*) the copper loss; (*b*) the electrical efficiency; (*c*) the total electromotive force generated.

Solution.

$$\frac{W}{V} = \frac{5000}{110} = 45.5 \text{ amperes.}$$

$.06 + .072 = .132$ ohm, total resistance.

(*a*) $A^2R =$

$45.5^2 \times .132 = 273$ watts, copper loss.

$5000 + 273 = 5273$, total power generated.

(*b*) $\frac{5000}{5273} = 95$ per cent, electrical efficiency.

(*c*) $V = AR =$

$45.5 \times .132 = 6$ volts, loss in armature and field coils.

$110 + 6 = 116$, total electromotive force generated.

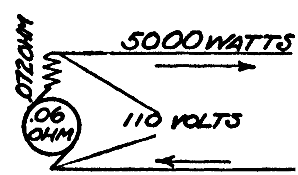

Fig. 100

2. A series-wound motor has a resistance of .68 ohm. When supplied with 15 amperes at a voltage of 105 volts, find

(*a*) the copper loss; (*b*) the electrical efficiency; (*c*) the volts lost in the motor.

Suggestion. Find the copper loss as in Problem 1 and subtract it from the number of watts supplied to the motor.

Electrical efficiency $= \frac{1422}{1575}$. Drop in voltage $= 15 \times .68$.

3. A shunt-wound dynamo furnishes 5 kw. at a voltage of 110 volts. The shunt resistance is 45 ohms and the armature resistance is .06 ohm. Find (*a*) the copper loss; (*b*) the electrical efficiency.

SOLUTION. $A = \frac{W}{V} = 45.5$ amperes.

In the shunt,

$$A = \frac{V}{R} = \frac{110}{45} = 2.44 \text{ amperes.}$$

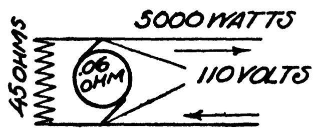

FIG. 101

$$45.5 + 2.44 = 47.9 \text{ amperes.}$$

$W = VA = 110 \times 2.44 = 268$ watts, loss in shunt.

$W = A^2R = 47.9^2 \times .06 = 137$ watts, loss in armature.

(*a*) 405 watts, total loss.

$5000 + 405 = 5405$ total watts.

(*b*) $\frac{5000}{5405} = 93$ per cent, electrical efficiency.

4. The armature of a shunt motor has a resistance of .02 ohm, and the shunt a resistance of 62 ohms. If the input is 5 h. p. at 124 volts, find (*a*) the copper loss; (*b*) the electrical efficiency.

SOLUTION. $5 \times 746 = 3730$ watts.

$$A = \frac{W}{V} = \frac{3730}{124} = 30.1 \text{ amperes.}$$

In shunt, $A = \frac{V}{R} = \frac{124}{62} = 2$ amperes.

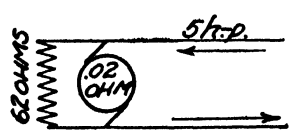

FIG. 102

$$30.1 - 2 = 28.1 \text{ amperes.}$$

$W = VA = 124 \times 2 = 248$ watts, loss in shunt.

$V = A^2R = 28.1^2 \times .02 = 16$ watts, loss in armature.

(*a*) 264 watts, total loss.

$3730 - 264 = 3466$ watts for useful work.

(*b*) $\frac{3466}{3730} = 93$ per cent, electrical efficiency.

Note that the current in the armature of a shunt motor equals the total current less the current in the field coils.

5. A 50-kw., 125-volt, compound-wound dynamo has a shunt resistance of 62.5 ohms, a series-coil resistance of .001 ohm, and an armature resistance of .002 ohm. Compute the copper losses and the electrical efficiency.

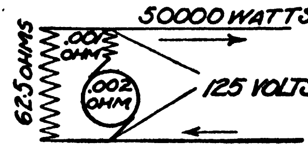

FIG. 103

SOLUTION. $\frac{50000}{125} = 400$ amperes.

In shunt, $\frac{V}{R} = \frac{125}{62.5} = 2$ amperes.

$$400 + 2 = 402 \text{ amperes, total current generated.}$$
$$402^2 \times .002 = 323 \text{ watts, loss in armature.}$$
$$402^2 \times .001 = 162 \text{ watts, loss in series coil.}$$
$$125 \times 2 = \underline{250} \text{ watts, loss in shunt.}$$
$$735 \text{ watts, total loss.}$$
$$\frac{50000}{50735} = 98.6 \text{ per cent, electrical efficiency.}$$

Note that the total current generated by a shunt dynamo equals the sum of the currents in the armature and in the field coils.

6. A compound motor is supplied with 50 amperes of current from 110-volt mains. If the armature resistance is .09 ohm, the series-coil resistance .078 ohm, and the shunt-coil resistance 55 ohms, find (*a*) the copper loss; (*b*) the electrical efficiency.

SOLUTION. $50 \times 110 = 5500$ watts.

$$\frac{V}{R} = \frac{110}{55} = 2 \text{ amperes in shunt.}$$
$$50 - 2 = 48 \text{ amperes in armature.}$$

Find loss in shunt, armature, and series coil to be 220, 207, and 180 watts respectively, and the electrical efficiency 89 per cent.

7. The output of a series dynamo is 20 amperes at 1000 volts. The resistance of the armature is 1.4 ohms and of the field coil 1.7 ohms. Find the copper loss, the electrical efficiency, and the volts lost in the dynamo.

8. The armature of a shunt motor has a resistance of .3 ohm, and the shunt a resistance of 120 ohms. When running at full load on a 110-volt circuit the motor takes a current of 8 amperes. Find the copper loss and the electrical efficiency.

Find the copper losses and electrical efficiency of the following dynamo-electric machines:

DYNAMOS

No.	Type	Resistance, ohms			Output	Volts	Amperes
		Armature	Series coil	Shunt coil			
9	Series	2	2.5		10 kw.	1000	
10	Compound	.003	.002	55	60 h. p.	110	
11	Shunt	.29		57.5	6.5 kw.	115	
12	Series	.15	.12			110	50
13	Compound	.04	.03	20	10 kw.	110	
14	Shunt	.006		12	50 kw.		500
15	Compound	.023	.012	19.4		111	220
16	Shunt	.0117		52.7		410	590

MOTORS

No.	Type	Resistance, ohms			Input	Volts	Amperes
		Armature	Series coil	Shunt coil			
17	Shunt	.15		48		110	10
18	Series	.39	.35		1 kw.	80	
19	Shunt	.14		44		110	50
20	Shunt	.018		200	30 kw.	400	
21	Series	.112	.113			220	100
22	Compound	.14	.02	55	5.5 kw.	110	

131. Commercial or net efficiency. The commercial efficiency of a dynamo or motor takes account of all the losses in the machine; it is equal to the output divided by the input.

$$\textit{Commercial efficiency} = \frac{\textit{Output}}{\textit{Input}}.$$

PROBLEMS

1. A motor is supplied with a current of 20 amperes at 110 volts. If 2.8 h. p. are developed at the pulley, find the commercial efficiency of the motor.

SOLUTION.

$$\text{Input} = 110 \times 20 \text{ watts.}$$
$$\text{Output} = 746 \times 2.8 \text{ watts.}$$
$$\text{Commercial efficiency} = \frac{746 \times 2.8}{110 \times 20}$$
$$= 95 \text{ per cent.}$$

Check. $110 \times 20 \times .95 = 2090 \text{ watts} = 2.8 \text{ h. p.}$

2. A motor generator takes a current of 14 amperes at 220 volts and supplies a current of 112 amperes at 25 volts. Find its efficiency.

3. A 220-volt electric hoist is raising coal at the rate of 1 T. 270 ft. per minute. If the current is 90 amperes, what is the efficiency of the hoist?

4. A 3-kw. motor is used to operate a lathe. Find its efficiency if it takes 30 amperes at 110 volts.

5. The output of a generator is 50 kw. If it requires 76 h. p. to drive it, what is its efficiency?

6. A 550-volt generator supplies a current of 300 amperes. If the generator has an efficiency of 85 per cent, how many horse power are required to drive it?

7. It takes 25 h. p. to operate a dynamo which supplies power for 40 arc lights in series at 7 amperes. The resistance of each lamp is 8 ohms and the line resistance is 25 ohms. Find the efficiency of the dynamo.

