Elements of LOGIC *and* FORMAL SCIENCE

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TO

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Preface

THIS TEXT offers an elementary study of the science of logic and the philosophy of formal science. It has seemed clearer to keep these two subjects separated, since they deal with different problems; formal science, of course, makes use of logical principles in its method, but it is not the science of logic. This scheme also avoids the usual bad procedure of making "inductive reasoning" or the methods of nonformal science a part of logic. The procedure is bad since it necessitates formulating some definition of logic which is totally lacking in precision.

Thus, the definition of logic as the science of correct inference involves the danger of including practically every science under logic, since every science states some of its laws in the form of inferences. In particular, probability-theory is not a branch of logic.

A very exact definition of logic, due to Professor E. A. Singer, Jr., is employed throughout this book. Questions of nonformal science could not be handled adequately without adding another volume; but a chapter discussing the general problems of this branch of philosophy has been included to show the contrast between the formal and nonformal aspects of science. Also, from the point of view of logic, considerable space is devoted to showing the applications of logic in other sciences, the conflicts which logic faces within itself and without itself, and the manner in which logical principles are sometimes altered to solve these difficulties. Such a discussion, of course, deals with the nonformal aspects of the science. Logic cannot be refuted by direct experimentation, needless to say; but it is indicated here how it may come to be altered in other ways.

There is usually either an overemphasis or an underemphasis on terminology in logic texts. The method chosen here consists in introducing new terms when the necessity arises, rather than including all terms in a list in the first chapter. Thus the distinction between "inference" and "implication" comes rather late in the book, since it is not required at an earlier stage.

While the text develops the elements of symbolic logic, a great part of the traditional logic is also included. The two are not incompatible; the best introduction to symbolic method is the symbolic Aristotelian logic, since this is simple in treatment but contains the fundamental laws of the class calculus. Further, in the deductive form, this logic provides a very fine application of the laws of the propositional logic, not to be found in the deduction of theorems in Boolean algebra. The method of making the "Rules for Invalidity" a set of postulates is not used here since the procedure is formally bad.

Parts of the work are for more or less mature students; e.g., parts of chapters XI, XV, XVII, and XVIII. Some of the exercises are also of a somewhat advanced nature. This material might be used for a second-term course. The remainder of the book requires little or no mathematical training, other than elementary (first-year) geometry and algebra, and requires no previous study of logic.

In general, the text attempts to avoid verbal generalizations and descriptions; the best way for a student to learn the science of logic, which is essentially a science of method, is to follow through the methods himself. He learns nothing if he is merely "told about it." Hence the best way of learning the logic of propositions is to apply its principles in deducing theorems, and the Aristotelian logic affords an excellent opportunity to do just this.

Reading suggestions are, of course, selective, being chosen because they are either historically important or are good elementary expositions of the subject.

I am indebted to my teachers, the late Professor H. B. Smith and Professor E. A. Singer, Jr.; chapter VIII is a résumé of material in Professor Singer's lectures on the philosophy of science; he has very kindly read part of the manuscript and made many valuable suggestions. My indebtedness to Professor Smith will be obvious to anyone acquainted with his *First Book in Logic.*

To all my colleagues I owe much thanks. They have all offered suggestions and have supplied me with many illustrations.

Very special mention must be made of Dr. Elizabeth F. Flower. Dr. Flower has written a section of chapter XI, on the

solution of the vitalist paradox, has contributed many illustrations, but, most of all, has been an indispensable aid in helping to formulate the plan of the work and bring it to completion.

Mr. P. C. Rosenbloom has gone over the symbolic part and corrected many errors. Mr. John Taylor has read the manuscript and has been an invaluable aid in the preparation of the proofs for printing.

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C. West Churchman

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Part I

Formal Science and Logic

Deductive Science 1

ALL SCIENCE may be divided into two parts, the "formal" and the "nonformal." Formal or deductive science has the following general structure: a set of statements are made, the so-called *assumptions* or *postulates,* and by means of these and certain definitions a new set of statements, the *theorems,* are "deduced." Formal science has a *hypothetical* character: *if* all the statements made are true, then all the theorems will be. Nonformal science, on the other hand, is interested in determining whether certain statements are actually true; the usual method employed in such determination is experience.

The distinction between the two sciences may be seen in many ways. When we say: "Suppose, for the sake of argument, that these statements are true, and let us see what follows," we are arguing formally; if we afterwards proceed to establish our suppositions as actual facts, we argue nonformally. The child who prefaces his game with "let's pretend" is, in a sense, a formalist; he assumes (in his imagination) that he is a cowboy or a great warrior. His less imaginative playmate who points out that these assumptions are not really true is a nonformalist.

It is important to note that every recognized branch of science has a formal and a nonformal aspect. The science of physics, for example, may be formalized; to do this we merely construct certain statements making use of physical terms, and, by assuming these, deduce theorems. In its nonformal aspect, physics tries to determine certain facts and laws by experiment in the laboratory. The science of logic likewise has its formal and nonformal parts; the former will be developed first in chapters IV and V, and the latter in chapters IX–XI.

We call a set of assumptions and definitions of a formal science a *formal system* or a *deductive system.* It is evident that while a given branch of science may have many formal systems, depending on the various assumptions which can be made, it can have only one *nonformal system,* which will represent the one true set of statements about the science.

Can one assume what one pleases in a formal or deductive system? The answer to this question is "no," and the study of what constitutes a formally correct set of assumptions constitutes the philosophy of formal science. Corresponding to this there is a philosophy of nonformal science, which endeavours to find the correct method for determining the actual truth or falsity of a given statement.

For the present, our interest lies in the philosophy of formal science. The following problem is to be investigated: "What criteria must be kept in mind in order that a given set of propositions form a correct deductive system?"

Historically, the father of deductive science is Euclid (*ca.* 300 B.C.), who, in his *Elements,* constructed a formal science of geometry. Euclid affords the best model to follow in that, with true Greek genius, he set down practically all the criteria for a correct formal system.

Since every deductive system begins with certain declarative sentences, we must examine first the general character of statements or judgments. According to grammar, every proposition contains nouns, adjectives, verbs, or adverbs. We shall do better if we reduce this number to two: "terms" and "relations." Every proposition, then, contains certain terms which are related to other terms. Thus in the proposition "This green table is larger than that brown table," the term "this green table" is related, by the relation "is larger than," to another term, "that brown table." Terms, then, include nouns or substantives, and modifiers, or adjectives, while relations are verbs, which may be modified by adverbs or, in the case of the verb "to be," may contain adjectives.

The so-called "traditional" logic, which was first developed by Aristotle and was more or less perfected during medieval times, classified terms in a number of ways. Thus all terms are either *abstract* or *concrete,* either *universal* or *singular.* Abstract terms are terms whose meaning is abstracted from any specific content, while concrete terms refer to some specific object or group of objects. Thus "beauty" is abstract, while "beautiful thing" is concrete; "humanity" is abstract, "human being" is concrete. "Beauty" does not refer to any object or any group of objects, while "beautiful thing" does refer to some definite, concrete thing. A good test for abstract terms is the following: suppose x is the term; then if we can say, "This is an x" (or "These are x's"), x is concrete; otherwise it is abstract. Thus, we cannot say, "This is a Beauty" or, "This is a Humanity," and these terms are consequently abstract; but we can say, "This is a beautiful thing" or, "This is a human being," and hence these terms are concrete. Sometimes a single word will have both an abstract and a concrete connotation; for example, "truth" is abstract in one sense, concrete in another. In the sentence "Beauty is Truth" it is abstract, while in the sentence "He just told a truth," it is concrete. In English, many abstract nouns end in "-ness" ("goodness").

Singular or particular terms are nouns representing one individual; e.g., "John Smith," "The Empire State Building," "The University of Paris," etc. Universal terms refer to a group of things; e.g., "man," "building," "university," etc. A term is singular if we can place a "the" before it, since this article indicates that the term following it is individual. Universal terms are terms which can be attributed to more than one individual.

Terms are also designated as *positive* and *negative*. Negative terms imply the negation of some quality; in English their first syllable is usually "un-," or "in-," or "non-." Thus "unhappy," "impractical," "nonaggressive" are negative terms; positive terms have no negative prefixes. One is apt to be deceived by this distinction, which is in part only arbitrary. Many negative terms may be stated as positives and vice versa. For example, "odd" is a positive term but means the same thing as "unusual," which is negative. Nevertheless, the study of negative terms forms a very important part of the science of logic (cf. pages 99 ff.).

The general properties of relations will be developed in chapter II. For the present we merely note that a relation is that which joins two or more terms together so that the result is a proposition or judgment.

(See Exercises, Group A, at end of chapter.)

In studying the nature of formal or deductive systems we must note first of all that the terms and relations are always restricted to a certain group. The best known deductive system, that of the geometry of Euclid, deals with the terms and relations of geometry. Examples of the terms would be: "line," "figure," "curve," "angle," and of relations: "is parallel to," "intersects," "lies between." A formal science of arithmetic would contain such terms as "number," "fraction," "minus," and such relations as "is equal to," "is greater than."

The first task of a deductive system, then, consists in making precise what terms it introduces and in defining these terms. After that, it must construct the assumptions which contain these terms (the axioms and postulates), and finally must deduce its theorems.

Definitions

Etymologically, the word "definition" signifies a process by which we limit the meaning of the word. Two methods have been suggested:

1. *The method of dichotomy:* this was proposed by Plato and examples of it are found in his *Sophist*. The famous "Tree of Porphyry" affords the best example. Suppose that we wish to define the singular term "Socrates" or the universal term "Athenian." We start first with the broadest class of all, the universe of things, and divide it into two parts, the corporeal and the incorporeal; we then designate into which class the object to be defined falls. In the case of the terms "Socrates" and "Athenian," the former class would contain the object. Proceeding to a further dichotomy, we divide the class of corporeal beings into two parts, the animate and the inanimate, the two examples we have chosen falling in the first class. Again, we divide the animate into the rational and the irrational, placing our terms in the first class again. If we proceed to dichotomize in this way indefinitely, it seems that we ought to be able to define any term; the more universal a term, the fewer dichotomies necessary to reach it. In the case of singular terms, the dichotomizing would presumably be quite lengthy; some philosophers, indeed, have felt that no amount of dichotomies would ever give us the individual as distinct from everything else. In other words, this method is only adequate for defining universal terms.

2. *The method of genus and species,* formulated by Aristotle, is a generalization of Plato's method. This starts with a broad, general class called the *genus,* and then "specifies" the parts of it, these parts being called the *species.* The method is essentially the same as Plato's, except that there may be more than two divisions. In the case of biological things, for example, the genus is "living things," and the various kinds of living things make up the species. If "minerals" were our genus, then the kinds of minerals would constitute the species: iron, copper, silver, etc. Often the species itself is a genus with respect to the forms within it, so that we have not necessarily defined a word completely when we have placed it in a genus and species. For example, "red rose" belongs first of all to the species "living" of the genus "things," then to the species "plant" of the genus "living," then to the species "rose" of the genus "plant," then to the species "red rose" of the genus "rose."

Those qualities which designate the divisions of a genus are called *differentia.* Thus, in the genus "line," the differentia would be "straight" and "curved," and these would designate the two species "straight lines" and "curved lines."

The same remarks concerning the definition of individuals made in the case of the method of dichotomy would also apply here.

One feature of definitions which is highly important with respect to deductive systems lies in the fact that all definitions presuppose that the meanings of certain terms are already known. Thus, it would do us little good to look up the meaning of "labor" in the dictionary if we did not know already the meaning of "toil" or "work." This is an assumption which all dictionaries make, namely, that the reader is already aware of the meaning of certain words. A foreigner who knew no English could hardly expect to learn vocabulary purely from the dictionary. For example, if, after looking up the meaning of "labor" and finding it to be "physical or mental toil," should he then go on to search for the meaning of "toil," he would find that the latter meant "labor with pain and fatigue," and now clearly his definition has become circular: "labor is physical or mental labor." The dictionary presupposes that we know either what "toil" or what "labor" means. No dictionary bothers to make a list of the words whose meanings it presupposes. Such a task would be far too laborious and hardly fruitful for the average reader. But the fundamental property of all deductive systems is that *they shall make explicit all presuppositions;* that is, everything which is assumed shall be set down, no matter how obvious. Hence, it is a part of the task of every deductive system to make a list of the terms or words the meanings of which it assumes or does not define. These terms or words are called *indefinables,* and by means of them are defined all other terms or words with which the deductive system deals.

There is, however, one important qualification of this point. As we go higher in the sciences, each science, without definition, may make use of terms of sciences which it "presupposes." The most basic science is logic, for every science presupposes logic, i.e., every science must be "logical." Hence, all sciences may make use of logical concepts without attempting to define them; only in the case of logic itself must they be defined or be considered as indefinables. Thus the word "is" appears in most definitions: "A dog *is* a canine," "A ton *is* two thousand pounds," etc. Again, the words "if . . . then" are frequently employed: "*If* a number has no factors other than itself and one, *then* the number is a 'prime' number"; "*If* a body of land is completely surrounded by water, *then* it is an island," etc. The term "not" (or "what is not") also appears in many definitions: "*What is not* living is 'inanimate,' " or "If x is *not* 0 or plus, then x is a 'minus' number," etc. As we shall see, all these terms belong to the science of logic.

But the science of geometry also makes use of arithmetical terms such as addition and subtraction without defining them, and rightly so, since geometry presupposes arithmetic, while the latter does not presuppose the former (arithmetic does not make use of geometrical terms such as "lines," "angles," etc.). It becomes necessary, then, to classify the sciences by way of priority so that when we formalize a given science we shall know exactly what terms of other sciences may be presupposed. Such a classification will be given in chapter VII.

To sum up, then, the following is a necessary criterion for good definitions in any formal system: All terms and relations of a deductive system (other than the indefinables) must be defined by means of a certain set of indefinable terms which, by their very nature, cannot themselves be defined.

(See Exercises, Group B, at end of chapter.)

This discussion of indefinables is essentially a modern contribution to the philosophy of deductive systems. Euclid does not seem to have been aware of the necessity of making explicit the words whose meanings are assumed. Examples of Euclid's definitions are:

1. A *point* is that which has no part.
2. A *line* is a breadthless length.
5. A *surface* is that which has length and breadth only.
23. *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Definitions 1, 2, and 5, which define the three fundamental concepts of geometry, presuppose as indefinables "part," "breadth," and "length." Ordinarily, "point," "line," and "surface" are taken as indefinables. But the presuppositions of 23, Euclid's famous definition of parallels, are much more extensive. Primarily, the definition assumes the fundamental concept of intersecting lines (lines that meet).

Axioms

Postulates and axioms comprise all the assumed statements of a given formal system. These statements must contain only the indefinables and the defined terms of the system. Just as each deductive system makes use of the terms of other sciences which it presupposes, so it also makes use of the assumptions and theorems of these sciences. If these assumptions are made explicit, they are called *axioms;* the assumptions which the given science introduces itself are called *postulates.* The test determining whether a certain assumption is an axiom or a postulate will be as follows: if the statement contains no indefinables or defined terms of the science itself, but only the terms of presupposed sciences, then it is an axiom; but if it contains a term which is an indefinable of the system or is defined by the indefinables of the system, then it is a postulate.

Thus every science after logic takes for granted the statements of logic. Arithmetic, for example, does not bother to assume or prove the statement that everything is either A or not A. Again, we find Euclid assuming under the name of "common notions" certain propositions of arithmetic, such as "If equals be added to equals, the results are equal." ("If $a = b$ and $c = d$, then $a + c = b + d$.") It can readily be seen that a science like physics would have a great many axioms, while a certain part of logic would have none. Thus again arises the importance of classifying the sciences into some hierarchy. It is noteworthy that a change in the laws of logic would be tantamount to a change in all sciences, since all sciences assume the laws of logic as axioms.

Postulates

The word "postulate" comes from the Latin *"postulare,"* which means, literally, "to demand." Thus the significant property of a postulate consists in this: it is a demand that we accept the given statement as true. No proof of this statement is to be given; the argument that it is "self-evident" or "obvious" is really no proof, but an appeal to the individual's intuition, and the question as to whether or not a postulate is valid is not left to individual opinion. It might be thought that the validity of the postulate is left to experience, so that if it contradicts some past experience which we or others have had, we call it false. But, though it may be exceedingly impractical to construct a deductive system many of whose theorems contradict experience, it is not impossible to do so. *Experimental validity is not a criterion for the postulates of a deductive system.*

But what shall we say are the requirements for a set of postulates? It seems evident enough that the choice of assumptions is not a haphazard matter. In the first place, we do not want any two of the postulates to contradict one another. Thus an arithmetic which postulates "$2 + 2 = 4$" and "$2 + 2 = 5$" has contradictory postulates (provided 4 and 5 and "$=$" are defined as usual). If we assume that a contradiction can exist, then we break down the principles of logic, and anything will follow from our postulates: every statement will be a theorem. We say, then, that the postulates of the deductive system must be "consistent" among each other.

Next, it is necessary that the postulates really be assumptions and not provable, for otherwise they are not postulates at all but theorems. That is, no postulate may follow from any of the others, no postulate may be a theorem. This property of the postulates we call their "independence"; each postulate is independent of all the others in the sense that it does not follow from them; it cannot be proved by means of them.

Last, there is the requirement of postulates, not so much logical as practical, that they shall be "sufficient" to prove all the theorems required, or, possibly, that they shall be sufficient to prove all statements containing the terms of the system. Thus the postulates of geometry are sufficient if they enable us to prove all the properties of points, lines, surfaces, and solids in space.

Traditionally, many other requirements have been offered. Thus, postulates should be "self-evident" according to some writers; but no one has ever been able to determine exactly what statements fall in this category. If the history of philosophy teaches us anything it is that there are no self-evident propositions; what has been considered "obvious" in one age is often considered to be false in the next. H. Sidgwick, in his *Methods of Ethics,* suggests that the postulates must meet with the approval of the experts in the field. Fortunately for the history of science, this criterion has frequently been neglected. It cannot be overemphasized that, from the point of view of formal science, the actual truth of the postulates, whether determined intuitively, experimentally, or dogmatically, is of no interest. The objection suggested here, that it seems very impractical to assume propositions which everyone knows to be false, besides introducing questions of value which have no place in the philosophy of formal science, also overlooks the fact only too frequently illustrated in the history of science that what everyone knows to be false in one generation everyone knows to be true in the next. A formal scientist, working as he does behind closed doors, may still be modeling the science of a future age, and if that age verifies his assumptions the work is done, for he has already completed by his theorems the remainder. The German mathematician, Gauss, for example, developed many mathematical theories which in his day (early nineteenth century) seemed to have no practical value, but today are extremely important for the physicist. The mathematician is continually developing instruments for future generations even though his labors today seem utterly useless. Indeed, these instruments are often the means for discovering new truths.

Euclid's postulates (as distinct from his "common notions" or axioms) are:

1. It is possible to draw a straight line from any point to any point.
2. Any finite straight line may be produced continuously in a straight line.
3. A circle may be constructed with any point as a center and any distance as a radius.
4. All right angles are equal.
5. If a straight line intersects two straight lines in such a manner that the interior angles on the same side are less than two right angles, the two straight lines, produced indefinitely, meet on this side on which the interior angles are less than two right angles.
6. Two straight lines intersect in only one point.

These postulates are generally recognized as being consistent, inasmuch as no one of them contradicts any other. Their independence, however, has not always been so readily granted. For centuries geometers believed that postulate 5, the so-called Euclidean Parallel Postulate, could be proved and hence was a theorem. Various methods were tried to accomplish this proof, among the most interesting of them being that of the Italian mathematician, Saccheri (1667-1733), who attempted to prove Postulate 5 by means of the "indirect proof." This method consists in assuming that the theorem to be proved is false and from this assumption deriving a contradiction. Hence Saccheri assumed the invalidity of Postulate 5 and attempted to show that a contradiction arose. As a matter of fact, though Saccheri, through an error, thought that he had arrived at such a contradiction, no such thing occurs, and it turns out that the contradictory of 5 is also consistent with the remaining postulates. This suggests, as it did to Lobatschewsky (1793-1856) and Bolyai (1802-1860), that it is possible to construct a geometry which assumes Postulates 1-4 but the contradictory of Postulate 5. Such a geometry, so-called non-Euclidean, has many theorems identical with the Euclidean geometry, but some quite important ones which are not. For example, "The sum of the angles of a triangle is less than 180 degrees," and "Through a point outside a line more than one line (actually an indefinite number) can be drawn in the plane parallel to the given line." Thus another geometry, as consistent as Euclid's and following the same deductive method, was constructed.

It is also possible to construct a consistent deductive system of geometry in which Postulate 6 is false. When this is the case, Postulate 5 becomes a theorem. This geometry was discovered by Riemann (1826-1866) and forms another non-Euclidean geometry. Here there are no parallel lines in a plane; that is, all lines in a plane intersect, or, through a point outside a line, no line in the plane can be drawn parallel to the given line.[1] The sum of the angles of a triangle is greater than 180 degrees. The best aid to the imagination in picturing these two non-Euclidean geometries [2] is to think of a plane in the first case (Lobatschewsky) as the surface of a napkin ring which curves inward. In Riemann's case, the plane is similar to the surface of a sphere, where the "straight lines" are "great circles," i.e., lines on the sphere which describe the circumference. All great circles meet in more than one point, there are no parallel great circles, etc. This analogy, however, breaks down when we try to conceive solid Lobatschewskian or Riemannian geometry, i.e., geometry of three dimensions.

The importance of this discussion from our point of view lies in the fact that it points a way to a method of demonstrating that a given postulate is independent of the other postulates of a deductive system.[3] For if we can show, as was shown in the case of Postulate 5 above, that the contradictory of the given postulate is consistent with the remaining postulates, then we can infer that it is not a theorem. For all theorems have the property of being true *whenever* the postulates are true and we have shown that this is not a property of the given postulate.

These three types of geometry also show that there are many possible deductive systems even in one field, and that the validity of a deductive system, as such, does not depend on the experimental truth of the postulates.

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[1] Hence the truth of Postulate 5.

[2] There are, of course, other non-Euclidean geometries.

[3] For such a method, see chap. XV.

Theorems

By means of the definitions and assumptions made by a deductive system, certain other statements are "deduced." The theorems have the character of being true if the postulates are true. The criterion for theorems in a deductive system is the following: All theorems must be proved by the definitions and the assumptions of the system only; that is, we cannot take for granted implicitly any assumptions in the proving of a theorem, no matter how obvious this assumption may be. This criterion is one of the most difficult of all to follow. It is usually almost impossible to be certain whether one has not unconsciously assumed some proposition in proving a theorem. But once a formal system has established a theorem on the basis of its assumptions, it may make use of this theorem in future proofs.

An example of Euclid's method of proof is the following, the first Proposition of the First Book:

THEOREM: Given any (finite) straight line AB, it is possible to construct an equilateral triangle on AB.

PROOF:
1. With A as the center and AB the radius, describe a circle. (This is possible by Postulate 3 above.)
2. Again, describe a circle with B as center and BA as radius.

Let C be one of the points where these circles meet; then

3. since AC and AB are radii of the same circle, AC = AB (by the definition of a circle).
4. Also, BC and AB are radii of the same circle: BC = AB.
5. Hence, AC = BC (since "Things equal to the same thing are equal to each other," an assumption of arithmetic.)
6. Hence AC = BC = AB, and the triangle ABC is equilateral by definition.

It will be noted that Euclid's proof, despite the fact that geometry has been held as a model for exact reasoning, makes many tacit (geometrical) assumptions. These assumptions are perhaps obvious enough, and many of them can readily be proved as theorems. But an exact deductive system would always make these assumptions explicit, and if they were theorems, it would prove them beforehand. The following are possible objections to this proof:

1. How can we be sure that the circles meet in order to make the point C? Of course it is impossible to imagine pictorially how they might not; but, as we must repeatedly emphasize, a deductive system should not appeal to imagination or experience; its only appeal must be to the postulates or axioms (i.e., the assumptions) it sets down, to the definitions it makes, or to the theorems it has proved.

2. How can we be sure that the lines AC and BC do not meet at some point D, before C? That is, how do we know that AC and BC do not have a segment DC in common, so that the real triangle under consideration will be ABD, which is not equilateral, since AD and BD are less than AB? This objection was pointed out (by Zeno of Sidon) at an early stage in the history of geometry. Again, pictorially, this possibility seems absurd. But it can only become absurd when some assumption or theorem of the system makes it so. Actually, the impossibility of the segment DC seems to be included in Postulate 6: "Two straight lines meet in only one point," which was itself, presumably, a tacit assumption of Euclid's in a later theorem. For this being the case, the straights AC and BC cannot meet at a point D distinct from C.

3. How do we know that ABC is really a triangle? Euclid's definition of a triangle is: "A rectilineal figure is one contained by straight lines, and a triangle is a rectilineal figure contained by three straight lines." Some further explanation, at least, is necessary to demonstrate that the given ABC actually conforms to this definition.

Other tacit assumptions may perhaps appear to the reader. The purpose of introducing this criticism is to show in what way even those proofs which appear most exact lack rigor.

In most detective stories of the "scientific" sort we are supposed to be able to "deduce" the identity of the villain from the facts given. In most cases, if one were to set down all the assumptions necessary to pin the guilt, they would far exceed the facts. The detective slides gracefully over these, and we, in no mood to be logical, follow.

In the deductive system for logic given in chapter IV an attempt will be made to avoid this looseness and to make every assumption explicit.

EXERCISES

GROUP A

1. Determine whether the following terms are abstract or concrete, universal or singular, positive or negative: boy, Abraham Lincoln, unhappiness, ability, good things, goodness, unknown, New York, animal, animality, uninteresting, disinterested, music, Beethoven's Fifth Symphony, term, godliness, dishonesty, house, street.
2. Name a term, other than those listed in (1), which is abstract and negative; concrete and universal; universal and negative.
3. Is there such a thing as an abstract, universal term? a negative, singular term?

GROUP B

1. Define by dichotomy the following terms: furniture, house, flower, goat.
2. Designate the genus, species, and differentia in the following definitions:

   a) A farce is a light dramatic composition of a satirical nature.
   b) Geography is the science of the earth and its life.
   c) A sculpin is a fish of the group *Loricati* of the family *Cottidae.*
   d) A maxim is a statement embodying a general truth.
   e) A bonfire is a large fire made in the open air.
   f) Two is a number obtained by adding one and one.
   g) Gravity is that force which attracts all bodies towards the center of the earth.

3. Place in their genera and species the following: man, logic, pleasure, triangle.
4. What are the principal species of the following genera: number, element, literature, trade.
5. What are the indefinables in the following set of definitions:

   a) x is an *immediate* cause of y, if x is the cause of y and there is nothing which is caused by x and causes y.
   b) x is a *mediate* cause of y, if x is the cause of y and there is something caused by x which causes y.
   c) x and y are *independent* if x does not cause y and y does not cause x.

The Logic of Propositions 2

All deductive systems draw some of their axioms from the material under consideration in this chapter. Here we are concerned with a branch of science known as the "logic of propositions." That is, we are interested here in the relations between propositions in general, in what sense one statement *implies* another, and the general rules for passing from the truth of one sentence to the truth of another by means of "reasoning."

As an example of the kind of statement which will form the subject matter of this discussion, take the following: "This proposition is either true or false." Now the truth of this statement does not depend on what proposition I have in mind, as it would if the statement were simply "This proposition is true." The truth of the former statement is a *universal* truth; it holds for all propositions. We say that its truth is *independent of* the content of the proposition, while the truth of such a statement as "If this (proposition) be so, then all men are created equal" does depend on the content. We may say, then, that our aim is to discover statements about propositions which are true no matter what propositions we may be talking about, or, to say the same thing more precisely, which are true independent of the meaning (content) of the propositions. Hence it can be seen in what sense this science is basic. For all sciences, in that they state their truths as propositions, must necessarily make use of the logic of propositions.

Before investigating the laws of this science, a few preliminary definitions and remarks are necessary. In the first place, it will be best to make somewhat definite the concept of "proposition." This term is really fundamental and no precise definition can be given. But we may distinguish the term "proposition" from other allied expressions by recalling the distinction made in elementary grammar between declarative sentences and other types of sentences, for a proposition is essentially the former. Hence, such expressions as "Shut up!" "What a nice day!" and "Isn't it raining?" are not propositions in our sense of the word.

But the logician makes a distinction between two different types of propositions. For example, compare these declarative sentences:

Jones is not honest.
Mr. A is not honest.

Two plus three is five.
$x + y = 5$.

Not all men are white.
Not all *a* is *b*.

12 is less than 7.
x is less than 7.

The first of each pair of propositions is a simple sentence which is either true or false. But the second in each case is more complex. None of these is either definitely true or false until we know what A, or *x*, or *y*, etc., are. Propositions of this latter type we call "propositional functions" (analogous to mathematical functions like the second and last examples given above; in fact, the latter are to be considered a special case of the former). Our interest will center particularly on the latter type, for we shall consider in the next chapter such propositional functions as "all *a* is *b*," "some *a* is *b*," etc. It is to be borne in mind that when we talk of propositions in this chapter we mean either type, i.e., simple propositions or propositional functions.

(See Exercises, Group A, at end of chapter.)

Important for the sequel are the relations which exist between a pair of propositions. We say that every pair of propositions may be classified under one (and only one) of the following: "contradictories," "contraries," "subcontraries," and "subalterns."

Contradictories

By far the most commonly recognized relation between two propositions is that of contradiction. The *contradictory* of a given proposition, *p*, is usually expressed "*p* is false." Thus, were we to contradict the expression, "This is a nice day," we would say, "It is false that this is a nice day," or, more rhetorically, "This is not a nice day." But we require some exact method for determining when two propositions may be said to be contradictories. We say, then, that two propositions are *contradictory* when it is impossible for them both to be true and when it is impossible for them both to be false (one of them at least must be true).

Thus the two contradictories "All men are equal" and "Some men are not equal" cannot both be true at once, but one or the other of them must be true. This property of contradictories is sometimes rather neatly expressed as follows: "Contradictories are mutually exclusive and together exhaust the possibilities." The Greek philosopher Aristotle (384-322, B. C.), who devoted an important part of his work to the foundations of logic, set down these properties of contradictories in two fundamental principles:

1. "No proposition can be both true and false" (the Principle or Law of Contradiction),

which can be seen to be equivalent to the property of contradictories that they cannot both be true.

2. "Every proposition is either true or false" (the Principle or Law of Excluded Middle),

which is equivalent to the statement that one or the other of two contradictory propositions must be true.

One other property of contradictories is important. This concerns the well-known method of "reductio ad absurdum," so often employed in the proof of geometrical theorems. We assume the contradictory of the theorem to be proved, and discover that this assumption leads to a contradiction. That is, we find that the contradictory of the given theorem is false, and hence infer that the theorem must be true. The assumption implicit in this argument may be phrased in a number of ways, perhaps the best of which is:

"The statement that it is false that a given proposition is false is equivalent to the statement that the given proposition is true."

But any of the following express the same fact:

"The contradictory of the contradictory of p is p itself."
"Two negatives make an affirmative."

The contradictory of a given proposition p may be symbolized by p' (often by $\sim p$ or $\bar{p}$). Hence the above three rules may be partially symbolized as follows:

1. It is impossible that p and p' both be true.
2. Either p or p' is true.
3. $(p')'$ is equivalent to p.

Contraries

Contrary propositions have the sense of "opposites" or "extremities." They are analogous to the end points of a line. Thus a contrary of "This is white" would be "This is black." With this hint we can define contrary propositions as two propositions which cannot both be true (are mutually exclusive) but may both be false (are not together all-inclusive). Note that while a proposition may have but one contradictory, it may have in general any number of contraries. Thus the proposition "There are exactly twenty people in this room" has as many contraries as there are whole positive numbers, for all such expressions as "There are exactly ten (or two, or eight) people in this room" will be contraries. There cannot be exactly twenty and exactly ten people in the room (both statements cannot be true), but there may be neither twenty nor ten, e.g., when there are exactly eight.

Subcontraries

Two propositions are said to be *subcontrary* when they may both be true (are not mutually exclusive) but cannot both be false (together exhaust the possibilities). For example, the propositions "Some books are good to read" and "Some books are not good to read" both may be (in fact, are) true, but they cannot both be false, for whether all books are good or all books are bad, one or the other of them will still be true.

Subalterns

Finally, we have *subalterns,* pairs of propositions which are such that they may both be false (are not mutually inclusive) and may both be true (are not mutually exclusive). These are of a class opposite to contradictories: For example, "All pigs have wings" and "Some pigs have wings" are subalterns.

(See Exercises, Group B, at end of chapter.)

Implication

By far the most important relation between propositions as far as logic is concerned is that of *implication.* The frequency of the use in implication is apparent from the observation of the number of synonyms for the word "implies." Thus, "All whales are mammals implies some whales are mammals" may be rendered *"If* all whales are mammals, *then* some are," or *"From the hypothesis that* all whales are mammals, *it follows that* some are."

Implication is the principal relation of logic or the science of the laws of reasoning. When we say that a certain statement p "implies" another statement q, we mean that there is a relation between the former and the latter, a relation which is such that whenever the former is true, the latter is true also. We might phrase this as follows: "The statement p implies the statement q" means the same thing as "q must be true when p is true," or "It is impossible that q be false when p is true." The word "implies," or one of its synonyms, is used constantly, but usually subconsciously. Here are some examples of everyday usage:

"If this administration stays in power, then the stock market will not go up." ("'This administration is in power' implies 'The stock market is not going up.'")

"It follows from all that has been said that war is imminent." ("All these statements imply 'War is imminent.'")

"If all men were as honest as he is, the world would be a better place to live in." ("'All men are as honest as he' implies 'The world is a better place to live in.'")

Since "p implies q" or "If p is true, then q is true" means the same thing as "p cannot be true while q is false," the contradictory of "p implies q" must be "p may be true while q is false." That is, one method of denying that "p implies q" is true consists in showing that there are cases where p is true and q is not true. Thus to show that the implication

"If it is Sunday, it is raining"

is false, we find a case of a rainless Sunday, i.e., a case where "It is Sunday" is true, but "It is raining" is false. We do not contradict the statement "*p* implies *q*" if we merely assert that *p* may be false while *q* is true, for contradictories cannot both be true, and there are cases where "*p* implies *q*" and "*p* is false while *q* is true" are both true statements. For example, "If all men are bald, then some men are bald" is a true implication; but it is also the case that "All men are bald" is false, while "Some men are bald" is true.

The importance of the relation of implication will become clear when we note that the theorems of a deductive science are all implied by the postulates; hence it is essential from the point of view of formal science to examine the properties of this relation.

A convenient method for examining the properties of any relation is to determine whether it is reflexive, symmetrical, or transitive.

Let us designate by *aRb* the fact that *a* is related to *b* by the relation *R*. Thus in the sentence "*a* is larger than *b*," *R* is the relation "is larger than," in the sentence "John is the brother of Bill," *R* is the relation "is the brother of," etc.

A relation is called *reflexive* if *aRa* is always true, i.e., if we can always relate an object to itself in the given way. Thus "equals" in mathematics is a reflexive relation, for the statement "$a = a$" is true for every quantity. Again, "looks exactly like" is a reflexive relationship in the realm of visible objects. For every such object looks exactly like itself. But "is greater than" is not reflexive, since it is not true that "*a* is greater than *a*."

However, it is peculiar of reflexive relations as defined above that they can be said to hold only in a restricted field. Thus, though "=" is reflexive in mathematics as was indicated, this relation is not reflexive if we extend our field to all things whatsoever, for if *a* is the term "triangularity," for instance, we have "triangularity = triangularity," and this cannot be said to be either true or false, since it is meaningless. Similarly, "implies" is reflexive for all propositions, since the statement "if *p* is true, then *p* is true" is itself universally true. But if *p* is the term "stones," then the statement "*p* implies *p*" is meaningless. Hence, it becomes more convenient to define a reflexive relationship in a slightly different manner so that its condition shall hold in certain cases universally: "A relation is said to be reflexive if when a given object holds this relation to some other object, it is so related to itself." For example, in the case of "looks like" we have: "If *a* 'looks like' some object *b*, then *a* can be said to look like *a*." This holds for all *a*'s whatsoever. For if *a* is "a minor chord," then "*a* looks like *a*" is nonsense, but the statement "*If* a minor chord looks like something, it looks like itself" is true; *if* we were to apply the relation "looks like" to such objects as minor chords, then we would have to say that they looked like themselves. In terms of the symbolism suggested above, this more precise definition becomes: "*R* is reflexive if *aRb* implies *aRa*."

Since "implies" is a reflexive relation, we have:

> PRINCIPLE 1. If *p* implies some other proposition *q*, then (*p* is a proposition and hence) *p* implies *p* (where the parenthetical part is for rhetorical purposes only).

That "implies" is reflexive may be shown to depend on the Law of Contradiction already explained. For the statement "*p* implies *q*" means "*p* cannot be true while *q* is false." Hence "*p* implies *p*" means "*p* cannot be true while *p* is false," and the latter is a form of the Law of Contradiction.

Next, a relation is said to be *symmetrical* if, when it holds between *a* and *b*, it also holds between *b* and *a*, that is, if *aRb* implies *bRa*. "Equals," for example, is symmetrical as well as reflexive, for if $a = b$, then $b = a$. Again "some—is—" is symmetrical, for if some *a* is *b*, then some *b* is *a*. But "all—is—" is not symmetrical, for we cannot infer from the fact that all *a* is *b* that all *b* is *a*.

If the relation of implication were symmetrical, we could assert that whenever the statement *p* implies the statement *q*, then *q* also implies *p*. But the reader will easily find instances where this is not the case. Suppose, for example, that *p* were the statement "The present temperature is 75" and *q* were the statement "The present temperature is over 70." Then if *p* is true, *q* must also be true, but the converse is hardly the case: if the temperature is over 70, we cannot necessarily infer that it is exactly 75. As we shall see later, this bad reasoning is called the Fallacy of Asserting the Consequent.

But there is a property of implication which, though not the symmetrical one, is very similar to it. Suppose we represent "p is false" or the contradictory of p by the symbol p' as above. Then we can assert:

PRINCIPLE 2*a*. "If p implies q, then q' implies p'."

That is, if p and q are two statements such that q is true whenever p is true (p implies q), then if q is false p must also be false. The above illustration will serve here as well. The statement that the temperature is now 75 implies the statement that the temperature is over 70; hence, by Principle 2*a*, we may assert that if the temperature is not over 70, then it cannot now be 75. Or, again, "If all men are rational, then some men are;" hence, we can logically deduce that "If it is false that (even) some men are rational, i.e., if none are rational, then it's false that all men are rational."

Principle 2*a* may be applied to the special case where p and q are "equivalent." When we say that two statements are equivalent we mean that they mutually imply each other. This property of equivalence appears frequently in mathematics, where many theorems have the form: "The necessary and sufficient condition for the statement q (e.g., 'This triangle has equal sides') is the statement p (e.g., 'This triangle has equal angles')." For the assertion that p is the necessary condition of q means that p must be true in order that q be true, that is, whenever q is true, p is true, or, as we put it, q implies p. And the assertion that p is the *sufficient* condition of q means that the truth of p is sufficient to guarantee the truth of q, or, p implies q. The statement, then, that p is the necessary and sufficient condition of q is precisely the same as the statement that p is equivalent to q (p implies q and q implies p). The equivalence of two statements p and q is also often expressed: "p is true if and only if q is true." In this special case, Principle 2*a* becomes:

PRINCIPLE 2*a'*. If p and q are equivalent, then if p is false, q is also false, and vice versa;

that is, if p and q are equivalent, then their contradictories are equivalent. Thus, since a necessary and sufficient condition that $a + 2 = b + 2$ is that $a = b$, we can say the statements "$a + 2 \neq b + 2$" ($a + 2$ "does not equal" $b + 2$) and "$a \neq b$" are equivalent.

Principle 2 may also be applied in the case where the implication is not valid. Thus, "If a man is honest sometimes it does not follow (necessarily) that he will be honest all the time." We may infer from this, by Principle 2, as follows: "If a man is not honest all the time it does not follow that he is not honest some of the time." In general, then:

PRINCIPLE 2*b*. If the statement p does not imply the statement q, then q' ("q is false") does not imply p' ("p is false").

Logic, like all sciences, often "generalizes" its principles. That is, it finds new principles which express what the old principles expressed and more as well. The generalization of Principle 2 can best be seen by means of an illustration: Suppose you were presenting a debate in which you had four points admittedly substantiating your argument. That is, numbers 1, 2, 3, and 4 imply number 5, the conclusion you wish to assert. If your opponent agrees that this implication is valid, how will he go about showing that number 5 is not the case? Simply by showing that you were wrong in one of your arguments. That is, granted that the above implication is correct, then if numbers 1, 2, and 3 are correct, but number 5 is false, then it must follow that number 4 is false.

In general then, if the statements $p, q, r \ldots$ imply the statement w, then if $q, r, \ldots$ are true, but w is false, then p must be false. If we shorten (*p and q and r* . . . but w') to ($pqr \ldots w'$) (note that the word "but" has the same logical significance as "and," and means, really, "and, however," where the word "however" has only a rhetorical significance) then we can write Principle 2 in this general form:

PRINCIPLE 2*c*. If $p\ q\ r \ldots$ imply w, then, $q, r \ldots w'$ imply p'.

Or, obviously, we could just as well write:

if $pqr \ldots$ imply w,
then $p, r \ldots w'$ imply q',
Or, if $p\ q\ r \ldots$ imply w, then $p, q, \ldots w'$ imply r', etc.

Note that Principle 2*c* becomes Principle 2*a* when there is but one premise implying w. Similarly, If $p\ q\ r \ldots$ do not imply w, then $w'\ q\ r \ldots$ do not imply p'.

As an example of the application of Principle 2*c* to arguments, suppose we grant that if

1. we are being attacked,
2. it is disgraceful to submit to an attack without resistance,
3. one ought not do anything disgraceful,

then,

4. we should go to war.

Hence if we grant

1. we are being attacked.
4′. we should not go to war.
3. one ought not do anything disgraceful.

then we must assert that it is not disgraceful to submit without resistance, 2′. That is, if 1, 2 and 3 imply 4, then 1, 4′ (4 is false), and 3 imply that 2 is false, or 2′.

Or again, suppose the hypothesis that there is life on Mars can be established if

1. The temperature there is comparable to that on the earth;
2. There is an atmosphere containing a specific amount of oxygen, comparable to the amount on the earth;
3. The geology of Mars' crust is comparable to that of the Earth;
4. Any planet comparable to the earth in temperature, atmosphere, and geology must contain life.

Then if later explorers should find no life on Mars, but 2, 3, and 4 are all found to be true, it must follow that the temperature on Mars is not like that of the Earth.

Again, suppose that the following premises do not establish the conclusion "There are more than one hundred elements":

1. The elements may be arranged in a series according to their atomic number.
2. Between any two supposedly adjacent atomic numbers, chemists have repeatedly found a third.

Then if there are less than one hundred elements and 2 is true, we cannot necessarily infer the falsity of 1.

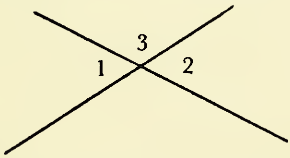
Principle 2*a* and its generalizations are very important, as the sequel will show, and for the sake of convenience we give them a name, the Law of Contradiction and Interchange. This name arises from the fact that the principle is best stated verbally as follows: "Given any valid or invalid implication between any number of premises (p, q, r, etc.) and a conclusion w, we may take any premise and contradict and interchange it with the conclusion and the validity of the resulting implication is unchanged."

This principle, under a different form, plays a very important role, especially in mathematics. The proofs of many theorems are possible only through an "indirect method." This consists simply in assuming that the theorem to be proven is false, and then demonstrating that an inconsistency with certain assumptions occurs. Thus, suppose we grant these theorems in geometry:

1. The sum of the angles around a point on a straight line is equal to two right angles.
2. Unequals added to equals give unequals.
3. All right angles are equal.

Then we prove by the indirect method that

4. If two lines intersect, the vertical angles are equal.



For suppose that 4 were not so; then in the accompanying figure, 1 and 2 are unequals, and hence $1 + 3$ and $2 + 3$ are unequals by our second premise. But this contradicts our first and third premises, which state that $1 + 3 = 2$ right angles $= 2 + 3$. In effect, what we have done is to show that if 2 and 3 are true and 4 is false, then 1 is false, i.e., 2, 3, and 4′ imply 1′; hence, by the Principle of Contradiction and Interchange, if 2 is true, and 1 is true, and 3 is true, then 4 is true, i.e., 1, 2, and 3 imply 4. Further illustrations of this principle under the Indirect Method are given in chapter VI.

Third, a relation is said to be *transitive* if when it holds between *a* and *b* and between *b* and *c*, it holds between *a* and *c*. Or, symbolically, *R* is transitive if *aRb* and *bRc* imply *aRc*. Thus, "equals" is also transitive, as is the mathematical relation $<$ ("is less than") ("If *a* is less than *b* and *b* is less than *c*, then *a* is less than *c*.") The relation "is the cause of" is usually thought to be transitive; that is, "If *a* is the cause of *b*, and *b* is the cause of *c*, then *a* is the cause of *c*." Similarly, "is the effect of" is also generally recognized as transitive. But "is one-half of" is not a transitive relation: If *a* is half of *b* and *b* is half of *c*, then *a* is not in general half (but a quarter) of *c*.

There are a great many relations, however, which at first glance appear transitive and are not actually so for a very special case. Thus, "is parallel to" is a relation commonly recognized as transitive, for if *a* is parallel to *b*, and *b* to *c*, then *a* must be parallel to *c*. However, the conclusion will be false in the particular case where *a* and *c* are the same lines, for according to the usual definition of parallel lines (lines in the same plane which never intersect), no line is parallel to itself. Thus, we might have *a* parallel to *b* and *b* parallel to *a*, but it would not follow that *a* is parallel to *a*. Mathematicians and scientists in general are prone to call relations "transitive" which fail to be so in this special case only, and because of their frequency, it will be convenient to designate such relations as "transitive in the restricted sense." In other words, a relation is transitive in the restricted sense when *a* is related to *b* and *b* to *c* in the given way, *a* will be related to *c* in the same way *provided a and c are not the same.* Note that such relations as "is greater than," "is equal to" do not require this proviso.

The reader perhaps already will have satisfied himself that implication is a transitive relation (in the general sense). In geometry, for instance, we often make use of a theorem already proved in order to prove another, and in so doing we really assume the transitivity of implication in a sense: we assume that if a given postulate implies a theorem which, in turn, implies another theorem, then the truth of the last can be said to depend directly on the first; i.e., the postulate implies the last theorem:

Principle *3a*. If when *p* is true, *q* is true, and when *q* is true, *r* is true, then when *p* is true, *r* is true.

Note that in Principle *3a* there are two premises ("If *p*, then *q*" and "If *q*, then *r*") which imply a conclusion ("If *p*, then *r*"). But Principle *2b* states that when two premises validly imply a conclusion we may contradict and interchange one of them with the conclusion and the resulting implication is still valid. That is, we may assert Principle *3b* as follows:

Principle *3b*. If (*p* implies *q*) but (*q* does not imply *r*), then (*p* does not imply *r*).

Principle 3c. If (p does not imply r) but (q does imply r), then (p does not imply q).

Thus, since "a is a factor of b" does not imply that "b is a factor of a," but "b is evenly divisible into a" does imply that "b is a factor of a," we can infer that "a is a factor of b" does not (necessarily) imply that "b is evenly divisible into a."

Principles 3a, 3b, and 3c and their related principles are easier to grasp if we introduce the terms "strengthen" and "weaken." Let us call the statement p a "strengthened" form of the statement q if p implies q, and let us call q a "weakened" form of p in this case. These terms agree to some extent with common usage, for the stronger a statement is, the more fraught it is with implications. Thus the proposition "All men are liars" is considerably stronger than the proposition "Some men are liars." But note that even though two statements are equivalent, i.e., even though they mutually imply each other, we can still say, by our definition of the term, that the one is "stronger" (or "weaker") than the other.

Principle 3a, which expresses the truth of the transitivity of implication, may now be expressed as follows: "Given any valid implication 'p implies q,' we may replace the conclusion q by a weakened form (in this case r) without altering the validity of the implication," or "Given any valid implication 'q implies r,' we may replace the premise q by a strengthened form (in this case p) without altering the validity of the implication." Again, 3c and 3b would become: "Given any invalid implication 'p does not imply r,' we may weaken the premise or strengthen the conclusion and the resulting implication is still invalid."

The foregoing statements illustrate the close analogy which the relation of implication bears to the relation "is (physically) stronger than or has the same strength as." Thus, if one thing is not stronger than another (p doesn't imply r), then something of equal strength or weaker than the former (something p implies) is not stronger than the latter (does not imply r). Or, if one thing is not weaker than another, then something of equal strength or stronger than the former is not weaker than the latter.

The results may be generalized in a manner analogous to the generalization of Principle 2. For suppose we have two premises,

p and q, which imply r; then we may replace p or q (or both) by stronger forms, or we may replace r by a weaker form, and the resulting implication is still valid. That is, in every-day language, we "strengthen" our argument by finding stronger premises upon which it rests; and we "weaken" our conclusion by not inferring so much from our premises.

PRINCIPLE *3d.* If the premises $p, q, r, \ldots$ validly imply w, and some statement x implies p (or q, or r, etc.), then we can assert that $x, q, r, \ldots$ validly imply w.

If the premises $p, q, r, \ldots$ validly imply w, and w implies x, then $p, q, r, \ldots$ validly imply x.

Similarly,

PRINCIPLE *3e.* If the premises p, q, r do not imply w, and p (or q, or r, etc.) implies x, then x, q, r, do not imply w.

If the premises $p, q, r, \ldots$ do not imply w, and some statement x does imply w, then p, q, r, do not imply x.

Thus from the premises "Harvard beat Yale," "Yale beat Princeton," "Princeton beat Penn," we cannot infer that "Harvard will beat Penn." But the statement "Harvard beat Yale" implies that "Harvard at one time was better than Yale," and the statement that "Yale beat Princeton" implies that "Yale was at one time better than Princeton," etc. Thus, from the premises "Harvard was at one time better than Yale," "Yale was at one time better than Princeton," "Princeton was at one time better than Penn," we cannot infer that "Harvard will beat Penn."

(See Exercises, Group C, at end of chapter.)

Constructive Hypothetical Syllogism (Modus Ponens)

A syllogism is an argument in which two premises are used to establish a certain conclusion. The fundamental law of implication, the principle which expresses the right to pass from the truth of one statement (p) to the truth of another (q) is called the Constructive Hypothetical Syllogism (or modus ponens):

Principle 4. "If p is true, then q is true (i.e., p implies q); but p (as a matter of fact) is true; hence, we may assert: 'q is true.'"

The foregoing syllogism is called "hypothetical" because of the first premise, the implication, which contains the conditional clause "If p is true." It is called "constructive" because the second premise "constructs" or asserts the truth of p.

The significance of the Constructive Hypothetical Syllogism, which perhaps seems overly trite, lies in the fact that it allows us to consider the statement q as true quite apart from p, i.e., to think of q as an independently true proposition. Thus, the following implication is true: "If there are ten people here, then there are at least six." But we cannot, on the basis of this, consider "There are at least six people here" as an independently true proposition. It is true *provided* the foregoing hypothesis ("There are ten people here"), or one like it, is true. It is pinned, as it were, to this condition. If the condition is verified, then the Constructive Hypothetical Syllogism allows us to unpin the foregoing phrase and consider "There are six people here" as an independently true proposition.

This syllogism may be generalized as follows:

"If (the premises) p, q, r, etc., are true, then the statement w is true; but p, q, r, etc., are all true; hence w is true."

The Constructive Hypothetical Syllogism plays a very important role in mathematics, since it is the logical principle behind the method of "mathematical induction." Inductive reasoning in general consists in establishing general principles from particular cases. In mathematics, we proceed as follows. Suppose that we have a certain proposition of algebra, e.g., "The expression $x^n - y^n$ is always (evenly) divisible by $x - y$, where n is any whole, positive number." We note first of all that this proposition is true when n has the value 1. We now proceed to show that *if* the proposition is true for a certain value of n, then it will be true for the value $n + 1$, that is, for the next number. Let us suppose then that $x^n - y^n$ is divisible by $x - y$, where n has a certain specific value, say 26, or 49, or 567. We may represent this specific value by the letter a; i.e., $x^a - y^a$ is divisible by $x - y$. Now $x^{a+1} - y^{a+1}$ is equal to

$$x^{a+1} - y^{a+1} + x^a y - x^a y$$

since we have merely added and subtracted the same quantity, $x^a y$. The latter may be written

$$x^{a+1} - x^a y + x^a y - y^{a+1}, \text{ or}$$
$$x^a (x - y) + y (x^a - y^a).$$

But this last expression is divisible by $x - y$, since the first part obviously is, and the latter part is by hypothesis. Hence, since the last expression is equal to $x^{a+1} - y^{a+1}$, we may say that this also is divisible by $x - y$. Hence we have: "*If* $x^a - y^a$ is divisible by $x - y$, then $x^{a+1} - y^{a+1}$ is also." Suppose a is the number 1: "If $x^1 - y^1$ is divisible by $x - y$, then $x^2 - y^2$ is also." But, as a matter of fact, $x^1 - y^1$, or $x - y$, is divisible by $x - y$. Hence, by the logical principle of the Constructive Hypothetical Syllogism, we have established the truth of the statement that $x^2 - y^2$ is divisible by $x - y$. Now suppose $a = 2$: "If $x^2 - y^2$ is divisible by $x - y$, then $x^3 - y^3$ is also." Since the premise is true, the conclusion is also. By carrying out this process far enough, we can show that $x^{23} - y^{23}$, $x^{125} - y^{125}$, $x^{2048} - y^{2048}$, etc. are all divisible by $x - y$. That is, we may assert that whatever whole positive number n may be, $x^n - y^n$ is divisible by $x - y$. (Cf. chapter XVIII for the role this principle plays in the postulates of arithmetic.)

It should be noted that mathematical induction comprises three distinct parts: (1) the verification of the proposition for some particular number, (2) the demonstration that if the proposition is true for some number, it must be true for the next, and (3) the logical principle of the Constructive Hypothetical Syllogism. There are many statements for which (1) is true, but (2) fails—e.g., the proposition "$6^a - a$ is divisible by a" is true for all whole numbers from 1 to 4, but (2) fails for this proposition, since even though it is true for $a = 4$, it is false for $a = 5$. Similarly, there are many propositions for which (2) holds, but which can never be verified for any number. Thus, if the statement "$x^n + y^n$ is divisible by $x - y$" is true for n, it will be true for $n + 1$, as may be demonstrated in a manner similar to that given. But this statement is not true for any values of x or y (greater than 0).

Closely connected with the Constructive Hypothetical Syllogism is a fallacious syllogism: "If p is true, then q is true; but

q is (admittedly) true; hence p is true." This is the argument known as the Fallacy of Asserting the Consequent. For example:

"Bacon was a great writer if he wrote Shakespeare's plays; but as a matter of fact, Bacon was one of the greatest writers of all times; hence Bacon must have written Shakespeare's plays."

A more complicated form of the Fallacy of Asserting the Consequent occurs in a paradox of Lewis Carroll. Suppose that three men, Allen, Brown, and Carr, are the three barbers in a certain shop. Allen, the conscientious proprietor of the shop, has a hard and fast rule that at least one man must always be there. But Allen, who suffers from the gout, insists that Brown accompany him whenever he goes out. From these two facts we can apparently deduce the conclusion that Carr must be in (despite the fact that there seems to be no contradiction in Allen's and Brown's being there while Carr is out). Suppose, for the sake of argument, that Carr is out. Then we know that *under this hypothesis* if Allen is out, Brown must be in (since at least one man must be there):

1. If Carr is out, then, if Allen is out Brown is in.

But by the second fact we know that

2. If Allen is out, Brown is out.

Hence on the hypothesis that Carr is out we have the proposition: "If Allen is out, Brown is in," and this contradicts the fact: "If Allen is out, Brown is out." Hence the hypothesis that Carr is out leads to a contradiction, and thus Carr must be in. What is the fallacy here?

The Destructive Hypothetical Syllogism (Modus Tollens)

Here the argument makes use of the Principle of Contradiction and Interchange:

PRINCIPLE 5. "If p is true, then q is true;
but as a matter of fact q is false;
hence p is (actually) false."

This argument allows us to assert the independent falsity of a given statement. For example:

"If the Cretans had visited the British Isles, we would have found some trace of their civilization; but as a matter of fact, no sign of it has been found; hence we may assert that the Cretans did not visit the British Isles."

Analogous to the Fallacy of Asserting the Consequent is the Fallacy of Denying the Antecedent: "If p is true, then q is true; but p is, as a matter of fact, false; hence q is false." For example: "If we had good housing conditions, the poorer people of our country would be happy; but as a matter of fact we have the poorest housing conditions; hence the poorer people cannot possibly be happy."

So far we have considered three of the relations of the logic of propositions: negation, implication, and equivalence. Traditionally, logicians have recognized two more. The first is called "conjunction," and is expressed by the statement "p and q are both true." The second, "disjunction," is expressed by the statement "Either p or q is true." The latter is ambiguous, since "either, or" may mean "either one or the other but not both," or "either one or the other and possibly both." The latter meaning is chosen here.

The properties of conjunction and disjunction have already been mentioned in part, e.g., in Principles 2*c*, 3*d*, 3*e*, etc. We mention a few more here, but the fuller explanation may be found in chapter XII. The most important law connecting these two relations is the following: "If we deny the conjunction 'p and q are both true,' then we assert the disjunction 'Either p is false or q is false.' " Thus if we deny that today is Monday and there will be no Sunday paper, we assert that either today is not Monday or there will be a Sunday paper. That is, if it is false that both propositions are true, then one or the other must be false. A similar principle is the following: "If we deny the disjunction 'p or q is true,' we assert the conjunction 'p is false and q is false.' " That is, if it is false that either one or the other is true, then both must be false. For example, if it is false that either all men are happy or that no men are happy, then some men are not happy and some men are happy. These laws may be stated as follows:

PRINCIPLE 6. The denial of a conjunction (pq) is a disjunction with the elements separately denied $(p'$ or $q')$,

and

The denial of a disjunction is a conjunction with the elements separately denied.

From the foregoing relations of the calculus of propositions we may construct several forms of argument which are often used. These are known under the common name of dilemmas.

The Complex Constructive Dilemma

This argument runs:

PRINCIPLE 7*a*. "If the statement p is true, then the statement q is true, and if r is true, then s is true; but either p is true or r is true; hence, either q is true or s is true."

For example:

"If you learn a great deal (p), then you forget a great deal (q), and if you don't learn much (r), then you never know much (s); but either you learn a great deal or you don't learn much (either p is true or r is true); hence, either you forget a great deal or you never know much (either q or s is true)." Or,

"If we sell to foreigners, we send valuables from our country; if we buy from foreigners, we send money from the country; hence, whether we buy or sell to foreigners we will either lose goods or else money."

The latter illustration may be "retorted"; i.e., a dilemma may be constructed which will argue on the other side in an analogous manner:

"If we sell to foreigners, we bring money to the country, and if we buy from foreigners we bring goods to the country; whether we buy or sell to foreigners, then, we either bring in money or goods."

A common form of the dilemma occurs when q and s are identical:

"If we surrender, they will hang us, and if we do not surrender, they will capture us and hang us. But we must either surrender or not. Hence, we must necessarily hang (we either hang or we hang)."

The third premise of the dilemma, the one containing the words "either, or" is in the form of a disjunction. The two alternatives of this disjunction in the first example above are "You learn a great deal" and "You do not learn much." Often the disjunction in question is not "complete," that is, the two alternatives do not exhaust the possibilities:

"If you tell the truth, men will hate you (for being candid), and if you tell lies, the gods will hate you (for lacking virtue). But either you tell the truth or you tell lies; hence either men or gods will hate you."

But we are not forced to accept the alternative in the third premise; a man may lie at times and be truthful at times and hence at least men will not hate him for this. When a third (or perhaps a fourth or fifth) alternative is possible, the dilemma is said to be refuted by "escaping through the horns."

The Complex Destructive Dilemma

Here the argument is the same as the above, but makes use of the Principle of Contradiction and Interchange:

PRINCIPLE *7b*. "If p is true, then q is true, and if r is true, then s is true; but either q is false or s is false; hence, either p is false or r is false."

For example:

"If the weather report is right, it will rain; if Jones is right, the game will be played; but either it won't rain or the game won't be played; hence either the weather report or Jones is wrong."

Obviously the following two mixed forms are also possible.

PRINCIPLE *7c*. "If p (is true), then q (is true), and if r, then s; but either p is true or s is false; hence either q is true or r is false."

PRINCIPLE *7d*. "If p, then q, and if r, then s; but either q is false or r is true; hence either p is false or s is true."

Fallacious dilemmas arise by use of the Fallacies of Asserting the Consequent and Denying the Antecedent. For example:

"If he has an inventive mind, he will do well as an engineer; but if he likes poetry, he will not make a good engineer; however, he will either make a good engineer or not; hence I conclude that he is either inventive or poetic."

(See Exercises, Group D, at end of chapter.)

The following offers a brief résumé of the material in this chapter and at the same time introduces certain symbols which have proved useful in the development of modern logic.

The logic of propositions is that science whose aim is to find all universal statements about propositions, i.e., to find all statements about propositions which hold regardless of the meaning or form of the propositions involved.

The fundamental relations of this science are four:

1. "p implies q," which we may symbolize by "$p \angle q$."[1]
2. "p is false," symbolized by p'.[2]
3. "p and q are both true," symbolized by pq, and called the "conjunction" of p and q.
4. "Either p is true or q is true (or both)," symbolized by $p + q$,[3] and called the "disjunction" of p and q.
5. By these four[4] fundamental relations we can define a fifth, "p and q are equivalent," which is symbolized $p = q$ and means simply "p implies q and q implies p," i.e., in symbols, $(p \angle q)(q \angle p)$.

With these symbols we can express the principles given in this chapter in this symbolic form:

1. $(p \angle q) \angle (p \angle p)$
2. *a*) $(p \angle q) \angle (q' \angle p')$
   $(p = q) \angle (p' = q')$
   b) $(p \angle q)' \angle (q' \angle p')'$
   c) $(pqr \ldots \angle w) \angle (qr \ldots w' \angle p')$
   $(pqr \ldots \angle w) \angle (pr \ldots w' \angle q')$, etc.

[1] In the history of symbolic logic, many symbols for this relation have been suggested; the following are some: $p \rightarrow q$, $p \supset q$, $p \prec q$, $p \dashv q$, $p < q$; the present symbol is a shortened form of the mathematical symbol $\leqq$, "less than or equal to;" there is some analogy between the two relations, for "p implies q" is sometimes thought of as "The cases where p is true are less than or equal to the cases where q is true," the former being included in the latter.

[2] Sometimes symbolized $\sim p$.

[3] Sometimes symbolized $p \vee q$.

[4] Ordinarily, logicians add a fifth relation: "p and q mean the same thing." When this is distinguished from "p and q are equivalent," the former is symbolized $p = q$ and the latter $p \equiv q$; but from the point of view of the elementary treatment here, it is not necessary to distinguish these relations, so that $p = q$ may mean either "p and q are equivalent" or "p and q mean the same thing."

3. *a*) $(p \angle q)\ (q \angle r) \angle (p \angle r)$
   b) $(p \angle q)\ (p \angle r)' \angle (q \angle r)'$
   c) $(p \angle r)' (q \angle r) \angle (p \angle q)'$
   d) $(pqr \ldots \angle w)\ (x \angle p) \angle (xqr \ldots \angle w)$
   $(pqr \ldots \angle w)\ (w \angle x) \angle (pqr \ldots \angle x)$
   e) $(pqr \ldots \angle w)' (p \angle x) \angle (xqr \ldots \angle w)'$
   $(pqr \ldots \angle w)' (x \angle w) \angle (pqr \ldots \angle x)'$
4. Constructive Hypothetical Syllogism: If $(p \angle q)\, p$ is true, then q is true.
5. Destructive Hypothetical Syllogism: If $(p \angle q)\, q'$ is true, then p is false, i.e., p' is true.
6. $(pq)' = (p' + q')$
   $(p + q)' = p'q'$
7. Constructive Dilemma:

$$(p \angle q)\ (r \angle s)\ (p + r) \angle (q + s)$$

   Destructive Dilemma:

$$(p \angle q)\ (r \angle s)\ (q' + s') \angle (p' + r')$$

   Mixed Dilemmas:

$$(p \angle q)\ (r \angle s)\ (p + s') \angle (q + r')$$
$$(p \angle q)\ (r \angle s)\ (q' + r) \angle (p' + s)$$

8. Aristotle's laws may be stated thus:

   a) Law of Contradiction: (pp')
   b) Law of Excluded Middle: $p + p'$

These three laws are also important:

9. $p'' = p$
10. $pq = qp$
11. $(p + q) = (q + p)$

The following give the symbolic definitions of the three classes of relations:

1. Reflexive: $(aRb) \angle (aRa)$
2. Symmetrical: $(aRb) \angle (bRa)$
3. Transitive: $(aRb)\ (bRc) \angle (aRc)$

EXERCISES

GROUP A

1. Which of the following statements belong to the logic of propositions, i.e., which are true for all propositions whatsoever?

a) If this proposition is true, then this same proposition is true.
b) This proposition is false.
c) If the wave-theory of light is correct, then this proposition is true.
d) If, when the proposition p is true, the proposition q is also true, then when q is false, p must also be false.
e) This proposition is either always true or always false. (Hint: Is this true for propositional functions?)
f) If the proposition p is true and the proposition q is true, then the proposition p is true.
g) This proposition implies that $x - z = 4$.
h) If it is false that this proposition is false, then this proposition is true.
i) This (proposition) does not imply that any felony has been committed.
j) This proposition implies (it follows from this proposition) that every proposition is true.

GROUP B

1. Tell whether these pairs of propositions are contradictories, contraries, subcontraries, or subalterns.
   a) It is raining.
   It is snowing.
   b) Greater love hath no man.
   Some man hath greater love.
   c) There are at least 20 people in this country.
   There are at least 30 people in this country.
   d) The tide is flowing out.
   The tide is flowing in.
   e) x is greater than ten.
   x is less than ten.
   f) All men are liars.
   All liars are men.
   g) Life's but a walking shadow.
   Life's a poor player who struts and frets his hour upon the stage.
   h) Some are born great.
   Some are not born great.
   i) If the proposition p is true, then q is true.
   If the proposition q is false, then p is false.
   j) Through a point outside a line, one and only one parallel in the plane can be drawn to the line.
   Through a point outside a line more than one parallel in the plane can be drawn to the line.

k) Motorists must drive under 40 miles an hour.
Motorists may drive as fast as they please.

l) $x + 2 = 6$.
$x < 4$ or $x > 4$ (where $x < y$ means "x is less than y" and $x > y$ means "x is greater than y").

m) These two propositions are contraries.
These two propositions are subalterns.

n) No one knows the truth about anything.
Everyone knows the truth about everything.

o) He is happy.
He is unhappy.

2. Given that *p* and *q* are contradictories, *p* and *r* are contraries, *p* and *s* are subcontraries, and *p* and *t* are subalterns, what are the relations subsisting between *q*, *r*, *s*, and *t*?
3. If *p* and q are contraries, *q* and *r* are contradictories, and *r* and *s* are contraries, what are *p* and *s*?

GROUP C

1. Classify the following relations by determining whether they are reflexive, symmetrical, or transitive.

a) "is higher than"
b) "is the cousin of"
c) "is parallel to"
d) "works for"
e) "has a greater velocity than"
f) "is the king of"
g) "Some—is—"
h) "is a part of"
i) "is a factor of" (in mathematics)
j) "belongs underneath"
k) "All—is—"
l) "loves"
m) "No—is—"
n) "is an analogue of"
o) "Some—is not—"

2. Give a relation which is reflexive, symmetrical, and transitive; one (other than "implies") which is reflexive and transitive, but not symmetrical; one which is neither reflexive, symmetrical, nor transitive; one which is only symmetrical; one which is only reflexive; one which is only transitive; one which is reflexive and symmetrical, but not transitive.

3. Prove that no relation can be transitive and symmetrical but not reflexive.
4. Determine whether the following relations are transitive in the restricted or in the general sense: "lies above," "is the brother of," "precedes," "is included in," "equals," "is a factor of," "is not the same object as." Find a relation which is transitive in the restricted sense and symmetrical, but not reflexive.
5. Let aRb denote that an object a has a given relation R to an object b (cf. page 22).

   R is irreflexive if aRa is never true.

   R is non-reflexive if aRa is sometimes true and sometimes false.

   R is asymmetric if aRb implies that bRa is false (no matter what a and b may be).

   R is non-symmetric if, when aRb, bRa is sometimes true and sometimes false.

   R is intransitive if aRb and bRc imply that aRc is false (no matter what a, b, and c may be).

   R is non-transitive if when aRb and bRc, aRc is sometimes true and sometimes false.

   Classify the following relations under the above six categories: a) "is parallel to," b) "equals," c) "has a point in common with," d) "is one-half of," e) "is less than."

   Prove that if a relation is symmetrical and intransitive, it is irreflexive; if a relation is transitive and irreflexive, it is asymmetrical; a relation cannot be asymmetrical and reflexive.
6. By applying Principle 2, the Law of Contradiction and Interchange, to the following statements, derive new statements:

   EXAMPLE: If $a = 0$, then $ab = 0$. If p represents "$a = 0$," and q "$ab = 0$," then we have an implication of the form "p implies q;" hence " 'q is false' implies 'p is false;' " that is, "$ab \neq 0$ implies $a \neq 0$."

   a) If we lower the tariff we lower our standard of living.
   b) Water is formed if hydrogen and oxygen are mixed in the proper quantities.
   c) If a body is falling free, its acceleration is 16 feet per second per second.
   d) From the fact that A and B are vertical angles, it follows that they are equal.
   e) If no man is honest, then no honest being is a man.
   f) If $a = 1$, then $ab = b$.
   g) A relation's being symmetrical and transitive implies that it is reflexive.

h) If you take a teaspoon of arsenic tonight, you will be dead tomorrow.

i) If two bodies having equal weight are hung at equal distances from the fulcrum, they will balance.

j) It does not follow that if a man performs a morally good act he is therefore virtuous.

k) "Some thinking is good" does not imply "Some thinking is not good."

l) If some of this is clear, it does not follow that all of it is.

m) Even though all rich men go to heaven, it does not follow that all people that go to heaven are rich.

n) If $a = 0$, then it is not necessarily the case that $a + b = 0$.

o) If you are bitten by a rattlesnake, it does not follow that you will die.

p) From the fact that some men are not Chinese, we cannot infer that some Chinese are not men.

q) If $a = b$, but $b \neq c$, then $a \neq c$. (Apply the principle to both premises.)

r) If two angles in the triangle A are equal respectively to two angles in the triangle B and the included sides are equal, then A and B are congruent.

s) If a is a factor of b and b is a factor of c, then a is a factor of c.

t) If a is greater than b and b is greater than c, then a is greater than c.

u) If two bodies are moving along a line towards each other, and their masses are equal and their accelerations are equal, they will come to a dead stop when they meet.

v) If all men are living, and no living thing is a stone, then no man is a stone.

w) If all cats are sly and some animals are not sly, then some animals are not cats.

x) If all tadpoles are secretive, and no secretive things are nice, and some fish are nice, then some fish are not tadpoles.

y) If some people are stupid and some donkeys are stupid, it does not follow that some people are donkeys.

z) If this is government of the people and this is government by the people, it does not follow that this is government for the people.

aa) From the fact that John is the father of Bill, and Bill is the father of Sam, we cannot infer that John is the father of Sam.

bb) If $a \neq b$ and $b \neq c$, it does not follow that $a = c$.

cc) Even though it is true that all men are living and all cats are living, we cannot say that some men are cats.

dd) The statements "No men are trees" and "No trees live underground" do not imply that all men live underground.

ee) The premises that many people have seen ghosts and some of the people who have seen ghosts are reliable and all reliable people tell the truth most of the time are not sufficient to establish the fact that ghosts exist.

7. Apply Principle 3 to the following and form new statements; then apply Principle 2 to the result.

EXAMPLE: If $a = b$ and $b > c$ ("b is greater than c"), then $a > c$. Now $a + 3 = b + 3$ implies that $a = b$, and $b = c + 2$ implies that $b > c$, and $a > c$ implies that $a \neq c$. If we let p be the statement $a = b$, q the statement $b > c$, r the statement $a > c$, s the statement $a + 3 = b + 3$, t the statement $b = c + 2$, and u the statement $a \neq c$, then we have the following: 1) p and q imply r, 2) s implies p (that is, s is a strengthened form of p), 3) t implies q, and 4) r implies u (i.e., u is a weakened form of r). Then it follows that, since we can strengthen either premise or weaken the conclusion, the following are true: s and q imply r, p and t imply r, s and t imply r, p and q imply u, s and q imply u, p and t imply u, s and t imply u; stating the last, which is the case where both premises are strengthened and the conclusion weakened, we have: If $a + 3 = b + 3$ and $b = c + 2$, then $a \neq c$.

a) If Clay had been president, the West would have benefited; but if the West had benefited, expansion would have taken place much more quickly.

b) If all of us are tried and true, then some of us are; but if some of us are tried and true, and no tried and true person ever surrenders, then some of us will not surrender.

c) If all statements are meaningful and nothing you say makes sense, then nothing you say is a statement. But if nothing you say is a statement, then some things you say are nonsense.

d) All men have reason and all angels have reason does not imply that some men are angels. But all angels have reason implies that devils have reason (the devils being fallen angels). Again, all men are angels implies that some men are angels.

e) No circles are squares and no circles are rectangles does not imply that some squares are not rectangles. But no circles are squares does imply that no squares are circles, and no circles are rectangles implies that some rectangles are not circles, and no squares are rectangles implies that some squares are not rectangles.

f) From the fact that $a + 2 = 3$ implies $a = 1$, what new statement can be inferred from *f*) in Example 6?

g) From the fact that if a man is a saint he is virtuous, what new statement can be inferred from *j*) in Example 6?

h) If "no stone is a living thing" implies that "no living thing is a stone," what can be inferred from *v*) in Example 6?

i) John is the father of Bill implies that Bill is the son of John; what follows from *aa*) in Example 6? What follows if we grant that Sam is the son of John implies that John is the father of Sam?

8. Restate Principles 3*d*) and 3*e*) by using the terms "strengthen" and "weaken" instead of "implies."
9. What is the strongest statement possible; what is the weakest statement possible?
10. Determine whether the following pairs of statements are contraries, contradictories, subcontraries, or subalterns:

a) p implies q.
p does not imply q.

b) p implies q.
q implies p.

c) p is a weakened form of q.
p is a strengthened form of q' ("q is false")

d) p and q are both weakened forms of r.
p is a weakened form of q.

GROUP D

1. Identify and criticize the following arguments (i.e., determine whether they are fallacious or not); in the case of the dilemma, determine the horns and if possible retort the dilemma or escape through the horns:

a) If there are a great many people unemployed in a country, there is something radically wrong with the economic structure there; but there are a great number of unemployed here; hence our economic structure leaves something to be desired.

b) If a man could count forever, he would never count all the numbers; but it is true that no one can count all the numbers; hence no one could count forever.

c) "If you can infer anything from what I have just said, it is certainly this, that I am as innocent as a newborn babe." "But it has been shown that what you have said is all lies; your innocence, therefore, is just as much a fiction."

d) "Believe me, sir, if you persist in this policy, we shall never speak again." "But I must speak to you often." "Then you must give up this policy."

e) Had Romeo known that Juliet was alive, he would not have killed himself; but he did not know this; hence his death was as inevitable as a law of mathematics.

f) Human freedom is without meaning if God is omniscient, for he is then aware of what our decision in any case will be before we decide. But man is a free agent. Therefore, it cannot be that God is altogether omniscient.

g) The world cannot have existed always, if it had a beginning in time. But it can have existed always. Therefore it had no beginning in time.

h) If logic were useful, it would teach us how to reason well; but it does not teach us to reason well; therefore it is useless.

i) If anything more is meant by the conclusion of an argument than was contained in the premises, then the argument must have been fallacious, since it made use of some unwarranted assumption; but if the meaning of the conclusion is contained in the premises, then there was no use in stating it; but either the conclusion asserts more than the premises, or its meaning is contained in them; hence the conclusion is drawn fallaciously or it is redundant.

j) If you obey the doctor's orders you will get well; if you do what you want to do, you won't get well; hence, either you won't obey the doctor or you won't do what you want to do. (Which premise of this dilemma is missing?)

k) As I was driving down the street the other day a little girl ran out in front of my car; I was in a bad situation, for if I put on the brakes, the car would slide into a nearby tree, and if I didn't put on the brakes I would hit the child.

l) A had a bet with B that he could prove that B was not there. A argued as follows: "You are not in Palm Beach; but if you are not in Palm Beach, you must be somewhere else; but if you are somewhere else, then you are not here." B then took the stakes; when A accused him of doing so unfairly, B argued: "If your argument was sound, then I was not here and hence had an alibi; if your argument is not sound, then I had a right to take the money."

m) LADY: Madam, we'll tell tales.
QUEEN: Of joy, or grief?
LADY: Of either, madam.
QUEEN: Of neither, girl:
For if of joy, being altogether wanting,
It doth remember me the more of sorrow;
Or if of grief, being altogether had,
It adds more sorrow to my want of joy.[5]

n) If $a = b$, then a is neither greater than nor less than b, if a is greater than b, then $a \neq b$; but either $a = b$ or $a \neq b$; hence a is neither greater than nor less than b, or else a is greater than b.

o) If you agree to the statement "All abbergesnarks are woo-woos" you are crazy, since the sentence is meaningless; but if you disagree with it, then you are not only crazy but also dogmatic; hence you are crazy, certainly, and may be dogmatic, too.

p) After the negotiations with France concerning the Louisiana purchase, Jefferson was in a dilemma. If he accepted these negotiations, he would go against the Constitution and hence violate his principles. But if he did not accept them, the purchase would be lost and his dream of United States expansion would be shattered; thus he was forced to violate his principles or his ambitions.

q) If the army advanced, it would meet destruction at the hands of the Hessians, and if it retreated, the British would annihilate it; it was obvious to all, then, that a terrible defeat was inevitable.

r) If a and b are whole numbers, then their sum, $a + b$, is a whole number; but if a is a whole number and b is not, then $a + b$ is not; now $a + b$ is either whole or not whole; hence, either a and b are both whole numbers, or a is whole but b is not.

s) If Black obeys orders, we will win the contract; but if you are there, Black will obey orders; hence we will either win the contract or you will fail to be there.

t) Those of us who are virtuous receive no praise, while those of us who are not are severely criticized; truly, it's the way of life to be ignored or blamed.

u) "But the Dogmatists are accustomed to retort by inquiring 'However does the Sceptic show that there is no criterion [of truth]? For he asserts this either without judging or with the

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[5] *Richard II.*

help of a criterion; but if it is without judging, he will not be trusted, while if it is with a criterion, he will be self-refuted.' "[6]

2. By mathematical induction, prove the following principles:

   a) If we add all the odd numbers up to $2n$-1 together, the result is n^2. (E.g., in the case of the number 5, we add 1, 3, 5, 7, and 9, and this gives 25, which is 5^2.)
   b) $2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \ldots + 2 \cdot n = n(n+1)$.
   c) $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
   d) $x^{2n} - y^{2n}$ is evenly divisible by $x + y$, where n is any whole, positive number.

3. By means of Principle 6 (De Morgan's Law) deny the following conjunctions and disjunctions:

   a) Either he is rich or I am a fool.
   b) We lose money and they gain it.
   c) We shall lose our money but (and) we shall not lose our reputation.
   d) In Revolutionary times, either you were not a Tory or you were not a rebel.
   e) They'll fight on and not lose their courage.
   f) Either p is true, or q is true, or r is true.
   g) p is true, and q is true, and r is true.
   h) Either a man is rich or he is wise or he is happy or he is an idiot.
   i) I came, I saw, I conquered.
   k) Either p and q are both true, or r is true.
   l) Either p and q are both true, or r and s are both true.
   m) Either p or q is true, and either r or s is true.
   n) The day is fair and we shall go sailing or we shall go swimming. (Does it matter how this sentence is punctuated?)
   o) Augustus is blind and Maecenas is dishonest, or Augustus is dishonest and Maecenas is a fool.
   p) He must either go on living this life or give it up, and he must either marry the girl or look like a fool.

4. Apply the Principle of Contradiction and Interchange to the following implications:

   EXAMPLE: If $a \neq b$, then either $a > b$ or $b > a$.
   Let p represent "$a \neq b$," and q "Either $a > b$ or $b > a$."

[6] Sextus Empiricus, *Against the Logicians.*

Then we have "p implies q," and hence "q' implies p'": "If Either $a > b$ or $b > a$ is false, then $a = b$," or, by De Morgan's Law, "If $a > b$ is false and $b > a$ is false, then $a = b$."

a) If two lines lie in the same plane, they either intersect or are parallel.
b) p implies "q and r are both true."
c) If the Stoics were right, some men are wise and some men are not wise.
d) p implies that either q is true or r and s are both true.
e) If the earth should suddenly stop, then either it would explode or else it would rush to the sun and there perish.
f) If all men are animals and all animals think, then all men think. (Take the two premises as one sentence, the conjunction "p and q," and hence deduce "r' implies either p is false or q is false.")
g) If no cats like dogs and Timmy does like dogs, then Timmy is not a cat.

5. What theorems can be deduced from the following assumptions by means of the principles of the logic of propositions set down in this chapter?

a) If a is the cause of b, and b is the cause of c, then a is the cause of c.
b) If a is the cause of b, then a and b are not the same.
c) If a is the cause of b, then either a is the immediate cause of b, or there is some event coming between a and c.
d) If a is the cause of b, then a comes before b.
e) If a comes before b, then b is not the cause of a.
f) If a and b are not the same, it does not follow that a is the cause of b.
g) If a is not the cause of b and b is not the cause of c, it does not follow that a is not the cause of c.

(Note: these propositions hold for all a's, b's, and c's, so that if a given statement holds for a and b, it will hold for a and c, or b and c, or any other letters.)

General Exposition of the Traditional Logic of Classes 3

ARISTOTLE OFTEN is called the founder of logic. So remarkable was his work that in the course of a single lifetime he laid down the necessary principles for a complete deductive system of classes. For centuries logicians considered Aristotle's work the final word, and, beyond a few simple changes for the sake of simplicity, left it unaltered. This great monument of the genius of one man is the subject of our present study.

Our immediate interest lies in the concept "class." We will not attempt to define the term accurately until later, but the following will offer an aid to the imagination: "A class is a group of objects having a common property." Thus, the class *men* is a group having the property of being rational animals; the class *triangle* is a group of objects having the property of three-sidedness.

But it is important to note that the above definition can be taken in two ways, depending on whether we emphasize the word "group" or "property." For example, the members of a club form a class which we may define in two ways; we may merely enumerate the members of the club, or we may describe its by-laws, its position, and other points. Similarly, any class may be defined in these two ways: (1) by exhibiting the members of the class, or (2) by describing the characteristics or properties of the members. A class defined in the first manner is called a class considered in *extension* (often called "denotation"), while a class taken in the second manner is called a class considered in *intension* (often called "connotation"). Certain important differences are noticeable. Two classes considered in extension (i.e., defined by denotation) may be exactly the same, but quite different in intension (defined by connotation): the two classes intensionally described as "rational animal" and "biped that can laugh" are extensionally equal since they both comprise only the human race. Again, two clubs may have exactly the same members and hence be equal extensionally, but they will not be equal intensionally (since they will have different by-laws and positions, for instance). Note that if we "increase" the intension, i.e., if we add more qualifiers or adjectives, or "connote" more, we decrease the extension, i.e., the resulting class has fewer members and hence we "denote" less. For example, the class "animal" is quite large extensionally; if, now, we increase the intension by connoting the adjective "white," we decrease the number of members considerably: there are fewer white animals than there are animals. Similarly, if we decrease the intension we generally increase the extension.

In this system we shall consider classes taken extensionally, so that the expression "All the class a is the class b" ("All vegetables are plants") means that all the members of a are also members of b, and $a = b$ means that the class a and the class b have exactly the same membership.

If these points are borne in mind, the objects under consideration may be thought of as "nouns," or "objects," and just as in the last chapter we were interested in statements about propositions which were true no matter what the propositions were, so here our interest lies in statements about nouns which are true no matter what the nouns may be. For example, the statement "All a is a" is true for every noun whatsoever: "All cats are cats," "All triangles are triangles." But such a statement as "Some a is b" is not true for every noun. If a is the noun "squares" and b the noun "circles," the statement is obviously false. It is for propositions of the first type that we are searching.

Examples of arguments which concern themselves with nouns are plentiful enough:

> "Jones says to Smith: 'I lent Brown, my ward leader, one thousand dollars'; whereupon Smith replies: 'That was foolish; don't you know that all politicians are dishonest?' "

There is an inference latent here, and the validity of this inference rests on reasonings about nouns. More precisely put, Smith's argument would run: "All politicians are dishonest; Brown is a politician; *therefore* (it is inferred that) Brown is dishonest." Here the nouns are "politician," "dishonest people," and "Brown" and the validity of the argument, i.e., the validity of the clause preceded by "therefore," rests on certain laws of reasoning which are concerned with nouns in general. That is, "All *b* is *a* and (all) *c* is *b*; *therefore* (all) *c* is *a*," and the above argument is only a special case of this reasoning which holds for all *a*'s, *b*'s, and *c*'s, i.e., for all nouns whatsoever.

But there are obviously innumerable statements about nouns which are true no matter what the nouns are, and the actual tabulating of them would be tedious beyond measure. Hence some means of classification is necessary. Aristotle cleverly recognized that all sentences which make statements about two nouns can be transformed into one type of proposition or sentence. This sentence has a subject (a noun, of course), a "copula" (the word "is" or "are," or "was" or "were"), and a predicate (which is again a noun). For instance, the sentence "The Greeks fight bravely" may easily be transformed into a proposition of the above type: "The (class of) Greeks are (in the class of) brave fighters."

The general type of proposition had, for Aristotle, four special cases, the so-called "categorical forms." For (1) it may assert that all of a certain class (noun) *a* belongs to another class *b*, and this we abbreviate by A (*ab*), "all *a* is *b*," or (2) it may assert that some *a* belongs to *b*, symbolized by I (*ab*), or (3), from a negative point of view, it may assert that some *a* does not belong to *b*, O (*ab*), or, finally, (4) it may assert that no *a* is *b*, E (*ab*):

1. A (*ab*) = "All *a* is *b*"
2. I (*ab*) = "Some *a* is *b*"
3. O (*ab*) = "Some *a* is not *b*"
4. E (*ab*) = "No *a* is *b*"

The small letters, *a* and *b*, stand for any noun or class whatsoever.

Examples of the manner in which various statements about nouns can be translated into one of the above four categorical forms are:

"In French some adjectives do not follow the noun." This is an example of O (*ab*): "Some French adjectives are not words which follow the noun."

"No one knows his secret." This is a case of E (ab) : "No one (person) is a knower of (person who knows) his secret."

"Everyone sins frequently." This is transformed into a case of A (ab) : "All men are frequent sinners."

"Some of us will achieve greatness"; I (ab) : "Some men are future achievers of greatness."

Note that the A- and E-forms apply to sentences in which the subject is singular. Thus, "Jones is a good man" is an A-proposition: " (all) Jones is good"; and "Smith is not a banker" is an E-form: " (none of) Smith is a banker."

(See Exercises, Group A, at end of chapter.)

We now examine certain of the properties of the four categorical forms, our main purpose being to lead up to a deductive system of classes. For just as geometry sets down certain postulates and axioms and from these deduces all its theorems, so we can find a deductive system of classes which will enable us to prove from postulates all the statements which hold true of classes in general. Our first step in this direction must be to discover just what propositions are true in general. Thus, for example, the sentence "A (ab) ," "All a is b," is not such a proposition, for we can find many nouns which make it false; e.g., let a be replaced by the noun "men" and b by "stones." But as an example of a proposition about nouns (or classes) which does hold true no matter what the nouns may be, take the following case of "Immediate Inference," i.e., of an inference drawn from one premise:

A (ab) implies I (ab).
"If all a is b, then some a is b."

No matter what nouns we may substitute for a and b, this statement remains true. It is obviously true in the case where the premise is verified: e.g., where a is "cats" and b "animals." Suppose, now, the premise is not true, e.g., if $a =$ "circles" and $b =$ "squares"; here, too, the *implication* is still valid: "*If* all circles are squares, then some circles are squares." We italicize the word "if," for the emphasis is there. *If* we lived in a world where all circles were squares, then in that topsy-turvy universe some circles would be squares. We do not grant the premise: we merely assert it as an hypothesis, much in the same manner in which we would start our childhood games with "Let's pretend." Hence, the inference is true even though the premise is false. Put otherwise, a formal system postulating "All circles are squares" must have as one of its theorems "Some circles are squares." There remains one other case to examine: namely, the case where *a* or *b* (or both) are meaningless. For example, suppose *a* were "toves" and *b* were "wabes." By arguments similar to those given, the inference is likewise valid in this case.

It is to be emphasized that this discussion constitutes no *proof* of the validity of the inference, for the arguments are based purely on intuition or else on very involved assumptions. Our clearest course will be to *assume* the validity of the above (or one like it), and these verbal arguments are to be thought of as little more than elucidations.

Suppose now we form all the cases in which one categorical form immediately implies another, i.e., all cases of "Immediate Inference" between any two of the four categorical forms given above. We shall keep the order of the "terms," i.e., the small letters *a* and *b*, the same in the premise and conclusion. That is, starting with A (*ab*), "All *a* is *b*," we are to inquire which of the forms this premise implies, and then continue this process with I (*ab*), O (*ab*), and E (*ab*).

In the first place, it must become apparent immediately that each form implies itself. E.g., it seems obvious enough that the statement "If 'All *a* is *b*,' then 'All *a* is *b*' " is true no matter what nouns *a* and *b* represent.

Next, "A (*ab*) implies I (*ab*)" is also true universally, but the two following forms of Immediate Inference, "A (*ab*) implies O (*ab*)" ("If all *a* is *b*, then some *a* is not *b*") and "A (*ab*) implies E (*ab*)" ("If all *a* is *b*, then no *a* is *b*") are obviously not true. Now, starting with I (*ab*) we are to discover whether it implies A (*ab*), whether it implies itself, etc.

The table below may be used in determining the valid cases. The implication sign and the small letters have been omitted, so that AE, for instance, is to be read "A (*ab*) implies E (*ab*)":

AA	IA	OA	EA
AI	II	OI	EI
AO	IO	OO	EO
AE	IE	OE	EE

One case of Immediate Inference will be found to contain some difficulties: If I (ab) (some a is b), then does it follow that O (ab) (some a is not b)? There is an ambiguity here in the words we employ to interpret the symbols, specifically, an ambiguity in the word "some." If I say "Some men will die," I mean "some, and possibly all" men. But if I say "Some numbers are prime numbers," I may quite well mean "some, but not all" numbers. It thus appears that in our everyday speech we use the word "some" with two quite distinct meanings. If we mean by "some," "some and possibly all," then the Immediate Inference "I (ab) implies O (ab)" will be false, since it will not be true for those values of a and b for which all a is b, e.g., if $a =$ "centaurs" and $b =$ "mythical monsters." On the other hand, if "some" means "some, but not all," then this inference will be valid: "If some (but not all) a is b, then some a is not b." It is to be noted that once we decide whether the inference is valid or not, we decide at the same time which meaning of the word "some" we shall choose. In other words, the system itself will make clear our precise meaning.

But, as a matter of fact, the meaning of "some" has already been decided, since we have taken as true the inference "A (ab) implies I (ab)," and this could not be true if "some" meant "some, but not all." Hence, when the accurate interpretation of I (ab) is given it reads "Some (and possibly all) a is b," and O (ab) reads "Some a is not b (and possibly no a is b)." Therefore "I (ab) implies O (ab)" is not true in general, i.e., not true for all nouns whatsoever. For example, let $a =$ "salts" and $b =$ "compounds." Then "Some salts are compounds" is true (for, as a matter of fact, all salts are compounds), but O (ab), "Some salts are not compounds," is false; hence the former cannot imply the latter, since such an implication would mean that *whenever* I (ab) is true, O (ab) is true, and we have shown that this is not the case.

There are six valid forms of Immediate Inference when the order of the terms is the same in the premise and the conclusion: namely, the four arising from the cases where a given form implies itself and "A (ab) implies I (ab)" and "E (ab) implies O (ab)."

We now examine the cases where the order of the terms is reversed in the conclusion, as, for example, A (*ab*) implies I (*ba*), "If all *a* is *b*, then some *b* is *a*."

For the sake of convenience, we shall call the form of Immediate Inference in which the order of the terms is the same in premise and conclusion the first figure, and the form in which the order of the terms is reversed the second figure:

EXAMPLE OF FIRST FIGURE: A (*ab*) implies I (*ab*).
EXAMPLE OF SECOND FIGURE: A (*ab*) implies I (*ba*).

These two figures exhaust the possible arrangements in Immediate Inference, for the only possible ways in which we can arrange two pairs of the same objects is to give them the same order or the reverse order. Hence, the Immediate Inference "A (*ba*) implies I (*ab*)" is also a case of the second figure, since the actual letters we use are not important but rather the *arrangement* of the letters (e.g., instead of *a* and *b* we could write numbers, or Greek letters, or meaningless symbols). Similarly, "A (*ba*) implies I (*ba*)" is in the first figure.

(See Exercises, Group B, at end of chapter.)

Each separate case of Immediate Inference will be called a "mood." Thus "A (*ab*) implies I (*ab*)" is a mood. There are, then, thirty-two moods of Immediate Inference. As a shorthand, we may write any mood by omitting the small letters and the word "implies" and merely indicating the figure. Thus, "AE in the first figure" means "A (*ab*) implies E (*ab*)," "IA in the second figure" means "I (*ab*) implies A (*ba*)." An examination of the table under the second figure will show that here are four more valid cases or "moods" of Immediate Inference. For, though A (*ab*) does not necessarily imply A (*ba*) (e.g., when $a =$ "houses" and $b =$ "buildings") nor does O (*ab*) imply O (*ba*) (e.g., when $a =$ "buildings" and $b =$ "houses"), it is true that I (*ab*) implies I (*ba*), and E (*ab*) implies E (*ba*), for all values of *a* and *b*. In effect this means that the order of the terms in the I- and E-forms is indifferent as far as their truth goes (though rhetorically, for the sake of emphasis, the order may be important).

But these two additional moods give us two more. For we already have as a valid mood "A (*ab*) implies I (*ab*)"; but since the order of the terms in an I-form may be reversed without altering its validity, we may likewise write "A (*ab*) implies I (*ba*)," or, viewed in another way, since if A (*ab*) is true, I (*ab*) is true, and if I (*ab*) is true, I (*ba*) is true, we can infer that if A (*ab*) is true, I (*ba*) is also true. Similarly, we can assert that "E (*ab*) implies O (*ba*)" is a valid mood.

There are, then, ten valid moods of Immediate Inference (six in the first figure and four in the second) and hence twenty-two "invalid" moods.

This result may be summed up by three convenient rules of thumb, the so-called "Rules for Invalidity of Immediate Inference." Suppose we call a categorical form "affirmative" if it contains an even number of "negations," i.e., words such as "no" or "not," and call it "negative" if it contains an odd number. Thus A (*ab*) and I (*ab*) are affirmative since they contain zero negations, and E (*ab*) and O (*ab*) are negative since they each contain one negation. Another and far more precise manner of defining an affirmative form would be: "A categorical form is affirmative if it becomes true when the terms are identified; otherwise it is negative." Thus A (*ab*) is affirmative, since when *a* and *b* are identified, i.e., when we have A (*aa*), a true proposition results, "All *a* is *a*" being true for all nouns whatsoever. That I (*ab*) is likewise affirmative and E (*ab*) and O (*ab*) negative by this definition is plain enough.

By means of these definitions of affirmative and negative forms we can formulate two rules:

RULE 1. An affirmative form does not imply a negative form.
RULE 2. A negative form does not imply an affirmative form.

For example, A (*ab*), which is affirmative, does not imply O (*ab*), which is negative (Rule 1), and E (*ab*) does not imply I (*ab*) (Rule 2).

But these rules are not sufficient for the detection of all invalid cases. For, though I (*ab*) ("Some *a* is *b*") does not imply A (*ab*) ("All *a* is *b*"), this mood is not declared invalid by either rule. Hence at least one (and actually only one) more rule is needed if we are to determine by means of the rules *all* invalid cases (i.e., if the rules are to be "sufficient").

In order to formulate this third rule we must introduce the concept of a "distributed" term. Let us call a term of a categorical form *distributed* if it is modified by the words "all" or "no." Thus, in the form A (ab), the subject, a, is distributed since it is modified by the word "all." Is the predicate? It is, if A (ab) really reads "All a is (all) b"; but this is not the case. If a is "dogs" and b "animals," then we have "All dogs are (some) animals"; that is, in the form A (ab) we are talking about all of the subject but not necessarily all of the predicate. Hence the predicate in A (ab) is not distributed. In the form I (ab) the subject is obviously not distributed, for it is modified by the word "some." The predicate likewise is so modified; I (ab) really reads "some a is (some) b," i.e., in the I-form we are not talking about *all* of either the subject or predicate; e.g., if $a =$ "animals" and $b =$ "white things": "Some animals are (some) white things." In E (ab) the subject is obviously distributed, since it is modified by the word "no." That the predicate is likewise distributed can be seen in a number of ways; perhaps the easiest is to point out the fact that E (ab) means that a is excluded from all of b, and hence b is implicitly modified by "all." In O (ab) the subject is not distributed, being modified by the word "some." To see that the predicate is distributed, we can rephrase O (ab) to read "All b is excluded from some of a" or, "all b is non-(some a)," where the fact that b is distributed becomes plain enough. Note that it is the *position* (subject or predicate) of the letter which determines whether it is distributed or not, and not the letter itself. Thus in A (ba), b is distributed but a is not.