8. A lighting circuit consists of 1200 ft. of No. 6 B. & S. gauge copper wire and eighty 16 candle power incandescent lamps in multiple, each having a resistance of 220 ohms. If the voltage is 110 at the lamps and 7.5 h. p. is supplied to the generator, find its efficiency.

9. In testing a motor the following results were obtained. Find the efficiency given by each test.

No.	Volts	Amperes	Brake horse power
1	224	96.5	24.6
2	221	101	25.7
3	222	103	27.2
4	230	109	29.1
5	227	123	32.6

10. The following data were obtained in a test of a motor generator.

Construct a curve showing the relation between output and efficiency.

Input	Volts	225	225	229	228	228	228
	Amperes	5.9	7.7	9.6	11.7	13.7	15.9
Output	Volts	21	20.8	21	20.6	20.2	20
	Amperes	0	20	40	60	80	100

CHAPTER XVIII

LOGARITHMIC PAPER

132. Description of logarithmic paper. In many engineering problems where it is necessary to compute a set of values from a formula, it is found that the required values can be secured quickly and easily by using paper ruled on the logarithmic scale. This paper is used both as a "ready reckoner," to read off tables of values and to find the law connecting the two variables in the problem. The advantage of logarithmic paper lies in the fact that many formulas which are represented by curves on squared paper are represented by straight lines on logarithmic paper. Hence while many pairs of values must be worked out to construct a curve on the former, only two or three pairs are required for the latter.

Fig. 104 shows the way in which logarithmic paper is ruled. The x-axis and the y-axis are laid off in divisions exactly like those of the slide rule. That is, OX and OY are each divided into 1000 equal parts; 2 is placed at the 301st division ($\log 2 = 0.301$); 3 is placed at the 477th division ($\log 3 = 0.477$); 4 is placed at the 602d division ($\log 4 = 0.602$), and so on.

Exercise. Construct a graph to read off the area of a circle of any given radius.

In order to learn the properties of logarithmic paper we will construct the graph by locating points. Later it will be shown that the whole graph can be constructed easily by locating only one point.

The formula for the area, $a = \pi r^2$, gives the following table:

Radius .	1	1.2	1.5	2	3	4	5	6	7	8	10
Area . .	3.14	4.52	7.07	12.6	28.3	50.3	78.5	113	154	201	314

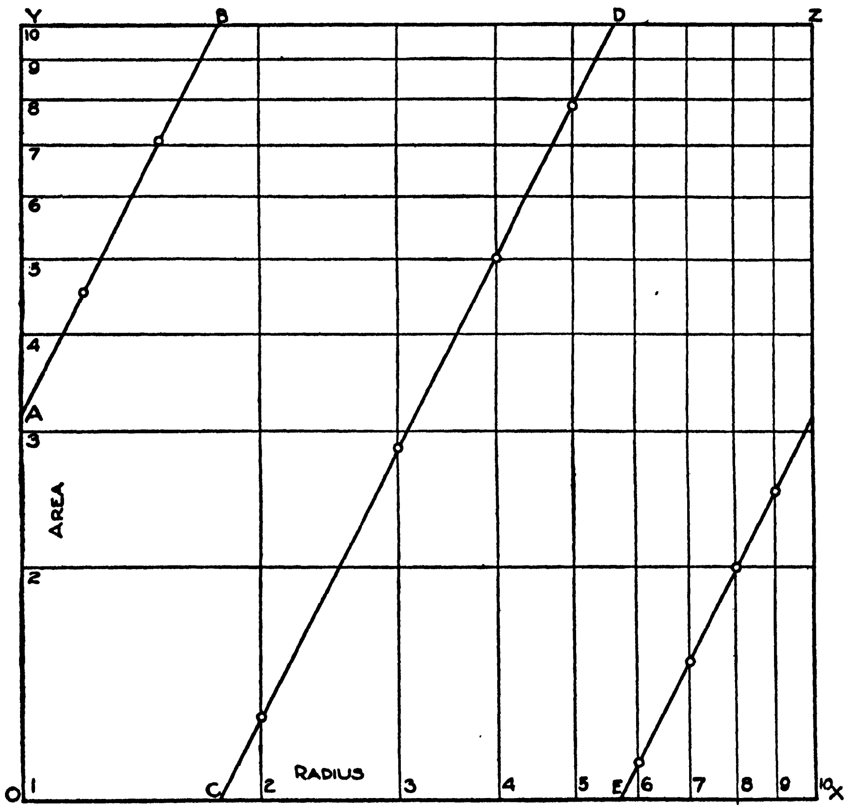

FIG. 104

Locating the points as shown in Fig. 104, we see that the points lie on the straight lines AB, CD, and EF. Hence $AB - CD - EF$ is the graph required. From it we see that when the radius is 2.5 the area is 19.6; when the area is 38.5, the radius is 3.5, and so on.

133. Properties of logarithmic paper. Some properties of the paper may now be noted. The equation $a = \pi r^2$ is in the form $y = mx^n$. AB, CD, and EF are parallel to one another. $BD = CE = \frac{1}{2} YZ$. $FX = 2\,EX$; hence $\frac{FX}{EX} = 2$, the exponent of r.

The graph can be drawn mechanically as follows: Find P, the mid-point of YZ. Tack the sheet of paper on a drawing board so that the T-square, in position, lies on O and P. Set the T-square on A (making $OA = 3.14$) and draw AB. Set the T-square at C on OX directly below B and draw CD. Similarly, draw EF. Check; F should be directly opposite A, that is, $FX = 3.14$.

It will be found that these are general properties of logarithmic paper, which may be used to construct graphs for formulas of the form $y = mx^n$; that is, a formula in which y equals an expression consisting of only one term in which the variable is raised to any power (n, being positive, negative, or fractional) and multiplied by any number. This form alone will be considered in the following discussion, and some of the properties of the paper which lead to simple and accurate constructions will be considered.

I. Equations of the Form $y = mx$

EXERCISES

1. Construct the graph of $y = x$.

x	1	2	3	4	5
y	1	2	3	4	5

Locating the points from the table, we see that they lie on the straight line OZ (Fig. 105). Hence OZ is the graph of $y = x$.

2. Construct on the same sheet of paper the graph of (1) $y = 2x$; (2) $y = 3x$; (3) $y = 4x$.

It is seen that all these lines are parallel. When we plot $y = x$ (1) and $y = 2x$ (2), we are really plotting the logarithmic equations $\log y = \log x$ (1′) and $\log y = \log 2x$, or $\log y = \log x + \log 2$ (2′). Comparing (1′) and (2′), we see that they

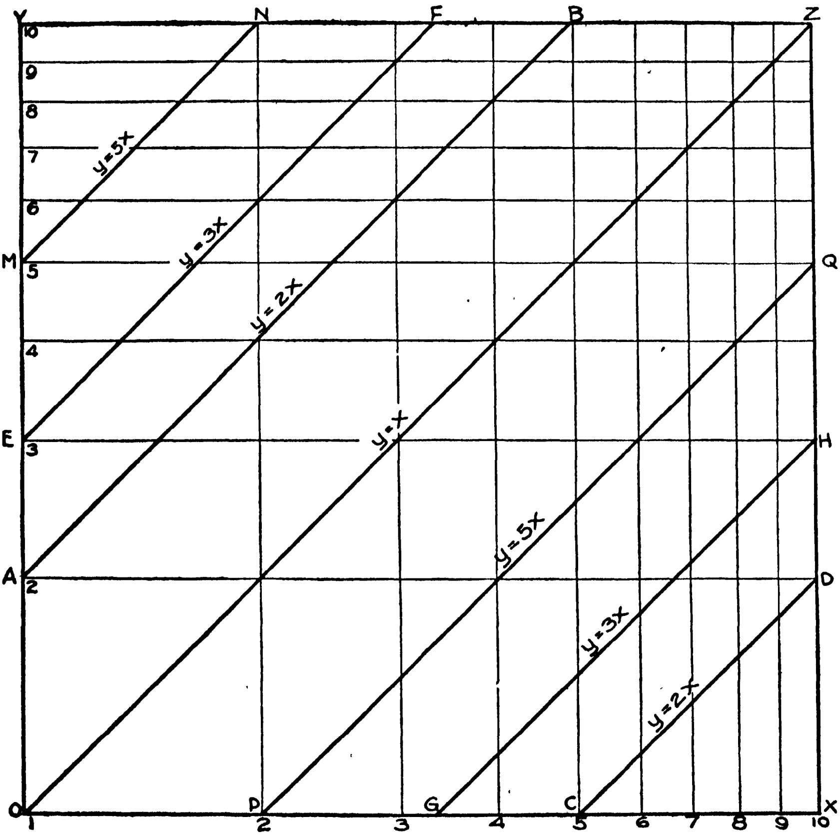

FIG. 105

differ only by the constant term log 2 on the right side; that is, every point of the graph of (2) is 2 above the corresponding point of the graph of (1). Note that the graph of each of these equations, except $y = x$, is made up of two lines; and all the lines are parallel to OZ. Hence to graph any equation of the form $y = mx$, for example, $y = 5x$, proceed as follows. From

5 on OY draw MN parallel to OZ. Take $OP = YN$ and draw PQ from P to 5 on XZ. $MN - PQ$ is the required graph.