The following scheme summarizes the foregoing, the underlined letters being the distributed terms: [1]

A $(\underline{a}b)$	E $(\underline{a}\underline{b})$
I (ab)	O $(a\underline{b})$

We are now in a position to state our third rule, which is:

RULE 3. A premise in which a given term appears undistributed does not imply a conclusion in which that term is distributed; or, if a term is distributed in the conclusion but not in the premise, the mood is invalid.

Thus "I (ab) implies A (ab)" is invalid since a is distributed

[1] Note that by means of the concept of distributed terms we might now define an affirmative form as one which does not distribute its predicate, a negative form as one which does. This definition, however, turns out to be inadequate later on. Cf. p. 110.

in the conclusion but not in the premise, "A (*ab*) implies A (*ba*)" is invalid since *b* is distributed in the conclusion but not in the premise. Note that the rule does not assert that a mood is invalid if a term is distributed in the premise but not in the conclusion. E.g., "E (*ab*) implies O (*ab*)" is valid.

These three rules are necessary and sufficient. They are necessary, i.e., indispensable, since if one were omitted we could not determine all the invalid moods of Immediate Inference. Thus, if Rule 2 were omitted the invalid mood "E (*ab*) implies I (*ab*)" could not be determined. The necessity of the other rules is similarly established by finding one mood which is made invalid by that rule and no other. The rules are sufficient since together they take care of all invalid cases. We might add that they are also "consistent" in that no valid mood is declared invalid by them.

(See Exercises, Group C, at end of chapter.)

In conformity with the principles of the logic of propositions set down in the previous chapter, a mood of Immediate Inference can only be shown false by exhibiting a case where the premise is true and the conclusion false. For any mood of Immediate Inference is an implication of the form "*p* implies *q*," and the contradictory of this is "*p* is true while *q* is false." (Cf. page 21.) Thus there are valid cases of Immediate Inference in which a false premise implies a true conclusion and valid cases in which a false premise implies a false conclusion; for example:

"All animals are men implies that some animals are men"

is a valid mood ["A (*ab*) implies I (*ab*)"] in which the premise is false but the conclusion true.

Again:

"If some squares are circles, then some circles are squares"

is a valid mood ["I (*ab*) implies I (*ba*)"], in which both premise and conclusion are false. But *we cannot find a valid mood of Immediate Inference in which the premise is true but the conclusion false,* for such an example would involve a contradiction in that by definition, if *p* validly implies *q*, *p* cannot be true when *q* is false.

We now proceed to a more general form than the last, namely, one in which two premises (instead of one) imply a conclusion. This form is commonly called the "syllogism," being a combination of two arguments (from the Greek σύν and λόγος). The following offers an example of a syllogism:

> "If no balloons are safe,
> and all dirigibles are balloons,
> then no dirigibles are safe."

Essentially, the syllogism is a form of argument which contains three terms; in the above argument these terms are "balloons," "safe (things)," and "dirigibles."

There is always one term ("balloons" in this case) which appears in both premises, but not in the conclusion. This is called the "middle" term. Other examples of syllogisms are plentiful enough:

> "All numbers divisible by two are even;
> the number 12 is divisible by two;
> ergo 12 is an even number."

(Here "numbers divisible by two" is the middle term; note that both premises are in the A-form.)

Very often, especially in everyday speech, one of the premises of a syllogism is omitted, being tacitly understood. When this is the case, the syllogism is called an *enthymeme.* For example:

> "He is a senator and therefore lives in Washington."

Here the premise "All senators live in Washington," which is necessary for the validity of the argument, is tacitly assumed.

Our next task, then, will be an examination of all forms of the syllogism, i.e., all possible "moods" constructible out of the four categorical forms.

First we must arrange all possible forms of the syllogism in some convenient array. That there will be more than two "figures" or possible arrangements of the terms here is perhaps obvious enough. The most convenient method of determining the exact number of figures will be to discover all the possible positions of the "middle" term, or term which does not appear in the conclusion. Let us call *a*, *b*, and *c* the three terms under discussion and let *b* represent the middle term. Then if *a* remains in the first premise and *c* in the second, there will be four possible ways of placing b. For example, take the syllogism which has A in each premise and conclusion. Then the middle term can appear

1. first in the first premise (the so-called "major" premise) and last in the second ("minor" premise):

   A (ba) A (cb) implies A (ca),[2]

2. last in both premises:

   A (ab) A (cb) implies A (ca),

3. first in both premises:

   A (ba) A (bc) implies A (ca),

4. (converse of 1) last in the major premise and first in the minor premise:

   A (ab) A (bc) implies A (ca).

It is a fact that the order of the premises of an argument may be reversed without altering the truth or falsity (cf. page 38); hence, these four cases represent all possible figures of the syllogism. The actual order of the premises is, then, arbitrary, and we fix on this arrangement: the predicate of the conclusion (the "major" term: in the above it is a) is to appear in the major or first premise and the subject of the conclusion (the "minor" term: here, c) is to appear in the second or minor premise. These four possible arrangements constitute the four figures of the syllogism, in the order given.

The explanation of this scheme of figures lies in the history of the syllogism. The most important figure for Aristotle and his medieval followers was the first. The typical syllogism was "A (ba) A (cb) implies A (ca)." The major premise of this form was always taken as a universal proposition, while the minor was a singular proposition (though still in the A-form). For example, the universal proposition might be "All men are mortal," a judgment arrived at through experience or definition or

[2] Note that the word "and" between the two premises is regularly omitted. If the symbolism suggested at the end of chap. II is used, this can be written: A (ba) A (cb) $\angle$ A (ca). Either the word "implies" or the word "imply" may be used in the case of the syllogism, depending on whether the two premises are taken as one sentence or as two sentences.

intuition. The singular proposition, usually determined by experience, might be "Socrates is a man," and hence, from these two judgments we infer a new fact: "Socrates is mortal." It is not going too far to say that a great part of Aristotle's philosophy of science was built around this syllogism.

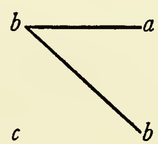
The remaining figures were shown to be modifications of the first. This explains the arrangement of the terms in the first figure (the arrangement in the third being perhaps the most logical as the first figure) as well as the fact that the term in the minor premise becomes the subject of the conclusion. Even in the nineteenth century (and perhaps today), logicians such as J. S. Mill in his *Logic* were inclined to share Aristotle's opinion as to the importance of AAA in the first figure; though it is true that all valid moods of the syllogism can be reduced to this form, they ignored the facts: (1) such reduction requires other assumptions, and (2) many other moods in other figures share this distinction.

The following method of determining figures may prove helpful. After arranging the premises in the correct order (subject of conclusion in minor premise), write down their terms one over the other as follows: If "I (*ba*) A (*cb*) implies A (*ca*)" is the given syllogism, write

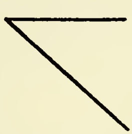
b *a*

c *b*

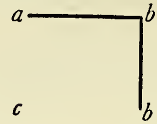
(i.e., write the terms of the major premise on top). Now join the middle term by a line and draw a line joining the two letters on top:



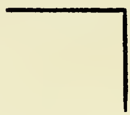
The resulting geometrical form



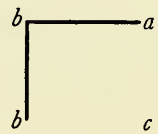
is characteristic of the first figure. In the second figure, as, for example, "I (*ab*) I (*cb*) implies O (*ca*)," we have



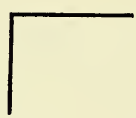
and the resulting form is



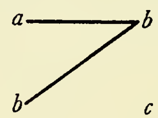
In the third figure we have



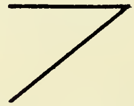
or



and in the fourth



or



The four geometrical forms, then, each characterize a certain figure, and we have only to associate the proper figure with the proper form to complete the determination. Note that *b* does not have to be the middle term, but whatever the letter representing this term may be (i.e., the letter appearing twice in the scheme), we draw a line joining its two positions.

1 2 3 4

The following cases illustrate the method of reducing any given syllogism to its proper figure.

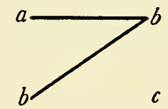
1. Determine the figure of the following syllogism:

A (bc) I (ab) implies I (ca).

Here the minor term, the subject of the conclusion, is c, and the major term is a. But c appears in the first premise, and our rule is that the minor term should appear in the second before we determine the figure. Hence, since we may reverse the order of the premises at pleasure, we may write this syllogism in the correct manner:

I (ab) A (bc) implies I (ca).

We are now in a position to determine the figure, and, since b is the middle term and comes last in the major premise and first in the minor, the figure is the fourth; or, by the second method, we write

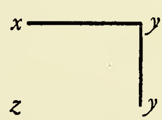


and this form is characteristic of the fourth figure.

2. Determine the figure of

E (xy) I (zy) implies O (zx)

Here the premises are in the proper order (z, the minor term, is in the second premise). The middle term is y (it does not appear in the conclusion) and this comes second in both premises. Hence the figure is the second. By the second method we have:



and this is characteristic of the second figure.

The general method of determining the figure of any syllogism is outlined thus:

1. Make certain that the premises are in the correct syllogistic order: the minor term, i.e., the subject of the conclusion, should lie

in the second or minor premise. If the order given is not correct, reverse the premises.

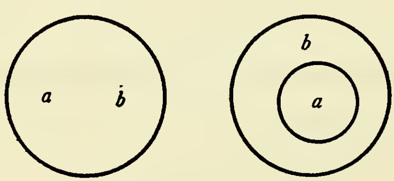
2. Determine the middle term, i.e., the term which does not appear in the conclusion; or, by the second method, write down the terms of the premises, with the terms of the major on top, and join the two positions of the middle term and draw the top horizontal.

3. Determine the figure by the position of the middle term, or else, in the second method, by the resulting geometrical form.

(See Exercises, Group D, at end of chapter.)

There are, as may be verified by a table, sixty-four moods of the syllogism in each figure, or a total of two hundred and fifty-six altogether.

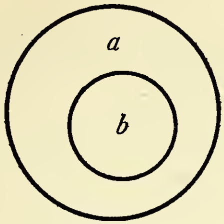
In order to determine which of these two hundred and fifty-six moods of the syllogism are valid, it will be found convenient to make use of the diagrams of the German mathematician, Euler. Euler's purpose was to set down in diagrammatic form the various properties of the categorical forms. Thus A (*ab*) may be represented by either of the following two diagrams:



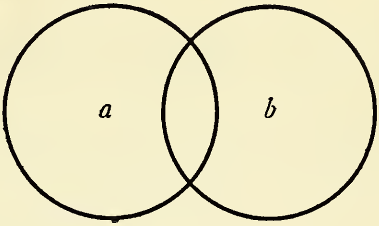
1 2

Of course, as we pointed out in the beginning, the *a*'s and *b*'s in the categorical forms are not necessarily areas in a plane; they may represent any noun or class whatsoever. But these two diagrams help the mind to picture what is meant when we assert "all *a* is *b*." For there are really two possibilities: *a* and *b* may be identical, i.e., they may have exactly the same members (e.g., in the assertion "Hunters are people who pursue game," where the subject and predicate are really identical in their membership); or, the predicate may be wider in extension (e.g., in the assertion "All ball-games are games").

For I (*ab*), "some *a* is *b*," the two foregoing diagrams would be possible as well as the two following, which would also be applicable to the case of O (*ab*), "some *a* is not *b*," but not to A (*ab*) or E (*ab*):

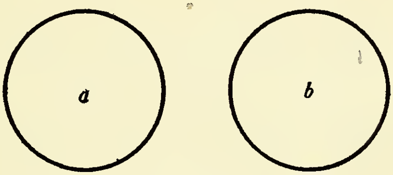


3



4

Finally, E (*ab*) has one possible diagram, which again is also possible for O (*ab*) but not for A (*ab*) or I (*ab*):



5

To illustrate the application of these diagrams, we will examine the four categorical forms in order to determine their properties with respect to the four relations, "contradictories," "contraries," "subcontraries," and "subalterns" (pages 18-21).

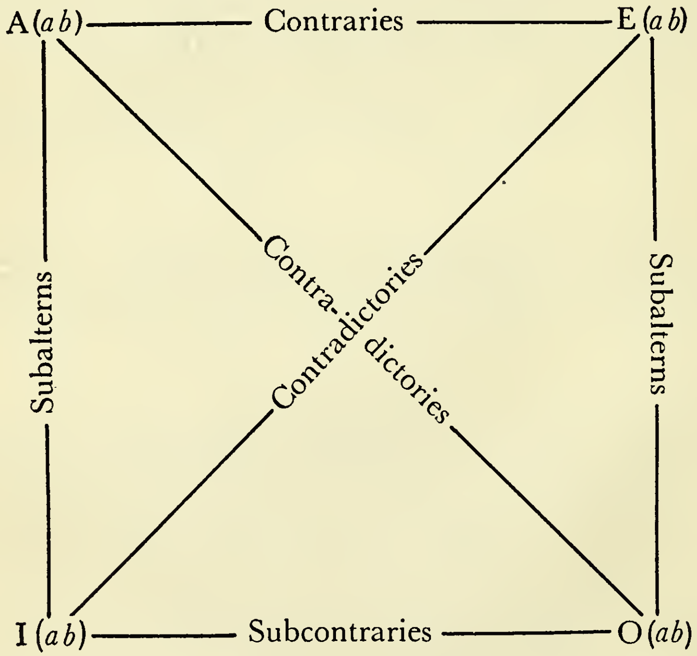
Two propositions are contradictory if they cannot be true together and cannot be false together. With respect to the diagrams, this definition would read: "Two categorical forms are contradictory when they have no diagram in common (cannot both be true) but together exhaust all the diagrams (cannot both be false)." Hence A (*ab*) and O (*ab*) are contradictories, for they have no diagrams in common, but since A (*ab*) covers diagrams 1 and 2, and O (*ab*) covers diagrams 3, 4, and 5, they together cover all the diagrams. Similarly, E (*ab*) and I (*ab*) are contradictories. (Note that E (*ab*) and I (*ba*) are also contradictories, but that A (*ab*) and O (*ba*) are not.)

Contraries have been defined as those pairs of propositions which cannot both be true but may both be false. With respect to the diagrams this means that they have no diagram in common but they do not exhaust all the diagrams. A (*ab*) and E (*ab*), then, are contraries (they do not cover diagrams 3 and 4).

Subcontrary categorical forms exhaust all possible diagrams (cannot both be false) but have some diagrams in common (may both be true). I (*ab*) and O (*ab*) have this property.

Finally, subalterns will be those categorical forms which have diagrams in common (may both be true) and do not cover all the diagrams (may both be false). A (*ab*) and I (*ab*) are subalterns since they have diagrams 1 and 2 in common but do not cover diagram 5. Similarly, E and O are subalterns.

The foregoing may be represented by the following diagram, the so-called "Square of Opposition":

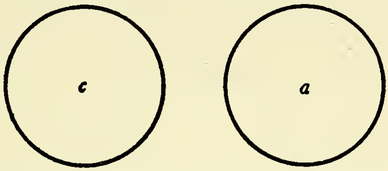


We return now to the syllogism and by the aid of Euler's diagrams we shall determine which of the two hundred and fifty-six moods are valid.

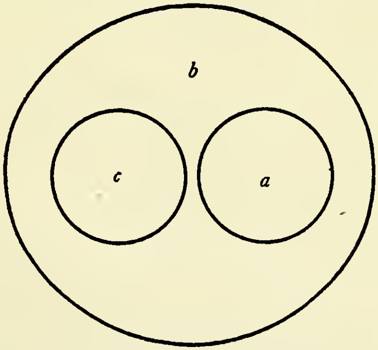
As an example of the method here, suppose we wished to determine the truth or falsity of the following syllogism:

A (ab) A (cb) implies I (ca) (AAI in the second figure, AAI_2.)

If we can construct Euler diagrams which verify these premises, but not the conclusion, then we can say that this syllogism is not true in general. The method, then, will consist in finding diagrams which make the conclusion false and then determining whether the premises may not be true under this case. Now there is only one diagram which covers the case where I (ca) is false, namely the diagram for E (ca):



We wish to find, if possible, an area, b, such that both A (ab) and A (cb) are true. This can be done as follows:

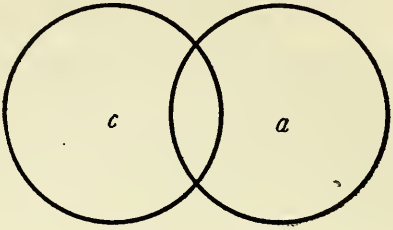


and this diagram illustrates the falsity of the syllogism. (Note that this method makes it a comparatively simple matter to find a concrete case of the invalidity of the syllogism; thus, let $a =$ "dogs," $b =$ "animals," and $c =$ "cats": "All dogs are animals and all cats are animals; but it does not follow that some cats are dogs.")

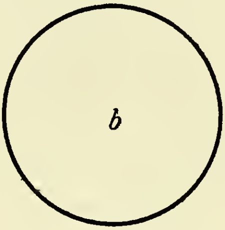
Again, consider the syllogism

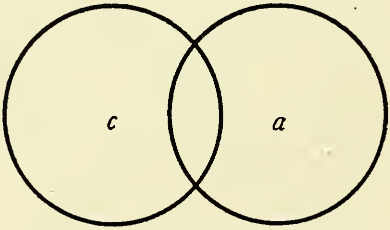
E (ba) E (bc) implies A (ca) (EEA in the third).

If the conclusion is false, then O (*ca*) is true, and here we have more than one choice. Let us take the following: [3]



We ask whether there is not an area, *b*, such that *b* has nothing in common with either *c* or *a*. This is easily found:





and this a diagram for the invalidity of EEA in the third.

The general method, then, is:

1. Draw a diagram which will make the conclusion false, i.e., draw a diagram representing the contradictory of the conclusion.
2. Determine an area for the middle term which will make both the premises true. If this is found, the mood is shown invalid. If not, then another diagram for the falsity of the conclusion must be tried and the same prócess repeated. If no case of invalidity can be found,

[3] In general, it will be better to take diagrams satisfying the contradictory of the conclusion, rather than just its contrary if it has one. Why?

the syllogism may be considered valid (i.e., true for all *a*'s, *b*'s, and *c*'s whatsoever).

Aristotle's method of determining the invalid moods of the syllogism differed from this. Suppose we wish to show that "E (*ba*) E (*cb*) implies I (*ca*)" is invalid. The method just explained consists in showing that there are cases in which the premises are both true and the conclusion false: "No squares are cats; no dogs are squares" are both true sentences, but "Some dogs are cats" is false. Aristotle shows that the premises E (*ba*) E (*cb*) are consistent with two contrary propositions, A (*ca*) and E (*ca*). For the statements "No squares are cats; no felines are squares" are consistent with "All felines are cats," and, as already shown, they are consistent with "No dogs are cats." Hence, since E (*ba*) E (*cb*) may yield either A (*ca*) or E (*ca*), they cannot universally imply any conclusion. Why? Thus, this method would immediately show the invalidity of four moods, rather than just one: EEA_1, EEI_1, EEO_1, EEE_1. Aristotle's method cannot always be applied. For, though A (*ba*) A (*cb*) do not imply O (*ca*), we cannot say that they are consistent with either of two contraries, since they are never consistent with E (*ca*).

(See Exercises, Group E, at end of chapter.)

The task of discovering the valid moods of the syllogism will be made simpler by making a table comparable to the table for Immediate Inference given above (page 53). That is, start with AAA in every figure, then AAI, AAO, AAE, AIA, AII, etc. The exercise of discovering the valid moods will not be so laborious as might be supposed, for one diagram will often suffice to show the invalidity of a score or more of moods. For example, the diagram for the invalidity of EEA in the third given here will also show the invalidity of EEA in all other figures, of EOA, OEA, OOA, EEE, OEE, EOE, OOE in every figure.

When the array of the moods of the syllogism has been examined by Euler's diagrams, it will be found that there are twenty-four valid moods, six in each figure. It must be understood that Euler's diagrams do not constitute any "proof" of the valid moods, since they make use of experience, and the assumptions upon which they are based are far too complex. Even though we can find no diagram which will make a certain mood invalid, we cannot say with certainty that there is not one, or, more important, we cannot say that this is any proof that the mood is always valid, for diagrams are only a special interpretation of the letters *a, b, c,* etc. As a matter of fact, Euler's diagrams are reliable, but we shall "prove" later on what they show, by means of certain definite assumptions and certain principles of logic, rather than by means of our eyes.

The twenty-four valid moods are arranged under their figures below:

1.	2.	3.	4.
AAA	AEE	AAI	AAI
AAI	AEO	IAI	AEE
EAE	EAE	AII	AEO
EAO	EAO	EAO	EAO
EIO	EIO	EIO	EIO
AII	AOO	OAO	IAI

The following lines [4] offer a convenient method of memorizing these valid moods. The vowels in the words with capital letters represent the four categorical forms and the words "prioris," "secundae," "tertia," and "quarta" give the figures. Thus *Barbara* is AAA in the first figure, *Cesare* is EAE in the second, etc.

> *Barbara, Celarent, Darii, Ferio*que prioris;
> *Cesare, Camestres, Festino, Baroko,* secundae;
> Tertia, *Darapti, Disamis, Datisi, Felapton,*
> *Bokardo, Ferison,* habet; quarta insuper addit
> *Bramantip, Camenes, Dimaris, Fesapo, Fresison.*

Note that valid moods ending in I or O are omitted if there are already moods ending in A or E. Thus AAI in the first (called "Barbari") is omitted, since AAA in the first (Barbara) is a valid mood: AAI in the first is, of course, an immediate consequence of AAA in the first, for if we can infer "All *c* is *a*" from the premises A (*ba*) A (*cb*), then certainly we can infer "Some *c* is *a*." Similarly, EAO in the first ("Celaront"), EAO in the second ("Cesaro"), AEO in the second ("Camestros"), and AEO in the fourth ("Camenos") are omitted. These lines are important since they offer convenient names or tags for the

[4] Reported by Petrus Hispanus (Pope John XXI).

valid moods of the syllogism, which actually form the heart of the Aristotlian system.[5]

These results can be summarized by five rules for Invalidity of the Syllogism, comparable to rules given in the case of Immediate Inference:

1. Two affirmative premises do not imply a negative conclusion (e.g., AIE in any figure is an invalid mood).
2. Two negative premises do not imply any conclusion (e.g., EOI in any figure is invalid, EOO is invalid.)
3. An affirmative and a negative premise do not imply an affirmative conclusion (e.g., IOI in any figure is invalid).
4. Two premises in which the middle term is not distributed at least once do not imply any conclusion; or, the middle term of a syllogism must be distributed at least once, or the mood is invalid (e.g., III in any figure is a false syllogism).
5. Two premises in which a given term is not distributed do not imply a conclusion in which that same term is distributed; or, if a term in a syllogism is distributed in the conclusion but not in its premise, the syllogism is invalid (e.g., IOE is false in any figure; note that, as in the case of the analogous rule for Immediate Inference, a term may be distributed in its premise and not in the conclusion; e.g., AAI in the first is valid).

These rules may be summarized as follows:

A valid syllogism must have (1) either two affirmative premises and an affirmative conclusion, or an affirmative and a negative premise with a negative conclusion; (2) a distributed middle term, and (3) all terms distributed in the conclusion distributed in the premises also.

As in the case of Immediate Inference, these rules are necessary and sufficient; they are necessary because if we omit one, then we cannot determine all the invalid moods. That is, for each rule there is at least one invalid mood which comes uniquely under this rule; the rules are sufficient because together they take care of all invalid moods. The determination of the unique cases is left as an exercise. In this respect, Rule 1

[5] The consonants in these words have a special significance. They refer to the manner in which a given mood may be "reduced" to Barbara or Celarent, such reduction being analogous to the deduction given in chap. IV.

has had a rather interesting history, since it has been maintained more than once that it is unnecessary on the ground that no invalid mood comes uniquely under it, and that the other four rules are consequently sufficient. As a matter of fact, there is but one mood which makes the rule necessary.

(See Exercises, Group F, at end of chapter.)

A syllogism is not necessarily false if either or both of the premises are false. (Cf. analogous statement for Immediate Reference, page 58.) The invalidity of a syllogism is only established when we can show a case where both premises are true and the conclusion is false. The following examples are all valid syllogisms in that the logical law allowing us to draw the conclusion from the premises is true, but the premises and conclusion exhibit every combination of true and false except the case where two true premises imply a false conclusion:

1. Two false premises may validly imply a true conclusion:

"All circles are squares;
all squares are loci of points equidistant from a given point;
hence, all circles are loci of points equidistant from a given point."

2. Two false premises may validly imply a false conclusion:

"Nothing which has four legs is an animal;
all four-legged things are horses;
therefore, some horses are not animals."

3. A false premise and a true premise may validly imply a true conclusion:

"Lions are carnivorous;
No dog is carnivorous;
hence, no dog is a lion."

4. A false premise and a true premise may validly imply a false conclusion:

"No Italian was ever wealthy,
but some wealthy people were natives of Rome;
hence, some Romans were not Italians."

5. Two true premises can validly imply only a true conclusion:

"Some statements are true;
all statements are meaningful;
hence, some meaningful things are true."

It might be asked whether these five rules (along with the rules for Invalidity in Immediate Inference) do not constitute a set of postulates from which we can derive all the valid moods of Immediate Inference and the syllogism. As a matter of fact, there are several serious objections to considering them as such. First, they have no "necessary" validity and have been constructed purely from an examination of the moods which were already taken as valid. Second, they are verbal, and strict definitions of their terms would be difficult to formulate. For example, the word "distributed" is a very difficult one to define unambiguously. But an even more serious objection will appear in the following discussion (pages 109 ff.), where it will be shown that these rules are not universally true.

In the next section we shall present a deductive system for the foregoing class "calculus" whose postulates are not open to these objections.

EXERCISES

GROUP A

1. Identify the categorical forms in the following sentences:

   a) No compound is an element.
   b) Some beings live forever.
   c) Not all books make good reading.
   d) Adams was an honest president.
   e) The Carthaginians fought courageously.
   f) Not all Shakespeare's plays were written by Shakespeare.
   g) The whole army climbed the ramparts.

2. Identify under the heads of "reflexive," "symmetrical," and "transitive" (cf. chapter II) the four categorical forms.
3. Identify the four categorical forms under the heads "irreflexive," "nonreflexive," "asymmetric," "nonsymmetric," "intransitive," "nontransitive." (cf. page 41, exercise 5.)

GROUP B

1. Identify the figure of the following moods of Immediate Inference:

   a) O (*ba*) implies E (*ab*).
   b) A (21) implies O (21).

c) A (*xy*) implies I (*yx*).
d) I ($\alpha\beta$) implies E ($\alpha\beta$).
e) E (56) implies O (65).
f) I (*cd*) implies I (*dc*).

GROUP C

1. Determine in the case of each of the Rules for Invalidity of Immediate Inference all the moods which make the rule necessary, i.e., all invalid moods which come uniquely under this rule.
2. In the following, if the mood is invalid name the rules which make it so, and if valid so indicate:

   a) AO in the second figure (i.e., "A (*ab*) implies O (*ba*)"; cf. page 55)
   b) AE in the second
   c) IA in the first
   d) OE in the first
   e) EO in the second
   f) IO in the first and second
   g) OO in the first and second
   h) OI in the second

3. A "particular" categorical form is defined as one which does not distribute its subject; thus I and O are particular; and a "universal" categorical form is one which does distribute its subject; A and E are universal. Show that a particular premise never implies a universal conclusion.

GROUP D

1. Write out the following syllogisms:

   a) AEO in the first figure.
   b) III_4 (III in the fourth figure)
   c) AIO_3
   d) EIO_2
   e) EEO_3
   f) AII_1
   g) EAO_4

2. Determine the figure of the following syllogisms:

   a) E (*bc*) O (*ca*) implies I (*ab*)
   b) A (*ca*) A (*bc*) implies A (*ab*)
   c) I (*ab*) A (*cb*) implies I (*ca*)

d) O (ca) A (cb) implies O (ba)
e) E (ba) I (ca) implies O (cb)
f) I (xy) O (yz) implies I (xz)
g) A (32) E (13) implies O (21)
h) I $(\beta\gamma)$ E $(\beta\alpha)$ implies O $(\alpha\gamma)$

3. Identify the syllogisms in the following arguments:

EXAMPLE: "All Cretans are liars;
Epimenides is a Cretan;
Therefore, Epimenides is a liar."

Let $a =$ Cretans, $b =$ liars, and $c =$ Epimenides. Then, since both premises and the conclusion are in the A-form, the argument becomes "A (ab) A (ca) implies A (cb)," which is AAA in the first figure.

a) No man is too rich, but some men are honest; hence some honest beings are not too rich.
b) All honest people are unhappy, and yet some honest people are clever; hence some clever people at least are not unhappy.
c) No sailboat is safe, but all safe things are expensive; hence no sailboat is expensive.
d) Some men achieve greatness, and of those beings who do, some are not saints; hence, some saints are not men.
e) All numbers have factors, and this thing also has factors and hence must be a number.
f) All triangles have angles which sum to 180 degrees. The angles of this figure sum to over 180 degrees, and hence this is not a triangle.
g) Some logical statements are trite, and all trite statements are best omitted; ergo, some logical statements are best omitted.

GROUP E

1. If the following syllogisms are false, give Euler diagrams to indicate their invalidity; also, wherever possible, apply Aristotle's method:

a) AAA_4
b) OAO_1
c) EEI_2
d) OAO_4
e) III in any figure.
f) EOE_1
g) AEE_1

h) EAO_2
i) EAE_3
j) EEE in any figure.

GROUP F

1. In the following syllogisms indicate all the rules which make the given mood invalid.

a) EEE_2
b) EAE_3
c) EIA_1
d) IAI_2
e) OIO_3
f) OOO_4
g) IIE_2
h) AEE_1
i) EII_3
j) IEE_3
k) AOO_1

2. Show that the total number of terms distributed in the premises of a valid syllogism must always be greater than the total number of terms distributed in the conclusion. (This result will be valuable in proving the theorems of the following exercises.)
3. Show by means of the rules for Invalidity of the Syllogism that two particular premises (see Exercise 3, Group C) do not imply any conclusion. (Hint: Show this in four parts: show (1) that OO does not imply any conclusion, (2) that II does not imply any conclusion, (3) that I and O do not imply an affirmative conclusion, and (4) that I and O do not imply a negative conclusion.)
4. Show (by means of the rules) that a mood in the third figure which has a universal conclusion is invalid.
5. Show that there can be an A-form in the conclusion in the first figure only.
6. If one of the premises of a valid mood is particular, then the conclusion must be particular.
7. In the first figure the middle term must be distributed in the major premise and must not be distributed in the minor premise if the mood is to be valid.
8. In a valid mood the major term must be distributed in the second figure.
9. In a valid mood of the third figure the minor term must not be distributed.
10. If the major premise of a mood in the first or second figure is not universal, then the mood is invalid.
11. If the minor premise is not affirmative in the third figure the mood is invalid.

12. A valid mood with the minor premise in the O-form can occur only in the second figure.
13. Determine the unique case which makes Rule 1 necessary; show that there can only be one such case. Determine unique cases for the remaining rules.
14. Show that if an I-form is in the premise of a valid mood, this may be replaced by an A-form (with the same terms, though the order is indifferent) and the mood is still valid; similarly, if the premise is an O-form, this may be replaced by an E-form without altering the validity. (This process is called "strengthening the premise"; cf., page 29). E.g.,

"A (*ba*) I (*cb*) implies I (*ca*) " may be changed to
"A (*ba*) A (*cb*) implies I (*ca*) " or to "A (*ba*) A (*bc*) implies I (*ca*) ."

15. Show that if the conclusion of a valid mood is an A-form, this may be replaced by an I-form (with the same terms, but the order is indifferent) and the mood is still valid; similarly, if the conclusion is an E-form, this may be replaced by an O-form, and the mood is still valid. (This process is called "weakening the conclusion"; cf., ibid.) E.g., "E (*ab*) A (*cb*) implies E (*ca*) " may be changed to "E (*ab*) A (*cb*) implies O (*ca*) ."
16. Show, by means of examination of the valid moods in the table above, that if the conclusion and either premise of a valid mood be interchanged and each replaced by its contradictory (for definition of contradictories, see page 65) another valid mood results. Thus from "A (*ba*) A (*cb*) implies A (*ca*) " we can infer "A (*ba*) O (*ca*) implies O (*cb*) " or AOO in the second.
17. By transferring the syllogisms in the exercise on page 75 into formal arguments, determine their validity or invalidity. E.g.,

"All honest people are unhappy,
and some honest people are clever;
hence, some clever people are not unhappy."

Let $a =$ "honest people," $b =$ "unhappy (people) ," $c =$ clever (people) ." Then the syllogism becomes:

"*A* ($\underline{a}b$) *I* (*ac*) implies O ($c\underline{b}$) ,"

which is an invalid syllogism by Rules 1 and 5.
18. Determine the validity or the invalidity of the following arguments:

a) All those who are aggressive are morally wrong; Hitler is aggressive, and hence he must be morally wrong.

b) Some bricklayers make money; Jones makes no money; hence Jones is not a bricklayer. (Does it affect the validity or invalidity of this argument to strengthen the first premise to "All bricklayers make money"?)

c) I love cats; hence I must be intelligent, since anyone who loves cats is intelligent.

d) No cats are amphibians, and this goes for dogs; hence some cats are dogs.

e) Some animals eat grass; no one who eats grass is very sensible; hence some animals are not sensible. (Can one infer that some sensible things are not animals?)

f) Jones is a liar and hence is not a good man. (What premise is missing?)

g) You are not an Athenian; all Athenians were humans; hence you are not a human.

h) Everyone is having a nice time, and some are playing bridge; hence some people who play bridge must enjoy it.

i) I am no musician; hence I don't like Bach.

j) All men are substances, and all animals are substances; hence, all men are animals.

k) All men are geometrical figures; some geometrical figures are liars; hence some men are liars.

l) Achilles was defeated for president in the 1916 election; everyone defeated in this election was fleet of foot; hence Achilles was fleet of foot.

m) Penguins fly very fast; nothing that flies very fast is without wings; hence no penguin lacks wings.

n) Any argument worthy of logical recognition must be such as would occur in ordinary discourse. It will be found that no argument in ordinary discourse is in the fourth figure. Hence, no argument in the fourth figure is worthy of logical recognition. (This syllogism is due to W. E. Johnson; it is a humorous answer to those logicians who, apparently following Aristotle, believe that the fourth figure is an "unnatural" method of argument.)

o) The French do not drink tea since only the English drink tea.

p) All sailors can read a compass; hence they must be physicists, for all physicists can read a compass.

19. What conclusions, if any, may be drawn from the following premises? E.g.,

"All pigs are greedy;
No pigs can fly." [6]

[6] Lewis Carroll.

Let a = "pigs," b = "greedy things," c = "things which can fly." Then the premises become:

$$A\,(ab)\;E\,(ac)\,.$$

From these premises we may validly infer $O\,(bc)$ (EAO_3 is valid):

"Some greedy things cannot fly."

a) Some numbers are prime numbers.
No prime number is evenly divisible by anything other than itself and one.

b) Some triangles are equilateral;
no triangle has four sides.

c) Some prime-ministers are not monkeys;
all monkeys live in trees.

d) Hydrogen always combines with oxygen;
hydrogen never combines with helium.

e) Chickens are bipeds;
horses are quadrupeds.

(Hint: Construct two syllogisms; what premise is missing?)

f) All totalitarian states must necessarily fail;
our state, however, is democratic.

g) To the victors belong the spoils;
We are the vanquished.

h) Tadpoles are not frogs;
but some tadpoles become frogs.

i) All Republicans wear beards at least three feet long;
No Democrat sports a beard of over two feet.

The Deductive System of the Aristotelian Class Calculus

4

WE NOW recall the characteristics of a deductive system as described in chapter I. In the first place, every deductive system must start with certain "undefined concepts" in terms of which it defines all other concepts under consideration.

Here we think of A (*ab*) (which can be read "all *a* is *b*") and E (*ab*) ("no *a* is *b*"), as undefined. We then define the other two categorical forms as follows:

DEFINITION: O (*ab*) means "A (*ab*) is false" (= "some *a* is not *b*"); i.e., O (*ab*) = [A (*ab*)]′.
DEFINITION: I (*ab*) means "E (*ab*) is false" (= "some *a* is *b*"). I (*ab*) = [E (*ab*)]′.

It will be noted that we make use of the logic of propositions in these definitions in assuming that the meaning of "false" is known. This will not be the only occasion in which we shall draw assumptions from this branch of logic (cf., the Axioms of the system).

The postulates necessary to prove the valid moods of Immediate Inference are:

POSTULATE 1. A (*ab*) implies I (*ab*). "If all *a* is *b*, then some *a* is *b*."
POSTULATE 2. I (*ab*) implies I (*ba*). "If some *a* is *b*, then some *b* is *a*."

The axioms which we shall assume here are borrowed from the logic of propositions. They are:

AXIOM 1. If *p* is definitionally equivalent to *q*, then *p* implies *q* and *q* implies *p*.

Thus, since O (ab) is definitionally equivalent to "A (ab) is false," we can say, by virtue of this axiom, that O (ab) implies [A (ab)]$'$ and [A (ab)]$'$ implies O (ab).

Axiom 2. If p is the contradictory of q, i.e., if p means "q is false," then q is the contradictory of p, i.e., q means "p is false."

Axiom 3. If p implies q, then the contradictory of q, q', implies the contradictory of p, p'.

That is, given any valid implication "p implies q," we may contradict and interchange the premise and the conclusion and the resulting implication is still valid (q' implies p'). (Cf. page 24.)

Axiom 4. If p implies q and q implies r, then p implies r.

That is, given any valid (immediate) inference, we may weaken the conclusion (by replacing it by a form which it implies) or strengthen the premise (by replacing it by a form which implies it) and the resulting implication is still valid. (Cf. page 28.) Note that any two (or three) of the statements p, q, and r may be the same. We shall make use of this fact later.

It might be asked whether we can be sure that A (ab) is a proposition and hence whether we can consider it as one of the p's, or q's, or r's of our axioms. But it is not necessary to show this (though it may seem evident enough), for each of the axioms is based on an hypothesis: *If* p implies q (or *if* p means q), *then* we can say, etc. Hence our postulates are enough, for they satisfy this hypothesis, and the conclusions or theorems drawn must be valid.

From the above two postulates and four axioms follow the remaining eight valid moods of Immediate Inference. But before proceeding to these, we must prove two preliminary theorems:

Theorem 1. A (ab) means "O (ab) is false"; A $(ab) =$ [O (ab)]$'$.

Proof: This follows immediately from the definition of O (ab) and Axiom 2. For, since O (ab) means "A (ab) is false," i.e., since O (ab) is the contradictory of A (ab), then, by Axiom 2, A (ab) is the contradictory of O (ab), i.e., A (ab) means "O (ab) is false."

Similarly,

THEOREM 2. E (ab) means "I (ab) is false"; E (ab) = [I (ab)]′.

THEOREM 3. E (ab) implies O (ab). "If no a is b, then some a is not b."

PROOF: Since, by Postulate 1, we know that

A (ab) implies I (ab) (1)

then, by Axiom 3, we may infer that

"I (ab) is false" implies "A (ab) is false,"
or [I (ab)]′ implies [A (ab)]′. (2)

But [I (ab)]′ is E (ab) by Theorem 2, and
[A (ab)]′ is O (ab) by the first definition. (3)

Hence, by replacing [I (ab)]′ and [A (ab)]′ in (2) by their equivalents in (3),[1] we have

E (ab) implies O (ab). Q.E.D.

THEOREM 4. E (ab) implies E (ba). "If no a is b, then no b is a."

PROOF: Postulate 2 asserts:

I (ab) implies I (ba).

Hence, again making use of Axiom 3, we can say that

[I (ba)]′ implies [I (ab)]′.

Hence, since [I (ba)]′ or "I (ba) is false" is E (ba) and [I (ab)]′ is E (ab), we can assert

E (ba) implies E (ab),

and this is the same implication as that which was to be proved, since both are EE in the second figure (cf. page 55).

THEOREM 5. A (ab) implies I (ba). "If all a is b, then some b is a."

PROOF: The following are true by Postulates 1 and 2 respectively:

A (ab) implies I (ab)
I (ab) implies I (ba).

[1] Strictly, we should assert also an axiom which gives us the right to replace any given expression by its equivalent without altering the validity of the whole.

But Axiom 4 asserts that if one proposition [here A (ab)] implies another [I (ab)], which in turn implies a third [I (ba)], then the first implies the third; i.e.,

A (ab) implies I (ba).

Or, we might phrase our proof as follows: Postulate 1 tells us that A (ab) is a strengthened form of I (ab) and hence (Axiom 4) we may replace the latter by the former in the valid implication "I (ab) implies I (ba)" (Postulate 2), and the resulting implication will still be valid. Obviously, we could also think of I (ba) as a weakened form of I (ab) and proceed in an analogous manner.

The following theorem is proved from Theorem 5 by a method similar to that used in Theorems 3 and 4:

THEOREM 6. E (ab) implies O (ba). "If no *a* is *b*, then some *b* is not *a*."

There remain, now, the four valid moods in which a given categorical form implies itself:

THEOREM 7. O (ab) implies O (ab).

PROOF: Since O (ab) is definitionally equivalent to [A (ab)]′, or "A (ab) is false," we may infer, by Axiom 1, the following two statements:

O (ab) implies [A (ab)]′
[A (ab)]′ implies O (ab).

But this allows us to apply Axiom 4 under the special case where p and r are the same. That is, if p implies q and q implies p, then we can say that p implies p:

O (ab) implies O (ab). Q.E.D.

Note that there are serious objections to simply asserting as an axiom: "p implies p." In the first place, such an axiom would not be true for every p, since if p is an object, such as "stone," the statement is nonsense. Hence, we would have to say "If p is a proposition, then p implies p." But then arises the second objection, namely, the difficulty of showing that O (ab) is a proposition. For, though this may be "obvious," the concept of a deductive system requires that even the most obvious statements be proved from the assumptions. The above method, which follows directly from the definition of O (ab), neatly avoids these difficulties.

But another method could have been employed here. For if we had asserted as an axiom the principle which expresses the reflexivity of implication (cf. page 23): If p implies q, then p implies p," then we could have shown that A (ab) implies A (ab) (and hence that O (ab) implies O (ab) by Axiom 3). For by Postulate 1, A (ab) implies another proposition (I (ab)), and hence by this additional axiom it implies itself. Either method is adequate.

In like manner we can prove:

Theorem 8. I (ab) implies I (ab).

Theorem 9. A (ab) implies A (ab).

(To prove this use Theorem 7 and Axiom 3.)

Theorem 10. E (ab) implies E (ab).

It is to be noted that we could have used other postulates than the ones chosen and that we could have proved our theorems is a different order. The other methods will prove valuable exercises (the proofs of Theorems 7-10 will remain the same in every case if the first method is chosen).

(See Exercises, Group A, at end of chapter.)

So far we have only proved that the valid moods of Immediate Inference are valid. It remains, of course, to prove that the other twenty-two invalid moods are actually invalid. We shall postpone this proof, however, until after we have proved the valid moods of the syllogism.

Here two additional postulates are necessary:

Postulate 3. A (ba) A (cb) implies A (ca).

This is the syllogism "Barbara" or AAA in the first figure: "If all b is a and all c is b, then all c is a."

Postulate 4. E (ba) A (cb) implies E (ca).

This is "Celarent," EAE in the first: "If no b is a and all c is b, then no c is a."

We shall require three additional axioms:

AXIOM 5. If p and q (pq) implies r, then pr' implies q'; also, if pq implies r, then $r'q$ implies p'.

That is, if two premises (p and q) validly imply a conclusion, r, then we may contradict and interchange either premise and conclusion and the implication is still valid (cf. p. 25).

AXIOM 6. If pq implies r, and s implies p, then sq implies r; or, if pq implies r, and s implies q, then ps implies r; or, if pq implies r, and r implies s, then pq implies s.

That is, if two premises validly imply a conclusion, then we may strengthen either premise (or both) or weaken the conclusion (or do all three at once) and the implication is still valid (cf. p. 30).

AXIOM 7. The statement "p and q (are true)" is equivalent to the statement "q and p (are true)": $pq = qp$.

We shall make use of this axiom in assuming that the order of the premises in an argument may be changed at will. It will often become necessary to change the order of the premises in the proofs of the syllogisms.

The Valid Moods of the Second Figure

We shall now prove all the six valid moods of the second figure:

THEOREM 11. A (ab) O (cb) implies O (ca). (Baroko) "If all a is b and some c is not b, then some c is not a."

PROOF: Postulate 3 asserts that

$$A(ba)\ A(cb) \text{ implies } A(ca).$$

But Axiom 5 allows us to contradict and interchange either premise and the conclusion of a valid syllogism without altering its validity. If we choose the second or minor premise in Postulate 3 we have the following valid implication:

$$A(ba)\ [A(ca)]' \text{ implies } [A(cb)]'.$$

But $[A(ca)]'$, or "It is false that all c is a," is O (ca), and $[A(cb)]'$ is O (cb). Hence the above becomes

$$A(ba)\ O(ca) \text{ implies } O(cb).$$

This is the required syllogism, since *a* has now become the middle term and occurs second in both premises (the latter being in the correct order: *c*, the subject of the conclusion, is in the minor premise). Hence, we have AOO in the second, which was to be proved. (To return to the exact lettering of the original theorem, replace *a* by *b* and *b* by *a* throughout in "A (*ba*) O (*ca*) implies O (*cb*).")

THEOREM 12. E (*ab*) I (*cb*) implies O (*ca*) (Festino) "If no *a* is *b* and some *c* is *b*, then some *c* is not *a*."

PROOF: The theorem is proved in exactly the same manner as Theorem 11, namely, by contradicting and interchanging the conclusion and minor premise of Postulate 4, and using the fact that [E (*ab*)]′, "It is false that no *a* is *b*," is equivalent to I (*ab*). It will turn out to be a general rule that if we contradict and interchange the minor and conclusion of a first figure syllogism we will derive a syllogism in the second figure. The brief proof is as follows:

POSTULATE 4: E (*ba*) A (*cb*) implies E (*ca*). Hence (Axiom 5): E (*ba*) [E (*ca*)]′ implies [A (*cb*)]′, or E (*ba*) I (*ca*) implies O (*cb*), which is EIO in the second. Q.E.D.

THEOREM 13. E (*ab*) A (*cb*) implies E (*ca*) (Cesare) "If no *a* is *b* and all *c* is *b*, then no *c* is *a*."

PROOF: Postulate 4 asserts that:

E (*ba*) A (*cb*) implies E (*ca*).
But (Theorem 4) E (*ab*) implies E (*ba*),

that is, E (*ab*) is a strengthened form of E (*ba*), and hence, by Axiom 6, we may strengthen the first or major premise of Postulate 4 and immediately derive the required syllogism.

The proof of this theorem suggests weakening the conclusion of Theorem 13 in the same manner, i.e., making use of the fact that E (*ca*) implies E (*ac*). If we do this, we derive

THEOREM 14. A (*ab*) E (*cb*) implies E (*ca*) (Camestres) "If all *a* and no *c* is *b*, then no *c* is *a*."

PROOF: Since E (*ab*) A (*cb*) implies E (*ca*) (Theorem 13), and E (*ca*) implies E (*ac*), we can say (Axiom 6) that

E (*ab*) A (*cb*) implies E (*ac*).

But now the premises are not in the proper syllogistic order, for a, which is the subject of the conclusion, is not in the second premise; hence, by virtue of Axiom 7, we change the premises around and derive

$$A(cb)\ E(ab) \text{ implies } E(ac),$$

which is the required AEE in the second.

The remaining two moods of this figure, EAO and AEO, follow immediately from Theorems 13 and 14 by weakening the E-forms in the conclusion to O-forms:

THEOREM 15. $E(ab)$ $A(cb)$ implies $O(ca)$ (Celaront)

PROOF: Theorem 13: $E(ab)$ $A(cb)$ implies $E(ca)$.
But by Immediate Inference (Theorem 3) $E(ca)$ implies $O(ca)$, and hence, by Axiom 6, we derive the required mood. Note that this theorem also follows from Theorem 12 by strengthening the minor premise, $I(cb)$, to $A(cb)$.

THEOREM 16. $A(ab)$ $E(cb)$ implies $O(ca)$.

The manner of proving the remaining valid moods of the syllogism is closely analogous to those already given. The general procedure is outlined below.

The Valid Moods of the Third Figure

These may all be derived directly from the first figure with the exception of EIO. The initial method consists in using Axiom 5 on AAA and EAE in the first by contradicting and interchanging the major premise and conclusion. For example:

THEOREM 17. $O(ba)$ $A(bc)$ implies $O(ca)$ (Bokardo)

PROOF: Postulate 3 asserts as valid the following implication:

$A(ba)$ $A(cb)$ implies $A(ca)$.

Hence, by Axiom 5, contradicting and interchanging the major premise, $A(ba)$, with the conclusion, $A(ca)$, we have $[A(ca)]'$ $A(cb)$ implies $[A(ba)]'$. Hence:

$$O(ca)\ A(cb) \text{ implies } O(ba),$$

which is the required OAO in the third.

Similarly, we derive:

THEOREM 18. I (*ba*) A (*bc*) implies I (*ca*) (Disamis) by contradicting and interchanging the major and conclusion in EAE in the first (Postulate 4).

We may now prove the following two theorems by strengthening the major premises in Theorems 17 and 18.

THEOREM 19. E (*ba*) A (*bc*) implies O (*ca*) (Felapton)
THEOREM 20. A (*ba*) A (*bc*) implies I (*ca*) (Darapti)

It is peculiar to both the second and third figures that if we change the order of the premises by weakening an I- or E-conclusion by reversing the order of its terms, the resulting syllogism is still in the same figure. For example, take Theorem 18: I (*ba*) A (*bc*) implies I (*ca*). Now weaken I (*ca*) to I (*ac*):

I (*ba*) A (*bc*) implies I (*ac*).

This change requires a change in the order of the premises, since now the subject of the conclusion is *a* and not *c*, and hence *a* must appear in the minor premise. Hence, by Axiom 7, we can write:

THEOREM 21. A (*bc*) I (*ba*) implies I (*ca*), (Datisi)

and this is another mood of the third figure.

The process of changing the terms in E or I by weakening if they appear in the conclusion, or strengthening if they appear in the premise, is called "conversion." Thus, we "converted" I (*ca*) to I (*ac*). The right to convert I and E depends on Postulate 2 and Theorem 4 respectively (plus, of course, Axiom 6), but for the sake of brevity we may omit to mention these references. (Note that if we convert I (*ca*) in Theorem 20 we derive Theorem 20 over again, so that this process does not help here.)

As already mentioned, EIO in this figure cannot be derived directly from the first figure. But since we have proved EIO in the second figure, it is a comparatively simple matter to prove it in the third by converting both major and minor premises:

THEOREM 22. E (*ba*) I (*bc*) implies O (*ca*) (Fresison)

PROOF: E (*ab*) I (*cb*) implies O (*ca*) (Theorem 12).
Hence, by strengthening (converting) both premises, we derive:

E (*ba*) I (*bc*) implies O (*ca*) Q.E.D.

Once we have EIO in any figure, we can prove it in all figures by converting one or both of the premises. Thus EIO in the first follows from EIO in the second by converting the major premise; EIO in the fourth, by converting the minor. Bear in mind that *only* E and I may be converted, for *it is not true that A (ab) implies A (ba) or that O (ab) implies O (ba)*.

The Valid Moods of the Fourth Figure

The moods of the fourth figure cannot be derived by contradiction and interchange in the first. Instead, we must use the method of conversion in the conclusion, since when this is applied to the first figure, the fourth figure results:

THEOREM 23. A (*ab*) E (*bc*) implies E (*ca*) (Camenes)

PROOF: E (*ba*) A (*cb*) implies E (*ca*) (Postulate 4)
But E (*ca*) implies E (*ac*) (Theorem 4).
Hence E (*ba*) A (*cb*) implies E (*ac*) (Axiom 6).
(The premises now are in the wrong order:)
Hence A (*cb*) E (*ba*) implies E (*ac*) (Axiom 7) and this is the required AEE in the fourth.

We cannot convert the conclusion of AAA in the first, since A (*ca*) does not imply A (*ac*). But we can convert the conclusion indirectly by passing to the I-form first (called "conversion per accidens"):

THEOREM 24. A (*ab*) A (*bc*) implies I (*ca*) (Bramantip)

PROOF: A (*ba*) A (*cb*) implies A (*ca*) (Postulate 3).
But A (*ca*) implies I (*ac*) (Theorem 5)
Hence A (*ba*) A (*cb*) implies I (*ac*) (Axiom 6)
Hence A (*cb*) A (*ba*) implies I (*ac*) (Axiom 7) Q.E.D.

It is peculiar to the fourth figure that if we contradict and interchange either premise and the conclusion, the fourth figure again results. Hence, all that is now required is to take Theorems 23 and 24 and apply Axiom 5 four times, first on the major and minor of 23, and then on the major and minor of 24. For example:

THEOREM 25. E (*ab*) I (*bc*) implies O (*ca*) (Fresison)

PROOF: A (*ab*) E (*bc*) implies E (*ca*) (Theorem 23)
Hence I (*ca*) E (*bc*) implies O (*ab*) (Axiom 5, contradicting and interchanging major and conclusion).
(The premises are not in the syllogistic order:)
Hence E (*bc*) I (*ca*) implies O (*ab*). Q.E.D.

It will be noted that whenever Axiom 5 is applied to a mood in the fourth figure in which the premises are in the correct order, the resulting mood will always have its premises in the wrong order as above. The proofs of the remaining three theorems follow by applying Axiom 5 to the minor of Theorem 23 and to the major and minor of Theorem 24:

THEOREM 26. I (*ab*) A (*bc*) implies I (*ca*). (Dimaris)
THEOREM 27. A (*ab*) E (*bc*) implies O (*ca*) (Camenos)
THEOREM 28. E (*ab*) A (*bc*) implies O (*ca*) (Fresapo).

The Valid Moods of the First Figure

These have been left to the last since two of them must be derived indirectly. But AAI and EAO in the first provide no difficulty since they follow immediately from Postulate 3 and 4 respectively by weakening the conclusion:

THEOREM 29. A (*ba*) A (*cb*) implies I (*ca*) (Barbari)
THEOREM 30. E (*ba*) A (*cb*) implies O (*ca*) (Celaront)

EIO in the first follows from EIO in the second in the manner indicated above, namely, by strengthening the major by conversion:

THEOREM 31. E (*ba*) I (*cb*) implies O (*ca*) (Ferio)

Finally, we require a proof of

THEOREM 32. A (*ba*) I (*cb*) implies I (*ca*) (Darii)

This follows from a good many of the theorems already proved. Perhaps the easiest method is to convert the minor in Theorem 21, AII in the third. Or, we could convert the conclusion of IAI in the fourth. Or, we could contradict and interchange the minor and conclusion of AEE in the second, or the major and conclusion of EIO in the third.

The discussion of the last proof suggests that there may be many ways of proving the valid moods of the syllogism and this is, of course, the case. The following exercises suggest some of these methods. Helpful in this respect will be the accompanying diagram which gives the method for passing from one figure to another. The Roman numerals refer to the figures; those on the top line are the ones operated on and those in the middle are the result. Thus, if we contradict and interchange (here abbreviated C. and I.) minor and conclusion in the third figure (III), we derive the second (II), or, if we convert the major in IV we obtain III.

	I	II	III	IV
C. and I. major and concl.	III	III	I	IV
C. and I. minor and concl.	II	I	II	IV
Convert major	II	I	IV	III
Convert minor	III	IV	I	II
Convert concl.	IV	II	III	I

Some abbreviation in the proofs will also be helpful. Thus the proof of Theorem 11 above may be stated thus:

A (*ba*) O (*ca*) implies O (*cb*)
(Postulate 1, Axiom 5 on the minor),

and the proof of Theorem 13 may be stated:

E (*ab*) A (*cb*) implies E (*ca*)
(Postulate 2, Axiom 6 on the major, EE_2),

where EE_2 indicates what mood of Immediate Inference was used in applying Axiom 6.

(See Exercises, Group B, at end of chapter.)

It is an interesting historical fact that Aristotle actually established the valid moods of the syllogism in a very similar manner. His method consisted simply in "reducing" all the valid moods (excepting the fourth figure, which Aristotle omits) to Barbara and Celarent. The "reduction" consists in either converting the terms in E and I, or else in showing that to deny the given mood is to deny Barbara or Celarent. The latter method is an example of the Indirect Proof (page 27) and this is another form of the Principle of Contradiction and Interchange. Thus, to show AOO in the second, Aristotle argues as follows: Suppose that the conclusion O (*ca*) cannot always be inferred from A (*ab*) O (*cb*), i.e., suppose that we can infer the contradictory of O (*ca*), or

A (*ca*) from these premises; but if A (*ab*) is granted and A (*ca*) follows, then by Barbara we can infer A (*cb*); but this conclusion contradicts the second premise, O (*cb*), and hence an absurdity has been reached; that is, granting A (*ab*) O (*cb*), we cannot have O (*ca*) false, or A (*ab*) O (*cb*) implies O (*ca*).

The Invalid Moods of Immediate Inference

Our deductive system would not be complete if we did not include postulates which will allow us to prove that the remaining moods of Immediate Inference and the syllogism are false. We require for the former four additional postulates:

POSTULATE 5. A (*ab*) does not imply A (*ba*).
POSTULATE 6. A (*ab*) does not imply O (*ab*).
POSTULATE 7. A (*ab*) does not imply O (*ba*).
POSTULATE 8. E (*ab*) does not imply I (*ab*).

Two additional axioms are needed as well:

AXIOM 8. Given any invalid (immediate) implication ("*p* does not imply *q*"), we may contradict and interchange the premise and conclusion and the resulting implication is still invalid.

That is, if *p* does not imply *q*, then *q′* does not imply *p′* (cf. page 25). (Note that Axioms 3 and 8 together give: "The contradiction and interchange of the premise and conclusion of an [immediate] inference does not alter the truth or falsity of the inference.")

AXIOM 9. Given any invalid implication, we may weaken the premise or strengthen the conclusion and the resulting implication is still invalid.

That is, "If *p* does not imply *q* but *p* does imply *r*, then *r* does not imply *q*," or "If *p* does not imply *q*, but *r* does imply *q*, then *p* does not imply *r*." (Cf. pages 28, 29.)

Rather than prove all the eighteen theorems we present examples of the method:

THEOREM 33. O (*ab*) does not imply O (*ba*).

PROOF: A (*ab*) does not imply A (*ba*) (Postulate 5)
Hence, [A (*ba*)]′ does not imply [A (*ab*)]′ (Axiom 8)

But [A (*ba*)]′ or "It is false that all *b* is *a*" is O (*ba*) and [A (*ab*)]′ is O (*ab*):
Hence, O (*ba*) does not imply O (*ab*), and this expresses the same fact as the theorem, both being OO in the second.

THEOREM 34. A (*ab*) does not imply E (*ab*).

PROOF: A (*ab*) does not imply O (*ab*) (Postulate 6)
But E (*ab*) does imply O (*ab*) (Theorem 3)
Hence A (*ab*) does not imply E (*ab*) (Axiom 9)

THEOREM 35. I (*ab*) does not imply A (*ab*).

PROOF: A (*ab*) does not imply A (*ba*) (Postulate 5)
But A (*ab*) does imply I (*ba*) (Theorem 5)
Hence I (*ba*) does not imply A (*ba*) (Axiom 9) Q.E.D.

THEOREM 36. E (*ab*) does not imply A (*ab*)

PROOF: E (*ab*) does not imply I (*ab*) (Postulate 8)
But A (*ab*) implies I (*ab*) (Postulate 1)
Hence E (*ab*) does not imply A (*ab*) (Axiom 9) Q.E.D.

The remaining theorems are left to be proved by the student. Note that Axiom 9 is used far more frequently than Axiom 8. In fact, with the exception of OO in the second, we could prove all the theorems by means of Axiom 9 and the postulates. Note also that if Axiom 8 is applied to Postulates 6-8, the resulting form is the given postulate over again; but Axiom 8 may be applied conveniently to such a form as IA above to obtain the invalid mood OE. Again, note that Postulate 5 (or theorems derived from it) must be used in all cases where the premise and conclusion are both affirmative or where they are both negative, Postulates 6 and 7 where an affirmative implies a negative, and Postulate 8 in all cases where a negative implies an affirmative.

The Invalid Moods of the Syllogism

Here there are seven postulates:

POSTULATE 9. A (*ba*) A (*cb*) does not imply O (*ca*)
AAO_1 is an invalid mood.
POSTULATE 10. A (*ba*) E (*cb*) does not imply I (*ca*)
AEI_1 is an invalid mood.
POSTULATE 11. A (*ba*) E (*cb*) does not imply O (*ca*)
AEO_1 is an invalid mood.

POSTULATE 12. E (ba) E (cb) does not imply I (ca)
EEI_1 is an invalid mood.

POSTULATE 13. O (ba) A (cb) does not imply O (ca)
OAO_1 is an invalid mood.

POSTULATE 14. A (ab) A (bc) does not imply A (ca)
AAA_4 is an invalid mood.

POSTULATE 15. A (ab) A (bc) does not imply O (ca)
AAO_4 is an invalid mood.

The following axioms are necessary:

AXIOM 10. If two premises do not imply a conclusion, then if either one of the premises is contradicted and interchanged with the conclusion the resulting implication is also invalid.

That is: "If pq does not imply r, then $r'q$ does not imply p'," or, "If pq does not imply r, then pr' does not imply q'." (Page 25.)

AXIOM 11. If two premises do not imply a conclusion, we may weaken either premise or strengthen the conclusion and the resulting implication is also invalid.

That is, "If pq does not imply r, but p implies s (or q implies s), then sq does not imply r (or ps does not imply r.)"; or, "If pq does not imply r, but s implies r, then pq does not imply s." [2]

(Axiom 7 will also be applied here; that is, if pq does not imply r, then qp does not imply r.)

The theorems are proved in a manner analogous to that given above for Immediate Inference. Here the rules for passing from one figure to another will again be useful (cf. page 91). Thus if we contradict major and conclusion in Postulates 9-13, we derive invalid moods of the third figure.

The following hints may prove helpful:

1. When the mood to be proved invalid has all affirmative premises and an affirmative conclusion, or one affirmative and one negative premise with a negative conclusion, use Postulate 11, 13, or 14. (Note that there are also valid moods of this form; in fact, all valid moods must be one or the other of these two forms: cf., Rules 1-3 of the Rules of Invalidity of the Syllogism, page 71. These two forms are, of course, closely allied, since if we contradict and interchange premise and conclusion in a mood

[2] Cf. p. 30.

in which both premises and the conclusion are affirmative, we derive the second form, and, conversely, if we contradict and interchange the negative premise with the conclusion in a mood of the second form, the first form results.)

EXAMPLE: A (*ab*) A (*cb*) does not imply I (*ca*).

PROOF: A (*ba*) E (*cb*) does not imply O (*ca*). (Postulate 11)
Hence, A (*ba*) A (*ca*) does not imply I (*cb*) (Axiom 10).
Q.E.D.

2. If the invalid mood has two affirmative premises and a negative conclusion use Postulate 9 or 15. (Note that A may be weakened to I in either figure and O may be strengthened to E, so that we can readily prove all invalid moods of this type from these two postulates.[3])

EXAMPLE: I (*ba*) I (*bc*) does not imply O (*ca*).

PROOF: A (*ba*) A (*cb*) does not imply O (*ca*) (Postulate 9).
But A (*ba*) implies I (*ba*) (Postulate 1)
and A (*cb*) implies I (*bc*) (Theorem 5).
Hence, I (*ba*) I (*bc*) does not imply O (*ca*) (Axiom 11).
Q.E.D.

(Note that this theorem follows from Postulate 15 as well.)

3. If the invalid mood has two negative premises implying an affirmative conclusion, use Postulate 12. (This postulate is sufficient for this type, since E may be weakened to E or O in either figure, and I may be strengthened to I or A in either figure.)

EXAMPLE: O (*ab*) O (*cb*) does not imply A (*ca*).

PROOF: E (*ba*) E (*cb*) does not imply I (*ca*) (Postulate 12).
Now, weakening E (*ba*) to O (*ab*), E (*cb*) to O (*cb*), and strengthening I (*ca*) to A (*ca*), we derive the invalid mood required by virtue of Axiom 11.

4. If the invalid mood has a negative and an affirmative implying an affirmative, or two negatives implying a negative (for the connection between these two types, cf. 1 above) use Postulate 10.

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[3] AAO_2, AAO_3 follow from Postulate 1 by Axiom 10.

EXAMPLE: E (ba) E (cb) does not imply E (ca)

PROOF: A (ba) E (cb) does not imply I (ca) (Postulate 10)
Hence, E (ca) E (cb) does not imply O (ba) (Axiom 10)
If now we weaken E (cb) to E (bc) and strengthen O (ba) to E (ba), we derive (Axiom 11):

E (ca) E (bc) does not imply E (ba) Q.E.D.

(See Exercises, Group C, at end of chapter.)

EXERCISES

GROUP A

1. Start with "E (ab) implies O (ab)" and "E (ab) implies E (ba)" as postulates, and, using the same axioms, deduce the remaining valid moods.
2. Start with "A (ab) implies I (ba)" and "E (ab) implies E (ba)" and prove the remaining moods.
3. Start with EO in the first and II in the second and prove the theorems in the following order: EE_2, EO_2, AI_2, AI_1.
4. Can all the theorems be deduced from AI_1 and EO_2?

GROUP B

1. Prove the moods of the second figure by assuming OAO and AII in the third as postulates.
2. Prove all the moods of the fourth figure by assuming only EAE in the first.
3. Prove all the moods possible from AAA in the first alone; from EAE_1 alone.
4. Prove the following from IAI in the fourth: IAI, AII, IAI, and AAI in the third, AII, AAI, in the first, EIO in all figures, EAO in all figures.
5. Prove the following in the order given from AAA and EAE in the first: IAI_3, IAI_4, AEE_4, OAO_3, EAO_4, AAI_1, AAI_3.
6. Prove all the valid moods from EIO in the fourth and OAO in the third; from AOO in the second and AEE in the fourth.
7. Can all the valid moods be derived from AOO in the second and EAO in the second? from AAA in the first and EIO in the third? from AAI in the fourth and AEE in the second? (If not, in each case determine the moods which cannot be proved.)

GROUP C

1. Prove that AE_2, OE_1, OA_2, EA_1, IA_2, OO_2 are invalid moods.
2. Prove that AAA_2 and AAA_3 are invalid moods.

3. Show that the first three rules for Invalidity of the Syllogism are true by giving, in each case, a method for proving moods of the given type invalid. (i.e., in the case of two negatives implying an affirmative, show how all moods of this type are shown invalid by Postulate 12.)
4. Prove that AAA_2, IAI_2, AII_4, III in all figures, are invalid moods; that AEE_1, EAE_3, EAE_4, AOO_3 are invalid.
5. Prove that IAO_1, IIO_1, AAO_3, AAO_2 are invalid moods.
6. Prove that OOI_1, EOI_3, OEI_4, OOA_3, EOA_2 are invalid moods.
7. Prove that EOE_2, OOE_3, EII_1, AOA_3, OOO_4, IOI_2, AEA_3, EAA_1 are invalid moods.
8. Demonstrate the invalidity of the following moods:

   AIE_2, EEE_4, AEE_1, AAE_4, OOI_1, IIA_3, AEA_1, OEI_1, AOO_4, OII_3, AII_2, IAO_3, EEA_3, OAO_4, AAA_2, AAA_4, EAE_2, AEA_2.

9. Making use of the principles of the logic of propositions that the denial of a conjunction is disjunction of the elements separately denied and the denial of a disjunction is a conjunction of the elements separately denied, give the contradictories of the following. What may be inferred if the contradictory is an absurdity, i.e., is self-inconsistent? What may be inferred if it is necessarily true?

   a) A (ab) O (ba) (For the symbols used here, see page 37)
   b) I (ab) O (ab)
   c) E (ba) I (ab)
   d) A (ab) + I (ba)
   e) E (ab) + I (ba)
   f) A (ab) + O (ba)
   g) A (ab) + E (ab)
   h) I (ab) O (ba) O (ab)
   i) A (ab) + I (ab) O (ba)
   j) A (ab) O (ba) + I (ab) + E (ab)
   k) A (ab) + O (cb) + O (ca)
   l) A (ab) A (ba) + O (cb) O (ca) O (cb)
   m) E (ab) + I (cb) + A (ca)
   n) O (ab) + O (bc) + E (ca)

Development of the Traditional Logic

5

In the following section we shall consider certain further aspects of the Aristotelian system. Among other things, it will be shown how the deductive system can be simplified (i.e., how the number of postulates can be reduced) and that the Rules for Invalidity of Immediate Inference and the Syllogism given on page 56 and page 71 cannot be considered as postulates for this Aristotelian system since they are not universally valid. Before entering on these discussions it will be necessary to define certain important concepts.

The Universe of Discourse

There is one important class we have not yet considered and this is the class commonly designated by the term "universe," i.e., the class that includes everything. In our everyday discourse we use the term "everything" in a somewhat ambiguous sense. If I answer a question regarding the whereabouts of some person: "He is nowhere around," I do not mean that he is nowhere at all (nowhere conceivable) but merely that he is nowhere in the building or nowhere in the neighborhood, or city. That is, my "universe" or "everywhere" may vary considerably, from a room to a country or the world.

In the same manner, when we say that a person is "not alive," we do not mean that he is possibly anything whatsoever except a living being. We do not mean that he may be a stone or a glass of water. We mean that he is something not living in that "universe" composed of beings who have lived sometime or other. Our "universe of discourse," the "everything" about which we are talking, is not "everything whatsoever," then. Again when we say "What is not less than or equal to zero is plus," we are talking only about numbers; we do not mean by "what" anything whatsoever (anything conceivable), but rather "any number"; i.e., our "universe of discourse" is made up solely of quantities or numbers.

(See Exercises, Group A, at end of chapter.)

Throughout this discussion and in the exercises we have made use of the expression "What is not something or other." This concept is very important in the "algebra" of classes, and we may symbolize the expression "what is not a" or "non-a" by a'. Then it is apparent that the classes a and a' are such that though they have no members in common, together they make up the universe. For example, what is white and what is not white together make up the universe of shades. Thus a' can be conceived as "what remains in the universe when a is removed." There is an analogy here between the class non-a and the expression "p is false," or p' in the logic of propositions, for "p is true" and "p is false" (i.e., the contradictory of p) were those expressions which were such that, though they could not both be true at once, together they exhausted all possibilities.

The class non-a also bears another analogy to the proposition "p is false." It is to be remembered that the expression "It is false that p is false" was equivalent to "p is true." Similarly, a'' or non-(non-a), "What is left in the universe when non-a is removed," is the class a. That is $a = a''$. The complete analogy breaks down, of course, since non-a is a class and "p is false" is a proposition. For example, though we can assert as valid "Either p is true or p is false," we cannot say "Either a or non-a," since the latter is not a statement at all but merely a class (the universe), and it is absurd to say that a class is true or false.

Non-a or a' also bears some analogy to the negative of a number in arithmetical algebra. On account of this, a' is often written $-a$ or $\bar{a}$, the "negation" sign being written on top in the latter; the notation a' is merely the result of placing the negation sign vertically. The analogy appears in a number of ways; for example, the expression $a = --a$ $(a = a'')$ is true in both the algebra of numbers and the "algebra" of classes.

Given the concept non-a or a' we can define the categorical form E(ab) in terms of A(ab). For obviously if "No *a* is *b*," then "All *a* lies in what isn't *b*," i.e., "All the members of *a* belong to non-*b*," and, conversely, if "All *a* lies in non-*b*," then "No *a* is *b*." Hence:

A(ab') is equivalent to E(ab).

"If all kings are unhappy, then no king is happy, and, conversely, if no king is happy, then all kings are unhappy (where 'unhappy' means 'non-happy')."

This principle is known as "obversion." It may also be given:

A(ab) is equivalent to E(ab').

"If all men are fools, then no man is sensible (non-foolish), and, conversely, if no man is sensible, then all men are fools."

Similarly, we may also assert that

O(ab') is equivalent to I(ab).

E.g., "Some men are not liars (non-truthtellers)" is equivalent to saying "Some men are truthtellers."

Or, we can assert that

I(ab') is equivalent to O(ab).

"Some books are hard to read" is equivalent to "Some books are not not-hard," or, "Some books are not easy."

Other relations which arise are:

A(ab) is equivalent to A$(b'a')$.

This is called the principle of "contraposition": "All Romans were brave" is equivalent to "Anyone who is not brave is not a Roman."

Similarly (and consequently):

O(ab) is equivalent to O$(b'a')$:

"Some thieves are not honest" is equivalent to "Some non-honest (dishonest) people are (not non-) thieves."

Note that the principle of contraposition does not hold for the E- and I-forms, just as the principle of conversion (simple reversal of terms) does not hold for the A- and O-forms.[1] E.g., the

[1] Indeed, as the diagrams show, the fact that E and I are not contraposable depends on the fact that A and O are not convertible.

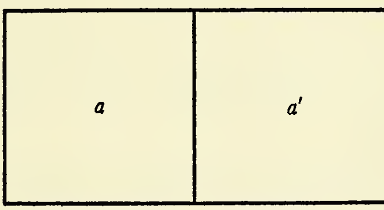
sentence "No squares are circles" is not equivalent to "No non-circle is a non-square," for the former is true, but the latter is false. The principles of contraposition can also be expressed:

A $(a'b')$ is equivalent to A (ba)
O $(a'b')$ is equivalent to O (ba),

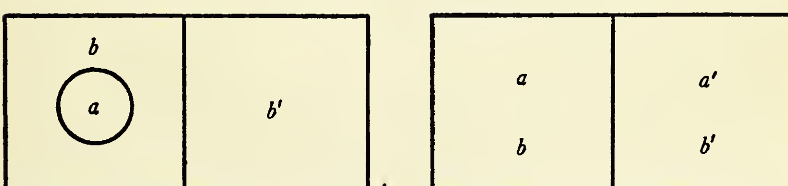
and also (since $a'' = a$):

A $(a'b)$ is equivalent to A $(b'a)$
A (ab') is equivalent to A (ba')
O $(a'b)$ is equivalent to O $(b'a)$
O (ab') is equivalent to O (ba').

Interesting with respect to the above is the extension of Euler's diagrams. Let us represent by a rectangle the universe of discourse. Then the relation between *a* and *a'* will be seen from this diagram:



A (ab) now can be represented by one of the following two diagrams:



A (ab)

where the equivalence of A (ab) and E (ab') is obvious. (Note that *a'* in the first diagram is everything in the square except the circle enclosing *a*.) The new diagrams for I (ab) and O (ab) may be constructed in a like manner.

(See Exercises, Group B, at end of chapter.)

It is a fact that the introduction of the "universe class" and the concept of "non-*a*" into the algebra simplifies this deductive system considerably. For example, we can reduce the postulates for the valid moods of the syllogism to one by use of obversion. Let us see how this is possible.

We take as undefined in the system the terms *a, b, c,* etc. (classes), the concept "non-*a*," or a', and A (ab). We then define the expression E (ab) by the principle of obversion:

E (ab) means A (ab').

I (ab) and O (ab) are defined already, the former being the contradictory of E (ab), the latter the contradictory of A (ab).

We postulate the original two postulates for Immediate Inference and the same axioms are assumed as above. But in the case of the syllogism we merely postulate Barbara:

POSTULATE 3. A (ba) A (cb) implies A (ca).

Since this is a postulate of the system, it must hold for *all a*'s, *b*'s, and *c*'s, that is, it must hold no matter what *a, b,* or *c* may be. Suppose now *a* is the "negative" of some class, i.e., suppose *a* represents "what isn't something else," say d'. Then in this case we can assert as true

A (bd') A (cb) implies A (cd').

But by definition A (bd') or "all *b* is non-*d*" is E (bd) and A (cd') is E (cd):

E (bd) A (cb) implies E (cd)

and this is EAE in the first figure, or Celarent.

The great simplification which occurs when "non-*a*" is introduced lies in the postulates for the invalid moods. For instead of the eleven postulates required above for the invalid moods of Immediate Inference and the Syllogism, here we require but one. This simplification, known as postulational economy, occurs quite frequently when new concepts are introduced into the system.

Simplification of the Aristotelian Class Calculus

INDEFINABLES:[2] 1. The "terms" ("classes") *a, b, c,* etc.[3]
2. A (ab) ("all *a* is *b*")
3. a' ("non-*a*")

---

[2] Cf. p. 7.
[3] More precisely, the indefinable is a class of elements K (which may be interpreted as the class of all classes).

DEFINITIONS:
1. E (ab) means A (ab')
2. O (ab) means [A (ab)]′ ("A (ab) is false")
3. I (ab) means [E (ab)]′ ("E (ab) is false")

POSTULATES:
1. A (aa) ("all a is a")
2. A (ab) implies I (ab)
3. I (ab) implies I (ba)
4. A (ba) A (cb) implies A (ca) (Barbara)
5. $a = a''$ "The class a is equivalent to non-(non-a)." (Here the symbol "=" has the significance that whenever a'' occurs in an expression we may replace it by a and vice versa.)
6. A (ab) A (cb) does not imply I (ca). (I.e., AAI in the second is an invalid mood.) [4]

The axioms mentioned for the valid moods are likewise assumed here and hence we can easily derive all the valid moods of Immediate Inference and the syllogism, since Celarent can be proved in the manner indicated above, and this plus Postulates 2, 3, and 4 is sufficient.

But the invalid moods require some additional axioms if they are all to follow from Postulate 6:

AXIOM 12. If a certain premise in an argument is an established truth, this premise may be "suppressed."

That is, if pqr . . . imply w, and p is actually true, then we may say the qr . . . imply w. The meaning of this axiom is best seen in the case of two premises. Suppose we have: "If p and q are true, then r is true." Suppose that p is granted; then the truth of r depends solely on q. That is, "If q is true, then r is true," and we have suppressed the first premise. Obviously, if all the premises are true, we suppress them all and merely assert r by itself (cf. page 31, the discussion of the Constructive Hypothetical Syllogism).

AXIOM 13. If a true statement implies or is equivalent to another statement, the latter is true. (This is the Constructive Hypothetical Syllogism.)

AXIOM 14. A true statement never implies a false statement; or, if a statement leads to (implies) a false statement, then the former cannot be universally true. (This axiom is the back-

[4] We might add, for the sake of "completeness":
7. If A (ab) and A (ba), then $a = b$.

bone of the method of *reductio ad absurdum* used later in the book.)

AXIOM 15. In any expression containing such variables as a, b, c, etc., these variables may be replaced by any other letters throughout, or the "primes" of any other letters (a', b', c') without altering the validity or the invalidity of the expression.

Thus Postulate 4 can be written "A (dc) A (bd) implies A (bc)," where c has been written for a throughout, d for b, and b for c; this could also be written "A $(b'a)$ A $(c'b')$ implies A $(c'a)$," or "A (fa') A (cf) implies A (ca')," etc. This axiom asserts the universal quality of the postulates and will be used implicitly.

THEOREMS: (We require several preliminary "lemmas"):

THEOREM 1. I (aa) ("some a is a.")

PROOF: A (ab) implies I (ab). (Postulate 2).
Hence, if we let b have the value of a, we have

$$\text{A}(aa) \text{ implies I}(aa).$$

But A (aa) is true (Postulate 1). Hence

$$\text{I}(aa) \text{ is true (Axiom 13).} \quad \text{Q.E.D.}$$

THEOREM 2. A $(ab) =$ E (ab')

PROOF: A $(ab') =$ E (ab). (Definition 1)
Hence, replacing b by b':

$$\text{A}(ab'') = \text{E}(ab'),$$

or, by Postulate 5, since $b'' = b$,

$$\text{A}(ab) = \text{E}(ab') \quad \text{Q.E.D.}$$

THEOREM 3. E (aa') ("No a is non-a.")

PROOF: A $(ab) =$ E (ab') (Theorem 2.)
Hence, letting $b = a$,

$$\text{A}(aa) = \text{E}(aa').$$

Hence, by Postulate 1 and Axiom 13, E (aa') is true. Q.E.D.

THEOREM 4. O (aa')

THEOREM 5. E $(a'a)$ ("No non-a is a.")

(Here we make use of Theorem 3 and the fact that E (ab) implies E (ba), the valid moods of Immediate Inference and the syllogism being proved already.)

THEOREM 6. O $(a'a)$

THEOREM 7. A (aa') is false, or $[\text{A}(aa')]'$.

PROOF: $\text{O}(ab) = [\text{A}(ab)]'$

Hence, replacing b by a', we have

$$\text{O}(aa') = [\text{A}(aa')]'.$$

But O (aa') is true (Theorem 4) and therefore (Axiom 13) $[\text{A}(aa')]'$, or "All a is non-a is false."

THEOREM 8. I (aa') is false, or $[\text{I}(aa')]'$. (This follows from the definition of I (ab) and Theorem 3.)

THEOREM 9. A $(a'a)$ is false, or $[\text{A}(a'a)]'$.

THEOREM 10. I $(a'a)$ is false, or $[\text{I}(a'a)]'$.

THEOREM 11. E (aa) is false, or $[\text{E}(aa)]'$.

PROOF: $[\text{E}(ab)]' = \text{I}(ab)$, and if $b = a$, then the theorem follows immediately by Theorem 1 and Axiom 13.

THEOREM 12. O (aa) is false, or $[\text{O}(aa)]'$.

We are now in a position to prove the invalid moods of Immediate Inference. For example:

THEOREM 13. A (ab) does not imply O (ba).

PROOF: We prove the theorem by showing that there is (at least) one case where the implication "A (ab) implies O (ba)" is false. For suppose that $b = a$ in this implication: "A (aa) implies O (aa)." But this latter is a false proposition, since the premise is true (Postulate 1) and the conclusion false (Theorem 12). Hence, by Axiom 14, AO_2 reduces to a false proposition and hence is not true in general.

In a similar fashion, we can prove invalid all forms of Immediate Inference in which we have an affirmative form implying a negative form. The converses can also be shown false. E.g.:

THEOREM 14. E (ab) does not imply I (ab).

Here we *cannot* let $b = a$, for this reduces E (ab) to a false proposition and we cannot infer anything.[5] But if $b = a'$, then E (ab) or E (aa') is true (Theorem 3), but I (ab) or I (aa') is false

[5] That is, even though we reduce the expression to the case of a false premise implying a true conclusion, the resulting form is not necessarily false; indeed, it is usually considered true. (Cf. Chap. XII.)

(Theorem 8), and hence EI reduces to a form in which a true proposition implies a false one, and hence EI_1 is false.

The general procedure here is to give b the value a or a', according as this *makes the premise true* (and the conclusion false). *We must have the premise true,* for otherwise no falsity results.

There still remain those invalid moods of Immediate Inference which follow from "AA_2 is false." Hence, if we prove this, we have proved all the invalid moods of Immediate Inference by merely making use of the axioms in the original system. Obviously we cannot use the same method as above in proving this theorem, since if $b = a$, both premise and conclusion are true, and if $b = a'$, both premise and conclusion are false, and nothing follows. We must make use of Postulate 6:

THEOREM 15. A (ab) does not imply A (ba).

PROOF: Suppose A (ab) implies A (ba). Then in Postulate 6, "A (ab) A (cb) does not imply I (ca)," we can weaken A (ab) to A (ba), since by Axiom 11 (page 94), in any invalid mood either premise may be weakened, without altering the invalidity, and hence deduce:

"A (ba) A (cb) does not imply I (ca)."

But this is absurd, since this is AAI in the first, which is a valid mood of the syllogism. Hence the assumption that A (ab) implies A (ba) leads to a contradiction (Postulate 6 contradicts Postulate 2 and 4), and hence A (ab) does not imply A (ba).

The invalid moods of the syllogism may be derived either by reducing the syllogism as above in the case of Immediate Inference to a form in which a true proposition implies a false proposition or to an invalid form of Immediate Inference, or else by employing Postulate 6. As an example of the first method, we take a syllogism in which two affirmative premises imply a negative conclusion:

THEOREM 16. A (ba) I (bc) does not imply E (ca).

PROOF: In the given syllogism, let $a = b = c$:

A (aa) I (aa) implies E (aa).

But now both the premises are true (Postulate 1 and Theorem 1), but the conclusion is false (Theorem 11). Hence AIE_3 is false in this case and hence

A (ba) I (bc) does not imply E (ca) Q.E.D.

Again, take

THEOREM 17. A (ab) A (bc) does not imply A (ca).

PROOF: Suppose A (ab) A (bc) does imply A (ca). Now let $a = b$:

A (aa) A (ac) implies A (ca).

But A (aa) is a true premise, and by Axiom 12 may be suppressed:

A (ac) implies A (ca),

which is false by the theorem already proved in Immediate Inference. Hence AAA_4 implies a false proposition and by Axiom 14 must itself be false.

In order to show that these postulates are sufficient to prove all the invalid moods we shall prove that the seven postulates for the invalid moods of the syllogism given above are theorems of this system. With these as theorems we can prove the remaining moods by Axioms 10 and 11 above (page 94), though it will often be simpler to apply the present method, especially in cases where two affirmative premises imply a negative. (Not all the invalid moods follow by the method of reduction described above. Some of the moods must be proved from Postulate 6. Thus Numbers 11 and 12 below must be proved by the latter method.)

POSTULATE 9. A (ba) A (cb) does not imply O (ca).

PROOF: The proof follows simply by letting $a = b = c$, and applying Postulate 1, Theorem 12, and Axiom 14.

POSTULATE 10. A (ba) E (cb) does not imply I (ca).

PROOF: The proof follows by suppressing the major premise by letting $b = a$:

A (aa) E (ca) implies I (ca),

which, by Axiom 12, reduces to

E (ca) implies I (ca),

a false mood of Immediate Inference. Hence by Axiom 14 we infer the invalidity of AEI_1.

POSTULATE 11. A (ba) E (cb) does not imply O (ca).

PROOF: A (ab) A (cb) does not imply I (ca) (Postulate 6). Hence, contradicting minor and conclusion, we infer

A (ab) E (ca) does not imply O (cb),

which is the required AEO in the first.

POSTULATE 12. E (ba) E (cb) does not imply I (ca).

PROOF: A (ab) A (cb) does not imply I (ca) (Postulate 6). Hence, by definition:

E (ab') E (cb') does not imply I (ca), or

E $(b'a)$ E (cb') does not imply I (ca)

(weakening E (ab') to E $(b'a)$ by Axiom 11); that is, we have shown that EEI_1 is not a valid mood in general.

POSTULATE 13. O (ba) A (cb) does not imply O (ca).

PROOF: We prove first that AAA_3 is an invalid mood; suppose

A (ba) A (bc) implies A (ca).

Now let $b = a$:

A (aa) A (ac) implies A (ca),

or, by Axiom 12 and Postulate 1:

A (ac) implies A (ca),

which is invalid by a theorem of Immediate Inference. Hence, A (ba) A (bc) does not imply A (ca). Now contradict and interchange major and conclusion of this, and by Axiom 10 we infer that

O (ca) A (bc) does not imply O (ba),

or that OAO_1 is an invalid mood. Q.E.D.

POSTULATE 14. A (ab) A (bc) does not imply A (ca).

PROOF: The proof follows simply by letting $b = a$.

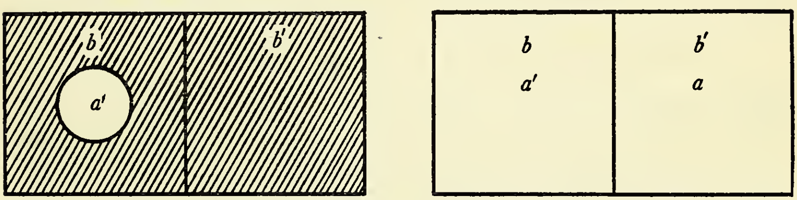
POSTULATE 15. A (ab) A (bc) does not imply O (ca).

PROOF: Let $a = b = c$.

(See Exercises, Group C, at end of chapter.)

We might now ask whether any categorical form which contains a "primed" term, i.e., a term such as a' or b', can be changed into a categorical form in which this is not the case. For example, E (ab') can be changed to A (ab), which has no primed terms, A $(a'b')$ to A (ba), and so forth. If this were the case, our categorical forms would be "complete," but it is not, for A $(a'b)$ (or E $(a'b')$), "All non-*a* is *b*" cannot be reduced further. Hence more categorical forms are required for the sake of this completeness. If we let U (ab) represent "All non-*a* is *b*" and let V (ab) be its contradictory: "some non-*a* is not *b*" ([U (ab)]$'$), then the categorical forms will be complete. The verification of this appears below.

The new categorical form U (ab) may be represented by Euler's diagrams as follows:



U (ab)

where the shaded portion in the first diagram is the area for *a*. V (ab) may be represented by the diagrams for I (ab), where the *a* and *b* in the latter are replaced by a' and b'; for V (ab) = I $(a'b')$, as may readily be gathered from the above reading of V (ab).

A peculiar fact arises with respect to U (ab) and V (ab); for when these categorical forms are added to the other four, and we determine the new valid moods of Immediate Inference and the syllogism, the old Rules for Invalidity in both cases are no longer entirely consistent.

In order to show this, let us first discover what terms in U (ab) and V (ab) are distributed. In U (ab) or "all non-*a* is (some) *b*" the predicate is not distributed. That the subject is not will appear when we note that U (ab) implies U (ba), as may be seen from the diagrams above. That is, we may reverse the order of the terms in U. Hence we may read for U (ab) "all non-*b* is (some) *a*" (note that the form of U is "all non-____ is ____," the "non" belonging to the form and not to the term); this clearly shows that the subject also is not distributed in U. In V (*ab*) both terms are distributed; for V (*ab*), "Some non-*a* is not *b*" is equivalent to O (*a'b*), where the predicate, *b*, is distributed. But since V (*ab*) is equivalent to V (*ba*),[6] we can assert that *a* is also distributed. The following is a valid mood of Immediate Inference:

"U (*ab*) implies O (*ab*)"
"If all non-*a* is *b*, then some *a* is not *b*."

Here *b* is distributed in the conclusion, but not in the premise, and hence Rule 3 of the Rules for Invalidity of Immediate Inference is inconsistent.

Again, in the case of the syllogism, the following valid moods demonstrate the invalidity of Rule 2, "Two negative premises do not imply any conclusion":

U (*ba*) E (*cb*) implies A (*ca*)
"If all non-*b* is *a* and no *c* is *b*, then all *c* is *a*."

Here U and E are both "negative," i.e., contain the word "not" (or "non-") an odd number of times (or, more precisely, when the terms are identified, they become false), and by the rule in question they should not imply any conclusion. Hence this rule is inconsistent. There are actually many such cases of valid syllogisms in which both premises are negative when the U- and V-forms are added: for example, EEV in all figures; EOV in what figures?

Again, Rule 4, "The middle term of a valid mood of the syllogism must be distributed at least once," is shown inconsistent by the following mood:

A (*ab*) U (*cb*) implies O (*ca*),
"If all *a* is *b* and all non-*c* is *b*, then some *c* is not *a*."

Rule 5, "A term distributed in the conclusion must be distributed in its premise" is shown inconsistent by the following valid mood:

A (*ba*) U (*cb*) implies O (*ca*),
"If all *b* is *a* and all non-*c* is *b*, then some *c* is not *a*."

Here *a* is distributed in the conclusion, but is not distributed in the major premise.

---

[6] "Some non-*a* is non-*b*" is equivalent to "Some non-*b* is non-*a*."

It might be thought that with the introduction of the U- and V- forms additional postulates were necessary. This is not the case. For example, the valid mood:

$$U\,(ab) \text{ implies } U\,(ba)$$

follows by the definition of U (ab), and the postulates. For

$$U\,(ab) = E\,(a'b').$$

This may be shown in the following way:

$$E\,(ab') = A\,(ab).$$

Hence, if we write a' instead of a, we have

$$E\,(a'b') = A\,(a'b) = U\,(ab).$$

Now E (ab) implies E (ba), and if we write for a and b, a' and b' respectively, we derive

E $(a'b')$ implies E $(b'a')$, or, by the above,
U (ab) implies U (ba).

In a similar manner we can prove the valid mood

$$U\,(ab) \text{ implies } O\,(ba).$$

For A (ab) implies I (ba). If, now, in place of a we write a', we have

$$A\,(a'b) \text{ implies } I\,(ba').$$

But A $(a'b)$ is U (ab) and I (ba') is O (ba); hence it follows that U (ab) implies O (ba).

From UU_2 and UO_2 we can prove the remaining valid moods of Immediate Inference (the moods UU_1 and VV_1 being proved in the same manner as AA_1 above). The required additional moods are, besides the original ten, AV_1, AV_2, VV_2, UO_1, VV_1, and UU_1.

In the case of the syllogism, AAA_1, Barbara, is sufficient for the proof of all the valid moods. From Barbara,

$$A\,(ba)\;\; A\,(cb) \text{ implies } A\,(ca),$$

we derive Celarent by writing a' for a as described above. If now, we write b' for b in Barbara, we have:

$$\mathrm{A}\,(b'a)\ \ \mathrm{A}\,(cb')\ \text{ implies } \mathrm{A}\,(ca),$$

or, since A $(b'a)$ is U (ba), and A (cb') is E (cb):

$$\mathrm{U}\,(ba)\ \ \mathrm{E}\,(cb)\ \text{ implies } \mathrm{A}\,(ca).$$

Again, writing c' for c in Barbara, we have:

$$\mathrm{A}\,(ba)\ \ \mathrm{A}\,(c'b)\ \text{ implies } \mathrm{A}\,(c'a), \text{ or}$$
$$\mathrm{A}\,(ba)\ \ \mathrm{U}\,(cb)\ \text{ implies } \mathrm{U}\,(ca).$$

From these four derived moods follow all the valid moods of the syllogism.

The invalid moods of Immediate Inference and the Syllogism are proved by a similar extension or else by the method of suppressing a premise already discussed.

(See Exercises, Group D, at end of chapter.)

Rules 1 and 3 of the syllogism are true even when the U- and V-forms are introduced. In fact, they are special cases of far more general rules which apply to forms which have any number of premises. This discussion aims at making clear the meaning of "generalization" or "generality" in a deductive system. The process of generalizing a formal science is quite common. For example, the plane geometry studied in most high-schools is a "special" case of the solid geometry studied afterwards. In other words, in the latter case we have extended the scope of our knowledge beyond the mere plane to solids such as spheres and cones.

The theorems of plane geometry still hold in solid geometry, provided they are prefixed with the clause: "If the following conditions take place on a plane." But many of the theorems of solid geometry are meaningless in plane geometry, because the concepts which the latter considers are so restricted. Thus, for example, we could not define a cube in plane geometry, for this requires the concept of planes cutting each other, a concept which does not appear here. This suggests the possibility that the three dimensions of solid geometry may also be restricted, and that there are geometries which consider four, five, six, n dimensions. This is actually the case: there are geometries which consider space (not picturable by us) of any number of dimensions.

As an example in another field, consider the law of gravitation. For centuries people had known that certain bodies (the "heavy" ones) fall, and certain bodies do not. But no one had been able to "generalize" his experience satisfactorily by giving the rule which would tell which bodies would fall and which would not. The attempt to discover such a law was the attempt to make the science of physics more general. Those acquainted with the theory of relativity will be aware that this theory aims specifically at generalizing the old laws of "classical" physics, which the scientists of the nineteenth century had considered the most general laws possible. Again, algebra might be thought of as a generalization of certain branches of arithmetic. For example, we discover in arithmetic that 4×3 is the same as $4 \times 4 - 4$, and that 8×9 is the same as $9 \times 9 - 9$. But in algebra we generalize these results by asserting that no matter what the number x may be, $(x - 1)\,x = x^2 - x$.

In an exactly analogous fashion, we can generalize the deductive system of classes which we have been considering. For obviously we are not restricted to one or two premises only, but we may have any number of premises implying a conclusion. Forms which involve more than two premises are called "sorites," and the extension of the system to sorites is a generalization of the logic. As an example of the manner in which we might "build up" a chain of premises which validly imply a conclusion, take first the valid syllogism:

$$\mathrm{E}\,(ba)\ \ \mathrm{I}\,(cb)\ \text{ implies }\mathrm{O}\,(ca)\,.$$

We have the right to strengthen either premise of a valid syllogism and the resulting implication is still valid. But the premises A (db) I (cd) validly imply I (cb), and hence in place of I (cb) we may write these:

$$\mathrm{E}\,(ba)\ \ \mathrm{A}\,(db)\ \ \mathrm{I}\,(cd)\ \text{ implies }\mathrm{O}\,(ca)\,.$$

This is a (valid) sorites of three premises. Again, E (ba) may be strengthened to the premises E (ea) A (be) (EAE in the first figure):

$$\mathrm{E}\,(ea)\ \ \mathrm{A}\,(be)\ \ \mathrm{A}\,(db)\ \ \mathrm{I}\,(cd)\ \text{ implies }\mathrm{O}\,(ca)\,.$$

This last sorites may be tested by Euler's diagrams. This process may be continued indefinitely, and a valid sorites of any number of premises may be constructed. Note that in the sorites no term appears more than twice and that if a term appears twice in the premises it does not appear in the conclusion. In the special case of the syllogism there is but one case of the latter, the "middle" term.

But our principal interest lies in determining whether a given sorites is valid or not. The following rules indicate the method of discovering which are the valid moods of the sorites:

1. If all the premises of a valid sorites are affirmative (A, I, or V) then the conclusion is affirmative (i.e., if a sorites has all affirmative premises and a negative conclusion, then the sorites is invalid.) For suppose the conclusion of the sorites were negative, as in the case of the sorites,

$$\mathrm{A}\,(ba)\ \ \mathrm{A}\,(cb)\ \ \mathrm{V}\,(dc)\ \ \mathrm{I}\,(ed)\ \text{ implies }\ \mathrm{E}\,(ea)\,.$$

The method of proving this false is simple enough, for we merely identify the terms in all the premises:

$$(a = b = c = d = e):$$
$$\mathrm{A}\,(aa)\ \ \mathrm{A}\,(aa)\ \ \mathrm{V}\,(aa)\ \ \mathrm{I}\,(aa)\ \text{ implies }\ \mathrm{E}\,(aa)\,.$$

Here the premises are true but the conclusion is false; hence the sorites is invalid. Note that this rule generalizes the first rule of Immediate Inference: "An affirmative premise does not imply a negative conclusion," and the rule of the syllogism: "Two affirmative premises do not imply a negative conclusion."