The slope of a graph. We shall find that each graph we are to consider (except $y = x$ and $y = x^{-1}$) consists of two or more parallel lines, and that one line in each graph cuts OX and XZ or OX and OY. Thus in the graph of $y = 5x$, PQ cuts OX and XZ. We will call $\frac{XQ}{XP}$ the *slope of the graph*; that is, the tangent of the angle which the line makes with OX, always taking the angle on the right-hand side of the line.

II. Equations of the Form $y = mx^n$

A. When n is a positive whole number.

EXERCISES

1. Construct the graph of $y = x^2$.

x	1	2	3	4	5	6	7	8	9	10
y	1	4	9	16	25	36	49	64	81	100

Locating these points, we get the graph $OA - BZ$ (Fig. 106). Note that A and B are the mid-points of YZ and OX respectively.

2. Construct the graph of $y = x^3$.

Locating points, we get $OD - FG - HZ$ (Fig. 106). Note that D and G, and F and H divide YZ and OX respectively into three equal parts.

3. Construct the graphs of $y = x^4$ and $y = x^5$ without locating points.

Roots of numbers. From the graphs of $y = x^2$, $y = x^3$, $y = x^4$, and so on we can read off roots of numbers. Thus in the graph of $y = x^2$, OA gives the square roots of numbers from 1 to 9, 100 to 999, 10,000 to 99,999, ...; that is, of numbers containing 1, 3, 5 ··· figures. BZ gives the square root of numbers containing 2, 4, 6 ··· figures. To find the square root of 2, read from 2 on OY to OA, 1.41; for the square root of 20 read from 2 on OY to BZ, 4.47. Similarly, $y = x^3$ gives cube roots; OD gives the cube root of numbers containing 1, 4, 7 ··· figures, FG

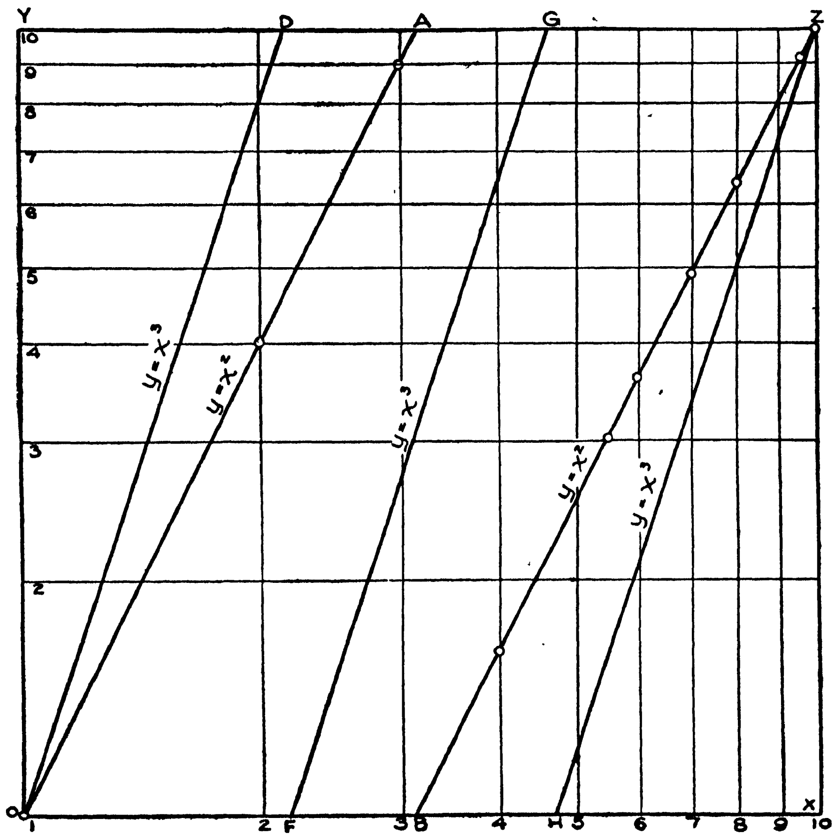

Fig. 106

of numbers containing 2, 5, 8 ··· figures, and HZ of numbers containing 3, 6, 9 ··· figures.

4. Construct the graph of $y = 2x^2$.

x	1	2	3	4	5	6	7	8	9	10
y	2	8	18	32	50	72	98	128	162	200

Note that each y is twice as great as the corresponding y in $y = x^2$. On locating the points and drawing the lines of the graph it will be seen that the lines are parallel to the lines of $y = x^2$ and 2 units above them. Hence the graph of $y = 2x^2$ may be constructed mechanically as follows: Tack the sheet of paper on a drawing board so that the edge of the T-square, in position, lies on O and the mid-point of YZ. Move the T-square up to 2 on OY and draw a line from 2 to YZ. Move the T-square to a point on OX directly below the point already determined on YZ and draw a line to YZ. Continue in the same manner and the graph will end at 2 on XZ if accurately drawn. This method holds for all cases where x^n has a coefficient. Note that the exponent of x is the slope of the graph.

5. Construct the graph of (*a*) $y = 2x^3$; (*b*) $y = .5x^4$; (*c*) $y = 1.68x^2$; (*d*) $y = .0625x^3$.

B. When n is a positive fraction.

EXERCISES

1. Construct the graph of $y = x^{\frac{3}{2}}$.

x	1	4	9	16	25	36	49	64	81	100
y	1	8	27	64	125	216	343	512	729	1000

Locating points from the table, we get the graph $OA - BC - DE - FZ$ (Fig. 107). A study of the graph shows that it could be drawn in the following manner: Divide OX and YZ each into three equal parts by the points F, B, E, and A; and OY and XZ each into two equal parts by the points D and C. Join O to A, the second point of division on YZ. This gives the correct slope, $\frac{3}{2}$. Directly below A is B, draw BC; opposite C is D, draw DE; below E is F, draw FZ.

A similar construction holds for any positive fractional value of n. Thus for $y = x^{\frac{3}{5}}$, divide OX and YZ each into 3 (the numerator of the exponent) equal parts, and OY and XZ each into 5 (the denominator of the exponent) equal parts, and join the points so as to make the slope $\frac{3}{5}$.

If x has a coefficient, for example, $y = 5\,x^{\frac{3}{2}}$, start the graph at 5 on OY and draw it parallel to OA, thus making the slope $\frac{3}{2}$.