2. If the conclusion of a valid sorites is universal (A, E, or U), then all the premises must be universal. This can best be seen in the case of affirmative premises. Suppose the sorites were

$$\mathrm{A}\,(ba)\ \ \mathrm{V}\,(cb)\ \ \mathrm{A}\,(cd)\ \text{ implies }\ \mathrm{A}\,(da)\,.$$

This is easily shown invalid by identifying terms in all the premises except the particular one [V (cb)]: (let $b = a$ and $d = c$):

$$\mathrm{A}\,(aa)\ \ \mathrm{V}\,(ca)\ \ \mathrm{A}\,(cc)\ \text{ implies }\ \mathrm{A}\,(ca)\,,$$

and the sorites reduces to

$$\mathrm{V}\,(ca)\ \text{ implies }\ \mathrm{A}\,(ca)\,,$$

an invalid mood of Immediate Inference.

3. If only one premise is negative, then the conclusion of the sorites must be negative. For suppose it were affirmative: e.g.,

A (*ba*) V (*cb*) E (*cd*) I (*de*) implies I (*ea*).

Then, by identifying the terms in all the affirmative premises (let $a = b = c$ and $e = d$), we derive:

A (*aa*) V (*aa*) E (*ad*) I (*dd*) implies I (*da*), or
E (*ad*) implies I (*da*),

an invalid mood of Immediate Inference. Note that this rule generalizes the Rule of Immediate Inference. "A negative premise does not imply affirmative conclusion," and the rule of the syllogism: "An affirmative and a negative premise do not imply an affirmative conclusion." [7]

(See Exercises, Group E, at end of chapter.)

Significance of the Aristotelian Class Calculus

This class calculus is usually and properly considered a branch of logic. The justification for this will be apparent when we recall that logic is defined as that branch of human science whose problem is to construct all propositions the truth of which is independent of the meaning of the terms. Thus the (true) proposition (of the science of biology) "No stones are animate" is not a proposition of logic, since its truth does depend on the meaning of its terms ("stones" and "animate"). In general, the truth of such an expression as A (*ab*), "all *a* is *b*" *depends* on what *a* and *b* are. But we have found statements about nouns or classes that are true no matter what the nouns may be, i.e., whose truth does *not* depend on (is independent of) the meanings of the terms. Thus "A (*ab*) implies I (*ab*)" is true and its truth does not depend on what we mean by *a* and *b*. The determination of all such propositions, then, is the central problem of logic. Hence, the separation of the moods of Immediate Inference, the syllogism, and the sorites into true and false may be properly thought of as answering a central problem of logic.

It can also be seen in what sense logic is the basic science. For

[7] These rules are by no means complete. For the most extended (and actually complete) study of the sorites, cf. H. B. Smith, "Abstract Logic" in *A First Book in Logic,* F. S. Crofts and Co., 1938.

all sciences must make use of terms or nouns, and hence the most general laws concerning terms must be the basic part of any science. Indeed, if we ask what an "object" or a noun is, the best answer that can be given is that it is anything satisfying this logic.

EXERCISES

GROUP A

1. Name the universe of discourse in the following statements:

a) Zero is less than anything.
b) Anything that goes up must come down.
c) What is not mortal is immortal.
d) What is not black or white has color.
e) What is not hopeless can be attained.
f) What is not right is wrong.
g) Wine, women, and song make up the universe.
h) "There is nothing but meeting and parting in this world."

GROUP B

1. Obvert and contrapose the following propositions:

a) All men are mortal = No men are immortal = All immortals are non-men.
b) Some men are not unhappy.
c) No man is honest.
d) Some theories are impractical.
e) All unusual things are interesting.
f) Some uncouth people are dishonorable.
g) What is not a color is not a shade.
h) No good soap is impure.
i) Unfriendly people are unsocial.

2. Determine the truth or falsity of the following syllogisms:

EXAMPLE: I (ab) A (cb') implies O $(c'a')$; Now A (cb') = E (cb), and O $(c'a')$ = O (ac), and hence the given syllogism becomes: I (ab) E (cb) implies O (ac); or, rearranging premises: E (cb) I (ab) implies O (ac), which is valid.

a) A (ab') A (cb) implies E (ca)
b) A $(a'b')$ I (bc) implies O (ca')
c) O (ba') E (bc') implies O (ca)
d) A (ba') A (cb') implies O (ca')

e) A $(a'b')$ A $(b'c')$ implies A $(c'a')$
f) E (ab') A $(c'b')$ implies I (ca')
g) A $(b'a)$ E (cb') implies O $(c'a)$
(Hint: Replace a by a' throughout)
h) E $(b'a')$ O (cb') implies I $(c'a')$
i) E $(a'b')$ I $(b'c')$ implies O $(c'a')$

3. By methods similar to those described, we may often reduce concrete syllogisms which would appear to have four terms to those having the required three terms. E.g.,

"All cows are contented;
all men are discontented;
therefore, no men are cows."

Here let $a =$ "cows," $b =$ "contented (things)," $c =$ "men"; but the term "discontented," or "non-contented," is b':

A (ab) A (cb') implies E (ca),

or, since A (cb') is E (cb):

A (ab) E (cb) implies E (ca),

which is a valid syllogism.

Determine the validity or invalidity of the following:

a) The Mongolians living in the United States are few in number, but the Japanese living there are plentiful; therefore, no Japanese in the United States is a United States Mongolian.
b) All acids are compounds; hydrogen is an element, and hence is no acid.
c) No man is omniscient, but some things which are not men are divine; hence all omniscient beings are divine.
d) Some climates are unhealthy, but no climate is worse than that at the North Pole; hence, there are some places other than the North Pole which are healthy.
e) All besides his band of unvirtuous friends were enemies of Cataline. Cajus was not unfriendly; is it not logical to suppose, then, that Cajus was in this band?
f) Dictionaries are useful; useless things are valueless; therefore, dictionaries are valuable.
g) No misers are unselfish; some selfish beings are unhappy; hence some misers are not happy.
h) Some healthy people are fat; no unhealthy people are strong; hence, some fat people are not strong. (Carroll)
i) All uneducated people are shallow, but students are all educated, and hence the latter cannot be shallow.

j) Improbable stories are incredible; none of his stories are probable; hence none of his stories are credible.
k) All dishonorable people are dishonest. Some impractical people are honorable. Hence some honest people are not practical.
l) Only meaningful propositions are true; everything he says is meaningless; hence, nothing he says is true.

4. What conclusions (if any) may be drawn from the following premises?

a) The things in Plato's World of Ideas are perfect; the things in the world of sense are imperfect.
b) Only the poor are happy; Brown is rich.
c) All planets revolve around the sun in ellipses; comets follow a path other than an ellipse.
d) Some unvirtuous men are unhappy; all saints are happy.
e) All rational numbers are expressible in the form of a fraction; no irrational number is evenly divisible into 2.
f) All pairs of lines which are such that, if they are cut by a transversal, the sum of the interior angles is less than two right angles, meet. All parallel (non-intersecting) lines meet at infinity.
g) Umbrellas are useful on a journey; what is useless on a journey should be left behind. (Carroll)
h) Sandwiches are satisfying. Nothing in this dish is unsatisfying. (Carroll)
i) All wise men walk on their feet; all unwise men walk on their hands.

5. We call a relation R "contra-reflexive" if, when $a\text{R}b$, $a\text{R}a'$; for example, E (ab) is contra-reflexive, since E (aa') holds. Which of the following relations have this property: A (ab), I (ab), O (ab), "lies inside of," "lies outside of," "is less than," "is either greater or less than," "is not the same as"?

GROUP C

1. Prove by the postulates given above that the following moods of Immediate Inference are invalid: OA_1, EI_2, AE_2, OI_1, IE_2, OE_1.
2. Show by the postulates that the following moods of the syllogism are invalid: IIO_1, III_3, AIA_2, EII_4, AIO_2, AEE_1, OIA_3, OOO_2, EOA_4, IOA_3, IOO_2, EOI_3, EEE_3, EAA_4, AAO_3, AEA_2, IAI_1, OEE_3, IEO_2.
3. Prove the principle of contraposition:

$$\text{A}(a'b') = \text{A}(ba).$$

(*Hint:* Prove (1) A $(a'b')$ = E $(a'b)$; (2) making use of the fact that E $(a'b)$ = E (ba'), i.e., simple conversion of the terms in E, show that A $(a'b')$ = E (ba'). The proof then follows by changing E (ba') back into an A-form.)

4. Prove O $(a'b')$ = O (ba).
5. Prove I (ab') = O (ab).
6. Prove O (ab') = I (ab).

GROUP D

1. From the postulates given here (page 103), prove all the valid moods of Immediate Inference, including those involving the U- and V-forms.
2. Given AV_1 and VV_2 as postulates in place of AI_1 and II_2, prove all the valid moods of Immediate Inference (keeping all the axioms intact) ; repeat the exercise for EO_2 and UU_2.
3. Prove by means of the above postulates the following valid moods of the syllogism: UEA, UEI, and UEV in all figures; UOI_2, UOI_1, UOI_4; OEV_3, OEV_1, OEV_4; EEV in all figures; AVV_2, AVV_1, AVV_4; AAV_1, AAV_2, AAV_4; VUO in all figures; AUO in all figures.
4. By means of the definitions of U (ab) and V (ab), and by the above postulates and theorems, prove the following:

   a) V (aa) "Some non-*a* is not *a*."
   b) U $(a'a)$ "All non- (non-*a*) is *a*."
   c) U (aa') "All non-*a* is non-*a*."
   d) [V (aa')]′ "It is false that some non-*a* is not non-*a*."
   e) [V $(a'a)$]′ "It is false that some non- (non-*a*) is not *a*."
   f) [U (aa)]′ "It is false that all non-*a* is *a*." Thus V is an affirmative form and U a negative (cf. page 56).

5. Show that the following moods of Immediate Inference are invalid: UA_1, VE_2, IU_2, AU_2, UI_1; UE_2 (hint: replace *a* by *a′* in the invalid mood AA_2) ; OU_1, OU_2, IV_1, IV_2, VI_1, VI_2, VA_1, VA_2.
6. Show that the following moods of the syllogism are invalid: VAO_1, VIA_2, OAV_3, UIV_4, UAE_2, VVV_2, IAU_3; UAO_2 (Hint: replace *a* by *a′* in Postulate 6, the invalid mood AAI_2) ; AUV_2, UUU_3, IUV_1, UIO_4; OOU_3, IIV_4, EEU_4, IUA_1.
7. Reduce the following to forms which have no primes. (This exercise establishes the completeness of the system.)

EXAMPLE: I $(a'b')$

Now I $(a'b')$ = [E $(a'b')$]′. But since E (ab') = A (ab), we have: E $(a'b')$ = A $(a'b)$ = U (ab). Hence, substituting, I $(a'b')$ = [U (ab)]′ = V (ab), we have reduced the given form to one without primes.

a) A $(a'b')$	*f*) O $(a'b')$	*k*) U $(a'b)$
b) I $(a'b)$	*g*) E (ab')	*l*) U $(a'b')$
c) I (ab')	*h*) E $(a'b)$	*m*) V $(a'b)$
d) O $(a'b)$	*i*) E $(a'b')$	*n*) V (ab')
e) O (ab')	*j*) U (ab')	*o*) V $(a'b')$

8. By suitably "priming" the proper terms in the valid moods of Immediate Inference and the Syllogism already proved, establish the validity of the following: (e.g., "E (ba) E (bc) implies V (ca)"; replace a by a' and c by c' in the valid mood "A (ba) A (bc) implies I (ca)," thus deriving A (ba') A (bc') $\angle$ I $(c'a')$, which, by definition, gives the required mood): UU_1, AV_2, UAU_4, VUO_4, UOI_1, EUI_4, UOI_2.
9. Certain nineteenth century logicians became interested in the problem of the "quantification of the predicate." That is, while Aristotle's A (ab) talks about all of the subject, it is not clear whether it refers to all of the predicate or merely some. Thus "All men are (all) rational animals" and "All men are (some) animals" are both examples of the categorical form $A\ (ab)$. Thus, in "quantifying" the predicate we ought to have two categorical forms in place of this one; these can be defined $\alpha\,(ab) = \mathrm{A}\,(ab)$ $A\ (ba)$, and $\beta\,(ab) = A\ (ab)\ \ O\ (ba)$. To make a complete set of categorical forms in which both subject and predicate are quantified, we require twelve forms, all of which are definable in terms of the six already given:

$$\alpha\,(ab) = \mathrm{A}\,(ab)\ \ \mathrm{A}\,(ba)$$
$$\beta\,(ab) = \mathrm{A}\,(ab)\ \ \mathrm{O}\,(ba)$$
$$\gamma\,(ab) = \mathrm{I}\,(ab)\ \ \mathrm{U}\,(ab)$$
$$\delta\,(ab) = \mathrm{I}\,(ab)\,\mathrm{O}\,(ab)\ \ \mathrm{O}\,(ba)\ \ \mathrm{V}\,(ab)$$
$$\varepsilon\,(ab) = \mathrm{E}\,(ab)\ \ \mathrm{U}\,(ab)$$
$$\theta\,(ab) = \mathrm{E}\,(ab)\ \ \mathrm{V}\,(ab)$$

The remaining forms are derived by taking the negatives of these six; thus $[\beta\,(ab)]'$ is $[\mathrm{A}\,(ab)\ \ \mathrm{O}\,(ba)]'$, or, by De Morgan's Law, "Either O (ab) or $A\ (ba)$." These forms completely quantify subject and predicate, since if any other forms are added, these are either redundant or make the whole expression an absurdity, and there are no other forms of this type. Thus if we try to further quantify the predicate of $\varepsilon\,(ab)$ by adding O (ba), we have E (ab) U (ab) O (ba), which is equivalent to E (ab) U (ab), since O (ba) is already contained in both E (ab) and U (ab); if we try to quantify the subject of $\delta\,(ab)$ by adding A (ab), the whole expression becomes false since it contains two contradictory elements, O (ab) and A (ab).

a) Show that the set of forms $\alpha(ab)$, $\beta(ab)$, . . . $\theta(ab)$ is "complete" in the sense used above by showing that "priming" one or both elements in any form always gives another form of the set; thus $\alpha(ab') = A(ab')$ $A(b'a) = E(ab)$ $U(ba) = \varepsilon(ab)$; $\beta(a'b') = A(a'b')$ $O(b'a') = A(ba)$ $O(ab) = \beta(ba)$.

b) Find all true cases of Immediate Inference between any two of the six forms $\alpha(ab)$, $\beta(ab)$, etc., in both figures, and prove these from Postulates 1-7 above.

c) What conclusions, if any, may be drawn from the following premises? Prove the resulting syllogisms:

(1) $\alpha(ab)$ $\epsilon(bc)$
(2) $\beta(ab)$ $\beta(cb)$
(3) $\beta(ab)$ $\beta(bc)$
(4) $\gamma(ab)$ $\theta(bc)$
(5) $\varepsilon(ab)$ $\theta(bc)$
(6) $\alpha(ba)$ $\delta(bc)$
(7) $\theta(ab)$ $\delta(cb)$

GROUP E

1. Prove the validity of the following sorites by constructing them from valid moods of the syllogism (by strengthening the premises of the latter in the manner just indicated):

a) A (*ce*) A (*eb*) E (*ad*) A (*bd*) implies E (*ca*).
b) A (*ba*) I (*db*) A (*dc*) implies I (*ca*).
c) A (*ab*) A (*bc*) E (*cd*) I (*ed*) implies O (*ea*).
d) E (*ba*) U (*cb*) E (*dc*) implies E (*da*).

2. Determine whether the following sorites are valid or not either by diagram or the above rules:

a) A (*ab*) E (*cb*) I (*dc*) implies I (*da*).
b) E (*ba*) I (*cb*) A (*cd*) O (*de*) implies O (*ea*).
c) A (*ab*) I (*bc*) A (*cd*) implies I (*da*).
d) A (*ba*) I (*bc*) A (*cd*) implies I (*da*).
e) O (*ab*) A (*bc*) I (*cd*) implies U (*da*).
f) A (*ba*) A (*cb*) A (*dc*) implies V (*da*).
g) E (*ba*) A (*cb*) I (*dc*) A (*de*) implies O (*ea*).

3. Prove (by the method of reduction) that the following sorites are invalid:

EXAMPLE: U (*ba*) O (*cb*) E (*dc*) implies O (*da*).
Here we suppress the U-premise by letting $a = b'$:

U (bb') O (cb) E (dc) implies O (db'); or, since U (bb') is a true premise, this reduces to:

O (cb) E (dc) implies O (db').

But O (db') is I (db):

O (cb) E (dc) implies I (db),

which is an invalid mood of the syllogism; hence the original sorites, since it implies this invalid mood in a special case, is not true in general.

a) I (ba) E (cb) V (cd) implies V (da).
b) A (ba) I (cb) V (cd) implies O (da).
c) E (ba) E (cb) E (dc) implies E (da).
d) A (ba) U (cb) U (cd) I (de) implies O (ea).
e) I (ab) V (cb) E (dc) U (ed) A (ef) implies O (fa).
f) U (ba) U (cb) U (cd) implies U (da).
g) V (ba) O (cb) I (cd) implies O (da).

4. Criticize the following arguments by determining whether the sorites is valid or not in each case (where necessary, supply the missing premises):

a) (1) The purpose of life is happiness.
(2) Happiness has as its condition prudence.
(3) Prudence consists in doing one's duty.
(4) Duty is an action which is eminently reasonable.

Therefore, the purpose of life is to do that which is eminently reasonable.[8]

b) The purpose of everyone in life is the acquirement of that which he most desires. Everyone desires pleasure, and what everyone desires most of all is his own pleasure. Hence the purpose of each and every one is his own pleasure (The Sorites of the Egoistic Hedonist).

c) The only absolutely good thing is a good will.
The good will is a will which legislates for itself (autonomous will).
The self-legislating will obeys the categorical imperative.
Therefore, the only absolutely good thing is a will which obeys the categorical imperative (The Sorites of Kantian Ethics).

d) That which everyone desires is power. The measure of power is the measure of one's chances of gaining a certain end. The increase in power is proportional to the increase in morals, art,

[8] The Stoic Sorites.

and science. Therefore, that which everyone desires is goodness, beauty, and truth.

e) No houses are fireproof; no boots are waterproof; but some fireproof things are not waterproof; hence, some houses are not boots.

f) No system of government is perfect, but all systems, no matter how bad, have some good qualities; now anything which is good at least in part should not be wholly condemned; hence some imperfect things are not to be wholly condemned.

g) No examinations are conclusive. But, as a matter of fact, all things in life are inconclusive; it seems plain, though, that examinations are less conclusive than most things; hence, examinations are not worth while.

Applications of Logic to Proofs of Theorems

6

In the previous chapters we have developed the fundamental laws of the logic of classes and the logic of propositions. We have also indicated the manner in which the laws of the latter may be applied to the postulates of the former to derive new theorems concerning classes. Thus, assuming as a postulate of the logic of classes the law that "All *a* is *b* implies some *a* is *b*," we may apply the law from the logic of propositions to the effect that if one proposition implies another, the contradictory of the second implies the contradictory of the first, and deduce a new law of the logic of classes: "No *a* is *b* implies some *a* is not *b*."

In this chapter we show how both branches of logic may be applied to another science, plane geometry, in the proof of theorems. This will substantiate what we shall say later regarding the relation between logic and the remaining sciences.

We take Euclid's postulates, previously enumerated (page 12), and show what theorems may be deduced from these by logical principles alone. To the five postulates listed, we add three more, required as assumptions for a strictly formal system of geometry, though usually omitted in most elementary treatments of the subject. Two of these, the "congruence postulates," are usually taken to be theorems in geometry textbooks, but their proof involves concepts and assumptions not mentioned in the postulates. For example, take the side-angle-side theorem, to the effect that if two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, then the triangles will be equal in every respect. This is usually proved by "placing one triangle on another" so that the equal angles coincide. It is then inferred that the remaining vertices will coincide and hence the remaining side. Aside from the fact that the geometrical operation of placing one figure on another has not been defined or assumed as an indefinable, the proof assumes that equal angles and equal segments will coincide when placed on one another. It may be noted that Euclid was doubtful about the congruence theorems, since he avoided them whenever possible, e.g., in the demonstration that two angles and a side determine a triangle. To avoid the introduction of new concepts, we merely postulate the theorems proved by this method in most treatises on plane geometry: [1]

1. Between any two points a straight line can always be drawn.
2. A straight line may be produced in a straight line indefinitely.
3. With any point as center, a circle of any radius may be constructed.
4. All right angles are equal.
5. If a straight line cuts two straight lines in such a manner that the interior angles on the same side are less than two right angles, the lines will ultimately meet on this side if produced.
6. Two non-coincident straight lines meet in only one point.
7. A triangle is uniquely determined by two sides and the included angle; or, if two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are equal in every respect.
8. A triangle is uniquely determined by its three sides; or, if three sides of one triangle are equal respectively to three sides of another, the triangles are equal in every respect.
9. If a straight line cuts one side of a given (bounded) figure composed of straight lines, it either has but one point in common with the figure, or else cuts another side of the figure (where one line "cutting another" excludes the possibility of coincidence of the two lines).

In addition to the laws of logic, geometry also assumes arithmetical laws; that is, the science of geometry applies the concept of numbers to its elements. Thus we speak of the measure of a line or angle, meaning that certain numbers are associated with certain (fixed) angles and line-segments. The properties of numbers in general are developed by the science of arithmetic, and geometry assumes the right to apply these properties in its own field. Examples of laws commonly assumed are: "If equals be

[1] Only the postulates necessary for this discussion are given and these are insufficient for Euclidean plane geometry. For a fuller treatment see D. Hilbert, *Foundations of Geometry.*

added to or subtracted from equals, the results are equal," "If a is less than b, and $c = d$, then $a - c$ is less than $b - d$."

In addition to employing logical and arithmetical laws in proving its theorems, geometry develops its own "proof-methods." The constructions that one can make in geometry are examples of a new method of proof. For example, the proof that the exterior angle (page 177) is greater than either opposite interior angle requires not only logical and arithmetical notions, but also the ability to bisect certain lines and produce others. Without the right to perform such constructions it would not be possible to deduce this proposition from the postulates alone. The basis of the construction-method of proof is provided in Postulates 1-3. Our interest here lies in theorems proved by logic alone or by logic and arithmetic, but significant applications of these two sciences can be made on theorems proved by the geometrical method.

Since four of these postulates are in the form of implications, the logical principle of contradiction and interchange can be applied to each one. For example, we may deduce

THEOREM 1. If two parallel lines are cut by a transversal, then the sum of the two interior angles on the same side will not be less than two right angles.

PROOF: If p is the proposition "A straight line cuts two straight lines in such a manner that the interior angles on the same side are less than two right angles" and q the proposition "The two lines described by p will meet," then, by Postulate 5, "p implies q" is true. In general, however, if p implies q, then the contradictory of q implies the contradictory of p, this being a logical law, valid for all propositions whatsoever. Hence, from Postulate 5:
"If the two lines do not meet, i.e., (by definition) if they are parallel, a straight line will not cut these lines so as to make the interior angles on the same side less than two right angles."

THEOREMS 2, 3, and 4 will be derived by applying the same logical principle to Postulates 7-9.

Another logical principle, drawn from the class logic, can be applied to Postulate 6. This postulate may be rephrased to become a categorical form in A: "All pairs of straight lines are lines meeting in only one point at most." The logical Principle of Contraposition, if A (ab), then A $(b'a')$ (page 100), gives

THEOREM 5: Two lines which meet in more than one point cannot both be straight.

Direct applications of logical and arithmetical principles are made in the "indirect proofs" of the theorems in geometry. The method of indirect proof consists simply in denying the theorem to be proved and observing the consequences; since the consequences result in a contradiction, the denial of the theorem is untenable, and hence the truth of the theorem may be asserted. As has already been pointed out (page 91), Aristotle used the indirect method in proving the valid moods of the syllogism, and his method was substantially the same as that which uses the Principle of Contradiction and Interchange. There is a peculiar application of this method in geometry. Suppose that the following theorems have been proved:

a) If two angles of a triangle are equal, then the opposite sides, will be equal. (I.e., if in the triangle ABC, $\measuredangle A = \measuredangle B$, then AC = BC.).

b) If one angle of a triangle is larger than another, then the side opposite the former will be larger than the side opposite the latter. (I.e., if $\measuredangle A > \measuredangle B$, then AC < BC, or if $\measuredangle A < \measuredangle B$, then AC > BC).

The converse of both *a*) and *b*) may now be demonstrated by the indirect method:

c) If AC > BC, then A > B (converse of *b*). For suppose that, given AC > BC, A > B were false. [Note that in order to deny the proposition "If *p*, then *q*," that is to derive its contradictory, we must assert the proposition: "*p* may be true while *q* is false." (Cf. p. 21).] Now, apply an arithmetical principle: if A > B is false, then either A = B or A < B. But if A = B, then AC = BC, and this is inconsistent with the hypothesis that AC > BC. Again, if A < B, then AC > BC, and this is again inconsistent with the hypothesis. Hence, to assume that A > B is false when AC > BC leads to a contradiction, and A > B must be true under this condition.

d) If AC = BC, then A = B. Employing the arithmetical principle that if $A \neq B$, then A < B or B < A, by the indirect method applied to *b*) we deduce *d*).

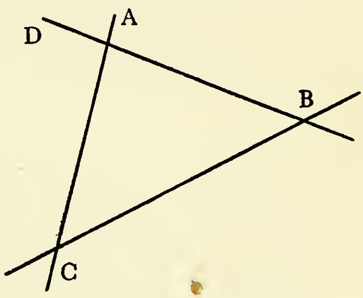
This method is generalized in order to discover the logical principles upon which it rests. Suppose that we have three mutually exclusive and all-inclusive propositions p, q, and r, i.e., three propositions such that at least one must be true while no two of them can be true together: the three propositions of arithmetic $a < b$, $a = b$, and $a > b$, form such a set. Suppose also that we have another set of the same kind, x, y, and z, and that we know that p implies x, q implies y, and r implies z. By means of these suppositions, it is possible to prove the converses of all these implications. For example, we can show that x implies p by showing that "p is false" implies "x is false." For if p is false, then either q or r is true. But if either q or r is true, then either y or z will be true; but if either y or z is true, then x cannot be true. Hence, if p is false, x is false, or, by the Principle of Contradiction and Interchange, if x is true, p is true. Or, by the direct method, if x is true, then y and z are both false. But if y is false, then q is false (since q implies y) and if z is false, then r is false. Hence, if x is true, both q and r will be false, and hence p will be true.

The whole procedure here is really a generalization of the Principle of Contradiction and Interchange. The set of three mutually exclusive and inclusive propositions is a generalization of the concept of contradictory propositions which form a set of two mutually exclusive and inclusive propositions. Also, this procedure has its counterpart for contradictories. Suppose p and q are contradictory, i.e., a pair of propositions such that the denial of one implies the assertion of the other. If x and y are also contradictories, and we know that p implies x and q implies y, we may easily prove the converses, that x implies p and y implies q, by means of the Principle of Contradiction and Interchange. For if p is false, then q is true, and hence y is true, and hence x is false. Therefore, since p is false implies x is false, x is true will imply p is true.

Exposition of this method indicates that the mathematician's dislike of the indirect method is groundless; the value of the direct method as compared to the indirect seems to rest at best on a point of elegance.

In many proofs of theorems in a deductive system, there is a choice in application between a law of the logic of classes of a law of the logic of propositions. For example, Euclid's Proposition 27, "If a straight line falling on two straight lines makes the alternate angles equal to one another, the straight lines will be parallel to one another" is a direct logical consequence of Proposition 16, "In any triangle, the exterior angle is greater than either of the interior and opposite angles."

This can be shown in two ways. First, the Principle of Contradictions and Interchange may be applied to 16 if we restate this in a weaker form as an implication: "If two intersecting lines are cut by a transversal the alternate angles cannot be equal (i.e., the exterior angle cannot equal and interior and opposite angle) ." The accompanying figure will clarify this proposition.



The exterior angle DAC cannot equal its alternate angle BCA. If we restate 16 thus, then it follows logically that if two lines are cut by a transversal so that the alternate angles are equal, the two lines do not intersect, i.e., are parallel, which is Proposition 27.

We may also state our proof as follows. Let *a* represent the class of straight lines forming a triangle with a transversal, and *b* the class of straight lines making with a transversal unequal alternate angles. Then Proposition 16 states the categorical form "All *a* is *b*." By the logical Principle of Contraposition, we can deduce that all non- *b* is non- *a*. "All lines not making with a transversal unequal alternate angles, i.e., all such lines making equal angles, are lines which do not form a triangle, i.e., are non-intersecting or parallel lines," and this Proposition 27.

In general, the statement *p* implies *q* can be rewritten "All things which make *p* true are things which make *q* true," [2] hence we may apply either the Principle of Contradiction and Interchange or the Principle of Contraposition to deduce a new theorem. Again, the syllogism Barbara, "If all *a* is *b* and all *b* is *c*, then all *a* is *c*," is closely allied with the principle of strength-

[2] Some logicians do not believe that this transformation can always be made. Cf. chap. XVII.

ening and weakening. "If p implies q, and q implies r, then p implies r." (Further analogies between the two branches of logic are examined in chapter XII.)

EXERCISES

GROUP A

1. Give examples of three mutually inclusive and exclusive propositions; of four.
2. What may one deduce by means of the method described above from the following sets of propositions:

   a) (1) If we win this game, I'll lose some money.
   (2) If we tie the game, I won't win or lose money.
   (3) If we lose the game, I'll win some money.

   b) If in the triangles ABC and A′B′C′, AB = A′B′ and BC = B′C′, then:

   (1) If $\measuredangle B = \measuredangle B'$, $AC = A'C'$.
   (2) If $\measuredangle B < \measuredangle B'$, $AC < A'C'$.
   (3) If $\measuredangle B > \measuredangle B'$, $AC > A'C'$.

   c) Let 1, 2, 3, represent the angles of a triangle, and a,b,c be the respective opposite sides, then:

   (1) If $1 = 2$ and $2 = 3$, then $a = b$ and $b = c$.
   (2) If $1 \neq 2$ and $2 = 3$, then $a \neq b$ and $b = c$.
   (3) If $1 = 2$ and $2 \neq 3$, then $a = b$ and $b \neq c$.
   (4) If $1 \neq 2$ and $2 \neq 3$, then $a \neq b$ and $b \neq c$.

3. Suppose p,q, and r are three mutually inclusive and exclusive propositions, and that x and y are contradictories. Suppose, also, that p implies x, and q implies x, while r implies y. What theorems can be deduced? Apply the results to the following set:

   a) (1) If $a > 0$, then $a \neq 0$.
   (2) If $a < 0$, then $a \neq 0$.
   (3) If $a = 0$, then $a = 0$.

   b) (1) If two lines are cut by a transversal so that the interior angles on the same side are equal to two right angles, the lines will not meet on this side.
   (2) Similarly, if the interior angles are greater than two rights, the lines will not meet on this side.

(3) If the interior angles are less than two rights, the lines will meet on this side.

4. What theorems may be proved from the following:
If this medicine contains both vitamins A and B, it will cure this disease, and if it contains A alone, it will cure the disease, but if it contains B alone or neither A or B it will not cure the disease.

Logic and the Philosophy of Formal Science

7

WE ARE now in a position to define the science of logic in a more accurate manner than heretofore and also to show its relation to the study of formal or deductive systems, i.e., to the philosophy of formal science.

The science of logic has already been defined by means of its central problem, that of finding all universal laws about objects and propositions. This definition is in a sense vague, however, since it might designate as logical many propositions which certainly do not belong to the science of logic. For example, the statement "If *a* is a cousin of *b*, then *b* is a cousin of *a*," really holds for all objects whatsoever, for even though our *a*'s and *b*'s may be nouns which, when related by "is the cousin of" yield nonsense, nevertheless, the entire statement remains true: "*If* beauty is the cousin of triangularity, then triangularity is the cousin of beauty"; that is, one willing to grant the premise, or a formal system asserting the premise, must also grant the conclusion.

To avoid this ambiguity, we must stipulate further what property the relations of logic must have. Logic is not restricted to any given set of objects (as arithmetic, for example, is restricted to numbers) but has universal application; hence it is reasonable to demand that logical relations have universal application as well. We may state this requirement more precisely thus: *a relation is said to be a relation of the science of logic if, when it is applied to any objects whatsoever, the resulting expression is meaningful.* That Aristotle's relation "all ____ is ____" has this property seems clear enough. No matter what nouns we may substitute for the blank spaces in "all ____ is (are) ____," we derive a meaningful statement: "All men are animals," "All humanity is beauty," "All triangularity is green." The resulting expression is not always true, of course, and is sometimes so obviously false as to seem nonsensical, but it is always possible to assert that the expression is true or is false, such possibility being a criterion of meaningfulness. When the object in question is singular, the "all" becomes redundant but not meaningless: " (all) John is a man." Aristotle's other three relations as well as the two additional ones mentioned [U (*ab*) and V (*ab*)] have the required property for logical relations as well. But the arithmetical relation "=", which means "is quantitatively identical with," yields a meaningless expression when applied to objects to which the attribute of quantity cannot be referred: "triangularity = beauty," "heaven = heaven." When "=" means "is identical with in all respects," then "=" is a logical relationship and is equivalent to the relation "All *a* is *b* and all *b* is *a*." We are apt to overlook the equivocal nature of the symbol "=".

Thus Euclid's axiom, "Things equal to the same thing are equal to each other," may be a law of logic or a law of arithmetic, depending on whether we use the latter or the former meaning of the relation "=". Again, the geometrical relation "is parallel to" can only be applied meaningfully to certain geometrical terms. It is meaningful to say that this line is parallel to that line, but it is nonsense to say that the number two is parallel to sixteen.[1]

This quality of generality required of the relations of the logic of objects may also be applied to the logic of propositions; thus, the relations of the latter science must be such that when applied to any proposition whatsoever the resulting expression is meaningful. For example, "implies" is a logical relationship, since "*p* implies *q*" is always meaningful (though not always true, of course) no matter what propositions *p* and *q* may be. But "declares" is not a logical relationship, for "*p* declares *q*" is mean-

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[1] Logicians have often criticized Aristotle's logic for its restrictions, arguing, probably justly, that not all propositions can be expressed in one of the four (or six) categorical forms; if it was Aristotle's aim to construct universal forms of propositions, then we might justly say that he failed. But the traditional logic is not restricted if we employ the definition of logic here suggested, for Aristotle's categorical forms were relations of logic and his laws of Immediate Inference and the syllogism, holding as they do for all objects, are perfectly general laws of logic. (For the justification of this statement in the face of arguments purporting to show the "breakdown" of traditional logic, cf. chap. XVI.)

ingless if p is a proposition. The remaining relations of the logic of propositions, given in chapter II: "is false," "both ____ and ____," "either ____ or ____," "is equivalent to," will all be found to share with "implies" the universal quality required of logical relations.

The philosophy of formal science attempts to solve the correct structure of a deductive system; it sets down certain criteria which all deductive sciences must follow. Every deductive system must fulfill five obligations. Each one must:

1. Set down a list of indefinable terms and relations.
2. Define all other terms and relations by means of the indefinables only.
3. Construct a set of propositions which are to be assumed, the so-called postulates, which contain only the indefinables and the terms and relations defined.
4. See to it that these postulates are consistent and independent, i.e., that no postulate contradicts any other, and no postulate follows from any other as a theorem.
5. Deduce theorems, making use only of the definitions and postulates explicitly set down.

In the case of rather advanced sciences, such as physics, astronomy, or psychology, the necessary indefinables and postulates are very many in number. Most of these are terms, relations, and assumptions of other sciences which the given science presupposes. For the sake of convenience, the formal sciences may assume implicitly these presuppositions, or make them explicit under the name of "axioms," so long as the form of the presupposed science is clear. Thus, if a given science presupposes the laws and terms of geometry, it must make clear whether it assumes Euclidean or non-Euclidean geometry.

If this simplification is to be effective, some scheme or classification of the sciences is necessary in order to determine just what sciences a given science may presuppose. Evidently, logic will be a science which all other sciences will have to presuppose, since a formal criterion for a science is that it be consistent, i.e., logical. In this sense, the logic of propositions seems more fundamental than the logic of classes, since the formal system of the latter makes use of principles of the former. Thus to deduce "E (ab) implies O (ab)" from "A (ab) implies I (ab)" we assumed the law of the logic of propositions that "If p implies q, then 'q is false' implies 'p is false.' "

The science of number apparently comes next,[2] i.e., arithmetic, for all sciences other than logic seem to presuppose the concept of number and arithmetical laws. Geometry is science presupposing the terms, relations, and assumptions of logic and arithmetic, as has been shown in chapter VI, but which is presupposed by all the more advanced sciences whose terms are always related to space in some way. After geometry comes the science which introduces the concept of "motion" or "time," generally known as kinematics. The simplest physical science is the one which describes the motion of a single particle (or of groups of particles taken distributively),[3] and this is called mechanics; its fundamental concept seems to be "mass." Physics may be defined as the science which studies the properties of groups of particles taken collectively.[3] It introduces many new terms such as "electricity," "magnetism." Astronomy, geology, and many other sciences are apparently special branches of mechanics or physics, since presumably they do not introduce any terms or make any assumptions other than physical ones. The unique concept which chemistry introduces is that of "valence"; whether this is a term belonging to physics has not yet been determined. If it is, then chemistry is a part of physics, otherwise it is not.

There are some, the "mechanists," who claim that the science of biology must be taken only as a part of physics. The fallacy in the usual argument purporting to establish this point of view is discussed in chapter XI. In the light of this discussion, we can say that it is perfectly consistent for the biologist to insist that the concept of "life" is not a physical one and that biology introduces nonphysical assumptions, though it does, of course, assume the terms and postulates of physics. The science of psychology clearly presupposes biology and introduces the concept of "mind." Sociology presupposes the concept of mind, but considers groups of individuals.

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[2] It is realized, of course, that this classification is not complete; not only are many of the more advanced sciences missing, but also there may be sciences intermediate to two mentioned here; thus the logic of propositional functions (Chap. XVII) would probably come between the logic of propositions and arithmetic.

[3] For precise definitions of these terms, cf. p. 169.

The following table outlines the suggestions made here:

Science	Terms and Relations Introduced
1. Logic of propositions	"implies," "and," "or," "not," "false"
2. Logic of classes	"thing," "all—is—," "non-" (all other terms of the Aristotelian class logic can be defined in terms of these; cf. p. 103)
3. Arithmetic	"number," "is less than," "is equal to quantitatively," "is greater than"
4. Geometry	"point," "line," "plane," "intersects," etc.
5. Kinematics	"time," "motion"
6. Mechanics	"mass," or "particles of matter"
7. Physics	"groups of particles (taken collectively)," "electricity," "atom"
8. Biology	"life," "animal," "plant"
9. Psychology	"mind," "intelligence"
10. Sociology	"groups of biological individuals" or "groups of men"

The close relation between the philosophy of formal science and the science of logic can readily be seen; some have been so impressed by this relationship that they have identified both under the name of "deductive logic." The philosophy of formal science requires logic in the investigation of the formal validity of any deductive system, since tests for the consistency and independence of postulates are purely logical problems. Further, all formal systems make use of logic in the deduction of their theorems. There naturally arises the problem, which has become one of the most important as well as the most difficult of modern logic, as to whether or not the criteria set down by the philosophy of formal science can be applied to the science of logic itself. We have already shown that they can in part, since we have formalized the traditional logic of classes, but in our formal system we have presupposed the other branch of logic in proving theorems. In other words, we have assumed such statements as "If p implies q, then p implies p" and have applied them to our assumptions. But the problem still remains concerning the whole of logic. How shall we demand that the postulates of logic be consistent if the concept of consistency depends on our postulates? Further discussion of this problem, especially with respect to modern symbolic logic, will be found in chapter XVII.

Formal sciences, i.e., deductive systems, are essentially hypothetical in character; that is, they are all of the form "*If* one grants these propositions we have taken as postulates, then so-and-so must also be granted." Opposed to every hypothetical statement is a categorical one: "These propositions are true and therefore so-and-so follows." Now the problem of the actual truth and falsity of the postulates does not fall in the scope of formal science. A given system is "formally true" if its postulates are consistent and independent, even though no scientist would recognize them as actually valid. We are thus faced with the problem: "When can we say that the postulates of a formal science are *actually* true (not merely consistent)?" This problem, which does not belong to the philosophy of formal science, forms the fundamental problem of the philosophy of non-formal science. The problem is discussed in the next chapter, where the value of formal science for the nonformal scientist is also explained.

Part II

Nonformal Science and Logic

Philosophy of Nonformal Sciences: Logic and the Problems of Scientific Method

8

IN THE preceding chapters we have considered the problems of the philosophy of formal science. Deductive systems, we have said, set down certain postulates or "demands," and from these demands, by the aid of logic and the laws of certain other sciences which the given deductive system presupposes, the system deduces another set of propositions, the theorems. Just what sciences a given system may draw upon is determined by classifying the sciences in such a way that the most basic science comes first, that is, logic, then arithmetic, geometry, and the others follow. This scheme has been developed in some detail already (cf. page 136). Every science may be formalized; for example, using the terms of physics we can construct a set of postulates and prove certain theorems from these.[1]

As far as the philosophy of formal science is concerned, there is no restriction on our choice of postulates for any science so long as these are consistent and independent (cf. page 10). For example, we might formulate postulates for logic which differ from those previously given. Any such set of postulates would constitute a formal science of logic so long as the assumptions were consistent and independent; that is, the set of propositions must not deny the Law of Contradiction, for such a set is formally bad. The criteria of formal science comprise the philosophy of formal science.

Since the choice of postulates in a formal system is determined solely by the criteria of consistency and independence (plus,

[1] Cf. chap. XVIII.

perhaps, sufficiency), a number of formal systems for any given science may occur. For example, in the case of geometry (cf. page 12), Euclid's formal science assumed, explicitly or implicitly, a certain set of propositions concerning geometrical entities. Later history showed that one or more consistent sets of propositions in geometry could be constructed which contradicted Euclid's assumptions. Similarly, a non-Euclidean arithmetic, denying certain arithmetical laws, might be constructed.

At this point, the question might be raised as to whether two formal systems really can contradict one another. For example, in Euclid's system of plane geometry, it follows that through a point outside a line, one and only one line can be drawn parallel [2] to the given line, while in the formal system of Lobatschewsky we apparently have a contrary of this, to the effect that more than one line can be drawn parallel to the given line (and hence, because of the continuity of space, an indefinitely large number can be drawn); finally, according to Riemann's geometry, no parallel lines can be drawn. It might be argued that these three propositions are not inconsistent with one another, despite the fact that verbally they sound so, for there may be an ambiguity in the terms used. Thus, if someone were to assert that all chairs must have four legs and four legs only, while a second insisted that some chairs can have less than four legs but not more, and a third declared that some chairs can have more than four legs but none can have less, we would merely say that the argument was futile in that by the term "chair" each means something different from the other. Hence their statements are not inconsistent with one another, for they now become "Chairs as A defined them have four legs and only four legs," "Chairs as B defined them have at most four legs," etc.

In the same manner, in the three systems of geometry in question we might suppose that each means something different from the rest by the concept "straight line." In fact we can find certain (curved) lines on surfaces in Euclidean three-dimensional space, called "geodesics," representing the shortest distance between two points along the surface, which obey all the properties of lines in Lobatschewskian geometry (in bounded areas). Conversely, we can find certain lines on surfaces in Lobatschewskian geometry that have all the properties of Eu-

[2] "Parallel" lines are defined as non-intersecting, coplanar, straight lines.

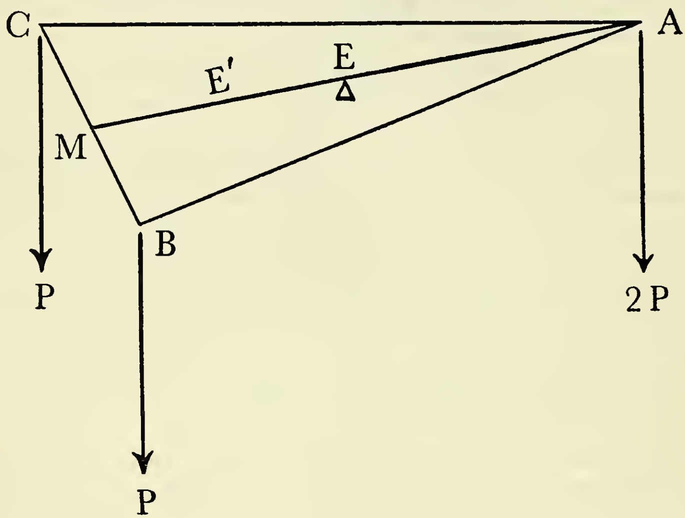
clid's straight lines. A similar transformation can be found in Riemann's geometry (e.g., Riemann's geometry of the plane holds on the surface of a sphere in Euclidean geometry). We might feel inclined to say, then, that all three geometries can be true, since they are all in agreement. Similar transformations can be made in the case of other sciences; a great many so-called non-Aristotelian logics can be made consistent with Aristotle's logic by properly defining the terms. When the given deductive system is expressed in symbols rather than words, transformations follow even more readily.

In general, it might be said that if the postulates of a given formal system are merely definitions, then two contradictory systems are impossible. Examples of such sets of postulates are given in chapter XVIII; they are very plentiful in mathematics.

But there is a sense in which formal systems are inconsistent with one another. If we no longer leave the concept "line" indefinable but identify it with some physical object in the world of experience or experiment, then we cannot employ transformations from one system to another. So long as our indefinable words are left undefined, we can show, by suitable definitions in another system, that two apparently contradictory formal systems are really not contradictory. But if the indefinable concepts are "tied down" to the physical world, then our various formal systems become contraries of one another. To just what physical concept a line corresponds is an extremely difficult problem to answer, though the answer is not necessarily unique. Rather than delve into a matter well beyond the scope of this book, let us suppose that by "straight line" we mean the path of light *in vacuo*.[3] Then Euclid's proposition states that through a point outside a beam of light, one and only one coplanar beam can be passed which will fail to meet the first if produced indefinitely; Lobatschewsky's geometry asserts that more than one beam can be passed, Riemann's that none can be. Assuming the consistency of nature, at most only one of these assertions can be correct.

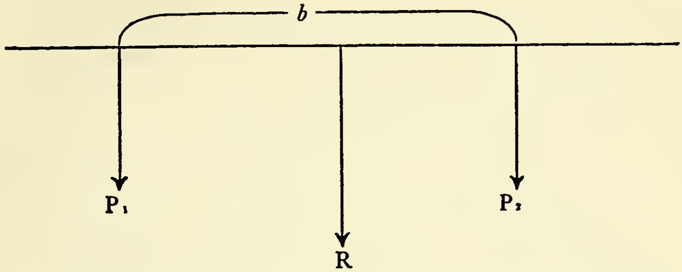
[3] We cannot define a straight line as a line which "looks straight," since such intuitive definitions are either too vague or else meaningless for the scientist. There are difficulties in defining a straight line as the shortest distance between two (physical) points, since the process of measurement of distances presupposes, apparently, that we have straight lines; presumably, a physicist cannot measure the "shortest distance" unless he presupposes a straight measuring rod or the means of passing from the rulings on a curved measuring rod to a straight one.

Again, we might take "straight lines" to mean lines of force. If this is done, significant changes occur in the formulas of mechanics when the geometries are changed. For example, if (as in the accompanying figure) a weight of two pounds is hung from point A of the (weightless) isosceles triangle ABC, and one pound weights are hung from B and C, then the entire system will balance in Euclidean geometry if the fulcrum is placed at the point E, the midpoint of AM, AM being the perpendicular



bisector of CB. But in the geometry of Lobatschewsky, the system will not balance if the fulcrum is placed at this point, but will balance only if the fulcrum is placed at a point E′ on AM which is closer to M than to A. In Riemannian geometry, the balancing fulcrum will lie nearer to A than to M. Again assuming the consistency of nature, only one of these results may be true. These remarks may be generalized by stating that if we have two forces acting on a line, in Euclidean geometry the magnitude of the resultant will be independent of the distance apart of the points of application of these forces (in the figure on the opposite page, the distance *b*), while in the other two geometries the magnitude of the resultant will depend on this distance.

Similarly, in the case of other sciences, if the indefinables are identified with physical objects and relations between physical objects (objects of experience), then we may have many formal systems for a given branch of science only one of which may be correct.



For example, in ordinary arithmetic addition is assumed to be "commutative," i.e., "$a + b = b + a$" is a true law. But there are important arithmetics which deny this law in general. So long as the a and b and "+" of this expression are not identified with physical objects and operations, $a + b = b + a$ can be made consistent with $a + b \neq b + a$ by taking the "+" in the latter to mean something different from the "+" in the former. But if a, b, and "+" are identified with physical concepts, then the two laws are inconsistent. Thus if a and b refer to numerical readings on a yardstick, then $a + b = b + a$ is a true law: if a table is 3 feet plus 2 feet long, it is also 2 feet plus 3 feet long. But if a and b represent chemical quantities, and $a + b$ represents the operation of adding a to b, then $a + b = b + a$ will not be true in general; for example, adding water to sulphuric acid does not yield the same result as the operation of adding sulphuric acid to water (since temperature differences occur). But whatever concrete operations we choose for a, b, and "+," the laws $a + b = b + a$ and $a + b \neq b + a$ are incompatible. Further examples of "non-Euclidean" arithmetics are given in chapter XVIII.

In the case of the science of logic, it has not been possible, apparently, to find contradictory systems. The problem of "alternative" logics is an extremely difficult one. Further mention of this is made later on. When the systems are expressed symbolically, other problems also arise because of the abstract nature of the symbols (cf. Chapter XVII).

It is noteworthy that two formal systems of geometry which are not consistent with each other will produce two different systems of kinematics, mechanics, and physics, etc., since these sciences all presuppose the science of geometry. That is, the physicist, in stating his laws and proving his theorems, makes use of the propositions of some geometry, and if one physicist assumes Euclid's geometry and another the geometry of Riemann, then despite the fact that they may start with the same postulates for physics, they will arrive at different theorems.

Hence we come to the concept of many descriptions of nature as a whole. These descriptions will all be in the form of deductive systems, but the deductive systems are all contradictory to one another at least in part. A "complete" description of nature would be a deductive system which set down the postulates for all sciences from logic to sociology (assuming this to be the last). Each deductive system would be formally correct in that its postulates were independent and consistent. But which description gives a true account of nature itself: which description is not merely consistent but actually true? An analogy with a particular instance will illustrate our question. Suppose ten persons are witnesses to an accident. Each gives an account of the happening, and each one's story is consistent in itself, yet disagrees in some point with all the rest. We may sensibly ask which account (if any) is the true one? This is no longer a problem of the philosophy of formal science, for each system is formally correct; we call the problem one of the philosophy of nonformal science.

This discussion can be treated in another manner by distinguishing between two meanings of the word "true." A given proposition is "formally" true if it follows as a theorem or is itself an assumption of a formally correct deductive system. Thus, the proposition $2 + 5 = 5 + 2$ is formally true in one formal system of arithmetic. In another deductive system of the same science, i.e., in another system assuming the contraries or contradictories of the assumptions in the first system, a given proposition may not be formally true. For example, the proposition "The sum of the angles of a triangle equals two right angles" is formally true in Euclid's geometry but is not formally true either in that of Riemann or that of Lobatschewsky. The determination as to whether a given proposition is formally true depends on showing that it follows from a given set of consistent assumptions. Formal truth is always hypothetical and is relative to the assumptions made: *if* these assumptions be true, then so and so is true.

On the other hand, a certain proposition is said to be "nonformally true" if it is "actually" true, i.e., true, not relative to other assumptions, but independently so. The proposition "All living men will sometime die" is true, not because it follows from certain assumptions arbitrarily made, but because it has been verified (by experience).

In a similar manner, a proposition is "formally false" if it is inconsistent with the assumptions of a deductive system, while it is "nonformally false" if its falsity has been actually established. Thus "The earth is round" is formally false in a formal system of astronomy which assumes the earth to be flat, but it is not nonformally or "materially" false.

Just as the philosophy of formal science attempts to answer the problem: "When can we say that a given proposition is formally true?" so the philosophy of nonformal science has as its problem: "When can we say that a given proposition is nonformally (actually) true?"

It may be seen readily that the nature of the problem at hand implies that it is one of method. What we are really asking is "How shall we go about determining which description of Nature is the actually true one, or how shall we go about determining whether this proposition is nonformally true?"

Historically, the answer to this problem has not been uniform. Four well-known solutions have been offered under the names of *dogmatism, intuitionism, rationalism, empiricism.* The dogmatist asserts that the correct method of determining the truth (at least in some cases) is to appeal to some external authority, written or verbal; the intuitionist asserts that some truths, at least, may be discovered through an inner intuition of their validity; the rationalist asserts that some truths (the absolutely certain ones) may be found simply by employing correctly the laws of reasoning (presumably, the laws of logic); the empiricist asserts that all truths are to be determined by the method of experience, i.e., by the use of the senses.

The method of dogmatism is not uniform. Appeals to authority beyond the investigator may take the form of (1) appeals to statements believed to be divinely inspired; (2) appeals to ancient or established authority, or (3) appeals to authority recognized by all or the majority of people. Illustrations of dogmatism of the first sort can be found in most religions; the validity of the Ten Commandments, which set down a code of morals, is sometimes made to rest on the fact that they were revealed to Moses on the mount. The famous Mohammedan dilemma, reported to have been the motive for the destruction by fire of the books in the library at Alexandria, is another example: "If a given book states the truth, it is superfluous, since the truth is already contained in the Koran; if it states a falsehood, it is perditious; therefore, all books other than the Koran are either superfluous or pernicious."

It may seem that examples of appeals to ancient or modern authorities generally recognized are much rarer today than in the darker period of medieval times, but such is hardly the case. Our belief in the validity of most social customs and manners, our credulity regarding newspaper and magazine articles and editorials, are all appeals to authority of some sort.

The weaknesses of the dogmatic method rest on the lack of criterion concerning the correct authority. If we are to appeal to authority, we must have some method of choosing the correct one. However, this method cannot itself be dogmatic; that is, we do not accept someone as an authority simply because he says so. But if some other method is employed, then this is more basic than dogmatism, which has consequently lost its claim to being the basic criterion of truth. In the case of many religions, dogmatism is sustained by another method: for example, it is argued that miracles have been observed to be performed by the authority in question, and here the method of experience is employed.

The school of intuitionism has had so many adherents in so many different fields that it is impossible to give it any historical date. There were certain English moralists of the seventeenth and eighteenth centuries [4] who insisted that the laws of ethics are intuitively known to us, so that we need only ask ourselves whether this act is morally correct and we receive the answer directly from within. Another form of intuitionism was prevalent among certain philosophers and scientists of the nineteenth century (Spencer and Whewell in particular). These men asserted

[4] Clarke, Cudworth and others.

that certain propositions are true if they are "obvious," or, perhaps more precisely, if we cannot conceive them to be false. For most of these thinkers, $2 + 2 = 4$ was such a proposition; the laws of Euclidean geometry and, in some cases, the laws of Newtonian mechanics, are other examples.

The principal difficulty of intuitionism lies in the ambiguity of its method of determining truth. We are asked to determine whether we can conceive the contradictory of the given proposition to be true. This seems to be a test of our imaginations. A child living in the tropics probably can't conceive of such a thing as frozen water, but we should hardly infer from this that no water is ever frozen. What has seemed inconceivable in one generation is very often an accepted fact in the next. Thus, for most nineteenth century physicists, it was inconceivable that Newton's mechanics should break down, but most physicists of today recognize that it is not universally valid.

The school of rationalism in modern times began with René Descartes (1596-1650). Descartes set out to find propositions which lay beyond the realm of any doubt. He thought that he had discovered two such propositions, and that these were true solely on the grounds of logical laws, so that anyone denying them necessarily contradicted himself. This was so, despite the fact that the propositions themselves were not laws of logic. The two statements were: "I exist" and "God exists." I cannot doubt that I exist without affirming the fact that I doubt and therefore that I exist; that is, the proposition "I exist" is necessarily true since its denial implies its affirmation. Again, God must exist, for by definition God contains all the attributes, and existence is an attribute.[5] Thus Descartes offered a method for determining some truths but, of course, not all, since he recognized that we distinguish a great deal by our senses. But these truths which we do derive from our reason are indubitable, a quality not belonging to any other propositions.

A critique of Descartes' arguments shows that they fail to accomplish their purpose. It is by no logical law that we pass directly from "I doubt" to "I exist"; that is, granting that "I doubt" is true, we cannot immediately infer that "I exist." Here

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[5] Both of these proofs are medieval in origin; the former is due to St. Augustine, the latter to St. Anselm. These men were not pure rationalists, however, since they did find other criteria for indubitable truth.

there are three terms: "I," "doubting thing," and "existent thing," and no valid mood of Immediate Inference can be constructed from three terms. The argument is really a syllogism with one premise missing. The missing premise might be "All doubting things are existent things." But this proposition cannot be proved by logical laws; hence, the conclusion "I exist," does not follow from purely logical propositions.[6] Even the proposition "I doubt" is not a logical necessity, for the word "I" is ambiguous. How can I be sure that this "I" which doubts is the same as the one doubting the doubt, unless it be assumed that the ego is an indivisible unit?

Concerning the proof for the existence of God, we need only point out that the proposition that existence is an attribute, however true it may be, is not a law of logic, and hence the result, "God exists" is not proved by the principles of logic alone.

With the failure of intuitionism and rationalism in supplying us with adequate answers to the problem of nonformal science, we turn to the last alternative mentioned, empiricism. Empiricism has had the widest acceptance among philosophers, but principally because of the vagueness of its fundamental tenet. The statement that all knowledge is derived from experience has been interpreted in many ways, depending on how one felt inclined to take the terms "derived" and "experience." The medieval philosopher, Roger Bacon, for example, is often heralded as being far ahead of his times in his insistence on the truth of the empirical postulate. But one of the meanings which Bacon gives to experience is "an inner reading of the soul," something quite close to intuition.

To make the empirical position more exact, we define it as that answer to the nonformal problem which states that the only method of arriving at truth is a method employing sensory experience.

This was the position taken by the English philosopher, John Locke (1632-1704), who declared that all knowledge arises from certain "immediate experiences"; by compounding and relating the impressions made on us by our senses, and later by "abstract-

[6] To defend Descartes, many have argued that this minor premise must be true since it is "clear and self-evident"; but such grounds for accepting this proposition would return us to the intuitionist method, explained above. On the grounds of rationalism alone, the minor premise ought to be a logical law, which it is not.

ing," we arrive at all our ideas, even the least concrete of them, as, for example, the idea of beauty or the idea of the soul. All facts and laws go back to these immediate gifts of our senses.

Philosophers following Locke found several objections to his theory:

(1) "If all we know is the impressions of our senses," said Bishop Berkeley (1685-1753), "then in answer to the question whether there exists an objective world ('reality') outside the mind we must reply in the negative, since a world apart from perception is absolutely meaningless for us."

(2) "If all universal propositions about nature are dependent for their verification on particular experiences," said David Hume (1711-1776), "then we must say that we can never be absolutely certain whether any universal law of nature is altogether true. For example, the law that every event must have a cause is a law that cannot be certain for us since even in the entire history of the human race it has only been shown to hold a finite number of times."

(3) "To say that we begin with certain immediate experiences and with these alone," said Immanuel Kant (1724-1804), "is to say that we have never experienced anything; for in order that an experience mean anything to us, it must first be individuated, then come under certain 'logical' relations which give it meaning. Hence, the mind has a certain equipment which allows it to experience and this equipment is necessarily prior to experience."

(4) "Not only is it erroneous to say that we can have certain knowledge of the universal laws of nature, but also our knowledge of the facts from which we build these laws is likewise uncertain," says the experimental scientist.

Solutions to these objections, all but the first of which are generally recognized as valid, are best made in the reverse order. Regarding the fourth, we observe that its claims are especially true of our everyday, "uncontrolled" experience. Our senses continually play tricks on us, and we often do not (and frequently cannot) bother to be sure that the experience we are having is reliable. The weaknesses of our usual observations are best seen in the illustration given of the various accounts which a group of persons will present regarding an auto accident, each apparently convinced that his story is the true one. They cannot all be true, yet each statement is presumably based on the experiences of the speaker; hence some experiences do not form a basis for determining truth.

If this point of view be true, and there can be little doubt that it is, then we must rephrase the empirical answer as follows: "Only those experiences which are somehow controlled in order to keep at a minimum the individual idiosyncracies and prejudices of the observer are of any value in determining scientific truth." Since the term "experiment" is used to designate controlled laboratory experience as opposed to uncontrolled, everyday experience, the empirical postulate becomes: "Experiment supplies the method of determining what descriptions of nature are closest to the truth." Our problem now becomes one of analyzing the method of experiment.

Even though we exclude vague experiences in our method, we are still forced to acknowledge that the facts of experiment are not fixed and certain. He who is measuring the length of a certain object in the laboratory does not expect to arrive at one answer, even though he recognizes that to a question concerning a given length at a given time there can be but one answer. Rather, the experimenter makes a number of tries at measuring, and each try gives him a certain "reading." Not all these readings agree; indeed, if they do, the experimenter is bound to regard them as worthless, realizing that he has not made his readings fine enough.

The problem which then arises is one of determining the most accurate reading. The experimenter might be inclined to take the average (or "arithmetical mean") of his readings and to regard that as his final answer. But this would be unsatisfactory since another set of readings would probably yield another average, and the experimenter would be at a loss as to which to choose. The Theory of Least Squares has as its object the answer to this question; a reading suggestion is appended at the end of the chapter. For the present, we merely note that the experimenter takes as the measurement of a given object some quantity lying within a certain range of values. For example, he may say that the length of this rod is to be taken as six centimeters, plus or minus a "probable error" of five millimeters. Hence every experimental fact has a range of error.

Stated more precisely, the experimenter is faced with some such problem as the following: "What is the distance from the point marking the center of gravity of the Earth to the center of gravity of Mars on the first of January, 1940?" He recognizes that in reality the answer to this question must be "single-valued," i.e., he cannot say that at one and the same moment the two points were at more than one distance apart. The assumptions upon which the experimental method proceeds, however, make it impossible for the experimenter to arrive at a single-valued answer in any finite number of readings. If, after measuring the required distance ten times, he finds that all his readings agree up to the third decimal place, then he has merely failed to carry out his readings to enough places, and he must make readings to the fourth, fifth, or sixth places. Since, in a given set of readings, certain ones will be different from others, these differences must be taken into account if the readings all have the same value. This is done in the Theory of Least Squares; as a result, there is a certain "range" within which the distance measured is to be taken. In the case of the rod mentioned, the range is from 5.5 to 6.5 centimeters; the length of the object measured is to be taken as falling somewhere within this range. The range, according to the theory, must be reduced as the number of readings increases, so that if the range is 5.5 to 6.5 centimeters after the first twenty readings, it must be reduced to approximately one half (say) at the end of the next eighty,[7] i.e., it must be reduced to 5.75 to 6.25. If this fails to happen, some "systematic" error in measurement has occurred; e.g., the measurer has failed to take into account changes due to temperature, atmospheric pressure, and other important factors.[8]

It might appear that, though such facts as the actual length of the table can never be determined without some error, there are nevertheless, certain facts which do enjoy unqualified certainty. An example of such a fact would be the proposition "I am now seeing brown" or "I am now having a pain." Apparently, no one can question the truth or falsity of this judgment; I cannot my-

[7] If $\frac{1}{n}$ represents the approximate reduction in the probable error, then this should take place when the readings have been increased n^2-fold.

[8] This account relates only one type of measurement; in its essentials it applies to all types of measurement, though these may differ in detail quite widely.

self be deceived on the matter, and no one else can ever determine its truth or falsity: I cannot be deceived on this question, for, though the object itself may not actually be brown, if brown is the color I think I perceive, there cannot be any question as to whether I do or do not see brown: if I think I see brown, I must be perceiving it; further, no one else but myself can determine anything regarding the matter, since this fact is strictly a private one. But the history of philosophy has shown the weakness of this point of view. The following are some of the objections which have been raised.[9] (1) The meaning usually given to an "immediate experience" is very vague; indeed, by the very nature of the case, no definition can be given, since such a defining makes the "immediacy" mediate; but if no definition of the concept can be given, then for the experimenter it can have no meaning, since no method can be devised for testing the truth or error of anything said about it. (2) It is not true that everyone is the best judge of his own mind; the lover who denies his love may be convinced that he does not really love, but we who observe him know that in his "heart of hearts" he really does. Hence not even these "private facts" about the impressions of our own mind are free from error.

The experimental method must also recognize the validity of Kant's objection. It is true that no experimenter would think of starting his laboratory work without certain instruments; the principles by which these instruments work are therefore presupposed by the experimenter as valid. Not all these "instruments" are physical; in any laboratory experiment the principles of arithmetic or mathematics in general are always presupposed. For example, in any reading of the measurement of an object, we assume a definite arithmetical law. If the rule reads 58.7 centimeters at one extremity of the object and 63.9 centimeters at the other, we assume that the required reading is 5.2 centimeters, since the arithmetical law $63.9 - 58.7 = 5.2$ is presupposed. Further, the experimenter must assume certain geometrical laws; at the very least, he must assume enough geometry to allow him to give a *position* (relative to the chosen coordinate system) to the object measured. Otherwise he would be without means of assuring himself that the measurements which yield

[9] For fuller explanation, see E. A. Singer, Jr., *Mind as Behavior.*

him his probable error had all been made of the same individual object.

The fact that all experimentation makes certain presuppositions is especially important from our point of view, since it is also the case that the experimenter assumes the laws of logic in any experiment. Thus he assumes that if one of his readings is 5.2 centimeters, it cannot also be 5.4 centimeters ("A proposition and one of its contraries cannot both be true."). Again, he assumes that if all his readings in measuring a certain object on one day fall in a certain range (say from 5.8 to 6.3) and all his readings on the next day fall in an entirely different range (say 6.5 to 6.9), either he cannot be measuring the same object, or else the given object must have changed somehow. (If the true length of A belongs in the range 5.8 to 6.3 and the true length of B to the range 6.5 to 6.9 and the two ranges have nothing in common, then the true length of A and the true length of B are different, i.e., A and B are not the same in all respects: this is the logical law, in the form of a sorites: A (*ab*) A (*cd*) E (*db*) implies E (*ac*).)

The question which now seems inevitable is: granted that the experimenter must assume the laws of logic, arithmetic, and geometry, before he can perform his experiment, then must it not follow that these laws, since they cannot be experimentally verified, are unconditionally true? Such a result would be unsatisfactory, since this would leave the question of certain truths forever undetermined; for if someone should assume another logic, or another arithmetic, or another geometry, we could not prove that he was in error, since our proofs would depend on a set of presuppositions which he would not recognize as valid. But this inference is bad. It does not follow that if we assume certain laws before experimenting, this experiment may not show the laws to be wrong.

Suppose an astronomer wishes to discover the laws governing the motion of the earth about the sun; to discover this, he measures the distance from the earth to the sun at certain points of time; but we have said that he must make a number of measurements in determining any given distance. These measurements cannot all be made at the same time, and since the given distance does not remain constant, he must have some law which will tell him how the distance varies, in order that he may have a certain number of readings for any given distance. But this law is exactly the one he is searching for. Actually, what the experimenter may do is to discover whether, on the presupposition of a certain law, as the number of readings increases, the range within which the distance measured is to be taken decreases. If the range does not decrease, then some new law must be tried. That is, even though he may presuppose a given law in his experiment, the experimenter may be led to reject the law because the range does not decrease with the increase in the number of readings.

Some of the laws of Euclidean geometry and the laws of classical kinematics have been rejected in recent years because, when they are presupposed in experimental procedure, they lead to certain contradictions. The manner in which this occurs in the case of kinematics is described in chapter XI.

The procedure in general is analogous to the "indirect proof" used in formal systems; a given theorem is often proved by assuming its contradictory to be true and then proceeding on the basis of this to show that a contradiction results.

From the point of view of this text, we are chiefly interested in the question whether logical laws may change or whether all logic must be immutable. The argument purporting to prove the latter view runs somewhat like this: "All logic depends on the Law of Contradiction, which states (in one of its forms) that no proposition can be both true and false at the same time; it is impossible to deny this law, for in effect it constitutes the definition of the word 'false'; that is, if one asserts that a certain proposition is both true and false, he does not mean the same thing by the term 'false' as most scientists and logicians do." The so-called "non-Aristotelian" logics are Aristotelian logics wearing a false mustache.

Whatever may be the other merits of this argument, it is erroneous in one point, namely, that all logic depends on the Law of Contradiction. No one so far has been able to deduce the whole of logic from this principle, and, indeed, it has been established within a fair degree of certainty that the task is an impossible one. For example, the law of the logic of propositions which allows us to convert the premises of an argument ("If p and q imply r, then q and p imply r") is independent of the Law of Contradiction. Further, there are many disputes among present day logicians which are independent of the validity or invalidity of the Law. The theory of types explained below (page 200) is not recognized as valid by all logicians, and those who do accept it are not in total agreement regarding all its principles; but this theory is essentially a logical one and sets down certain logical laws. Another point of dispute is concerned with the statements "A false proposition implies any proposition" and "A true proposition is implied by any proposition." These are recognized as valid laws by some logicians and by others are rejected as false. One may accept or reject them, however, without denying the Law of Contradiction.

One important modification of the science of logic which has occurred during the past century was caused by a conflict or difficulty in logical theory. This modification took the form of the introduction of the concept of the "universe of discourse" (cf. page 98). One way in which this new concept solves an ancient difficulty is explained on page 230.

The foregoing discussion leads us to the second question, asked by David Hume: on the basis of the experimental method, can we ever say that we have obtained a law of science about which there can be no doubt? Hume's contention that we can never arrive at absolute certainty regarding any law might be called the kernel of modern experimental philosophy. The contention is certainly a sound one, for if it is true that scientific "fact" cannot be given without some degree of error, then certainly scientific law has no claim to unquestionableness.

The history of the sciences has borne out this contention; probably nothing has ever seemed so certain to the scientist as did Newton's mechanics to the physicist of the nineteenth century. For him future experimentation could only bring more confirmation of the basic Newtonian principles. Yet the turn of the century brought a revolution from which emerged the more or less generally recognized fact that the older, classical mechanics is only true in a restricted sense. Similar revolutions in other sciences are well known: the doubt and for the most part the rejection of the ancient principles of biology during Darwin's time; the complete rejection in chemistry of the theory of combustion on the principle of *phlogiston,* i.e., the explanation of combustion by means of a separate, material element; the constant revision of theories concerning the nature of the atom; the rejection of the Ptolemaic theory of the solar system; and a great many other such principles.

Must we say, then, that the scientist's hopes are futile ones, that we can never answer those riddles which nature continually poses?

The answer which saves science from such a tragic end was long ago given by the Greek sceptic, Carneades (214-129, B. C.). The Sceptic School, flourishing as it did during an age of extremely contrary doctrines, urged the thesis that man can never attain to the truth about anything. The absolute sceptic, believing that there was no criterion of truth at all, asserted that as far as man was concerned, the true and the false were absolutely indistinguishable. Carneades, however, was a more conservative sceptic and asserted that while it is true that we can never say with certainty that a given proposition is true, we can say that one proposition is more likely to be true than another, or that one proposition has a greater degree of probability than another. Thus science's end is not a futile one; the object of the scientist's labors is to increase the probability of his laws. His goal is an ideal one, since science never reaches absolute certainty, but his goal is meaningful, nevertheless, since science can always approach it. By successive experiments we increase the probability of a given law's being true, even though the possibility of its being false always remains. Unattainable ends are not necessarily futile ones if they are approachable; indeed, it is characteristic of humanity that it soon wearies of attainable goals. The scientist's task, although too gigantic for completion in any finite time, is for all this and perhaps on account of this, a task no scientist would dream of relinquishing.

Carneades' answer, however, is hardly precise. What shall we mean when we say that one proposition has a greater degree of probability than another? The statement "This proposition is more probable than not" is usually meant quite ambiguously.

There is a branch of mathematics called the "Theory of Probability" which considers the term "probable" in a quantitative sense. An example of the type of problem solved by this theory would be: "What is the probability that in ten throws of a coin heads will appear each time, where it is equally likely that heads or tails will appear on any throw?" If we have an "ideal" coin (i.e., a coin as likely to fall one way as another) a definite fraction can be assigned to the proposition "Heads will appear in each throw." (The fraction here is $1/(2)^{10}$, or there is one chance in 1024.)

But it follows from the previous discussion that the mathematical theory of probability cannot be applied to scientific propositions, for this theory presupposes that we are given a series of events all of which are equally likely to occur. In the coin-tossing example, the chances that the coin will fall heads on any throw were equal to the chances that it would fall tails, and it is certain that it will fall either heads or tails; i.e., the probability that heads will appear in the first throw is ½. In two throws there are four things that can happen, any one of which is equally likely to occur: we could have two heads, or heads then tails, or tails then heads, or two tails; hence the chances of heads appearing twice is ¼. In ten throws, 1024 permutations are possible.

But it would be impossible to find such an ideal coin by the very nature of the experimental method, for the construction of such a coin would depend (among other things) on certain measurements, and such measurements would be open to the inevitable error attached to all experimental procedure. It is true that in most games of chance the gambler makes use of the mathematical theory, since the possible "deviations from the ideal" are not sufficiently apparent or significant to worry him. In many games, for example in cards, there operates a "Principle of Sufficient Ignorance"; since the laws governing the distribution of the cards in a shuffle are completely unknown, we can consider that any permutation is as likely to occur as any other. But though this procedure is adequate for the gamester, it hardly suffices for the scientist whose aim is above all things precision.

There are other, and perhaps more practical, cases where a given fraction is attached to a certain proposition, which purports to represent its probability. One may ask "What are the chances that a man of thirty today will live to be sixty?" and the answer is an important one for insurance companies. The manner in which this fraction is determined depends on statistics; by comparing figures of survivals and nonsurvivals in the past, a table is constructed for future use. That this method has little in common with the theory of mathematical probability is clear enough. That it lacks precision perhaps also is clear. The probability assigned is not one which is established beyond any error; indeed, actuarial tables are constantly being changed. Further, the actual data which go to make up the tables are open to error.

Whatever may be the difficulties involved in assigning a definite fraction to represent the probability of certain scientific laws, it seems that in many cases such an assignment is impossible. Suppose, for example, that we are endeavoring to discover the acceleration of a body as it falls freely. According to the old "inductive logic," the scientist discovers that one second from the start the velocity is 32 feet per second, two seconds from the start the velocity is 64 feet per second, after three seconds it is 96 feet per second, and so on. He then tries another body and discovers that the same "facts" hold; after a sufficient number of experiments, he "induces" the proposition that the acceleration is 32 feet per second per second.

But this description is inaccurate. The experimenter does not arrive at the "fact" that the body's velocity is 32 feet per second after the first second; rather, he obtains a reading which involves a certain probable error; that is, his readings "range" from, say, 31.7 to 32.2 feet. Similarly, each of his remaining readings involve a certain probable error. Since there is always a "range" for any given point of time, the scientist has a choice of laws, all of which will conform to his discoveries. The proposition that the acceleration is 32.001 feet per second might also fit the facts, or he might "induce" that the acceleration was not constant but was increasing or decreasing. We cannot say that there is more probability that the one law is correct than the other unless we introduce certain other, non-quantitative criteria.

This result applies even more forcibly to those laws of science which are presupposed by the experimenter. It is difficult to see how one might assign a certain degree of probability to the laws of logic, say. The laws of presupposed science become "less probably true" for the scientist when they lead to certain conflicts with other laws or with experimental procedure; but it does not seem possible to make some such statement as "The chances that Euclid's Fifth Postulate will lead to difficulties is ¼."

But one may say, that the proposition "The earth goes around the sun" has a greater degree of probability than its contradictory, even though a definite fraction cannot be assigned to represent this probability. The basis for this assertion would depend on certain criteria. In the history of the philosophy of science several such criteria have been mentioned: (1) If the supposition of a certain law leads to a "simplification" of a given science, then this law has a greater degree of probability than another which makes the science more complex (e.g., the supposition that the earth goes around the sun leads to much simpler laws than the supposition that the sun goes around the earth) ; such a criterion demands a more exact definition of the term "simplification"; (2) If the supposition of a certain law explains more phenomena than another law, the former is more probable.[10] The terms in this proposition, again, are in need of a more precise definition.

In general, present solutions of the problems relating to probability as applied to scientific method are at best tentative. There is an obvious need for more thorough research in this field, a research that cannot be carried on by the philosopher or the scientist alone, but requires the cooperation of both; the former has behind him the history of attempts to answer the epistemological problem and the resulting failures. Hence the philosopher is aware of the difficulties involved, while the scientist is conscious of the obstacles which must be met in the application of any theory of experimental method. Books on the philosophy of science written by "pure" philosophers show an ignorance of laboratory practice, while books written by the scientist display an ignorance of the history of the failures of the theories they propose. It is not too much to say that a better solution of the problem of probability as applied to scientific method is fundamental to every line of research, be it in logic, or physics, or morals, or sociology.

We have finally to consider Berkeley's argument, which really is concerned with the value of scientific research. If it is true, and it seems to be so by the very nature of the case, that science is concerned only with the relation between experimental objects, i.e., sensory objects of a certain type, then what guarantee

[10] This was Newton's criterion in his *Principia*.

have we that these experiences correspond at all with reality as it exists apart from the mind? As Berkeley claimed, not only must we fail (on the basis of the empirical method) in showing a correspondence between sensory data and reality, since such a demonstration for the empiricist can only take place through the senses and the nature of the problem excludes this method, but also the very concept of a reality unrelated to experience is meaningless.

If we leave problems of "ultimate reality" to the imaginations of the metaphysicians, we can adequately define scientific reality in such a manner that the method of science, i.e., the experimental method, becomes valuable, for we may say that the "scientifically real" length of this table at a certain moment of time is that reading whose probable error is zero. By the very nature of the method, probable error cannot be eliminated completely in any finite number of readings; hence, the real length of the table will be reached only after an infinite number of experiments; it is, therefore, a "limiting concept," ever approachable by the scientist but never attainable. Reality in general represents the answers to all scientific problems, answers which at any finite stage are never certainties. Hence reality "lies beyond" experiment, but depends on experiment. Thus we would answer Berkeley's argument by asserting that my experience or my experiment at this moment is not the "real." The true answer to the question, "What are the real properties of the world at this moment?" can be answered only by an infinite number of experiments. Such an answer, then, is never attainable, so that no "finite" experiment gives the real, but each finite (significant) experiment allows us to approach it.

READING SUGGESTIONS

a) Philosophical Background:

1. Scientific Intuitionism:
   Whewell, *Philosophy of the Inductive Sciences.*
2. Rationalism:
   Descartes, *Discourse on Method* and *Meditations.*
3. Empiricism:
   Locke, *Essay Concerning Human Understanding.*
   Berkeley, *Principles of Human Knowledge.*
   Hume, *Treatise on Human Nature.*

4. Criticism:
   Kant, *Prolegomena.*

b) Philosophy of Science:

Mill, *System of Logic.*
Pearson, *Grammar of Science.*
Mach, *Analysis of Experience.*
Singer, *Mind as Behaviour.*
Jevons, *Principles of Science.*
Bridgman, *Logic of Modern Physics.*
Poincare, *Foundations of Science.*
Enriques, *Problems of Science.*
Lenzen, *Physical Theory.*

c) Theory of Least Squares:

Merriman, *Textbook on the Method of Least Squares.*

d) Probability Theory:

Nagel, "Principles of the Theory of Probability" (in the *Encyclopaedia of Unified Science;* this monograph contains reading suggestions for more detailed research in this subject.)

Fallacies

9

THE STUDY of fallacious reasoning has been regarded traditionally as a part of the science of logic. Aristotle, founder of the science, classified [1] the many ways in which the mind falls into error. These classifications have remained practically unchanged for more than two thousand years, despite the fact that they are probably not the best that could have been devised, for they overlap, are often vague, and do not exhaust all possibilities.

For purposes of clarity, we have arbitrarily divided these slips of the mind into two classes; one, which, following tradition, we label "fallacies," comprises errors in reasoning that are more or less easily detected and placed under some given classification; the other, called "paradoxes," is a class of arguments that have caused serious difficulty for scientists and philosophers throughout the ages. Paradoxes are, of course, errors in reasoning as well as fallacies, but the difficulty has been to show wherein their error lies.

It might appear that the only fallacies which the science of logic considers are those fallacious arguments that assume a certain proposition of logic to be true when in reality it is false. Thus, were I to argue that since some men are reasonable, all men must be, I would argue fallaciously in the logical sense, for I would assume implicitly the validity of the proposition: "If some *a*'s are *b*'s, then all *a*'s are *b*'s."

Most of the traditional fallacies do not arise because of errors in the laws of reasoning, but rather because of extra-logical mistakes. Those fallacies which are purely logical we already have considered in their context, e.g., the Fallacies of Asserting the Consequent and Denying the Antecedent (pages 32-34). The following argument, however, would be classed as a fallacy also: "Lincoln was right; right is opposed to left; therefore, Lincoln

[1] Principally in his *De Sophisticis Elenchis.*

was opposed to left." No law of logic has been directly violated here; provided the term "right" means the same thing in both premises, the syllogism is valid. The argument is fallacious because it assumes that "right" has the same meaning in both sentences; yet the fact that it does not is no logical truth, but depends on the assumptions of some other science. Consequently, the error is extra-logical. The older logicians, indeed, reduced the fallacy to a logical one by insisting that the argument made use of a "four-term" syllogism: "If all *a*'s are *b*'s, and all *c*'s are *d*'s, then all *a*'s are *d*'s," which is clearly an invalid proposition of logic. But that the argument in question actually contains four terms, and not just three, can only be shown by showing that there exist two meanings of the word "right," and hence basically the fallacy is extra-logical.

However, there is some justification for including the study of fallacies in the study of logic, for they illustrate the extreme difficulty in the application of general laws. The knowledge of the laws of reasoning does not necessarily make us good reasoners; the correct application of the syllogism requires a great deal more than the formulation of the formal laws of syllogistic reasoning, as the examples later in this chapter will show.

Before we classify the fallacies, it is worth noticing the close relation between the types of fallacies and the types of humor. This relationship is rather natural, since the basis of humor is incongruity and the most obvious slips of the mind are incongruous to the observer. It will be convenient to illustrate some of the fallacies by means of jokes, though the jokes are not, of course, fallacies.

Equivocation

The Fallacy of Equivocation,[2] probably the most common type of fallacious argument, occurs when a single term is used in an ambiguous manner. The validity of the logical law to which the one arguing presumably appeals rests on the fact that a certain term used in two or more premises is the same, whereas in reality it is not. For example, an ancient case of verbal quibbling, mentioned so disparagingly by Seneca, runs: "Mouse is a syllable, and a mouse eats cheese; therefore some

[2] From the Latin "aequivocari" meaning "to have the same sound."

syllables eat cheese." [3] The law of logic here involved is the valid mood of the syllogism Darapti, "If all *b* is *a* and all *b* is *c*, then some *c* is *a*," which has been incorrectly applied: the term "mouse" is not the same in the two premises, however identical the two uses may appear to the ear or eye.

Terms which have more than one meaning are called *equivocal;* the correlative of equivocal is *univocal,* which indicates that the given term is used in only one sense.

The term "nothing" in English is equivocal, and fallacies of this sort frequently are committed as a result. Thus (apparently) an apple is better than heaven, since an apple is better than nothing, and nothing is better than heaven ("is better than" being a transitive relation). Another and more ingenious equivocation on the same term is involved in the solution of Lewis Carroll's problem: "It is required to distribute twenty-four pigs in four pens placed in a circle, so that as one goes around, each pen he comes to will have the number of its pigs nearer ten than the preceding." This problem would be easy if the traveler was supposed to stop at the end of one round, but the stipulation is that he may begin where he pleases and keep going around forever.

Plato, in his dialogues, enjoyed making fun of those Sophists who delighted in word-quibbling. The following anecdote from the *Euthydemus* illustrates the manner in which these traveling teachers made use of the fallacy of equivocation. Cleinias, an unsuspecting youth of Athens, is about to be ensnared in the traps of two bloodthirsty sophists, Euthydemus and Dionysodorous. Euthydemus begins by proposing a question:

" 'Tell me, Cleinias, are those who learn the wise or the ignorant?' Cleinias answers that those who learn are the wise.

"Euthydemus proceeds: 'There are those whom you call teachers, are there not?' The boy assents.

" 'And they are the teachers of those who learn, and their pupils are the learners?'

" 'Yes.'

" 'And when you were learners you did not as yet know the things which you were learning?'

" 'No,' he says.

[3] In the original: "Mus syllaba est; mus autem caseum rodit; syllaba ergo caseum rodit."

" 'And were you wise then?'
" 'No, indeed,' he said.
" 'But if you were not wise, you were unlearned?'
" 'Certainly.'
" 'You, then, learning what you did not know, were unlearned when you were learning?'
"The youth nods assent.
" 'Then the unlearned learn, and not the wise, Cleinias, as you imagine.'
"At these words the followers of Euthydemus like a chorus at the bidding of their director, laugh and cheer. Then, before the youth has time to recover, Dionysodorus takes him in hand and says: 'Yes, Cleinias; and when your grammar teacher dictated to you, were they the wise or the unlearned who learned the dictation?'
" 'The wise,' says Cleinias.
" 'Then after all the wise are the learners and not the unlearned; and your last answer to Euthydemus was wrong.'
"Then follows another peal of laughter and shouting, which comes from the admirers of the two heroes, who are ravished with their wisdom . . ." [4]

Equivocation gives rise to the most common type of joke, the pun. Punning was a favorite pastime of the Elizabethan theater, and the solemn context in which puns often appear would lead us to conclude that they were regarded as a form of wit rather than ribald humor. Thus Richard II, in one of the most tragic parts of the play, exclaims,

"Swell'st thou, proud heart? I'll give thee scope to beat, since foes have scope to beat both thee and me."

But puns, as a type of humor, go back much further than the sixteenth century. In the *Iliad,* Odysseus takes good advantage of equivocation to outwit the giant one-eyed Cyclops, Polyphemus, who keeps the hero and his followers imprisoned in a cave. Odysseus tells the giant that his name is "Noman," and then in the dark of night he and his band drive a stake through the evil giant's eye. The bellowing that ensues causes all the neighboring Cyclopes to rush to Polyphemus's cave, but when they ask him who has done this horrible deed, he can only answer, "Noman has done it"; in disgust the others return home, and Odysseus and his band escape.

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[4] Condensed from Jowett's translation.

While many puns depend upon the fact that two distinct words have the same pronunciation, though different spellings, as, for example, in the famous German saying: "Man ist was er isst," [5] other puns occur when two words have the same spelling but different pronunciations, an example of which is the Latin "Mater tua est sus." [6]

Unintentional puns seem to be the most delightful. Thus Wordsworth is reported to have said that, had he a mind to, he could write like Shakespeare.

Not only the humorous, but the serious, and even the tragic, often have their source in the ambiguities of terms. The Athenians were placed in serious predicament when the oracle at Delphi informed them that their only salvation against the Persians lay in their wooden walls. One party insisted that "wooden walls" meant a fortification, another that the term must mean ships. And there is a story to the effect that one of the captains of the guard in the palace of Louis Phillippe, during the time of the mob uprisings, having a bad cold, exclaimed "Ma sacré toux!" This the guard took to mean "Massacrez tous!" and they fired on the mob, killing many, with disastrous results.

Because of an ambiguity in the term "person" in the Constitution of the United States, the Bill of Rights did not have to be changed after the freeing of the slaves; indeed, in the history of the country, many terms and phrases of the Constitution have acquired new meanings, such flexibility probably being more of an advantage than a disadvantage.

Equivocation is frequently used to give rhetorical effect to what might otherwise seem a very trite phrase. For example, the proposition "Business is business" means more than the simple tautology "*a* is *a*." Lincoln is said to have made the following remark during the inspection of some trenches after a hard battle:

"Anyone who likes this sort of thing must enjoy it very much."

Lewis Carroll shows the implied ambiguity in such phrases in *The Three Voices*. An epicure, in defence of his philosophy, urges that

"Dinner is dinner and tea is tea."

[5] "Man is what he eats."
[6] "ĕst" = "is," but "ēst" = "eats."

But the lady of the poem, in reply, overturns his position by taking his statement literally:

> . . . "Yet wherefore cease,
> Let thy scant knowledge find increase;
> Say men are men, and geese are geese."

Composition and Division

The Fallacies of Composition and Division are really special cases of equivocation in which the term "all" is used ambiguously. In English this term has two distinct meanings: used *collectively*, "all" refers to the total collection in question, as, for example, in the phrase "all the house" ("the whole house"); used *distributively*, "all" means "each and every one," as in the expression "all the numbers" (of which there is no collection). The two fallacies are defined as follows: the Fallacy of Composition consists in taking collectively what should be taken distributively (separately), while the Fallacy of Division consists in taking distributively what should be taken collectively. Since most illustrations of these fallacies are cases in which one premise uses "all" in the collective sense, the other in the distributive sense, these may be classified under either head depending on which premise is taken to be true. Thus the following propositions appear to be contradictories, despite the fact that they are both valid in Euclidean geometry:

> "All the angles of a triangle are less than two right angles."
> "All the angles of a triangle are equal to two right angles."

The "all" in the first proposition is the distributive "all," the "all" in the second, the collective.

In many cases, the word "all" is implicit, as in the following syllogism, which might appear to be formally true:

> Peter was an apostle;
> The apostles were twelve men;
> Therefore, Peter was twelve men.

We cannot say that the argument is bad because the term "apostle" is used in the singular in one premise and in the plural in the other, since such an argument as "This is a table and all tables are furniture, and therefore this is furniture" is perfectly valid. But the syllogism does contain four terms, for "apostle" is used in the one case as a class taken distributively, in the second as a class taken collectively, i.e., as a singular term. All finite classes may be taken in either of these two ways: our interest may be centered on the members or on the entire group taken as a unit. Thus, all the people over twenty-one in the United States (taken as a unit) form the voting public, but we cannot infer that John Smith forms the voting public because he is a member of the class of people over twenty-one (taken distributively).

Fallacies are frequently committed in discussions of the concept of probability. One might argue as follows: Given an "ideal" coin, i.e., one as likely to fall heads as tails, the probability that in five throws heads will appear each time is quite small (actually the chances are 1 in 2^5), yet the chances that in any particular throw heads will appear is exactly one-half. Suppose that in four throws heads appeared each time; then on the fifth throw (apparently) the chances that heads will appear will be 1 in 2^5 and also 1 in 2. This argument confuses the collective and distributive; distributively, i.e., one by one, on each throw the chances of head's appearing is exactly one-half, and this is as true of the fifth, or the fiftieth throw as it is of the first. But we may take a certain number of cases and consider them as a collection, asking ourselves how the distribution of heads and tails will occur throughout this collection; the chances that we will have a collection of five heads in a collection of five throws is 1 in 2^5, though the chance that any particular element will be heads is one-half.

The Fallacy of Composition is most frequently committed in discussions regarding the concept of the infinite. Most beginners feel inclined to treat the symbol ∞ as though it were another number, obeying the usual properties of numbers. That it cannot be so treated is demonstrated in many ways. If ∞ were a number the equation $12 \cdot \infty = 3 \cdot \infty$ would be true, since to multiply an infinite quantity by any finite quantity yields an infinite quantity. In general, if $ab = cb$, then $a = c$, and hence $12 = 3$, which is absurd. But ∞ cannot be considered as another number because it does not represent any complete quantity. A variable x is said to become infinite ($x \rightarrow \infty$) if for any finite number we may choose, we can find larger values of x.

"Infinity," then, must be taken in the distributive sense. When we speak of such infinite quantities as the set of "all" numbers, we cannot mean this "all" in the collective sense, since there is no collection of all the numbers.

The Italian mathematician, Saccheri (*ca.* 1700), forfeited the distinction of being the founder of non-Euclidean geometry because he confused the properties of the finite and the infinite. He argued (correctly) that if we deny Euclid's postulate concerning parallel lines (page 12), then two straight lines can become asymtotic, i.e., can approach each other indefinitely but never meet at any finite distance. Hence, argued Saccheri, these lines must meet at infinity and there become one line; but since two straight lines cannot have a segment in common, a contradiction (he thought) was reached.

It should not be assumed from this, however, that in cases where an infinite number of objects is under consideration, we cannot assert general laws concerning them. Thus, when I say "All propositions are either true or false," I might seem to be asserting something absurd, since there is no such thing as a collection of all propositions, there being an infinity of meaningful statements. But "all" is used here in the distributive sense, to mean "Each and every proposition which might arise is true or false." Or, as we have put it earlier, the expression "All propositions are either true or false" is equivalent to the statement " 'This proposition is either true or false' is true no matter to what proposition 'this' refers."

In a later chapter more difficult cases of these fallacies will appear.

Amphibology

Ambiguities in language frequently occur, not from the equivocation in a single term, but from the ambiguous construction of the sentence as a whole, and fallacies resulting from such ambiguities we call Fallacies of Amphibology.[7] The ambiguities involved in the syntax of indirect discourse in Greek is said to have been the chief cause of the success of the Oracle at Delphi. In Greek, as in Latin, the subject and direct object

[7] From the Greek *ἀμφιβάλλειν*, "to doubt": *ἀμφι* = "on both sides," *βάλλειν* = "to throw."

in the dependent clause in indirect discourse ("he says that—") are both in the same case (accusative), and since word-order does not signify which noun is to be taken as subject, an ambiguity often results. Thus, were the anxious general to ask the Oracle whether the Greeks would win the battle he proposed to fight, the oracle might reply: "Apollo says that the Persians the Greeks will defeat." If perchance the Greeks lost, one argued that the Oracle must have intended "Persians" to be the subject, "Greeks" the object; if the Greeks won, the opposite was intended.

One of the greatest difficulties for a writer in English is the problem of avoiding ambiguities in sentence structure caused by the paucity of relative and demonstrative pronouns. The following sentences exhibit such ambiguities:

"No one is allowed to see the patient who is very ill."

"He carried the dog across the room and tenderly placed it on the sofa; it was an overstuffed specimen."

"Dangling participles" frequently yield humorous cases of amphibology:

"Sailing gracefully along under her own power, Jane was fascinated at the spectacle of the ship before her."

"Sitting in a comfortable Pullman seat, the scenery flashed by."

One of the rare pieces of humor which occurs in Kant is a case of amphibology: "In this manner, then, results a harmony like that which a certain satirical poem depicts as existing between a married couple bent on going to ruin, 'O, marvellous harmony, what he wishes, she wishes also' "; or like what is said of the pledge of Francis I to the Emperor Charles V, 'What my brother Charles wishes that I wish also. (viz. Milan)' "[8]

This type of fallacy often occurs in misapplications of rules. The now-classic example took place when a young Chinese student at an American university, having carefully memorized the proper phrases in his English-conversation book, replied to the dean as he accepted his cup of tea: "Thank you, Sir, or Madam, as the case may be."

Lewis Carroll made frequent use of amphibology.

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[8] *Fundamental Principles of the Metaphysic of Morals*, Abbott's translation.

Alice, slightly overcome by all the running she has had to do, asks the king: "Would you be good enough to stop a minute to get one's breath?" To which his majesty replies: "I'm *good* enough, only I'm not *strong* enough. You see, a minute goes by so fearfully quick. You might as well try to stop a Bandersnatch."

Accent

The fallacy of accent really falls under amphibology in the general sense. Practically every sentence changes its meaning when different words are accented, as one may easily verify for himself by emphasizing the various words in so every-day an expression as "I hope you will come tonight." To accent the initial word is to give the impression that others do not share in the wish, while to accent the third implies that the one invited might well leave others at home. Often proper accent will give meaning to an apparently meaningless sentence. The most astonishing example of this, which also illustrates the peculiarities of the English language, is the following, which, if properly punctuated, is meaningful: "John, where Jim had had had had had had had had had had had the teacher's approval." [9] The philosopher's dogma, when unpunctuated, also appears to be nonsense: "That that is is that that is not is not."

Apparent contradictions are often resolved by correct punctuation or a shift of accent. One might truthfully assert both the following propositions: "Three times two plus four is eighteen" and "Three times two plus four is ten."

Serious predicaments may arise over the ambiguity of accent. The warden of a Russian prison, having demanded instructions regarding a prisoner, is reported to have received this telegram from the Czar: "Pardon impossible jail him."

Amusing situations and remarks resulting from the ambiguity of accent are plentiful enough. Perhaps the most ancient example is to be found in I Kings, 13:27: "He spake to his sons, saying, Saddle me an ass, and they saddled him." There is also the story of an amateur of the stage, whose dramatic moment consisted in the cry "My Heavens, I'm shot!" To add a little realism to the moment, his colleagues secretly smeared red on his shirt. The unfortunate actor, having given

[9] ". . . had had 'had,' had had 'had had'; 'had had' had had . . ."

his all to the rendition of his line, happened to look down, and with more realism than drama, repeated his line: "My Heavens, I *am* shot!"

The Fallacy of Accent is a favorite tool of the rhetorician in persuading his audience. By omitting or passing lightly over the weaknesses in his position, he emphasizes and enlarges upon the errors of his opponents. That is, he emphasizes one point of view, so that we become convinced by an argument that in itself is hardly sufficient grounds for the validity of his entire position. Thus, in attempting to discover the nation guilty of the First World War, one might accent the murder in Austria and show how this led to the conflict. Such an argument, however, would neglect entirely the militarism and imperialism that had engulfed all the great European powers and that were certainly responsible in part for the tragedy. Again, in attempting to show the values of democracy, one might accent the freedom of speech which this form of government allows and underemphasize the difficulties which arise when a legislative body is forced to shift its policy continuously as public opinion changes. But the defender of totalitarianism accents the orderliness of governmental procedure and ignores the pain and suffering that are the lot of the dissenter.

In general, most of our strong prejudices and dislikes are based on the Fallacy of Accent. We think little of a particular man because he smokes evil-smelling cigars; a certain book is bad because it has a weak ending; Jones cannot be a good Senator since a certain dishonest politician is a close friend of his.