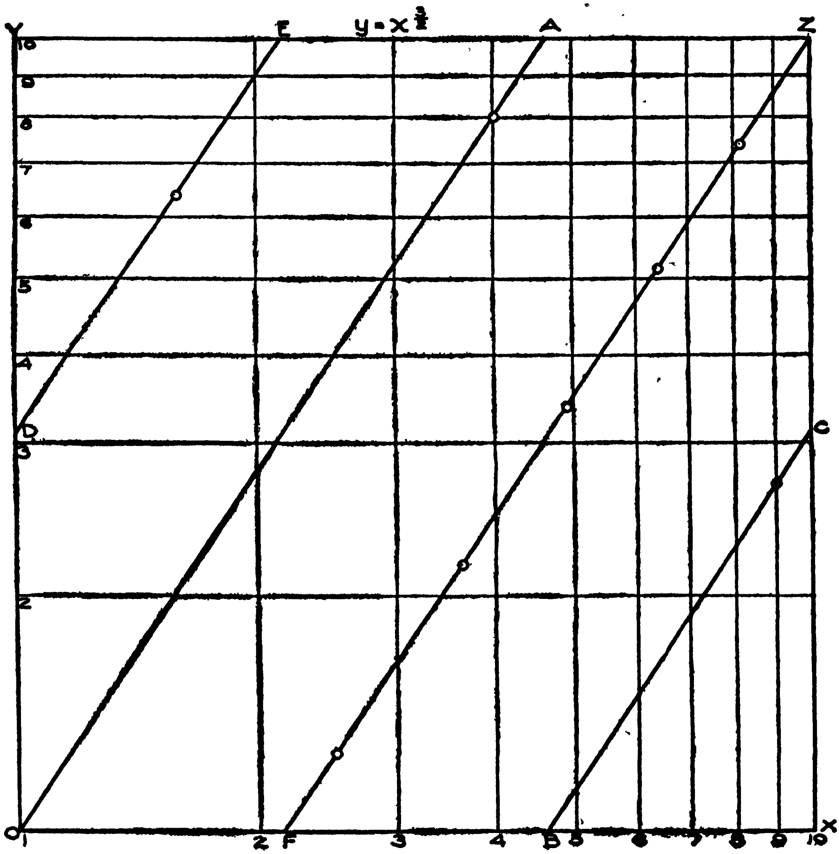

Fig. 107

2. Construct the graphs of (*a*) $y = x^{\frac{3}{4}}$; (*b*) $y = 2\,x^{\frac{3}{4}}$; (*c*) $y = 3\,x^{\frac{5}{2}}$; (*d*) $y = 5\,x^{1.4}$; (*e*) $y = 2.5\,x^{3.2}$; (*f*) $y = .06\,x^{1.1}$.

3. Construct a graph to show the distance passed over by a falling body in 1 to 10 sec.

4. Construct graphs to find (*a*) the surface, (*b*) the volume of spheres of radii from 1 to 10 in.

C. When n is negative.

EXERCISES

1. Construct the graph of $y = x^{-1}$ or $y = \dfrac{1}{x}$.

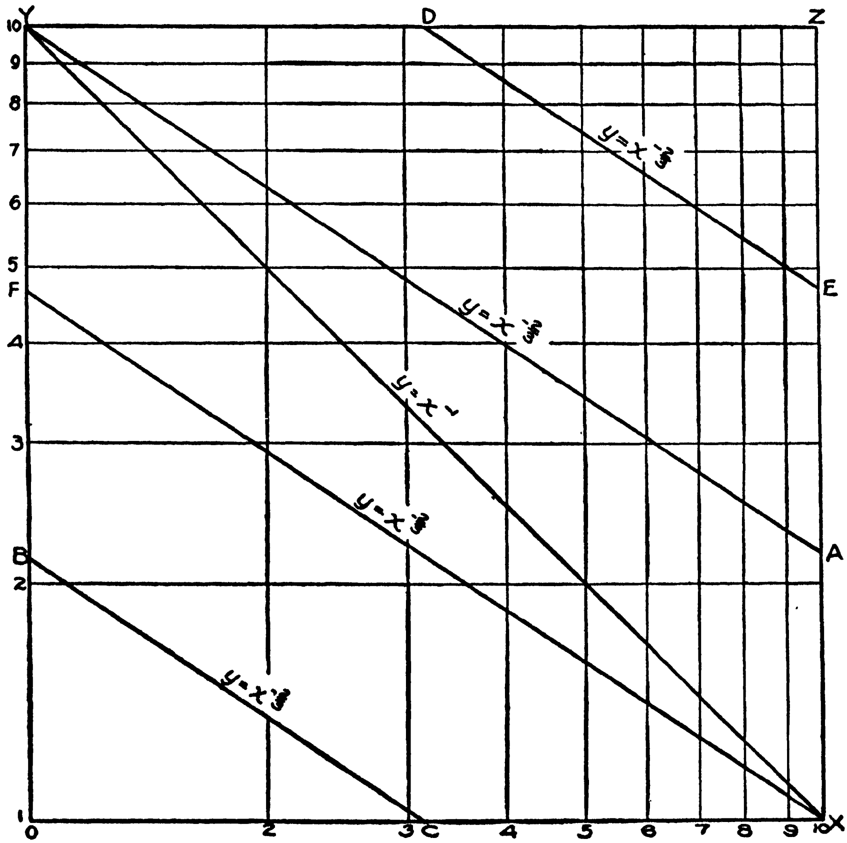

FIG. 108

x	1	2	4	8	10
y	1	.5	.25	.125	.1

Locating the points from the table, we get the graph YX (Fig. 108). The graph of $y = mx^{-1}$ is parallel to YX, and we begin to draw it from m on OY. Thus, to graph $y = 4x^{-1}$, from 4 on OY draw a line parallel to YX cutting OX at a point A; from B on YZ directly above A draw a line parallel to YX cutting XZ at C.

2. Construct the graph of $y = x^{-\frac{2}{3}}$.

Divide OX and YZ into 2 (numerator of the exponent) equal parts, and OY and XZ into 3 (denominator of the exponent) equal parts. Draw lines as shown in Fig. 108, and we get the graph $YA - BC - DE - FX$.

3. Construct the graphs of:

(a) $y = 3x^{-1}$.
(b) $y = .5x^{-2}$.
(c) $y = 4x^{-\frac{1}{2}}$.
(d) $y = 8x^{-\frac{3}{2}}$.
(e) $y = 28x^{-.6}$.
(f) $y = 125x^{-3.5}$.
(g) $y = .006x^{-2.4}$.
(h) $y = 2800x^{-1.12}$.

PROBLEMS

1. If in a gas engine the gas expands without gain or loss of heat, the law of expansion is found to be $pv^{1.28} = 3060$. Construct the curve to show the pressure as the volume increases from 10 cu. in. to 26 cu. in.

Locate only one point (Fig. 109); when $v = 10$, $p = 180$. Mark this point by A on OY. The exponent of v is $-\frac{128}{100}$, when the equation is in the form $p = 3060\,v^{-1.28}$.

Measure $OM = 123$ mm. on OY, and $ON = 100$ mm. on OX. Tack the paper on a drawing board so that the T-square, in position, lies on M and N. Move the T-square to A and draw AB. Move the T-square to C on YZ directly above B and draw CD. $AB - CD$ is the graph; from this graph pressures can be read off for volumes from 10 cu. in. to 100 cu. in.

Given that steam expands without gain or loss of heat; construct graphs on logarithmic paper for volumes from 10 to 100 cu. in.:

2. $pv^{1.11} = 3000$.
3. $pv^{1.25} = 2840$.
4. $pv^{1\frac{7}{8}} = 3200$.
5. $pv^{1.81} = 3420$.
6. $pv^{.9} = 250$.

7. The diameter d of wrought-iron shafting to transmit h horse power at 100 r. p. m. is given by $d = .85\,h^{\frac{1}{3}}$. Construct the graph and make a table for horse power from 10 to 80.

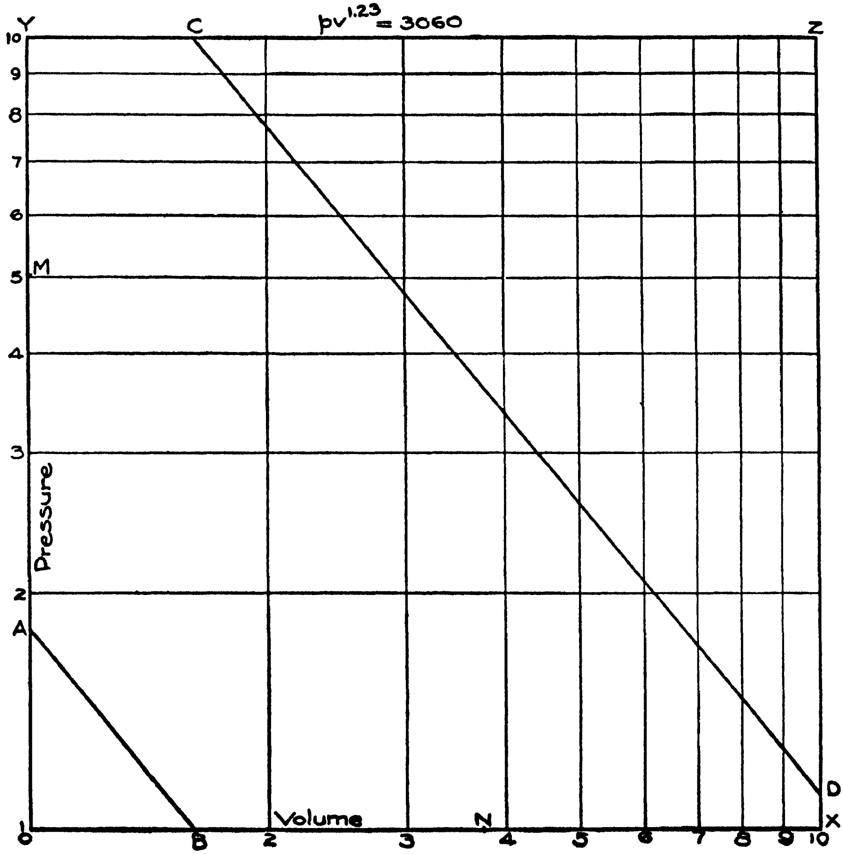

FIG. 109

8. The number of gallons of water per minute flowing over a rectangular weir 6 in. wide is given by $g = 17.8\,h^{\frac{3}{2}}$, where $g =$ the number of gallons per minute, and $h =$ the depth in inches from the level of free water to the sill of the weir. Construct the graph and make a table showing the number of gallons per minute for depths 1, 1.5, 2, 2.5, $\cdots$, 6 in.

9. The number of cubic feet of water per minute discharged over a V-notch, or triangular weir, is given by $Q = 18.5\, bh^{\frac{3}{2}}$, where $Q =$ the number of cubic feet per minute, $b =$ breadth of notch in feet at the free surface, and $h =$ depth in inches from the free level to the bottom of the notch. Construct a graph and make a table for the quantity of water discharged for depths from 6 to 15 in. when $b = 1$ ft.

10. The diameter of a copper wire which will be fused by an electric current is given by $d = .00212\, A^{\frac{2}{3}}$, where $d =$ the diameter in inches, and $A =$ the number of amperes. Construct a graph and make a table of diameters of wire which will be fused by currents of 10, 20, 30, $\cdots$, 100 amperes.

11. The weight in pounds that a rectangular steel beam, supported at both ends, can sustain at its center, is given by $w = 890 \dfrac{bd^2}{l}$, where $w =$ the weight in pounds, $b =$ the breadth of beam in inches, $d =$ the depth of beam in inches, and $l =$ the length of beam in feet.

Find the number of pounds that can be supported at the middle of a steel beam 4 in. in breadth and 15 ft. long for depths from 4 to 10 in.

12. In accordance with the building laws of Chicago the safe load in tons, uniformly distributed, for yellow-pine beams is given by $w = \dfrac{.08\, bd^2}{l}$, where $w =$ load in tons, $b =$ breadth of beam in inches, $d =$ depth of beam in inches, and $l =$ length of beam in feet between the supports.

Find the safe load for yellow-pine beams 25 ft. long, 4 in. in breadth, and depths from 8 to 18 in.

13. The number of cubic feet of air transmitted per minute in pipes of various diameters is given by $q = .327\, vd^2$, where $q =$ number of cubic feet of air per minute, $v =$ velocity of flow in feet per second, and $d =$ diameter of pipe in inches.

Make a table showing the volume of air transmitted in pipes of diameters from 2 to 10 in. with a flow of 12 ft. per second.

14. The following formula is used for computing the surface curvature in paving streets: $y = \frac{b}{a^2}x^2$, where $x =$ horizontal distance in feet from center of street, $y =$ vertical distance in inches below grade, $a =$ one half the width of the street in feet, $b =$ depth of gutter in inches below center of street.

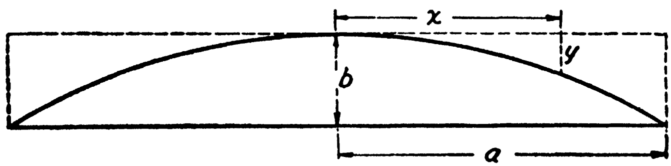

FIG. 110

Construct a graph to read off the vertical distances below grade at points 2, 4, 6 ft. · · · from the center of a street 60 ft. wide, if the gutter is 15 in. below the center of the street.

Find the equation connecting x and y when the following corresponding values are given:

15.

x	2	2.5	3	3.5	4
y	14	21.9	31.5	42.9	56

Suggestion. Locate the points and draw a line through them, cutting OX at A and YZ at B. From C on YZ directly above A draw a line parallel to BA, cutting OY at D. $OD = 3.5 = m$. The slope of AB is 2; hence the required equation is $y = 3.5\,x^2$.

16.

x	2	3	4	5	6
y	32	108	256	500	864

17.

x	4	5	6	7	8
y	4	4.47	4.90	5.29	5.66

18.

x	1.61	2.01	3.05	4.48	7.59
y	220	230	250	270	300

Suggestion. The line through the points cuts OY at 2. The values of y, however, suggest that it should be read 200, and this will be found to be correct on checking.

19.

x	20	30	40	50	60
y	1099	2248	3826	5717	7943

Suggestion. Let the line through the points cut OX at A and YZ at B. From C on OX directly below B draw CD to XZ parallel to AB; and from E on YZ directly above A draw EF to OY parallel to AB. $FE - AB - CD$ is the part of the graph for values of x from 10 to 100. To find m construct the part of the graph for values of x from 10 to 1.

20.

x	15	20	25	30	64
y	486	589	684	772	1280

Find the law connecting the two variables in the following:

21. In a test of cast-iron columns 6 ft. long, both ends rounded, the following results were obtained, where d = diameter of column in inches, and t = load in tons under which the column broke by bending.

d	2	2.5	3	3.5	4
t	10.7	24.9	49.4	88.2	146

22. The bearing end of a vertical shaft is called a pivot. For slow-moving steel pivots the following table of values is given, where d = diameter of pivot in inches, and p = total vertical pressure on the pivot in pounds.

d	1	1.5	2	2.5	3	3.5	4	4.5
p	816	1836	3265	5102	7347	10,000	13,061	16,530

23. The following table gives the absolute temperature (F.) of air at different pressures when it is compressed without gain or loss of heat. t = absolute temperature (F.), and p = pounds per square inch.

p	15	30	45	60	90
t	530	649	730	792	892

24. The following results were obtained in a test in towing a canal boat. p = pull in pounds, and v = speed of boat in miles per hour.

p	76	160	240	320	370
v	1.68	2.43	3.18	3.60	4.03

In the following examples find the law connecting p and v. The expansion is without gain or loss of heat, and p and v are corresponding values of the pressure and volume.

25. Steam.

v	1	2	3	5	7	9
p	100	37.7	21.3	10.4	6.48	4.54

26. Steam.

v	3	4	6	8	10
p	118	90.8	63.3	48.9	40

27. Superheated steam.

v	2	3	5	7	9
p	105	61.8	52	20.7	15

28. Mixture in cylinder of a gas engine.

v	2	4	6	8	10
p	57	21.2	11.8	8.1	5.9

Wire Table — Copper Wire

Brown and Sharpe Gauge	Area in circular mils	Diameter in mils	Resistance, ohms per 1000 ft.	Weight, pounds per 1000 ft.
	2,000,000	1414	.00519	6044
	1,750,000	1323	.00593	5289
	1,500,000	1225	.00692	4533
	1,250,000	1118	.00830	3778
	1,000,000	1000	.01038	3022
	950,000	974.7	.01093	2871
	900,000	948.7	.01153	2720
	850,000	922.0	.01221	2569
	800,000	894.4	.01298	2418
	750,000	866.0	.01384	2266
	700,000	836.7	.01483	2115
	650,000	806.2	.01597	1964
	600,000	774.6	.01730	1813
	550,000	741.6	.01887	1662
	500,000	707.1	.02076	1511
	450,000	670.8	.02307	1360
	400,000	632.5	.02595	1209
	350,000	591.6	.02966	1058
	300,000	547.7	.03460	906.5
	250,000	500.0	.04152	755.5
	225,000	474.3	.04614	680.0
0000	211,600	460.00	.04906	639.33
000	167,805	409.64	.06186	507.01
00	133,079	364.80	.07801	402.09
0	105,592	324.95	.09831	319.04
1	83,694	289.30	.12404	252.88
2	66,373	257.63	.15640	200.54
3	52,634	229.42	.19723	159.03
4	41,742	204.31	.24869	126.12
5	33,102	181.94	.31361	100.01
6	26,251	162.02	.39546	79.32
7	20,816	144.28	.49871	62.90
8	16,509	128.49	.62881	49.88
9	13,094	114.43	.79281	39.56
10	10,381	101.89	1.0000	31.37
12	6,529.9	80.808	1.5898	19.73
14	4,106.8	64.084	2.5908	12.41
16	2,582.9	50.820	4.0191	7.81
18	1,624.3	40.303	6.3911	4.91
19	1,288.1	35.890	8.2889	3.89
20	1,021.5	31.961	10.163	3.09
22	642.70	25.347	16.152	1.94
24	404.01	20.100	25.695	1.22
28	159.79	12.641	64.966	.48
32	63.20	7.950	164.26	.19
36	25.00	5.000	415.24	.08
40	9.89	3.144	1049.7	.03

TABLES

UNIT EQUIVALENTS

PRESSURE

1 pound per square inch . . .	2.042 inches of mercury at 62° F.
1 pound per square inch . . .	2.309 feet of water at 62° F.
1 atmosphere	14.7 pounds per square inch.
1 atmosphere	30 inches of mercury at 62° F.
1 atmosphere	33.95 feet of water at 62° F.
1 foot of water at 62° F. . . .	.433 pound per square inch.
1 inch of mercury at 62° F. . .	.491 pound per square inch.