If one issue is much more important than any other in deciding a question, then no fallacy is committed, since this kind of error rests on improper accent.

Accident

The Fallacy of Accident consists in assuming that what is accidental in a certain application of a law must be true in general also (*a dicto secundum quid ad dictum simpliciter*). For example, we agree to the general law that all Athenians are Greeks. Plato, being an Athenian, must have been a Greek. It is an accidental property of Plato that he is a philosopher. Hence all philosophers are Greeks. Again, one might argue as follows: "You are not what I am; I am a man; therefore, you are not a man." Or, "He who calls you a man speaks truly; he who calls you a fool calls you a man; therefore, he who calls you a fool speaks truly."

Another form of this fallacy is committed when one assumes that a statement made about an object when considered simply will hold for this object when accidental attributes have been added (*a dicto simpliciter ad dictum secundum quid*). Thus, "Man is a rational animal" is a statement made about the term "man" taken "simply," i.e., the term is considered only with regard to its essential attributes; one might then try to argue that a dead man, or a drunken man, is rational, i.e., that the former statement will be true when the term is no longer considered simply, but under accidental conditions. Again, the statement that one should not thrust a knife into another seems to be a sound moral axiom; but it would be fallacious to argue that a surgeon is immoral, since his whole business consists in this activity.

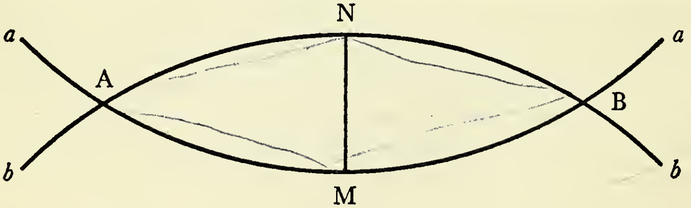
Petitio Principii

A fallacy that has played a peculiar role in the history of logic is that of Petitio Principii, or Begging the Question. This fallacy is often supposed to occur when one assumes in his argument the proposition to be proved. Thus a moralist might argue: "I ought not to do this act, because it is wrong," but if asked how he knows the act to be wrong, he replies, "Because I know that I ought not to do it."

But the error in thus defining the fallacy is that, far from violating a logical law, it would actually be an application of the most fundamental of all logical laws, the law of tautology. This law in its most general form may be expressed: "If the proposition p be true, then p is true," or, more generally, "If the propositions p, q, r, etc., are all true, then p is true." That is, we do not contradict any logical law in assuming a given proposition in our argument and then "proving" it later.

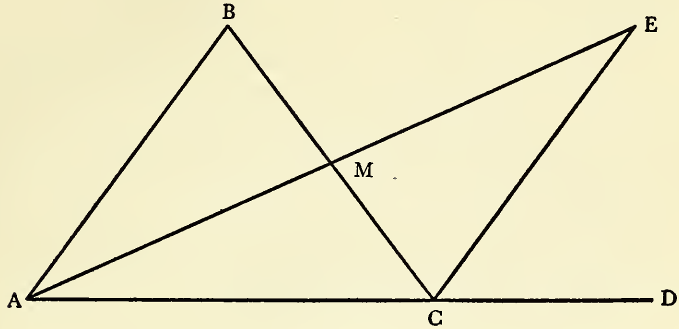
The fallacy really arises in the following manner: let us suppose a certain set of propositions to be granted. Someone wishes to show that when these propositions are true, another proposition follows. However, in his demonstration he assumes (usually tacitly) some other proposition, as a rule the one to be proved. Hence, while pretending to show that the statements p, q, r, etc., imply w, he has merely shown that $p, q, r,$---, *and* x imply w. As originally intended, the Fallacy of Begging the Question was committed when this x was the same as the conclusion, w, but it is easy to see that the fallacy in this form is really a restricted case of a more general one.

The fallacy in question as often as not deceives the arguer as well as others, as it often is committed unintentionally. A significant example of this occurs in geometry. Some of the followers of Euclid took Postulate 6, "Two straight lines meet in only one point" to be a theorem of his system, since, they claimed, this follows from Proposition 16: "The exterior angle of a triangle is always greater than either opposite interior angle." For suppose that two lines, a and b, did meet in two points, as in the accompanying figure. Let these points be A and B. Bisect the length AB on the line a, calling the midpoint M, and bisect AB on b, calling this midpoint N. Join MN. Then triangle AMN is congruent with triangle MNB, since the three sides of one are equal respectively to the three sides of the other.



Hence the angles at M are right angles, as are the angles at N. Therefore the exterior angle NMB of the triangle ANM is equal to one of the interior and opposite angles, ANM, and this contradicts Proposition 16 and hence the assumption that two lines meet in two points, since it leads to this contradiction, must be false.

But this argument is fallacious in that the proof of Proposition 16 assumes that two straight lines meet in but one point. This proof proceeds as follows: Suppose triangle ABC is any given triangle with the exterior angle BCD, as in the accompanying figure. Bisect BC at M, draw AM, and extend it to E, so that AM = ME. Draw EC. Then triangles ABM and ECM are congruent (side, angle, side), and hence angle ABC is equal to MCE, i.e., the interior angle of ABC is equal to only a part



of the exterior angle BCD. But this proof assumes that the two lines AE and AD meet in only one point. What happens if they do not?

Karl Pearson in his *Grammar of Science* commits the fallacy of Begging the Question. Comparing the human mind to a clerk in a telephone exchange who receives his messages at the brain terminals of the sensory nerves, he says: "Messages in the form of sense impressions come flowing in . . . But of the nature of things-in-themselves, of what may exist at the other end of our system of telephone wires, we know nothing at all."

If this analogy has any meaning, it is clear that we know a great deal about the external world. We have assumed, for example, that the external world exists and does make impressions on the sensory nerves, that the nerves are like telephone wires, that they do "convey" messages, etc.

John Locke commits the same fallacy in his argument that we derive the idea of the succession of events in time from our senses and reflection alone: "It is evident to anyone who will but observe what passes in his own mind, that there is a train of ideas which constantly succeed one another in his understanding as long as he is awake. Reflection on these appearances of several ideas one after another in our minds, is that which furnishes us with the idea of succession . . ." [10] If we interpret this argument correctly, it seems to say that we gain the idea

[10] *Essay concerning Human Understanding,* Book II, chap. XIV, 3.

of succession by observing succession, i.e., it assumes that we do recognize the meaning of the train of ideas succeeding one another in order to show that we have the idea of succession.

A modified form of this fallacy occurs in certain definitions. An old maxim with regard to proper defining was to the effect that the *definens* must not contain any part of the *definendum*; that is to say, any definition which assumes for its complete meaning the meaning of the thing to be defined is an unsatisfactory definition. Thus the Italian mathematician, Veronese, defined equality between two numbers thus: "Numbers whose units correspond to one another uniquely and in the same order and of which the one is neither a part of the other nor equal to a part of the other are equal."

Ignoratio Elenchi

This fallacy is a common one in everyday discussions; it consists, literally, in an ignorance of the point at issue, and he who commits it attempts to refute the argument of his opponent by proving something quite irrelevant. There are many subdivisions of the fallacy, depending on what course the irrelevant response may take. A common form is to be found in arguments employed during a trial. The real point at issue is to determine the prisoner's guilt or innocence, but counsel for the defence will very often devote the majority of its time showing the high type of character the accused has, while the prosecution, on the other hand, attempts to establish his perfidy, dishonesty, and other negative qualities. Neither lawyer argues to the point, but rather argues concerning the individual in question; this form of *ignoratio elenchi* is called *argumentum ad hominem*. Thus De Morgan reports the case of a lawyer who handed his brief to the pleading attorney with the comment: "No case—abuse the plaintiff's attorney."

The Fallacy of *Ipse Dixit* is really a special case of *argumentum ad hominem*. A argues that a certain proposition must be true because B says it's so. Appeals to authority are so common that illustrations are hardly necessary. But this type of argument is often offered as a way of determining truth, in which case its proponents would hardly consider it to be a fallacy (cf. page 147).

If one speaks to arouse the emotions of his hearers, rather than their reason, he commits this fallacy under the form of *argumentum ad misericordiam*. A beautiful example of this occurs in Socrates' speech at his trial before the Athenian Assembly; note that the speaker, though insisting that he will not commit the fallacy, really does so in a subtle fashion:

"Well, Athenians, this and the like of this is all the defense which I have to offer. Yet a word more. Perhaps there may be some one who is offended at me, when he calls to mind how he himself on a similar, or even a less serious occasion, prayed and entreated the judges with many tears, and how he produced his children in court, which was a moving spectacle, together with a host of relations and friends; whereas I, who am probably in danger of my life, will do none of these things. The contrast may occur to his mind, and he may be set against me, and vote in anger because he is displeased at me on this account. Now if there be such a person among you,—mind I do not say that there is,—to him I may fairly reply: My friend, I am a man, and like other men, a creature of flesh and blood, and not 'of wood or stone,' as Homer says; and I have a family, yes, and sons, O Athenians, three in number, one almost a man, and two others who are still young; and yet I will not bring them hither in order to petition you for an acquittal. And why not? Not from any self-assertion or want of respect for you . . . There seems to be something wrong in asking a favor of a judge, and thus procuring an acquittal, instead of informing and convincing him. For his duty is not to make a present of justice, but to give judgment; and he has sworn that he will judge according to the laws, and not according to his own good pleasure; and we ought not to encourage you, nor should you allow yourselves to be encouraged, in this habit of perjury —there can be no piety in that. Do not then require me to do what I consider dishonorable and impious and wrong, especially now, when I am being tried for impiety. . . . For if, O men of Athens, by force of persuasion and entreaty I could overpower your oaths, then I should be teaching you to believe that there are no gods, and in defending should simply convict myself of the charge of not believing in them. But that is not so—far otherwise. For I do believe that there are gods, and in a sense higher than that in which any of my accusers believe in them. And to you and to God I commit my cause, to be determined by you as is best for you and me." [11]

A vicious form of the Fallacy of *Ignoratio Elenchi* occurs

[11] Plato, *Apology*, Jowett translation.

when one attempts to discard the arguments of another by giving them a label. "Such feelings are decidedly un-American," he declares. "This is sheer communism," "Opinions like this are heretical," or even the trite "These thoughts are dangerous for a young man of your age," are all common enough. That is, we pretend to refute the argument by declaring it to be something reputedly unsound, though we fail to define what we mean by "communism," "heretical," and other such terms, or to show what is wrong with these beliefs.

Many Questions

A fallacy of the humorous sort is that of Many Questions. In its strict form, this fallacy consists in assuming that every question of the direct sort may be answered by "yes" or "no." Unfortunately, to answer some questions in either the affirmative or negative presents awkward difficulties. One would hardly feel inclined to answer "yes" to the question "Have you stopped beating your mother?" but to answer "no" would be even worse! There are a multitude of questions of a similar sort: "Has your town another horse yet?" "Have you given up your drinking habit?"

Despite its ludicrous aspect which has caused many to disregard its importance, this fallacy does present a logical problem. It has been asserted that the Law of Excluded Middle, "All propositions are either true or false," is a fundamental law of logic. But apparently some such statement as "I have stopped doing thus-and-so" in many cases is neither true nor false.

The solution of the difficulty can be found by properly defining the word "stop." The equivalent of "I have stopped beating my mother" is "I have been beating my mother and I am not beating her now." The latter is in the form of a conjunction and asserts that "p is true and (but) q is true." A conjunction is false if one of the elements is false; hence, if p is actually false, as it would be if I had never indulged in the habit of beating my mother, then the entire statement "p is true and q is true" will be false. There are two ways, then, of denying the statement "I have stopped doing so-and-so"; one may deny that one has ever done the action or one may deny that he is doing it now. The fallacy lies in assuming that when one says he has not stopped doing so-and-so, this must imply that he is continuing to do it.

False Cause

Most superstitions are grounded in the Fallacy of False Cause, which consists merely in attributing the wrong cause to a certain event. The most common error, called by the schoolmen *post hoc ergo propter hoc*, lies in inferring that a certain event must be the cause of another because it precedes the other. Thus, after a black cat crosses my path, I leave my pocketbook somewhere and deduce that the cat caused me to lose it, henceforth considering black cats as augurs of certain ill luck. Many people consider the number thirteen so unlucky that they refuse to work or sleep on the thirteenth floor, believing dire disaster will result. Many hotels and other large buildings commit a fallacy of another sort by simply having the floor above floor number twelve numbered fourteen.

The instinct associated with this fallacy is demonstrated in certain psychological experiments on animals. A bell precedes the intermittent feeding of a dog, and after a certain length of time, the animal's natural processes of salivary excretion and digestion can be initiated solely by the ringing of the bell, regardless of whether the meal follows.

One must not be too critical of the fallacy, however. If I walk under a ladder and a paint can falls on my head, I would not be arguing falsely if I ascribed my ill luck to walking under a ladder. Indeed, scientific procedure takes very much the same path (though greatly refined, of course, by other considerations) in determining whether A is the cause of B. Those eager but inexact scientists of old, who declared that a change in weather is due to a change in the moon, that a plague is caused by an eclipse of the sun, and other like prophecies, were committing the fallacy, though many were trying to apply good scientific procedure. A more accurate definition of the fallacy would be the inference that A is the cause of B, since A has been found in one case (or in an indefinite number of cases not experimentally controlled) to precede B.

False Analogy

The argument from analogy is not usually classified as a fallacy, principally because it has proved to be a very useful tool for the scientist. But many serious errors in science have occurred because of this method of reasoning. The fallacy consists in assuming that if two phenomena are closely alike in certain respects, they will be alike in most if not all respects, since they are "analogous."

Thus, when scientists began to learn more about the structure of the atom, they were amazed to discover the close similarity between this minute piece of the universe and the solar system, for the atom contained a nucleus about which revolved a certain number of electrons, much in the same manner as the planets revolve above the sun. Nor was there such a great discrepancy in relative distances; the distance of an electron from its nucleus, relative to the size of the electron, was comparable to the distance of the earth from the sun, relative to the size of the earth. But the argument that the path of an electron about the nucleus is a special case of the same law as that governing the motions of the planets seems to have been false, and the attempts to construct atomic theories on this basis have failed.

An interesting example of the Fallacy of False Analogy occurred in antiquity, when the same type of argument was used by Plato to prove the immortality of the soul and by Lucretius to prove its mortality. Plato's argument runs thus:

"Let us consider this question, not in relation to man only, but in relation to animals generally, and to plants, and to everything of which there is a generation, and the proof [of the soul's immortality] will be easier. Are not all things which have opposites generated out of their opposites? . . . I mean to say, for example, that anything which becomes greater must become greater after being less, . . . and the weaker is generated from the stronger, and the swifter from the slower. . . . This holds of all opposites, even though not expressed in words—they are generated out of one another, and there is a passing or process from the one to the other of them. Since life and death are opposites, these are generated the one from the other and have their intermediate processes, too; hence, just as the waking are generated from the sleeping, so are the living from the dead. Hence we arrive at the inference that the living come from the dead, just as the dead come from the living; and if this be true, then the souls of the dead must be in some place out of which they come again." (Plato, *Phaedo,* Jowett translation, condensed)

But analogy is also used by Lucretius to show that the soul is mortal:

"We feel that the mind is begotten along with the body, and grows up with it, and with it grows old. For as children running about have a body infirm and tender, so a weak intelligence goes with it. Next when their age has grown up into robust strength, the understanding and the power of the mind is enlarged. Afterwards, when the body is wrecked with the mighty strength of time, and the frame has succumbed with blunted strength, the intellect limps, the tongue babbles, the intelligence totters, all is a-wanting and fails at the same time. It follows, therefore, that the whole nature of spirit is dissolved abroad like the smoke into the high winds of the air, since we see it begotten along with the body, and growing up along with it, and, as I have shown, falling to pieces at the same time, wearied with age. Add to this that just as the body itself is liable to awful disease and harsh pain, so we see the mind liable to carking care and grief and fear; wherefore it follows that the mind also partakes of death." [12]

The philosophical school of empiricism is based on the postulate that we can only know those things that we perceive through our senses (cf. page 150). Several consequences seem to result: if we can only know the things of our perceptions, then we cannot know anything of the world apart from our minds, since such knowledge would have to be free of our perceptions. The manner in which an empiricist, Karl Pearson, tried to explain this difficulty by means of analogy is given under the Fallacy of Begging the Question: the example cited there may be considered as an illustration of false analogy as well. Another difficulty which faces the empiricist, a difficulty that many have tried to overcome by the analogy argument, is the problem of the existence of other minds; if perception is the basis of all knowledge, then how shall we prove that minds other than our own exist, since we never perceive another's mind, but only his body and its actions. The "analogy argument" here runs as follows:

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[12] Lucretius, *De Rerum Natura,* III, pp. 445-462, Loeb translation.

"I am aware, and I alone am aware, that certain of my bodily acts are accompanied by mental states. When I observe similar acts in other bodies I infer that they too are accompanied by like states of mind. No experience can be brought to confirm this inference, but then nothing can transpire to refute it. Meanwhile, my feelings are spared a severe strain by risking it—the loneliness of not risking it is too tragic to be faced."

The objectionable points of this line of argument are just all the points of its make-up. To begin with, it is so far from self-evident that each man's mental state is his own indisputable possession, no one hesitates to confess at times that his neighbor has read him better than he has read himself . . . No one finds fault with Thackeray for intimating that the old Major is a better judge of Pendennis's feeling for the Fotheringay than is Pendennis himself . . . Next, the analogy argument calls its procedure an inference. Now, everyone knows an inference from a thousand cases to be more valuable than one drawn from a hundred, an anticipation based on a hundred observations to be safer than one with only ten to support it. But there are those who, knowing all this, would conclude that an inference from one instance has *some* value. If in my case mental states accompany my body's behavior, there is at least *some* ground for supposing like acts of another's body to be in a manner paralleled. This illusion, for it is one, springs I think from a failure to catch the meaning of inference. An inference from a single case, if it be really an inference from a single case, has exactly no value at all. No one would be tempted to attribute eight planets to every sun because our sun has eight such satellites. The reason a single observation is sometimes correctly assumed to have weight is that the method of observing has been previously tested in a variety of cases. The shopkeeper measures his bit of fabric but once; he has however measured other fabrics numberless times, and has a fairly clear idea of the probable error of his result. But the principle holds absolutely of all results: no series of observations, no probable error; no ground for inference; no meaning as a datum.[13]

The method of analogy, it should be noted, is not to be condemned simply because when applied without restrictions it leads to a fallacy. Science has found that analogies are an indispensable aid in suggesting and formulating theories; but however much of an aid the argument may be, it does not establish any conclusion on its own right, and inferences which are based on analogy alone are fallacious.

[13] E. A. Singer, Jr., *Mind as Behavior,* chap. I.

Non Sequitur

The preceding classification of the fallacies is evidently not complete. For the most part, Aristotle did not seem to have a very definite scheme of classification, but rather chose the common types of error. The result was that many fallacies are either identical with a special case of others, or else do not fall under any of the prescribed classes.

To obviate the latter difficulty, logicians commonly include the class of fallacies called *Non Sequitur*; such a fallacy is committed if one infers the truth of p from the truth of q when the inference is false. Thus, many have been inclined to assert that God does not exist because they have been convinced that his existence cannot be proved.

This fallacy, needless to say, is so general as to include all others as special cases. All fallacies fundamentally consist in invalid inferences. But in general, only those fallacies which cannot be classified otherwise are called cases of *Non Sequitur*. Often, however, *Non Sequitur* is supposed to occur when certain necessary premises have been omitted in an argument, and in this case the fallacy is closely allied to *Petitio Principii*. Sherlock Holmes constantly committed the *Non Sequitur* fallacy. Thus he astonished Watson on their first meeting by "deducing" that he must have come from Afghanistan. His deduction ran: "Here is a gentleman of a medical type, but with the air of a military man. Clearly an army doctor, then. He has just come from the tropics, for his face is dark, and that is not the natural tint of his skin, for his wrists are fair. He has undergone hardship and sickness, as his haggard face says clearly. His left arm has been injured. He holds it in a stiff and unnatural manner. Where in the tropics could an English army doctor have seen much hardship and got his arm wounded? Clearly in Afghanistan." But one can think of many other conclusions which would be consistent with these premises; for example, Watson might have been a retired army doctor of private means, who spent his time hunting. As far as "deduction" goes, Holmes committed a *Non Sequitur,* though his reasoning does involve a certain degree of probability. But questions of probability and questions of deduction are entirely different.

Non Sequitur is committed in those examples of syllogistic arguments given (page 77) which make use of invalid moods of the syllogism.

EXERCISES

GROUP A

Where possible, classify the passages below under one of the fallacies. In the case of arguments, criticize the reasoning involved:

1. No news is good news; war in Europe is no news; therefore, war in Europe is good news.
2. Wanted: woman to sew buttons on third floor.
3. Lightning is caused by the terrible intonation which clouds make when they collide.
4. Socrates was an Athenian. The Athenians were a nation. Therefore, Socrates was a nation.
5. There cannot be any vacuum in the world. For suppose there were a vacuum, i.e., a space containing nothing. Then if we had a vacuum, say, within a box, this would mean that the opposite sides of the box would touch, since there is nothing between them.[14]
6. "You shall do marvelous wisely, good Reynaldo, before you visit him to make inquiry of his behavior."[15]
7. Vaine Andum, the famous movie actor, smokes Hayo cigarettes.
8. Prosecution: "How long had you been in the murdered man's room before you murdered him?"
9. Oxygen supports combustion. Hence, since water contains oxygen, water must support combustion.
10. You should not lie. Therefore, all doctors should tell their patients the exact truth about their illnesses.
11. What you have not lost, you still have; you have not lost your horns; therefore, you still have them.
12. What is true of the whole must be true of its parts. It is true of the whole jury that it contains twelve members; hence, this must be true of each member.
13. What is true of the parts must be true of the whole; the income of no one in the United States exceeds one billion dollars; hence, the income of all must not exceed one billion dollars.
14. What is true of the parts must be true of the whole; it is true that every particle of matter in the world is completely deter-

[14] Descartes.
[15] *Hamlet.*

mined as to its motion and position by inexorable law; hence, a human being, who is made up of particles of matter, must be completely determined with respect to his motion and position.

15. "I do think I'd like another helping of pork."
"Well, cut a piece off yourself."
16. If A (*ab*) is true, then I (*ab*) is true; but if I (*ab*) is true, then O (*ab*) may be true; therefore, if A (*ab*) is true, O (*ab*) may be true.
17. The worst conceivable thing in the world must exist. For suppose X is the worst thing one can imagine. If a thing is bad, it is worse if it exists than if it is merely imaginary, as a real murder is worse than one which is just contemplated. Therefore, if X is the worst thing possible, it must exist.
18. Are we going to keep on submitting ourselves to the influence of foreign powers?
19. "EUTHYDEMUS: Do those who learn, learn what they know or what they do not know?
CLEINIAS: Those who learn, learn what they do not know.
EUTH.: Don't you know your letters?
CL.: Yes.
EUTH.: All letters?
CL.: Yes.
EUTH.: But when the teacher dictates to you, does he not dictate letters?
CL.: Yes.
EUTH.: Then if you know all letters, he dictates that which you know.
CL.: Granted.
EUTH.: Then you do not learn that which he dictates; but he who does not know his letters learns?
CL.: Nay, but I do learn.
EUTH.: Then you learn what you know, if you know all your letters."[16]
20. The world could not have had a beginning in time, since this would imply that there was a time when there was nothing, and this result is absurd for two reasons: (1) the notion of time is dependent on motion, but if nothing exists there can be no motion and hence no time; (2) if there were nothing at one time and something at a later time, then something must have come out of nothing, and this is absurd. On the other hand, the world cannot have existed forever, for then there would have been an infinite series of events up to this moment; i.e., this

[16] Plato, *Euthydemus*, Jowett translation.

moment would mark the end of an infinite series, which is absurd, since an infinite series has no end. (Attributed to the Sophist, Gorgias.)

21. Is a stone a body? Yes. Is not an animal a body? Yes. And you an animal? Yes. Then you are a stone.
22. The White King: Just look down the road and tell me if you see either of my messengers.
    Alice: I see nobody on the road.
    King: I only wish *I* had such eyes. To be able to see Nobody! and at this distance, too! Why, it's as much as I can do to see real people in this light. (The messenger arrives; the King addresses him:) Whom did you pass on the road?
    Messenger: Nobody.
    King: Quite right—this young lady saw him too. So of course Nobody walks slower than you.
    Messenger: I do my best. I'm sure nobody walks much faster than I do.
    King: He can't do that, or else he'd have been here first.[17]
23. Patrick was accused of stealing a neighbor's pig.
    "Well, now, Patrick," said the judge, "when you are brought face to face with Widow Maloney and her pig on Judgment Day, what account will you be able to give of yourself when she accuses you of stealing?"
    "You said the pig would be there, sir?" said Pat. "Well, then, I'll just say: 'Mrs. Maloney, there's your pig!' "
24. "The only proof capable of being given that an object is visible, is that people actually see it. The only proof that a sound is audible, is that people hear it: and so of the other sources of our experience. In like manner, I apprehend, the sole evidence it is possible to produce that anything is desirable, is that people desire it." [18]
25. "No reason can be given why the general happiness is desirable, except that each person, so far as he believes it to be attainable, desires his own happiness. This, however, being a fact, we have not only all the proof which the case admits of, but all which it is possible to require, that happiness is a good; that each person's happiness is a good to that person, and the general happiness, therefore, a good to the aggregate of persons." [19]
26. "Mr. Owen, again, affirms that it is unjust to punish at all; for the criminal did not make his own character; his education, and

[17] *Through the Looking Glass,* Lewis Carroll.
[18] *Utilitarianism,* J. S. Mill.
[19] *Ibid.*

the circumstances which surround him have made him a criminal, and for these he is not responsible."[20]

27. Improbable events happen almost every day; but whatever happens almost every day is probable. Therefore, improbable events are probable.
28. Our minds are much like wax tablets upon which our various senses write; he who has a retentive mind will remember well what he has learned just as good wax retains its impressions, but the forgetful mind is like a tablet made up of soft wax, since the marks readily disappear. But just as a tablet is blank before anyone writes upon it, so must our minds have been empty at birth before we had any sensations, so that everything we ever come to know we must learn through our senses, just as every mark a wax tablet contains must have been put there by some stylus.
29. The following argument, which appears in De Morgan's *Budget of Paradoxes,* purports to show that the world cannot be round: "How is't that sailors, bound to the sea, with a 'globe' would never start,
    But in its place will always take Mercator's LEVEL chart?"
30. De Morgan delights in exposing pseudo mathematicians[21] who attempt to prove the quadrature of the circle, i.e., that the circumference of a circle (or its area) can be expressed as some rational multiple of the radius; in other words, they want to prove that the value of π is a fraction whose numerator and denominator may be expressed as whole numbers. A certain Mr. James Smith demonstrated the required quadrature in the following manner: he supposes (*ex hypothesi*) that there is a circle whose circumference is $^{25}\!/_{8}$ of the radius; he then finds that certain consequences which follow are not inconsistent with the supposition upon which they were made.
31. De Morgan has an account of an itinerant lecturer in physics whose speech went something as follows: "You have heard what I have said of the wonderful centripetal force, by which Divine Wisdom has retained the planets in their orbits around the sun. But, ladies and gentlemen, it must be clear to you that if there were no other force in action, this centripetal force would draw our earth and other planets into the Sun, and universal ruin would ensue. To prevent such a catastrophe, the same wisdom has implanted a centrifugal force of the same amount and directly opposite." De Morgan remarks that if Divine Wisdom had just let the planets alone, it would have come out to the same thing "with equal and opposite troubles saved."

[20] *Ibid.*
[21] De Morgan, *op. cit.*

32. "Diderot paid a visit to Russia at the invitation of Catherine the Second. At that time he was an atheist, or at least talked atheism . . . His lively sallies on this subject much amused the Empress, and all the younger part of the Court. But some of the older courtiers suggested that it was hardly prudent to allow such unreserved exhibitions. The Empress thought so, too, but did not like to muzzle her guest by an express prohibition: so a plot was contrived. The scorner was informed that an eminent mathematician had an algebraical proof of the existence of God, which he would communicate before the whole Court, if agreeable. Diderot gladly consented. The mathematician, who was not named, was Euler. He came to Diderot with the gravest air, and in the tone of perfect conviction said, '*Monsieur!*

$$\frac{a + b^n}{n} = x,$$

donc Dieu existe; répondez!' Diderot, to whom algebra was so much Hebrew and whom we may suppose to have expected some verbal argument of alleged algebraical closeness, was disconcerted; while peals of laughter sounded on all sides. Next day, he asked permission to return to Paris, which was granted. An algebraist would have turned the tables completely by saying '*Monsieur! vous savez bien que votre raisonnement demande le développement de* x *suivant les puissances entières de* n.' "[22]

33. Lucretius, as a proof of the mortality of the soul, argues: "Moreover, when the piercing power of wine has penetrated into a man, and its fire has been dispersed abroad, spreading through the veins, why does heaviness come upon the limbs, why are his legs impeded, why does he stagger, his tongue grow tardy, his mind bemused, his eyes swim, noise and hiccups and brawls arise, and all the rest of this kind of thing follow, why is this, I say, unless it be that the vehement fury of wine is wont to confuse the spirit while yet in the body? But if anything can be confused and impeded, this indicates that if some cause a little more compelling should penetrate, the thing would perish and be robbed of its future life."[23]

34. "Again, do any seeds of spirit remain or not in the lifeless body? Now if any are left and are in it, it will be impossible rightly to consider the spirit immortal, since it has gone away diminished by the loss of some parts. But if it has departed and fled forth from its component parts so intact that it has left in the body no

[22] De Morgan, *op. cit.*
[23] *De Rerum Natura,* III, Loeb translation.

particles of itself, how do corpses exhale worms from flesh already grown putrid, whence comes all the great mass of living creatures boneless and bloodless that surge through the swelling limbs?" [24]

35. "The nature of mind and spirit is bodily; for when it is seen to drive forward the limbs, to arouse the body from sleep, to change the countenance, to rule and sway the whole man, and we see that none of these things can be done without touch, and further, that there is no touch without body,—must we not confess that mind and spirit have a bodily nature? Besides, you perceive the mind to suffer along with the body, and to share our feeling in the body. If the grim force of a weapon driven deep to the dividing of bones and sinews fails to hit the life, yet a languor follows and a backward fall to the ground, and upon the ground a turmoil that comes about in the mind, and sometimes a kind of hesitating desire to rise. Therefore the nature of the mind must be bodily, since it suffers by bodily weapons and blows." [25]
36. What has always been true in the past will probably occur again. What probably occurs may not occur. $2 + 2 = 4$ has always been true, and hence may possibly fail to be true.
37. An hour of study doesn't do us much good. Why bother?
38. Cutting down trees is wicked; for how would you like it if someone came along with an axe and cut you down?
39. This table is white; white is a color; therefore, this table is a color.
40. It is a tale told by an idiot full of sound and fury signifying nothing.
41. Lucretius attempts to prove that space must be infinite as follows: "If space were finite and if anyone should run to the very end of it and there throw a spear, this would fly beyond the place where it was thrown or something would stop it; in either case there would be some space beyond the 'end of space.'"
42. There is something immaterial in the world. For in the case of some people, it does not matter what food they eat, so long as it is nourishing; that is, the food they eat is immaterial.
43. I still have the car I bought last year; the car I bought last year was brand new. Therefore, the car I have now is brand new.
44. "I beg pardon?" said Alice.
    "It isn't polite to beg," said the king.[26]
45. I oppose this bill advocating naval rearmament because its sponsor knows nothing about the navy.

[24] *Ibid.*
[25] *Ibid.*
[26] Carroll, *op. cit.*

46. "Do you know languages?" asked the Red Queen. "What's the French for fiddle-de-dee?"
"Fiddle-de-dee's not English," Alice replied gravely.
"Whoever said it was?" said the Red Queen.[27]

47. "Experiment has shown that the heat or force necessary to decompose water would raise it to the height of 5,314,200 ft. This is only 210 feet less than 10 times the square of 729, which is the cube of 9. This shows that the foot as a measure of length is very nearly exact and that the yard is more appropriate than the meter."[28]

48. Night is the cause of the extinction of the sun; for as evening comes on, the shadows arise from the valleys and blot out the sunlight (early Greek physics).

49. "All smug people are bad; I'm glad I'm not like them." This man's statement must be wrong, for he shows himself to be smug when he criticizes smug people.

50. "Try another subtraction sum," said the Red Queen. "Take a bone from a dog and what remains?"
Alice considered: "The bone wouldn't remain, of course, if I took it—and the dog wouldn't remain; it would come to bite me—and I'm sure *I* shouldn't remain!"
"Then you think nothing would remain," said the Red Queen.
"I think that's the answer."
"Wrong as usual," said the Red Queen. "The dog's temper would remain."
"But I don't see how——?"
"Why look here," the Red Queen cried. "The dog would lose its temper, wouldn't it?"
"Perhaps it would," Alice replied cautiously.
"Then, if the dog went away, its temper would remain!" the Queen exclaimed triumphantly.
Alice said, as gravely as she could, "They might go different ways."[29]

51. "I've been telling you for the last five minutes his name's John Smith."
"Nonsense, he must have been John Smith longer than five minutes."

52. The assassination of the Archduke in Austria-Hungary was undoubtedly the cause of the First World War, since up to that point Europe had been at peace.

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[27] Carroll, *op. cit.*
[28] W. M. Malisoff, *New Budget of Paradoxes.*
[29] Carroll, *op. cit.*

53. A thing cannot be lost if we know where it is; hence the Lusitania is not lost, since we can point to its exact location.
54. You cannot name one thing which this man has done which would prove that he is capable of the position of President; I cannot see, therefore, any grounds for voting for him.
55. We may assume as a fundamental law of evolution that the fittest survive. For if a given living being is not fit, then, lacking some power necessary for life, it will die.
56. "I exist" is a proposition which is necessarily true; for if I doubt or deny this proposition, then I reaffirm it, since I must exist to do the doubting or denying.
57. "I think" is a proposition which is necessarily true; for if I doubt or deny this proposition, then I reaffirm it, since the act of doubting (or the act of denying) is an act of thinking.
58. We start with the following true statement:

$$4 - 28 + 49 = 64 - 48 + 9.$$

Now since each side is in the form of a perfect square $(a^2 - 2ab + b^2)$, we may extract the square root of both sides and derive:

$$(2 - 7) = (8 - 3);$$

that is,

$$-5 = +5.$$

59. The Stoics attempted to prove that the world must be endowed with reason as follows: "What has reason is better than that which has not. Nothing is better than the world. Therefore, the world must have reason." The sceptic, Carneades, attempted to refute this argument by pointing out that the same reasoning might be used to prove that the world is musical.
60. Cardan Swan offered the following argument to show that life can arise spontaneously: "Scoop out a hole in a brick, put into it some sweet basil, crushed. Lay a second brick upon the first so that the hole may be perfectly covered. Expose the two bricks to the sun, and at the end of a few days, the smell of sweet basil, acting as a ferment, will change the earth into real scorpions."
61. Man's soul is immortal, for when a man dies his soul goes to heaven and there is no death in heaven.
62. SHE: Don't you dare kiss me again!
HE (relenting): All right, I'll stop.
SHE: Don't you dare! Kiss me again!

63. Thibaut, a German mathematician, argued as follows to prove that the sum of the angles of a triangle must be equal to two right angles: "Let ABC be any triangle whose sides are traversed in order from A along AB, BC, CA. While going from A to B we always gaze in the direction AB*b* (AB being produced to *b*), but do not turn around. On arriving at B we turn from the direction B*b* by a rotation through the angle *b*BC, until we gaze in the direction BC*c*. Then we proceed in the direction BC*c* as far as C, where again we turn from C*c* to CA*a* through the angle *c*CA; and at last arriving at A, we turn from the direction A*a* to the first direction AB through the external angle *a*AB. This done, we have made a complete revolution,—just as if, standing at some point, we had turned completely round; and the measure of this rotation is 360 degrees. Hence the external angles of the triangle add up to 360 degrees, and the internal angles $A + B + C = 180$ degrees." Q.E.D.[30]

64. "GENERAL: Tell me, have you ever known what it is to be an orphan?

    KING: Often!

    GENERAL: Yes, orphan. Have you ever known what it is to be one?

    KING: I say, often.

    GEN.: I don't think we understand one another. I ask you, have you ever known what it is to be an orphan, and you say "orphan." As I understand you, you are merely repeating the word "orphan" to show that you understand me.

    KING: I didn't repeat the word often.

    GEN.: Pardon me, you did indeed.

    KING: I only repeated it once.

    GEN.: True, but you repeated it.

    KING: But not often.

    GEN.: Stop: I think I see where we are getting confused. When you said "orphan" did you mean "orphan"—a person who has lost his parents, or "often"—frequently?

    KING: Ah! I beg your pardon—I see what you mean—frequently.

    GEN.: Ah! You said often—frequently.

    KING: No, only once.

    GEN.: (Irritated) Exactly—you said often, frequently, only once!" [31]

65. There is an anecdote about Gilbert which tells of a man who rushed up to him at the door of a theatre and, mistaking him

[30] Bonola, *Non-Euclidean Geometry*, p. 63.

[31] Gilbert, *Pirates of Penzance*.

for a porter, shouted to him: "Call me a two-wheeler!" "All right," replied Gilbert. "You're hansom."

66. "It is absolutely and undeniably certain that something has existed from all eternity . . . For, since something now is, 'tis manifest that something always was: otherwise the things that now are must have been produced out of nothing, absolutely and without cause: which is a plain contradiction in terms. For, to say a thing is produced, and yet to say there is no cause at all of that production, is to say that something is effected when it is effected by nothing; that is, at the same time when it is not effected at all. Whatever exists, has a cause, a reason, a ground of its existence either in the necessity of its own nature, and then it must have been of itself eternal: or in the Will of some other Being; and then that other Being must, at least in the order of Nature and causality, have existed before it. That something, therefore, has really existed from eternity, is one of the certaintest and most evident truths in the world." [32]
67. Samuel Clarke offers the following proof to show that there cannot have been an infinite series of changeable and dependent beings produced from one another in endless progression, without any original cause at all: "But if we consider such an endless progression as one endless series of dependent beings; 'tis plain this whole series can have no cause from without, of its existence; because in it are supposed to be included all things that ever were in the universe: and 'tis plain that it can have no cause within itself, of its existence; because no one being in this infinite succession is supposed to be self-existent or necessary, but every one dependent on the foregoing: and where no part is necessary, 'tis manifest that the whole cannot be necessary, etc." [33]
68. Nothing is too good for you.
69. Sam is a sheep. Sheep circulate among themselves. Therefore, Sam circulates among himself.
70. Plato, in order to define justice, first defines it in the case of the state, and then applies his definition to the individual man with the following argument: "When two things, a greater and a less, are called by a common name, they are like in so far as the common name applies, so that a just man will not differ from a just state so far as the idea of justice is involved. But we resolved that a state was just when the three classes of characters present in it were severally occupied in doing their proper work: that it was temperate, and brave, and wise in consequence of certain

[32] Samuel Clarke, *A Demonstration of the Being and Attributes of God.*
[33] *Ibid.*

affections and conditions of these same classes. Hence we shall also adjudge, in the case of the individual man, that, supposing him to possess in his soul the same generic parts, he is rightly entitled to the same names as the state, in virtue of affections and conditions of these same classes." [34]

71. Is three many or few? A few. Then if three is a few, four is a few? Yes. And hence five is a few? Yes. And hence six? Yes. Etc., etc.
72. Club Bore: On one side of me, a lion was creeping up; on the other, a tiger approached stealthily. When they were about a yard from me, what do you think I did?
    New Member: Woke up?
    Club Bore (indignantly) : No, sir!
    New Member (in admiration) : Gee! I couldn't have slept on after that.
73. In the papal bull of Boniface VIII (1294-1303) the following is given as the reason why we cannot say that there are two distinct principles (the spiritual and the temporal) governing the church: ". . . quia, testante Moyse, non in principiis, sed *in principio coelum Deus creavit et terram* (cf. Gen. 1:1) ."

[34] *Republic*, Davies and Vaughn translation.

The Logical Paradoxes 10

ALL SCIENCES must face the serious problems that threaten their foundations. That logic is no exception to this rule was recognized early in the history of the science. Those arguments which pretend to show the invalidity of certain principles of logic were called "insolubilia" in the medieval or scholastic period, but in modern times these arguments are generally known as paradoxes. The same description might be applied to the fallacies as well; the distinction is purely one of degree. The weakness of the argument in the case of the fallacies is more readily detected than in the case of the paradoxes.

Medieval logicians recognized the importance of the "insolubilia," but because of their apparently trivial nature, the early modern writers classed them with the insignificant verbal quibbles of the scholastics. Much as modern philosophy is to be admired for clearing away the deadwood in medieval thought, there can be no doubt that in some instances it went to the other extreme and ignored much that was valuable, especially in the case of the science of logic. Practically no important contributions were made in this science between the fifteenth and the eighteenth centuries, and it was with the birth of modern symbolic logic a hundred years ago that a reawakening of interest in the subject took place. Indeed, only the last forty years have seen the construction of more or less firm foundations. With the increasing interest in logic came a return to the ancient problems which had beset the science.

The "Epimenides" or "Liar" paradox presents one of the fundamental problems of logic. In its less exact but more picturesque formulation it tells of a certain Cretan, Epimenides, who is reported to have said that all Cretans are liars. Besides the obviously unpatriotic tone of this utterance, there is involved an apparent denial of the Law of Contradiction, which states that no proposition can be both true and false at the same time. For Epimenides' statement certainly has meaning and is thus a proposition; but it must be true and it must be false. Assuming that "liar" means "one who never tells the truth," then if Epimenides is right, no Cretan ever tells the truth, including the said Epimenides, whose statement, therefore, cannot be right. There is no contradiction yet, of course. We deny no logical law when we show the existence of statements such that to assume them true is to infer their falsity; for example, the proposition "No proposition is true" has this property. Such propositions are, of course, necessarily false, and hence so must be the statement of Epimenides. But now, suppose Epimenides' statement is false. Then Cretans become truth-tellers, of which Epimenides is one, and what he has said must be true. Hence, from the denial of his statement we have inferred its truth; but statements which are such that to deny them is to assert them, are necessarily true (cf. discussion of indirect method of proof, page 27). Hence Epimenides is both right and wrong, and the Law of Contradiction no longer holds.

There is no difficulty discovering an obvious fallacy in this argument. It is true that Epimenides cannot be right; but if we assume him wrong, i.e., if we deny "All Cretans are liars," we merely assert "Some Cretans are not liars," the denial of a form in A being the assertion of a form in O. The argument, however, assumes that this denial means that "No Cretans are liars" and further assumes that if a man is not a liar (i.e., is not one who always tells a falsity), he must always tell the truth, all these unwarranted assumptions being necessary for concluding that Epimenides always tells the truth.

A more exact formulation of the paradox, and one which avoids these difficulties, is: Suppose one were to assert, "This proposition is false." No difficulty occurs if "this proposition" means "$3 + 5 = 7$," or "Green is a color." In these cases the speaker is either right or wrong, and that is the end of the matter. But if "this proposition" means "this one I am now stating," then the given statement is apparently both true and false (or, if you will, neither true nor false). For if the proposition is true, then its assertion that it is false is true, or it is false; if it is false, then its assertion that it is false is itself false, or it is true. Another, and perhaps clearer, formulation of the paradox may be made by considering the three propositions:

1. Number 2 is false.
2. Number 3 is false.
3. Number 1 is false.

To assume 1 is false is to prove it true, and to assume 1 is true, is to prove it false. For if number 1 is false, then the statement "Number 2 is false" must be false, i.e., number 2 must be true. But number 2 says that number 3 is false, i.e., that the statement "Number 1 is false" is a false statement; hence number 1 is true. That is, assuming number 1 false, we are led to the conclusion that it must be true; but if we assume that number 1 is true, then number 2 really is false, hence number 3 is true, hence number 1 is false. Number 1, then, is a proposition which is such that, if we assume it false, we infer it to be true, and if we assume it true, we infer it to be false; hence, it is both true and false at the same time.

A similar scheme may be constructed for any odd number of propositions. An even number yields no contradiction, but a paradox will result if we vary the form slightly. Thus, the following dialogue (a medieval illustration) displays the paradoxical character:

SOCRATES: What Plato is about to say is false.
PLATO: Socrates has just spoken the truth.

In the face of these difficulties, several courses are open. We may regard the arguments as trivial and discard them. But despite their simplicity of form, these paradoxes should not be passed over lightly. If we allow them to remain we admit defeat; we admit that logic contains contradictions. To assert that these contradictions will appear only in trivial cases and will not appear in the important ones, would be comparable to a salesman who would try to sell us a car with the remark that so far the machine has exploded only when insignificant people have been riding in it. Either logic is a coherent science or it is not. To assert that it is not is to assert the inconsistency of all sciences, since all sciences are necessarily logical.

One possible solution would be to accept the argument at its face value and admit that the Law of Contradiction is not universally true; this explanation was offered by the ancient sceptics. The opponents of Carneades (214-129 B.C.) had argued that we cannot be sceptical concerning the principles of logic, and the latter answered by displaying the above paradox (or one like it). This escape from the difficulty, however, should only be a last resort, for if we deny that the Law of Contradiction is universally true, we must proceed to try to show in what cases it fails and in what cases it does not. Such restrictions would demand a re-examination of all science, for science has so far proceeded on the assumption of the universal validity of the law, that is, an inconsistent law is assumed to be false.

Another solution, and one which medieval and modern logicians share, is to deny that the expression "This proposition is false" is really a proposition, that it has any meaning, when "this proposition" refers to the whole expression. Hence no denial of the Law of Contradiction follows, since this holds only for meaningful statements. This solution requires a definition of "meaningful," or at least certain criteria for meaningfulness. A statement is said to be meaningless if one of its terms is or includes the entire statement itself. Such statements are called "self-referring" propositions. Thus if I say "All propositions are false," I am asserting something meaningless if by "all propositions" I mean all propositions including the one I have just stated. Proponents of this solution sometimes argue that such a self-referring statement must be meaningless, in that "all propositions" refers to some definite, "closed" set which cannot be definite and must be incomplete if statements made about such a collection are a part of the collection. I cannot say *this* is the set of all propositions if I can then proceed to construct meaningful statements about this set. E.g., I cannot say, "These are all the true propositions of physics" and refer to some definite list, for the statement I have just made or its contradictory must be contained in the list and hence my claim that I have included all true statements is invalid. Hence, "all propositions" cannot mean "all, and this one too." But this argument is apparently fallacious, for it seems to commit a fallacy of composition. If "all" is used in the collective sense, then it is true that "all propositions" means a definite collection; but in this sense there is no such thing as "all propositions" since supposedly the set of all propositions is infinite. Rather, when I say "all propositions are false" I mean "all" in the distributive sense: "Each and every proposition (anyone may think of) is false," and this may be meaningful, without contradiction even though I include the statement in question as one of the propositions I may meet.

But however incorrect this reasoning may be, it may be assumed arbitrarily that self-referring propositions are meaningless; this is perfectly sound scientific procedure. If a certain assumption leads to the destruction of a fundamental principle of a science, the scientist is perfectly justified in rejecting this assumption, provided such rejection does not also destroy fundamental principles. When both the assertion and the denial of a statement lead to difficulties, then the given science must change its fundamental principles. In the next chapter we shall consider examples from the history of science in which such changes have been forced.

The assumption that self-referring propositions are meaningful apparently leads to difficulties. For if "This proposition is false" is meaningful when "this" refers to this proposition I am in the act of stating, we presumably have a meaningful statement which is both true and false, and we have denied a fundamental law of logic.

Does the assumption that self-referring propositions are meaningless likewise lead to insurmountable difficulties? If so, then there seems to be no alternative but to change our logic. At first sight a contradiction apparently does occur, for under this assumption the entire logic of propositions appears to become false. For example, among the fundamental laws of this logic there is the Law of Excluded Middle: "Every proposition is either true or false." If our assumption that self-referring propositions are meaningless is true, this law cannot refer to itself; that is, every proposition is either true or false except the Law of Excluded Middle, which is therefore meaningless.

To save logic from the chaos which would result from such an assumption, proponents of the solution have divided all propositions into "types." Propositions which talk about individual objects are placed in the first type. Examples of such statements are "Jones is a Senator," "The Cardinals won the World Series," or "I hate potatoes." Propositions which talk about propositions of the first type belong to the second type. For example, "Jones says that Greene is dishonest," or "All statements about your honesty are false." Propositions which have propositions of the second type as terms are of the third type.

We assume that propositions of the nth type must talk about propositions of the $(n - 1)$ th type or less; if a statement includes as one of its elements a proposition of its own type or higher, it is meaningless. The "Theory of Types," as thus formulated, requires a separate Law of Excluded Middle for every type, but the Law of Excluded Middle is saved from oblivion. Certain difficulties still remain, such as, to what type belongs the proposition which forms the principle assumption of the theory itself? The account given here includes only a sketch of this solution. For more complete expositions, see the reading list at the end of this chapter.

Another solution of the difficulties of the Epimenides paradox consists in defining accurately the concepts in question in a search for an ambiguity in terms. A possibly successful solution of this type is that of H. B. Smith.[1] Such a solution, if sound, avoids all the complexities of the theory of types.