LENGTH

1 mil	.001 inch.
1 inch	2.54 centimeters.
1 mile	1.609 kilometers.
1 centimeter	.3937 inch.
1 kilometer	3280.8 feet.

AREA

1 circular mil	.7854 square mil.
1 square mil	1.273 circular mils.
1 square inch	645.16 square millimeters.
1 square centimeter	.155 square inch.

VOLUME

1 cubic inch	16.387 cubic centimeters.
1 cubic foot	7.48 gallons (liquid, U. S.).
1 pint (liquid, U. S.)	473.18 cubic centimeters.
1 pint (liquid, U. S.)	28.875 cubic inches.
1 gallon (liquid, U. S.)	231 cubic inches.
1 bushel	2150.4 cubic inches.
1 cubic centimeter	.061 cubic inch.
1 liter	61.02 cubic inches.
1 liter	2.113 pints (liquid, U. S.).

Weight

1 ounce (avoirdupois)	437.5 grains.
1 ounce (avoirdupois)	28.35 grams.
1 pound (avoirdupois)	453.6 grams.
1 ton (2000 pounds)	907.185 kilograms.
1 cubic centimeter of water . .	1 gram.
1 gram	.0353 ounce (avoirdupois).
1 cubic foot of water	62.4 pounds.
1 cubic inch of water	.0361 pounds.
1 gallon of water (liquid, U. S.) .	8.345 pounds.

Energy, Work, Heat

1 British thermal unit (B. t. u.) .	1 pound water 1° F.
1 British thermal unit	778 foot pounds.
1 British thermal unit	.293 watt hour.
1 horse power hour	746 watt hours.
1 horse power hour	2544.7 British thermal units.
1 kilowatt hour	3412.66 British thermal units.
1 kilowatt hour	1.341 horse power hours.

Power

1 watt	44.25 foot pounds per minute.
1 watt	.0569 B. t. u. per minute.
1 horse power	33,000 foot pounds per minute.
1 horse power	746 watts per minute.
1 horse power	42.41 B. t. u. per minute.

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II. The Natural Sines, Cosines, Tangents, and Cotangents of Angles differing by Ten Minutes, and their Four-place Logarithms

1	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
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21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
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23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
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25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
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31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	0	1	2	3	4	5	6	7	8	9

50	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0	1	2	3	4	5	6	7	8	9

Angle	Sines		Cosines		Tangents		Cotangents		Angle
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
0° 00′	.0000	∞	1.0000	0.0000	.0000	∞	∞	∞	90° 00′
10	.0029	7.4637	1.0000	0000	.0029	7.4637	2.5363	343.77	50
20	.0058	7648	1.0000	0000	.0058	7648	2352	171.89	40
30	.0087	9408	1.0000	0000	.0087	9409	0591	114.59	30
40	.0116	8.0658	.9999	0000	.0116	8.0658	1.9342	85.940	20
50	.0145	1627	.9999	0000	.0145	1627	8373	68.750	10
1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	1.7581	57.290	89° 00′
10	.0204	3088	.9998	9999	.0204	3089	6911	49.104	50
20	.0233	3668	.9997	9999	.0233	3669	6331	42.964	40
30	.0262	4179	.9997	9999	.0262	4181	5819	38.188	30
40	.0291	4637	.9996	9998	.0291	4638	5362	34.368	20
50	.0320	5050	.9995	9998	.0320	5053	4947	31.242	10
2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	1.4569	28.636	88° 00′
10	.0378	5776	.9993	9997	.0378	5779	4221	26.432	50
20	.0407	6097	.9992	9996	.0407	6101	3899	24.542	40
30	.0436	6397	.9990	9996	.0437	6401	3599	22.904	30
40	.0465	6677	.9989	9995	.0466	6682	3318	21.470	20
50	.0494	6940	.9988	9995	.0495	6945	3055	20.206	10
3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	1.2806	19.081	87° 00′
10	.0552	7423	.9985	9993	.0553	7429	2571	18.075	50
20	.0581	7645	.9983	9993	.0582	7652	2348	17.169	40
30	.0610	7857	.9981	9992	.0612	7865	2135	16.350	30
40	.0640	8059	.9980	9991	.0641	8067	1933	15.605	20
50	.0669	8251	.9978	9990	.0670	8261	1739	14.924	10
4° 00′	.0698	8.8436	.9976	9.9989	.0699	8.8446	1.1554	14.301	86° 00′
10	.0727	8613	.9974	9989	.0729	8624	1376	13.727	50
20	.0756	8783	.9971	9988	.0758	8795	1205	13.197	40
30	.0785	8946	.9969	9987	.0787	8960	1040	12.706	30
40	.0814	9104	.9967	9986	.0816	9118	0882	12.251	20
50	.0843	9256	.9964	9985	.0846	9272	0728	11.826	10
5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	1.0580	11.430	85° 00′
10	.0901	9545	.9959	9982	.0904	9563	0437	11.059	50
20	.0929	9682	.9957	9981	.0934	9701	0299	10.712	40
30	.0958	9816	.9954	9980	.0963	9836	0164	10.385	30
40	.0987	9945	.9951	9979	.0992	9966	0034	10.078	20
50	.1016	9.0070	.9948	9977	.1022	9.0093	0.9907	9.7882	10
6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	0.9784	9.5144	84° 00′
10	.1074	0311	.9942	9975	.1080	0336	9664	9.2553	50
20	.1103	0426	.9939	9973	.1110	0453	9547	9.0098	40
30	.1132	0539	.9936	9972	.1139	0567	9433	8.7769	30
40	.1161	0648	.9932	9971	.1169	0678	9322	8.5555	20
50	.1190	0755	.9929	9969	.1198	0786	9214	8.3450	10
7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	0.9109	8.1443	83° 00′
10	.1248	0961	.9922	9966	.1257	0995	9005	7.9530	50
20	.1276	1060	.9918	9964	.1287	1096	8904	7.7704	40
30	.1305	1157	.9914	9963	.1317	1194	8806	7.5958	30
40	.1334	1252	.9911	9961	.1346	1291	8709	7.4287	20
50	.1363	1345	.9907	9959	.1376	1385	8615	7.2687	10
8° 00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	0.8522	7.1154	82° 00′
10	.1421	1525	.9899	9956	.1435	1569	8431	6.9682	50
20	.1449	1612	.9894	9954	.1465	1658	8342	6.8269	40
30	.1478	1697	.9890	9952	.1495	1745	8255	6.6912	30
40	.1507	1781	.9886	9950	.1524	1831	8169	6.5606	20
50	.1536	1863	.9881	9948	.1554	1915	8085	6.4348	10
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	0.8003	6.3138	81° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
Angle	Cosines		Sines		Cotangents		Tangents		Angle

ANGLE	SINES		COSINES		TANGENTS		COTANGENTS		ANGLE
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	0.8003	6.3138	81° 00′
10	.1593	2022	.9872	9944	.1614	2078	7922	6.1970	50
20	.1622	2100	.9868	9942	.1644	2158	7842	6.0844	40
30	.1650	2176	.9863	9440	.1673	2236	7764	5.9758	30
40	.1679	2251	.9858	9938	.1703	2313	7687	5.8708	20
50	.1708	2324	.9853	9936	.1733	2389	7611	5.7694	10
10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	0.7537	5.6713	80° 00′
10	.1765	2468	.9843	9931	.1793	2536	7464	5.5764	50
20	.1794	2538	.9838	9929	.1823	2609	7391	5.4845	40
30	.1822	2606	.9833	9927	.1853	2680	7320	5.3955	30
40	.1851	2674	.9827	9924	.1883	2750	7250	5.3093	20
50	.1880	2740	.9822	9922	.1914	2819	7181	5.2257	10
11° 00′	.1908	9.2806	.9816	9.9919	.1944	9.2887	0.7113	5.1446	79° 00′
10	.1937	2870	.9811	9917	.1974	2953	7047	5.0658	50
20	.1965	2934	.9805	9914	.2004	3020	6980	4.9894	40
30	.1994	2997	.9799	9912	.2035	3085	6915	4.9152	30
40	.2022	3058	.9793	9909	.2065	3149	6851	4.8430	20
50	.2051	3119	.9787	9907	.2095	3212	6788	4.7729	10
12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	0.6725	4.7046	78° 00′
10	.2108	3238	.9775	9901	.2156	3336	6664	4.6382	50
20	.2136	3296	.9769	9899	.2186	3397	6603	4.5736	40
30	.2164	3353	.9763	9896	.2217	3458	6542	4.5107	30
40	.2193	3410	.9757	9893	.2247	3517	6483	4.4494	20
50	.2221	3466	.9750	9890	.2278	3576	6424	4.3897	10
13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	0.6366	4.3315	77° 00′
10	.2278	3575	.9737	9884	.2339	3691	6309	4.2747	50
20	.2306	3629	.9730	9881	.2370	3748	6252	4.2193	40
30	.2334	3682	.9724	9878	.2401	3804	6196	4.1653	30
40	.2363	3734	.9717	9875	.2432	3859	6141	4.1126	20
50	.2391	3786	.9710	9872	.2462	3914	6086	4.0611	10
14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	0.6032	4.0108	76° 00′
10	.2447	3887	.9696	9866	.2524	4021	5979	3.9617	50
20	.2476	3937	.9689	9863	.2555	4074	5926	3.9136	40
30	.2504	3986	.9681	9859	.2586	4127	5873	3.8667	30
40	.2532	4035	.9674	9856	.2617	4178	5822	3.8208	20
50	.2560	4083	.9667	9853	.2648	4230	5770	3.7760	10
15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	0.5719	3.7321	75° 00′
10	.2616	4177	.9652	9846	.2711	4331	5669	3.6891	50
20	.2644	4223	.9644	9843	.2742	4381	5619	3.6470	40
30	.2672	4269	.9636	9839	.2773	4430	5570	3.6059	30
40	.2700	4314	.9628	9836	.2805	4479	5521	3.5656	20
50	.2728	4359	.9621	9832	.2836	4527	5473	3.5261	10
16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	0.5425	3.4874	74° 00′
10	.2784	4447	.9605	9825	.2899	4622	5378	3.4495	50
20	.2812	4491	.9596	9821	.2931	4669	5331	3.4124	40
30	.2840	4533	.9588	9817	.2962	4716	5284	3.3759	30
40	.2868	4576	.9580	9814	.2994	4762	5238	3.3402	20
50	.2896	4618	.9572	9810	.3026	4808	5192	3.3052	10
17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	0.5147	3.2709	73° 00′
10	.2952	4700	.9555	9802	.3089	4898	5102	3.2371	50
20	.2979	4741	.9546	9798	.3121	4943	5057	3.2041	40
30	.3007	4781	.9537	9794	.3153	4987	5013	3.1716	30
40	.3035	4821	.9528	9790	.3185	5031	4969	3.1397	20
50	.3062	4861	.9520	9786	.3217	5075	4925	3.1084	10
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	0.4882	3.0777	72° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
ANGLE	COSINES		SINES		COTANGENTS		TANGENTS		ANGLE

Angle	Sines		Cosines		Tangents		Cotangents		Angle
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	0.4882	3.0777	72° 00′
10	.3118	4939	.9502	9778	.3281	5161	4839	3.0475	50
20	.3145	4977	.9492	9774	.3314	5203	4797	3.0178	40
30	.3173	5015	.9483	9770	.3346	5245	4755	2.9887	30
40	.3201	5052	.9474	9765	.3378	5287	4713	2.9600	20
50	.3228	5090	.9465	9761	.3411	5329	4671	2.9319	10
19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	0.4630	2.9042	71° 00′
10	.3283	5163	.9446	9752	.3476	5411	4589	2.8770	50
20	.3311	5199	.9436	9748	.3508	5451	4549	2.8502	40
30	.3338	5235	.9426	9743	.3541	5491	4509	2.8239	30
40	.3365	5270	.9417	9739	.3574	5531	4469	2.7980	20
50	.3393	5306	.9407	9734	.3607	5571	4429	2.7725	10
20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	0.4389	2.7475	70° 00′
10	.3448	5375	.9387	9725	.3673	5650	4350	2.7228	50
20	.3475	5409	.9377	9721	.3706	5689	4311	2.6985	40
30	.3502	5443	.9367	9716	.3739	5727	4273	2.6746	30
40	.3529	5477	.9356	9711	.3772	5766	4234	2.6511	20
50	.3557	5510	.9346	9706	.3805	5804	4196	2.6279	10
21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	0.4158	2.6051	69° 00′
10	.3611	5576	.9325	9697	.3872	5879	4121	2.5826	50
20	.3638	5609	.9315	9692	.3906	5917	4083	2.5605	40
30	.3665	5641	.9304	9687	.3939	5954	4046	2.5386	30
40	.3692	5673	.9293	9682	.3973	5991	4009	2.5172	20
50	.3719	5704	.9283	9677	.4006	6028	3972	2.4960	10
22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	0.3936	2.4751	68° 00′
10	.3773	5767	.9261	9667	.4074	6100	3900	2.4545	50
20	.3800	5798	.9250	9661	.4108	6136	3864	2.4342	40
30	.3827	5828	.9239	9656	.4142	6172	3828	2.4142	30
40	.3854	5859	.9228	9651	.4176	6208	3792	2.3945	20
50	.3881	5889	.9216	9646	.4210	6243	3757	2.3750	10
23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	0.3721	2.3559	67° 00′
10	.3934	5948	.9194	9635	.4279	6314	3686	2.3369	50
20	.3961	5978	.9182	9629	.4314	6348	3652	2.3183	40
30	.3987	6007	.9171	9624	.4348	6383	3617	2.2998	30
40	.4014	6036	.9159	9618	.4383	6417	3583	2.2817	20
50	.4041	6065	.9147	9613	.4417	6452	3548	2.2637	10
24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	0.3514	2.2460	66° 00′
10	.4094	6121	.9124	9602	.4487	6520	3480	2.2286	50
20	.4120	6149	.9112	9596	.4522	6553	3447	2.2113	40
30	.4147	6177	.9100	9590	.4557	6587	3413	2.1943	30
40	.4173	6205	.9088	9584	.4592	6620	3380	2.1775	20
50	.4200	6232	.9075	9579	.4628	6654	3346	2.1609	10
25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	0.3313	2.1445	65° 00′
10	.4253	6286	.9051	9567	.4699	6720	3280	2.1283	50
20	.4279	6313	.9038	9561	.4734	6752	3248	2.1123	40
30	.4305	6340	.9026	9555	.4770	6785	3215	2.0965	30
40	.4331	6366	.9013	9549	.4806	6817	3183	2.0809	20
50	.4358	6392	.9001	9543	.4841	6850	3150	2.0655	10
26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	0.3118	2.0503	64° 00′
10	.4410	6444	.8975	9530	.4913	6914	3086	2.0353	50
20	.4436	6470	.8962	9524	.4950	6946	3054	2.0204	40
30	.4462	6495	.8949	9518	.4986	6977	3023	2.0057	30
40	.4488	6521	.8936	9512	.5022	7009	2991	1.9912	20
50	.4514	6546	.8923	9505	.5059	7040	2960	1.9768	10
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	0.2928	1.9626	63° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
Angle	Cosines		Sines		Cotangents		Tangents		Angle

Angle	Sines		Cosines		Tangents		Cotangents		Angle
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	0.2928	1.9626	63° 00′
10	.4566	6595	.8897	9492	.5132	7103	2897	1.9486	50
20	.4592	6620	.8884	9486	.5169	7134	2866	1.9347	40
30	.4617	6644	.8870	9479	.5206	7165	2835	1.9210	30
40	.4643	6668	.8857	9473	.5243	7196	2804	1.9074	20
50	.4669	6692	.8843	9466	.5280	7226	2774	1.8940	10
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	0.2743	1.8807	62° 00′
10	.4720	6740	.8816	9453	.5354	7287	2713	1.8676	50
20	.4746	6763	.8802	9446	.5392	7317	2683	1.8546	40
30	.4772	6787	.8788	9439	.5430	7348	2652	1.8418	30
40	.4797	6810	.8774	9432	.5467	7378	2622	1.8291	20
50	.4823	6833	.8760	9425	.5505	7408	2592	1.8165	10
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	0.2562	1.8040	61° 00′
10	.4874	6878	.8732	9411	.5581	7467	2533	1.7917	50
20	.4899	6901	.8718	9404	.5619	7497	2503	1.7796	40
30	.4924	6923	.8704	9397	.5658	7526	2474	1.7675	30
40	.4950	6946	.8689	9390	.5696	7556	2444	1.7556	20
50	.4975	6968	.8675	9383	.5735	7585	2415	1.7437	10
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	0.2386	1.7321	60° 00′
10	.5025	7012	.8646	9368	.5812	7644	2356	1.7205	50
20	.5050	7033	.8631	9361	.5851	7673	2327	1.7090	40
30	.5075	7055	.8616	9353	.5890	7701	2299	1.6977	30
40	.5100	7076	.8601	9346	.5930	7730	2270	1.6864	20
50	.5125	7097	.8587	9338	.5969	7759	2241	1.6753	10
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	0.2212	1.6643	59° 00′
10	.5175	7139	.8557	9323	.6048	7816	2184	1.6534	50
20	.5200	7160	.8542	9315	.6088	7845	2155	1.6426	40
30	.5225	7181	.8526	9308	.6128	7873	2127	1.6319	30
40	.5250	7201	.8511	9300	.6168	7902	2098	1.6212	20
50	.5275	7222	.8496	9292	.6208	7930	2070	1.6107	10
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	0.2042	1.6003	58° 00′
10	.5324	7262	.8465	9276	.6289	7986	2014	1.5900	50
20	.5348	7282	.8450	9268	.6330	8014	1986	1.5798	40
30	.5373	7302	.8434	9260	.6371	8042	1958	1.5697	30
40	.5398	7322	.8418	9252	.6412	8070	1930	1.5597	20
50	.5422	7342	.8403	9244	.6453	8097	1903	1.5497	10
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	0.1875	1.5399	57° 00′
10	.5471	7380	.8371	9228	.6536	8153	1847	1.5301	50
20	.5495	7400	.8355	9219	.6577	8180	1820	1.5204	40
30	.5519	7419	.8339	9211	.6619	8208	1792	1.5108	30
40	.5544	7438	.8323	9203	.6661	8235	1765	1.5013	20
50	.5568	7457	.8307	9194	.6703	8263	1737	1.4919	10
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	0.1710	1.4826	56° 00′
10	.5616	7494	.8274	9177	.6787	8317	1683	1.4733	50
20	.5640	7513	.8258	9169	.6830	8344	1656	1.4641	40
30	.5664	7531	.8241	9160	.6873	8371	1629	1.4550	30
40	.5688	7550	.8225	9151	.6916	8398	1602	1.4460	20
50	.5712	7568	.8208	9142	.6959	8425	1575	1.4370	10
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	0.1548	1.4281	55° 00′
10	.5760	7604	.8175	9125	.7046	8479	1521	1.4193	50
20	.5783	7622	.8158	9116	.7089	8506	1494	1.4106	40
30	.5807	7640	.8141	9107	.7133	8533	1467	1.4019	30
40	.5831	7657	.8124	9098	.7177	8559	1441	1.3934	20
50	.5854	7675	.8107	9089	.7221	8586	1414	1.3848	10
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	0.1387	1.3764	54° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
Angle	Cosines		Sines		Cotangents		Tangents		Angle

Angle	Sines		Cosines		Tangents		Cotangfnts		Angle
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	0.1387	1.3764	54° 00′
10	.5901	7710	.8073	9070	.7310	8639	1361	1.3680	50
20	.5925	7727	.8056	9061	.7355	8666	1334	1.3597	40
30	.5948	7744	.8039	9052	.7400	8692	1308	1.3514	30
40	.5972	7761	.8021	9042	.7445	8718	1282	1.3432	20
50	.5995	7778	.8004	9033	.7490	8745	1255	1.3351	10
37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	0.1229	1.3270	53° 00′
10	.6041	7811	.7969	9014	.7581	8797	1203	1.3190	50
20	.6065	7828	.7951	9004	.7627	8824	1176	1.3111	40
30	.6088	7844	.7934	8995	.7673	8850	1150	1.3032	30
40	.6111	7861	.7916	8985	.7720	8876	1124	1.2954	20
50	.6134	7877	.7898	8975	.7766	8902	1098	1.2876	10
38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	0.1072	1.2799	52° 00′
10	.6180	7910	.7862	8955	.7860	8954	1046	1.2723	50
20	.6202	7926	.7844	8945	.7907	8980	1020	1.2647	40
30	.6225	7941	.7826	8935	.7954	9006	0994	1.2572	30
40	.6248	7957	.7808	8925	.8002	9032	0968	1.2497	20
50	.6271	7973	.7790	8915	.8050	9058	0942	1.2423	10
39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	0.0916	1.2349	51° 00′
10	.6316	8004	.7753	8895	.8146	9110	0890	1.2276	50
20	.6338	8020	.7735	8884	.8195	9135	0865	1.2203	40
30	.6361	8035	.7716	8874	.8243	9161	0839	1.2131	30
40	.6383	8050	.7698	8864	.8292	9187	0813	1.2059	20
50	.6406	8066	.7679	8853	.8342	9212	0788	1.1988	10
40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	0.0762	1.1918	50° 00′
10	.6450	8096	.7642	8832	.8441	9264	0736	1.1847	50
20	.6472	8111	.7623	8821	.8491	9289	0711	1.1778	40
30	.6494	8125	.7604	8810	.8541	9315	0685	1.1708	30
40	.6517	8140	.7585	8800	.8591	9341	0659	1.1640	20
50	.6539	8155	.7566	8789	.8642	9366	0634	1.1571	10
41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	0.0608	1.1504	49° 00′
10	.6583	8184	.7528	8767	.8744	9417	0583	1.1436	50
20	.6604	8198	.7509	8756	.8796	9443	0557	1.1369	40
30	.6626	8213	.7490	8745	.8847	9468	0532	1.1303	30
40	.6648	8227	.7470	8733	.8899	9494	0506	1.1237	20
50	.6670	8241	.7451	8722	.8952	9519	0481	1.1171	10
42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	0.0456	1.1106	48° 00′
10	.6713	8269	.7412	8699	.9057	9570	0430	1.1041	50
20	.6734	8283	.7392	8688	.9110	9595	0405	1.0977	40
30	.6756	8297	.7373	8676	.9163	9621	0379	1.0913	30
40	.6777	8311	.7353	8665	.9217	9646	0354	1.0850	20
50	.6799	8324	.7333	8653	.9271	9671	0329	1.0786	10
43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	0.0303	1.0724	47° 00′
10	.6841	8351	.7294	8629	.9380	9722	0278	1.0661	50
20	.6862	8365	.7274	8618	.9435	9747	0253	1.0599	40
30	.6884	8378	.7254	8606	.9490	9772	0228	1.0538	30
40	.6905	8391	.7234	8594	.9545	9798	0202	1.0477	20
50	.6926	8405	.7214	8582	.9601	9823	0177	1.0416	10
44° 00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	0.0152	1.0355	46° 00′
10	.6967	8431	.7173	8557	.9713	9874	0126	1.0295	50
20	.6988	8444	.7153	8545	.9770	9899	0101	1.0235	40
30	.7009	8457	.7133	8532	.9827	9924	0076	1.0176	30
40	.7030	8469	.7112	8520	.9884	9949	0051	1.0117	20
50	.7050	8482	.7092	8507	.9942	9975	0025	1.0058	10
45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	0.0000	1.0000	45° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
Angle	Cosines		Sines		Cotangents		Tangents		Angle

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