The Epimenides paradox does not by any means exhaust the number of logical paradoxes. Some arguments which modern logicians classify as paradoxes are not purely logical but result from an apparent conflict between the principles of logic and the principles of another science. These are undoubtedly important, both from the point of view of logic and the other science involved. A discussion of these will be found in the next chapter.

There are, however, other paradoxes of the purely logical order. Many of these are merely different forms of the Epimenides paradox, and such solutions as have been offered for the latter are usually taken in their most general form to apply to the former as well. Hence, in the exposition of these given below, no further suggestions regarding solutions will be made.

One such paradox is that of Russell.[2] This, like the Epimenides paradox, has many forms. One relates the predicament of the librarian at Alexandria whose task it was to make a catalogue of just those books which did not mention themselves in their contents. The question arose whether the catalogue, which was to stay in the library, should include itself in the list of works. If not, then, since it fails to mention itself, it fulfills the rule of

[1] Cf. Chap. XVII.

[2] Named for Bertrand Russell, modern founder of the theory of types; an exposition of the fallacy appears in his *Principles of Mathematics,* though the fallacy was known in a special form by medieval logicians.

belonging in the catalogue and should be included; if so, then it does mention itself and fails to fulfill the requirements for being included.

Another form of the paradox divides the class of adjectives into "homonyms" and "heteronyms." A homonym is any adjective which describes itself. Thus "small" is a small word, "tremendous" is a tremendous word, "old" is an old word. We ask whether the adjective "heteronym," which asserts that a given adjective is not a homonym, is itself a homonym or not. If so, then it must describe itself, in which case it would be a heteronym. If not, it would be heteronymous hence self-descriptive, hence a homonym.

Russell's paradox may be generalized to include all these and similar examples. Among the set of classes, we recognize that some include themselves, some do not. Thus, the class of all classes is itself a class, but the class of men is not a man. We define a class M as: to M belong all those and only those classes which do not include themselves as members. Does M include itself or not?

Another type of argument which seems to yield a contradiction is that perhaps best designated as the "dilemma paradox." [3] One form of this paradox is the tale of the beautiful maiden and the conscientious alligator. This maiden, the fairest in all Africa, overstepped the bounds of safety and propriety, and fell into the hands of the king of the alligators. It is a part of the moral code of all alligators never to take *complete* advantage of another. So to the plea of the maiden's mother for the return of her daughter, the alligator replied that he would free her, provided the mother would tell him one true proposition (possibly realizing the terrible feminine propensity for exaggeration). The mother was about to say "Grass is green" or "$2 + 2 = 4$," but thought, at the last moment, that the situation might become embarrassing were the alligator to ask "Why?" So, with true feminine intuition, she said, "You are going to keep my daughter." The alligator is caught on the horns of an awful and inescapable dilemma. If he keeps the maiden, then her mother told the truth, and he has broken the sacred word of the alligators. If, however, he relinquishes her, the mother has told a lie, and by rights he ought

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[3] So-called because the argument always takes the form of a dilemma (cf. p. 35).

to keep her. Here the interest of logic ends, but the tragedy of the alligator begins.

There is a similar tale about a king who chose as husbands for his daughters all suitors who could answer truthfully the question he asked them and complete successfully the task he proposed. This left the choice of sons-in-law entirely in the hands of the wise monarch, for to those who did not please him he proposed the following question: "What will you be doing tomorrow at high noon?" and followed this by offering the following task: "Tomorrow at high noon you are to do the opposite of your answer to my question."

In the case of the dilemma paradox, a solution may be offered different from the theory of types or allied solutions. One may simply ask, concerning either of these stories, "What consequence is this to logic?" A paradox should try to show the falsity of some fundamental law of logic. What law is violated here? The alligator, we say, must either release the maiden or keep her, and this much is logically true. The argument goes on to point out that in either case he breaks his promise. But no logical law is involved. I may promise someone that I will be at this spot tomorrow at three o'clock and also that I will be somewhere else at three o'clock. Now, no matter what I do I must break my promise. Similarly, the alligator has proposed an impossible task for himself *provided the mother answers as she does.* What he really promises in this case is both to return and to keep the maiden, a promise that cannot be fulfilled. Likewise, the story of the king and the suitors clearly displays the impossibility of the task required.

There is another form of argument which appears to involve a logical paradox, but in reality does not. This is generally known as the argument leading to an "infinite regress." An example of a regress occurred several years ago (it is said) when the government proposed to transform Pecos County, Texas, into a huge map of the United States. Every single object in the Union, houses, trees, roads, rivers, etc., was to be reproduced in scale on the map. The engineers were faced with the problem of Pecos County itself; in representing Pecos County accurately on the large map, a miniature map of the United States was required, which included Pecos County, necessarily requiring another, even smaller map, etc., etc. A similar situation would occur if a manufacturer were to print on the box containing his merchandise a picture of someone carrying the box.

Another tale, with a like moral, is told of a certain emperor who resolved that no suitor should ever win the hand of his beloved daughter. To each one who came seeking marriage he assigned the following task: to copy into one volume all the words of the books in the emperor's library, including any duplications caused by two books being the same or having sections in common. Since this volume, when finished, became a part of the emperor's library, the hapless suitor found himself involved in an endless task, for he had to copy out the words of the volume he had just completed, and when he had finished this, he had to start all over again, this process being repeated endlessly.

Often the infinite regress brings us around to the starting point again, so that the process is infinite but circular. Thus, two snakes in the act of consuming each other would involve themselves in an infinite task, provided each could transform what he had just eaten to a new body. Again, a certain four brave soldiers were contemplating a daring feat. A declared that he would go if B went, while B made his going dependent on C's going also, while C would go only if D went, and D, finally, agreed to go if A would. Here the procedure of determining on these grounds whether the party went or not is an infinite and also a recurrent one, since we keep repeating the given propositions.

The misanthropists of a certain country, we are told, disgusted with man's delight in the companionship of others, decided to form a club whose by-laws read as follows:

1. Anyone who belongs to no other club automatically is elected a member of the Misanthropist Club.
2. If any member of the Misanthropist Club is a member of some club, he is automatically expelled from membership in this club.

Mr. A, who belonged to no club at the time the misanthropists organized, found himself involved in an infinite series of elections to and expulsions from the new club. By-law number 1 elected him to membership but By-law 2 immediately expelled him, when, since he again belonged to no club, he automatically regained his membership, whereupon he was again expelled, etc.

Many cases which appear to involve an infinite regress do not actually do so, some because a contradiction results in a finite number of steps, others because the process may stop (without contradiction) at some finite place. Thus it might appear that the proposition "I am now lying" or "The statement I am now making is false" involves an infinite regress; for if it is true that I am now lying, then I am now saying something false, i.e., it is false that I am lying, or I am telling the truth. What I say, however, is that I am lying, hence I must actually be doing so, in which case I am saying something false, and so it goes on. But, though the series described is infinite, a contradiction has been reached on the second step as has been shown above. If the statement "I am now lying" is true, then it must be false; but any statement which implies its own negation is a false statement and this proposition must be false also. But if "I am now lying" is false, it must be true; hence the statement is both true and false. Here the interest of logic stops, even though the argument may proceed indefinitely, for the series has involved us in a contradiction. Thus, in general, all cases of the Epimenides paradox are not treated as cases of infinite regress.

An example of a series apparently infinite, but not actually so, occurs in the story of Euthalus and Protagoras. Euthalus, a law pupil of Protagoras, made an agreement with his teacher: he was to pay half his tuition at the end of his course and the other half when he won his first case, this being some sort of a guarantee on the teacher's part for the practical value of his instruction. However, Euthalus fell heir to a considerable fortune, forsook the practice of law, and turned to more pleasant if less lucrative activities, whereupon Protagoras, wearying of waiting for the remainder of his fee, took the matter to court. Euthalus now decided to argue his own case. The judge at the trial appeared to be involved either in an infinite regress of decisions or a contradiction. If he grants the verdict to Euthalus, then the lad has won his first case and according to the stipulation of the contract must pay Protagoras. But if he does have to pay, he really did not win the trial, and having won no other case, he need not pay, in which case, having after all won the trial, he must pay, etc. A similar regress appears to occur if the verdict is given to Protagoras.

Actually the series must stop at the second step. The only question before the judge is "Must Euthalus pay Protagoras his fee?" and this question, according to the terms of the contract, depends solely on the answer to another question "Has Euthalus won any cases as yet?" Since, at the time of the trial, the answer to the latter question is in the negative, then the judge must answer the first question in the negative as well, and grant the verdict to Euthalus. The trial completed and won by the pupil, he must then pay his master, for if the master took the matter to court again he could establish the fact that Euthalus had won a case. Thus Euthalus wins the first trial, but since he had elected to plead his own case, he must lose the second. The loss of the second trial does not erase the victory of the first, though it does make this victory one of small value for the victor.

It is important to note that the process of infinite regress involves no logical difficulties and hence is not a logical paradox as we have defined the term. However impractical an infinite series of steps may be in deciding the truth or falsity of some proposition or in completing some task, no contradiction is involved. The absolute sceptic declares that nothing is certain, not even this declaration he has just made; and that the entire statement that nothing is certain, not even this, is itself uncertain, and the declaration that this is uncertain is also uncertain and so on. Such a philosopher involves himself in no contradiction even though he requires an infinite series to state his whole position. Absolute scepticism is bad, since in practice we require some hypothesis upon which to proceed, and for the sceptic such an hypothesis is usually taken to be the statement that nothing is certain. If he were faithful to his position, apparently he would be forced to spend his days defining his philosophy. Rather than do this, he forsakes his theory and acknowledges one certainty, namely, the certainty that nothing other than this is certain. But questions of practice are not questions of logic; it may be inconvenient to be involved in an infinite regress, but it is not inconsistent.

A paradox which seems to fall within the scope of logic, though it draws on certain propositions from other sciences, is that which concerns itself with the "nameability" of the integers. We note that as we go higher in naming the series of whole numbers we require more and more syllables, so that "one thousand three hundred and forty" requires more syllables than "one thousand and ten." It seems evident enough that among the integers there will be a least integer not nameable in fewer than nineteen syllables. In the usual scheme of naming integers, this number is 111,777. Hence "the least integer not nameable in fewer than nineteen syllables" is a name for 111,777. But this name contains but eighteen syllables. Hence 111,777 both is and is not nameable in fewer than nineteen syllables.

Proponents of the theory of types mentioned here attempt to solve this paradox by pointing out that "the least integer not nameable in fewer than nineteen syllables" cannot be included among the names implied in the word "nameable," otherwise we would have the case of a self-referring proposition. Thus there appears to be an equivocation in the word "name." Opponents of this theory might urge a similar equivocation without accepting the theory in its other applications.

Not all the difficulties of the science of logic center around the Principle of Contradiction. One such paradox which apparently destroys the validity of another important logical law was given by C. L. Dodgson (Lewis Carroll). We cannot do better than to let him tell the story in his own incomparable way:

Achilles had overtaken the Tortoise, and had seated himself comfortably on its back.

"So you've got to the end of our racecourse?" said the Tortoise. "Even though it *does* consist of an infinite series of distances? I thought some wiseacre or other proved that the thing couldn't be done?"

"It *can* be done," said Achilles. "It *has* been done! *Solvitur ambulando.* You see, the distances were constantly *diminishing:* and so——"

"But if they had been constantly *increasing?*" the Tortoise interrupted. "How then?"

"Then I shouldn't be *here,*" Achilles modestly replied; "and *you* would have got several times around the world, by this time!"

"You flatter me—*flatten,* I mean," said the Tortoise; "for you *are* a heavyweight, and *no* mistake! Well now, would you like to hear of a racecourse, that most people fancy they can get to the end of in two or three steps, while it *really* consists of an infinite number of distances, each one longer than the previous one?"

"Very much indeed!" said the Grecian warrior, as he drew from his helmet (few Grecian warriors possessed *pockets* in those days) an enormous notebook and a pencil. "Proceed! And speak *slowly,* please! *Shorthand* isn't invented yet!"

"That beautiful First Proposition of Euclid!" the Tortoise murmured dreamily. "You admire Euclid?"

"Passionately! So far, at least, as one *can* admire a treatise that won't be published for some centuries to come!"

"Well, now, let's take a little of the argument of that First Proposition—just *two* steps, and the conclusion drawn from them. Kindly enter them in your notebook. And, in order to refer to them conveniently, let's call them A, B, and Z:

(A) Things that are equal to the same are equal to each other.

(B) The two sides of this Triangle are things that are equal to the same.

(Z) The two sides of this Triangle are equal to each other.

"Readers of Euclid will grant, I suppose, that Z follows logically from A and B, so that any one who accepts A and B as true, *must* accept Z as true?"

"Undoubtedly! The youngest child in a high school—as soon as high schools are invented, which will not be till some two thousand years later—will grant *that.*"

"And if some reader had *not* yet accepted A and B as true, he might still accept the *Sequence* as a *valid* one, I suppose?"

"No doubt such a reader might exist. He might say 'I accept as true the Hypothetical Proposition that, if A and B be true, Z must be true; but I *don't* accept A and B as true.' Such a reader would do wisely in abandoning Euclid, and taking to football."

"And might there not *also* be some reader who would say 'I accept A and B as true, but I *don't* accept the hypothetical'?"

"Certainly there might. *He,* also, had better take to football."

"And *neither* of these readers," the Tortoise continued, "is *as yet* under any logical necessity to accept Z as true?"

"Quite so," Achilles assented.

"Well, now, I want you to consider *me* as a reader of the second kind, and to force me, logically, to accept Z as true."

"A tortoise playing football would be—" Achilles was beginning.

"—an anomaly, of course," the Tortoise hastily interrupted. "Don't wander from the point. Let's have Z first, and football afterwards!"

"I'm to force you to accept Z, am I?" Achilles said musingly. "And your present position is that you accept A and B, but you *don't* accept the Hypothetical——"

"Let's call it C," said the Tortoise.

"—but you don't accept:

(C) If A and B are true, Z must be true."

"That is my present position," said the Tortoise.

"Then I must ask you to accept C."

"I'll do so," said the Tortoise, "as soon as you've entered it in that notebook of yours. What else have you got in it?"

"Only a few memoranda," said Achilles, nervously fluttering the leaves: "a few memoranda of—of the battles in which I have distinguished myself!"

"Plenty of blank leaves, I see!" the Tortoise cheerily remarked. "We shall need them *all!*" (Achilles shuddered.) "Now write as I dictate:

(A) Things that are equal to the same are equal to each other.

(B) The two sides of this Triangle are things that are equal to the same.

(C) If A and B are true, Z must be true.

(Z) The two sides of this Triangle are equal to each other."

"You should call it D, not Z," said Achilles. "It comes *next* to the other three. If you accept A and B and C, you *must* accept Z."

"And why *must* I?"

"Because it follows logically from them. If A and B and C are true, Z *must* be true. You don't dispute *that,* I imagine?"

"If A and B and C are true, Z *must* be true," the Tortoise thoughtfully repeated. "That's *another* Hypothetical, isn't it? And, if I failed to see its truth, I might accept A and B and C, and *still* not accept Z, mightn't I?"

"You might," the candid hero admitted; "though such obtuseness would certainly be phenomenal. Still, the event is *possible.* So I must ask you to grant one more Hypothetical."

"Very good. I'm quite willing to grant it, as soon as you've written it down. We will call it

(D) If A and B and C are true, Z must be true.

"Have you entered that in your notebook?"

"I have!" Achilles joyfully exclaimed, as he ran the pencil into its sheath. "And at last we've got to the end of this ideal racecourse! Now that you accept A and B and C and D, *of course* you accept Z."

"Do I?" said the Tortoise innocently. "Let's make that quite clear. I accept A and B and C and D. Suppose I *still* refuse to accept Z?"

"Then Logic would take you by the throat, and *force* you to do it!" Achilles triumphantly replied. "Logic would tell you 'You can't help yourself. Now that you've accepted A and B and C and D, you *must* accept Z.' So you've no choice, you see."

"Whatever *Logic* is good enough to tell me is worth *writing down,*" said the Tortoise. "So enter it in your book, please. We will call it

(E) If A and B and C and D are true, Z must be true.

"Until I've granted *that,* of course, I needn't grant Z. So it's quite a *necessary* step, you see?"

"I see," said Achilles; and there was a touch of sadness in his tone.

Here the narrator, having pressing business at the Bank, was obliged to leave the happy pair, and did not again pass the spot until some months afterwards. When he did so, Achilles was still seated on the back of the much-enduring Tortoise, and was writing in his notebook, which appeared to be nearly full. The Tortoise was saying "Have you got that last step written down? Unless I've lost count, that makes a thousand and one. There are several millions more to come. And *would* you mind, as a personal favor—considering what a lot of instruction this colloquy of ours will provide for the Logicians of the Nineteenth Century—*would* you mind adopting a pun that my cousin the Mock-Turtle will then make, and allowing yourself to be renamed Taught-Us?"

"As you please!" replied the weary warrior, in the hollow tones of despair, as he buried his face in his hands. "Provided that *you,* for *your* part, will adopt a pun the Mock-Turtle never made, and allow yourself to be re-named A Kill-Ease!" [4]

Carroll's paradox deals with the problem as to how a theorem in a deductive system may be asserted apart from the propositions which are used in its proof. That is, suppose in our deductive system that p and q imply r, and also that p and q are true. How may he claim that r is true without adding to our system the proposition "If p and q imply r, and p and q are both true, then r is true"? And after this has been added, we require another like it, and so forth, ad infinitum.

The problem is really one belonging to the philosophy of formal science. One might answer the paradox simply by stating that in a formal system we never do disassociate the assumptions necessary to prove a given proposition from the assertion of that proposition. When we say "p is a theorem in this system" we mean "p is true if the postulates of this system are true." Such an answer may be adequate in the realm of formal science, but if we have actually established the truth of the postulates, say by some experiment, we would like to assert the theorems as inde-

[4] Lewis Carroll, "What the Tortoise Said to Achilles," *Mind,* Dec., 1894.

pendent truths; i.e., we would like to "suppress" the hypotheses necessary to establish a given theorem if these hypotheses are verified.

One solution of the problem lies in making a distinction between "implication" and "inference." The statement "If A, then B" expresses an implication between two propositions, while the statement "A is true, and therefore B" is an *inference,* the truth of B having been inferred from that of A. We then assert that the truth of B may be inferred from the statement "A is true and A implies B." This assertion is not an implication and must be distinguished from a similar law of logic which is in an implicative form: "If A is true and A implies B, then B is true." We cannot say that the validity of the former, the inference, rests solely on the validity of the latter, the implication, without involving ourselves in some such paradox as the above. Rather, the two statements are independent laws.

The paradoxes enumerated here are not exhaustive, of course, nor is there any apparent method of showing that logic is forever safe from contradiction or that someone may not invent an insoluble paradox sometime. Further, some paradoxes that many logicians believe to be purely logical, have been placed under the heading of "Conflicts between Logic and Some Other Science." Such conflicts arise when one science sets down as true a proposition which is false for another science. When this is the case, one of the sciences must change its laws in some respect. Hence, in the paradoxes enumerated in the next chapter, all of which show cases of conflicts between the science of logic and some other science, there is always the question as to whether or not we should change logic.

EXERCISES

GROUP A

Determine, if possible, what type of logical paradox is exhibited by each of the following; can you suggest solutions?

1. Jones bets Smith that Smith will win all his bets during the next month, the agreement being that whenever Smith wins, he must pay Jones the amount won plus fifty dollars, while if Smith loses, Jones must pay him his loss plus fifty dollars. At the end of the

month the question arises as to who won the given bet. If Jones won, then Smith lost and must pay Jones fifty dollar; but then Jones must pay Smith his loss plus fifty dollars, etc. If Smith won, then he must pay Jones fifty dollars, etc.

2. There was a barber in a certain town who shaved all those and only those who shaved themselves. Did he shave himself?
3. "I said in my heart, all men are liars."
4. A man invented a machine to detect whether any bells were ringing in a nearby city; if none were ringing, a bell was automatically started in his machine to register this fact, while if any started to ring, the bell in the machine instantly stopped. One day the inventor decided to carry his invention within the city limits, thus placing it in an inextricable dilemma when all the bells of the city were at rest.
5. Some ancient scientists, raising the question as to what held the earth in position, declared that it was Atlas. Now Atlas, to be successful, must stand on something; this something must stand on something else, etc., etc.
6. A certain philosopher asserts that every proposition is true. Is this assertion self-defeating?
7. The number of words in the English language is finite, and hence the number of meaningful statements made out of these words must be finite, since no meaningful statement contains an infinite number of words. Hence, the number of true propositions must be finite. But this is not possible, for if I gather together all the true propositions I may assert "This is the list of true statements" adding another true statement to the list already made. Hence for any given number of true propositions there is always another true proposition and so the number must be infinite; therefore, so must be the total number of meaningful statements in English.
8. It is impossible to know anything beyond any question of doubt. For suppose that you know that the proposition p is unquestionably true. In order to know this, you must know that you know that p is true. But to know this last statement completely, you must know that you know that you know p is true, etc. In other words, to know something completely and beyond question, one must know an infinite number of things, and this is impossible.
9. "So natural, indeed, to the morbid activity of man are these revolving forms of alternate repulsion, where flight turns suddenly into pursuit, and pursuit into flight, that I myself, when a schoolboy, invented several. This, for instance, which once puz-

zled a man in a wig, and I believe he bore me malice to his dying day, because he gave up the ghost by reason of fever, before he was able to find out satisfactorily what screw was loose in my logical conundrum; and thus, in fact, 'all along of me' (as he expressed it) the poor man was forced to walk out of life *re infecta,* his business unfinished, the sole problem that had tortured him unsolved. It was this. Somebody had told me of a dealer in gin, who, having had his attention roused to the enormous waste of liquor caused by the unsteady hands of drunkards, invented a counter which, through a simple set of contrivances, gathered into a common reservoir all the spillings that previously had run to waste. Saint Monday, as it was then called in English manufacturing towns, formed the jubilee day in each week for the drunkards; and it was now ascertained (i.e. subsequently to the epoch of the artificial counter) that oftentimes the mere 'spilth' of Saint Monday supplied the entire demand of Tuesday. It struck me, therefore, on reviewing the case, that the more the people drank the more they would *titubate,* by which word it was that I expressed the reeling and stumbling of intoxication. If they drank abominably, then of course they would titubate abominably; and titubating abominably, inevitably they would spill in the same ratio. The more they drank, the more they would titubate; the more they titubated, the more they would spill; and the more they spilt, the more, it is clear, they did *not* drink. You can't tax a man with drinking what he spills. It is evident, from Euclid, that the more they spilt, the less they *could* have to drink. So that, if their titubation was excessive, then their spilling must have been excessive, and in that case they must have practiced almost total abstinence. Spilling nearly all, how could they have left themselves anything worth speaking of to drink? Yet, again, if they drank nothing worth speaking of, how could they titubate? Clearly they could not; and, not titubating, they could have had no reason for spilling, in which case they must have drunk the whole—that is, they must have drunk the whole excess imputed, which doing, they were dead drunk, and must have titubated to extremity, which doing, they must have spilt nearly the whole. Spilling the whole, they could not have been drunk. *Ergo,* could not have titubated. *Ergo,* could not have spilt. *Ergo,* must have drunk the whole. *Ergo,* were dead drunk. *Ergo,* must have titubated. And so round again, as my Lord the bishop pleasantly expresses it, *in secula seculorum.*" [5]

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[5] De Quincey, *Sir William Hamilton.*

10. The great philosopher, Leibnitz, constructed a theory of reality in which all substance is made up of perceiving minds, called "monads." Each monad mirrors the entire universe of monads, though most monads are not conscious of all that they perceive. Now, since every monad perceives the universe, it must also perceive itself, since it is a member of the universe; but since it is perceiving everything, it must perceive itself perceiving everything, and since that "everything" includes itself, it must perceive itself perceiving itself perceiving everything, etc., etc. Further, since each monad perceives every other monad, and the other monad is perceiving everything, and this "everything" is every other monad, each monad perceives an infinite number of monads which are perceiving itself, etc., etc.

11. Dear Sir:

I am a customer of yours and I say that you are wrong in saying that the customer is always right. Now if I am right in saying that, then I am wrong, and if I am wrong then you are liars. Therefore, I am withdrawing my account.

Yours truly,

12. "It's true, isn't it, John, that a man is always a tyrant in his own home?"

"Well, my dear, I hardly think——"

"It's true, isn't it?"

"Yes, my dear."

13. "A more pleasant example of the same logical see-saw (as the Epimenides paradox) occurs in the sermons of Jeremy Taylor. That man, says the inimitable bishop, was prettily and fantastically troubled, who, having used to put his trust in dreams, one night dreamed that all dreams were vain. He considered if so, then *this* dream was vain, and the dreams might be true for all this. (For who pronounced them *not* true except a vain dream?) But if *they* might be true, then *this* dream might be so upon equal reason. And dreams *were* vain, because this dream, which told him so, was true; and so round again. In the same circle runs the heart of man. All his cogitations are vain, and yet he makes especial use of this—that that thought which thinks so, *that* is vain. And if *that* be vain, then his other thoughts, which are vainly declared so, may be real and relied upon. You see, reader, the horrid American fix into which a man is betrayed, if he obeys the command of a dream to distrust dreams universally, for then he has no right to trust this particular dream, which authorizes his general distrust. No; let us have fair play.

What is sauce for the goose is sauce for the gander. And this ugly gander of a dream, that notes and protests all dreams collectively, silently, and by inevitable consequence notes and protests itself." [6]

14. "Let T be the relation which subsists between two relations R and S whenever R does not have the relation R to S. Then, whatever relations R and S may be, 'R has the relation T to S' is equivalent to 'R does not have the relation R to S.' Hence, giving the value T to both R and S, 'T has the relation T to T' is equivalent to 'T does not have the relation T to T.' " [7]

REFERENCES

THEORY OF TYPES:

a) Russell, *Principles of Mathematics,* Appendix B.
b) Russell and Whitehead, *Principia Mathematica,* 2nd Edition, Introduction, chap. II, and Appendix C.
c) Ramsey, *The Foundations of Mathematics.*
d) Lewis and Langford, *Symbolic Logic,* chap. XIII.

[6] De Quincey, *op. cit.*
[7] Whitehead and Russell, *Principia Mathematica.*

Conflicts Between Logic and Other Sciences

11

In this chapter we consider cases in which the accepted laws of a given science and those of logic somehow come in conflict. By "accepted" we mean that the laws have been established by some such method as that described in chapter VIII.

To solve these difficulties, three paths, at least, are open. We may (1) change one or more of the laws of the science in question, (2) change the laws of logic, or (3) search for an equivocation or some other fallacy in the argument purporting to show a conflict.

I. It would be a mistake to follow our classification of the sciences in enumerating these conflicts, since some of the paradoxes of arithmetic are far more difficult than those of sciences higher in the scale. It will be best to begin with one of the earliest examples of scientific paradox, that which appears to contradict the basic assumption of kinematics. Kinematics is the first science to introduce the concept of time and consequently the concept of velocity or, in general, the concept of the motion of a particle. Very early in history of the occidental philosophy it occurred to certain Greek philosophers that the assumption that there exists any such thing as motion at all was false. That is, our senses, to which we appeal for the validity of the proposition that certain bodies move, are deceiving us.

Parmenides (*ca.* 502 B. C.) was the first thinker who arrived at this conclusion, and he came to it after he had set out to answer a problem as old as Greek philosophy: "Of what is the world made?" The problem of the correct recipe for the matter of the world had been answered in various ways by the predecessors of Parmenides: one said that the world was made of water, which underwent far more changes than one realized, to form every material object. Another, with more caution but less simplicity, insisted that the world was made of four irreducible elements: air, earth, water, and fire. But Parmenides thought he saw the solution to the problem in the answer to another question: "What does everything in the world have in common?" Surely, considering all the diverse objects of the world, there can be but one reply, namely, that all the objects in the world have in common only one thing, their existence. *Being* is common to everything that is, this much is certainly tautological. The primary element in the world, then, must be Being.

Are there any other elements? At first sight we are inclined to reply "yes," for though this table and that man are both existent, they are different nevertheless, hence there must be something which makes man a man and not a table, and something else that makes a table a table and not a man. In the case of this illustration, there must be an element other than being to explain wood and another element to explain flesh. But, says Parmenides, in this we are deceived: there cannot be anything in the world other than Being. For suppose there were another element, say x, which is not the same as Being. Then x is non-Being, since it differs from Being. But to say that x is non-Being is to say that x is nonexistent; hence x does not exist and our supposition that there exists in the world another element besides Being is absurd.

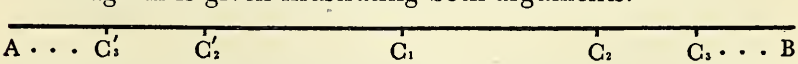
A similar argument allows us to prove that nothing ever moves in the world. For the only thing in the world is Being, as we have already shown. If something moves, it must move from the place where it is to the place where it is not, for otherwise there is no motion. Hence, if Being moves, it must move to a place where it is not. But the place where Being is not, is, by definition, a nonexistent place. By similar arguments Parmenides supposedly proved that change is impossible.

In the face of these paradoxes, most of us are apt to insist that the whole argument is a verbal quibble, though exactly where the equivocation lies we are not so certain. It is doubtful whether the contemporaries of Parmenides took him very seriously; indeed, certain commentators would have us believe that Parmenides himself was not convinced that the world was such a static and homogenous place as he had pictured it, and he felt that he was obliged to give another account of the world in which motion and change do take place.[1] But however ill-disposed we may feel towards the reasoning of Parmenides, the arguments of his pupil, Zeno, certainly seem to settle the matter.

Zeno, like Parmenides, attempts to show that motion is impossible, though his arguments are altogether of a different nature. There are four proofs which have been handed down through Aristotle:

1. The first is also the most general. If there were such a thing as the motion of a particle, it would take place with respect to two points, the beginning and the terminus of some line, straight or curved, connecting these points. If the given points are A and B, then the path of motion may be represented by AB. Now for the particle to move from A to B, e.g., for a book to fall from the table to the floor, it must first go half the distance, i.e., it must pass through the middle point of the path AB, say C_1. Here the argument may take one of two courses. We may point out that if the particle arrives at C_1, it must still pass through the midpoint of C_1B, i.e., the three-quarter point of AB, say C_2. However, even though it arrives at C_2, it still has to pass through C_3, the midpoint of C_2B, or the seven-eighths point of the whole path; and similarly it must pass through C_4, the midpoint of C_3B, and through C_5, and through C_6, ad infinitum. Since it is impossible that it pass through all these midpoints, there being an infinite number of them, and since it must do so to reach B, the particle never can arrive at B. Or, we may point out that before coming to C_1, the particle must already have passed through C'_2, the midpoint of AC_1, and that, before arriving at C'_2, it must have passed through C'_3, the midpoint of AC'_2. That is, before arriving at any point we like, the particle must already have passed through an infinity of points, an absurdity, and hence the particle cannot even start. Either argument is adequate, since in the first case AB may be taken as small as we please, so that the particle cannot move any distance.

A diagram is given illustrating both arguments:



[1] However, the best authorities seem to take Parmenides' *Way of Opinion* as representing the belief of the naive scientist who is unaware of the *Way of Truth*.

Some have insisted that only the second form of Zeno's argument has any force, on the grounds that the first assumes that there is such a think as motion. But such procedure is perfectly adequate and is used constantly in mathematics under the name of the "Indirect Proof." We assume the validity of a certain proposition throughout an argument with the ultimate intention of showing that such an assumption leads to a contradiction, and hence must be false.

2. The second of Zeno's arguments is identical with the first for all practical purposes but has more romantic appeal since it is a case of the weak overcoming the strong. The story is told so admirably by De Quincey that it is quoted verbatim:

Achilles, most of us know, is celebrated in the Iliad as the swift-footed; and the tortoise, perhaps all of us know, is equally celebrated among naturalists as the slow-footed. In any race, therefore, between such parties, according to the equities of Newmarket and Doncaster, where artificial compensations as to the weight of riders are used to redress those natural advantages that would else be unfair, Achilles must grant to the tortoise the benefit of starting first. But if he does *that,* says the Greek sophist, then I, the sophist, back the tortoise to any amount, engaging that the goddess-born hero shall never come up with the poor reptile. Let us see. It matters little what exact amount of precedency is conceded to the tortoise; but say that he is allowed a start of one-tenth part of the whole course. Quite as little does it matter by what ratio of speed Achilles surpasses the tortoise; but suppose this ratio be that of ten to one, then, if the race course be ten miles long, our friend the slow-coach, being by the conditions entitled to one-tenth of the course for his starting allowance, will have finished one mile as a *solo* performer before Achilles is entitled to move. When the *duet* begins, the tortoise will be entering on the second mile precisely as Achilles enters on the first. But, because the Nob runs ten times as fast as the Snob, whilst Achilles is running his first mile, the tortoise accomplishes only the tenth part of the second mile. Not much, you say. Certainly not very much, but quite enough to keep the reptile in advance of the hero. This hero, being very little addicted to think small beer of himself, begins to fancy that it will cost him too trivial an effort to run ahead of his opponent. But don't let him shout before he is out of the wood. For, though he soon runs over that tenth of a mile which the tortoise has already finished, even this costs him a certain time, however brief. And during that time the tortoise will have finished a corresponding subsection of the course—viz., a tenth part of a tenth part. This fraction is a hundredth part of the total distance. Trifle as that is, it constitutes a debt against Achilles, which debt *must* be paid. And whilst he *is* paying it, behold our dull friend in the shell has run the tenth part of a hundredth part, which amounts to a thousandth part. To the goddess-born what a fleabite is that! True it is so, but still it lasts long enough to give the tortoise time for keeping his distance and for drawing another little bill upon Achilles for a ten-thousandth part. Always, in fact, alight upon what stage you will of the race, there is a little arrear to be settled between the parties and always *against* the hero. "Vermin, in account with the divine and long-legged Pelides, Cr. by one billionth or one decillionth of course," much or little, what matters it, so long as the divine man cannot pay it off before another installment becomes due? And pay it off he never will, though the race should last for a thousand centuries.[2]

3. Zeno's third argument is quite different from the first two, the philosopher apparently feeling obliged to establish his point in as many ways as possible. The arrow in flight, says Zeno, must either be in motion at a point A on its path or else not.[3] But an object cannot be moving at a point, for motion involves the concept of distance or length, and a point has no length. Put otherwise, if we say that the object is moving at the point A, then since its velocity is a function of length and the length here is zero, it must have no velocity at A and hence is not in motion. But if the body does not move at any point A of its path, the body is in motion nowhere, and hence it is absurd to suppose that it moves along its path at all.

4. The fourth argument is concerned with the relativity of motion. A modern example will be better than Zeno's. Imagine three trains of an equal number of cars at rest side by side in a station. Train A and train B start off, one north, the other south, at the same time and at the same speed, while C remains stationary. A person on the observation car of A will pass all the cars of train B in the same time in which he has passed but half of the cars of C. Hence if the time it takes him to pass the cars of B is one minute and the trains have ten cars, his velocity will be ten cars a minute and also five cars a minute, an obvious contradiction.

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[2] De Quincey, *op. cit.*

[3] The argument is usually given in the form: "The arrow is either moving where it is or where it is not," but the form above seems clearer and more cogent.

With the exception of Zeno's first and second arguments, the Eleatic[4] paradoxes must be solved by different means; this is an illustration of the ingenuity of the Greek mind, that it could devise so many different kinds of arguments to establish one point. No one has felt inclined to agree with the Eleatic thesis that motion is impossible, chiefly because such a thesis destroys all science from kinematics on; indeed, Parmenides' arguments concerning the homogeneity of the world would lead us to deny any applications of arithmetic or geometry as well, so that all that remains is logic, if even that. Hence, if the foundations of kinematics are not at fault, we must find solutions to the paradoxes. As in the case of the logical paradoxes, despite the simplicity of their form, the Eleatic arguments are not trivial; we can fairly say that there is no unanimity of opinion regarding their correct solution. We will suggest possible solutions of the difficulties involved, but such suggestions are by no means to be considered as final answers.

In general, most paradoxes may be solved by properly defining the terms used and setting forth the principles involved. Parmenides' argument asserts the following principles to show that only the attribute of existence is meaningful:

1. If something has an attribute other than existence, the attribute must differ from existence somehow.
2. By "differing from existence" we must mean "nonexistence."
3. Hence, something having an attribute differing from existence does not exist.

It is clear that if ambiguities are to be found, they must be found in number 2. If I say that this table is brown and "brown" means something other than existence, it does not follow that "brown" means "nonexistence"; when I describe something by one quality and then by another, if the second differs from the first I do not mean that the first is lacking if the second is there.

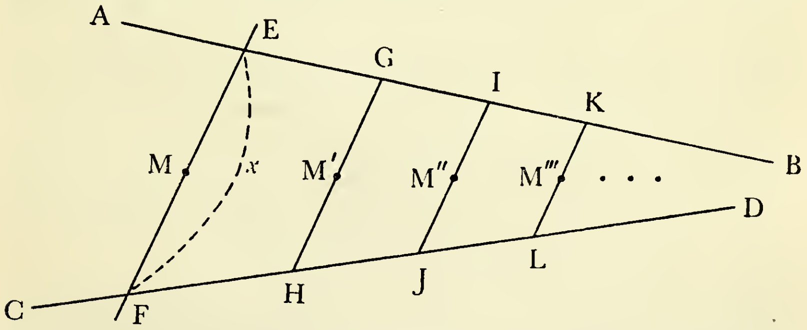
The difficulty is solved by showing that there are many universes of discourse (cf. page 98). The universe of discourse of brown is color, and "not brown" means some color other than brown. The universe of discourse of "existent" is all things and "nonexistent" means something which does not exist. When we

[4] Zeno and Parmenides were called Eleatics after the name of their country, Elea.

say that something is brown and exists, we mean by "brown" something "other than 'existent,' " but we do not mean by brown "nonexistent." Rather we mean that the terms "brown" and "exists" belong to different universes of discourse.

In effect, what is done here is to change a law of logic, to modify its range of application. The Aristotelian law "Nothing is both A and non-A" cannot be applied without certain restrictions: a table may be white at one time and nonwhite at another; a man may be having both pleasure and pain at the same time. The present solution indicates the manner in which the term "non-A" is to be considered.

Many solutions to Zeno's paradoxes have been offered in the history of science. Some mathematicians have been prone to reject the paradoxes as either trivial or else lying beyond the scope of mathematics. That this is hardly the case can be seen from an application of the paradox to geometry made by an ancient geometer sometime after Euclid. Euclid had asserted as his fifth postulate (cf. page 12) that if two lines are cut by a transversal so that the interior angles on the same side sum to less than two rights, the lines will meet. This geometer set out to prove that, granted the hypothesis, the lines will not meet, and this despite the fact that we can apparently construct such lines which do meet, as, for example, in the case of any triangle. For suppose AB and CD are two lines cut by the transversal EF so that the sum of angle BEF and angle DFE is less than two right angles.



Now take the midpoint of EF, M. Draw EG = EM on AB and FH = FM on CD. Then AB and CD do not meet in the rectangle EGHF, as the reader may easily verify for himself from the construction of the figure. We now take M′ the midpoint of

GH and construct GI = GM′ and HJ = M′H, and show that AB and CD do not meet in the rectangle GIJH; since we may go on constructing indefinitely such rectangles where AB and CD fail to meet, these lines never do intersect.

We cannot refute Zeno's argument by the method chosen by one ancient philosopher, who "without attempting to meet and dissolve the argument, rose up from his seat, and walked; *redarguebat ambulando;* according to his own conceit, he refuted the Sophist by moving his spindle shanks, saying, *thus* I refute the argument. I move, as a fact, and if motion is a fact of the experience, then motion, as an idea, is conformable to the reason. But to me it is plain that this philosopher little comprehended the true incidence and pressure of the difficulty. . . .

"The case was briefly this: Reason, as then interpreted, said, This thing cannot be. Nature said, But though impossible, it is a fact. Metaphysics (or logic) denied it as conceivable. Experience affirmed it as actual. There was, therefore, war in the human mind, and the scandal of an irreconcilable schism. Two oracles within the human mind fought with each other. But in such circumstances, to reaffirm or to exalt either oracle, is simply to reinforce and strengthen the feud. . . . The man who simply parades the strength and plausibility investing one of the arguments, without attempting in the smallest degree to invalidate the other, does in fact, only publish and repeat the very ground of your perplexity. That argument, strong as the centrifugal force which so tauntingly and so partially he causes to coruscate before your eyes, you know but too well. Knowing *that,* however, does not enable you to hide from yourself the antagonist argument, or to deny that in power it corresponds to a centripetal force. How needless to show that motion exists as a fact! Too sensible you are of that, for what else is it than this fact which arms with the power of perplexing and confounding the metaphysical scruples affecting the idea of motion?" [5]

Some philosophers have been willing to accept the logic of Zeno's argument and to modify the fundamentals of kinematics, insisting the Zeno has not destroyed the hypothesis that motion exists but rather that motion is continuous. Roughly, the statement that a body moves continuously between two points A and B means that there exists a line between A and B, called

[5] De Quincey, *op. cit.*

the "path" of the moving object, such that a point in the object can be said to have passed through any point on this line. This assumption is certainly required by Zeno's first and second arguments. Hence Zeno's paradoxes are avoided if we assert that motion is not continuous so that bodies move by "spurts." More precisely, there is no path between A and B which is such that the body can be said to pass through every one of its points; if the body is at point D at a certain time, then it passes to another point E in no time, or else, if a time does elapse, we can say that there is a time when the body is nowhere. This solution seems a happy one to the philosophical school called "sensationalism," whose proponents believe that the only meaningful concepts are ones which are based on actual sensations. Since no one has ever seen a body moving at every point on its path, the conception is a false one. As far as our senses go, motion is discontinuous; since our eyes are merely taking rapid snapshots of the moving body, such snapshots necessarily leave out parts of the path of motion.

But, besides the weaknesses of sensationalism as a philosophy, one may object to denying the hypothesis of continuity of motion on the grounds that all this is too expensive a price to pay for a solution of the difficulty, especially if something simpler lies at hand. The "expense" involved lies in finding a new method for determining the identity of a point in a body as it moves in space; in "classical" physics the identity of a moving point is determined by the continuous path which it traverses. This point is said to be the same point as one occurring at an earlier time if there is a continuous path between the two locations at which the points appear; but if no continuous path exists, i.e., if motion is discontinuous, then we are obliged to find some other criterion by which we may be allowed to call a point A in a body the "same" as point B.

This solution follows the first of the courses we suggested above by proposing a change in the science in question. But, instead, we may look to the third avenue and seek to find an equivocation. Such a solution seems to have been offered by Aristotle.[6] Zeno's argument, in brief, insists that an object mov-

[6] The formulation of Aristotle's solution given here is not the philosopher's own, but is due to Professor E. A. Singer, Jr. References in Aristotle are: *Physics*, 233a, 13f., and *Metaphysics*, 1048b, 9f. What Aristotle calls the "potential" infinite we interpret as the infinite taken distributively, and the "actual infinite" (which does not exist for Aristotle) means the infinite taken collectively.

ing along a path must go through all the points of the path; but of these there are an infinite number, and hence the object must pass over an infinity of objects, an impossibility. Essentially, what Aristotle finds here is a case of the fallacy of composition. When we say the object must pass over all the points of the line, we mean "all" in the distributive sense, not the collective. There is no such thing as the collection of "all" the points on a line; a line is not made up of points, as certain Greek philosophers insisted, and hence we are under no obligation to consider it as an infinite collection. When we say that a body passes through "all" the points, we mean that it passes through each and every point, or, there exists no point on the line which one might name through which the given body does not pass. Thus, the traveler to the door must pass through all the halfway points in the sense that we can name no halfway point through which he does not pass, but we do not say that he passes through them "all" in the sense that he passes through an infinite collection. The fallacy is reduced to a case of taking collectively what should be taken distributively (Fallacy of Composition).

Those who believe Aristotle's argument to be sound are still faced with the problem of answering Zeno's third argument.[7] This necessity does not exist in the case of those who favor denying the law of continuity, since the latter would agree that the arrow is at rest at every point of its path. But other answers may be found. Actually, the velocity of a moving object at a point is not necessarily zero, even though the distance moved may be so. For, velocity being defined as the ratio of distance to time, the velocity at a point must be the indeterminate fraction $\frac{0}{0}$, since time as well as distance has a numerical value of zero. Mathematicians recognize that this fraction is "indeterminate," for were a fixed value assigned to it, certain contradictions would follow. Thus, one might be inclined to say that $\frac{0}{0} = 1$, but this would cause difficulties; for it is true that $12 \cdot 0 = 1 \cdot 0$, and hence, dividing both sides by 0 and using the assumption that $\frac{0}{0} = 1$, we would have the absurdity $12 = 1$. The problem of assigning values to such indeterminate quantities as $\frac{0}{0}$ and $\frac{\infty}{\infty}$ ("infinity divided by infinity") is solved by differential

[7] The second argument is a special case of the first.

calculus, where 0 is recognized as the limit of an infinite series of values of a variable which becomes as small as we please.[8] This calculus enables us to assign a velocity to a moving object at every point, such velocity depending on the nature of the path.

Zeno's fourth argument, concerning the relativity of motion, is far more important than a great many scientists and philosophers have been willing to believe.[9] It leads to a change from absolute kinematics to relative kinematics. Those who, like Newton, believe in an absolute kinematics, assert that though it is true that the velocities of bodies may vary, depending on what "point of reference" we may take, so that the velocity of a man walking down the car of a moving train may be four miles an hour with respect to the car, but sixty miles an hour with respect to the ground, nevertheless there is an "absolute" velocity and this is taken with respect to "absolute" space, which remains fixed. Hence, the true velocity of the earth as it moves on its path is not to be measured with respect to the sun, which may, and probably is, moving itself, but with respect to "space." Though it would be inaccurate to say that Zeno was the founder of the theory of relativity, nevertheless, since the discovery of this theory, physicists have come to regard absolute kinematics as a false picture of the world. In a sense accepting the import of Zeno's argument, they have insisted on a relative kinematics, where velocities may be taken with any point of reference, there being no "absolute" set of co-ordinates. More cogent arguments for accepting a relativistic kinematics are given in the next section.

II. We begin by stating a fundamental law of Newtonian mechanics, the *law of inertia:* "All bodies remain in a state of rest or uniform motion in a straight line (if sufficiently far removed from other bodies)." Now in order to determine what kind of motion a body is undergoing, we must have some "co-ordinate" or "reference" system relative to which we make our measurements. Thus a man walking along a sidewalk of a straight road would be traveling in a straight line relative to a stationary tree,

[8] More accurately: x is said to approach zero (from the positive side) if, given any number $\epsilon > 0$, there exists another quantity $\delta > 0$, such that $(\epsilon - x) > \delta$.

[9] Whether Zeno actually realized the full import of his argument is very doubtful, but this hardly detracts from its importance.

but would be traveling in a curved line relative to an object which is itself moving in a curve. We shall call a reference system "permissible" if the motions of bodies relative to it obey the law of inertia. The earth is not a permissible system of reference, since relative to the earth the visible fixed stars, which ought to obey the law of inertia to a high degree of approximation, do not move in straight lines, but in immense curves.

Suppose that K is a permissible co-ordinate system. Imagine someone on a body which is traveling with uniform (nonaccelerated) motion relative to K. Then if an airplane moves in a straight line relative to K, it will also travel in a straight line for someone on the moving body, though the airplane's velocity and direction may be different. Hence, if K is a permissible system, i.e., if the law of inertia holds relative to K, then K′, K″, etc., will all be permissible systems if they are all moving uniformly relative to K.

This result may be generalized, its generalization being the (special) principle of relativity. Suppose, again, that K is a permissible reference system. Suppose, also, an immense box traveling with uniform motion relative to K. The inhabitants of this box include a group of scientists, who begin experimenting, and finally arrive at a set of laws for physics. As we have remarked above, they will come to the conclusion that the law of inertia holds. They will also derive many other laws. We say that these laws will not differ fundamentally from the laws of an experimenter who is fixed relative to K. More precisely, the principle of relativity (in the restricted sense) states that if K′ is in uniform motion relative to a permissible K, then phenomena will obey the same general laws with respect to both K and K′.

Now there occurs another fact, experimentally verified, which seems to conflict with this principle. Suppose that the box we have pictured is moving along at the rate of 60 miles per hour relative to K. Then if one of the inhabitants walks in the direction of the line of motion with a speed of 4 miles an hour, he will travel at the rate of 64 miles an hour relative to K, by the simple law of addition of velocities. In general, if v is the velocity of a body moving in the direction of motion for an observer in the box, $60 + v$ will be the velocity for an observer stationary relative to K.

The velocity of light is approximately 186,000 miles per second. There is a principle of physics called the law of constancy of the velocity of light which states that the velocity of light does not depend on the velocity of motion of the source of the light. But this principle comes in conflict with the principle of relativity. For, by the law of addition of velocities, if the velocity of light is 186,000 miles per second for a scientist who is fixed relative to K, it ought to be 186,000 + 1/360 for an observer in the box. For systems traveling at higher speeds, the difference would be even more pronounced. Hence, we seem to have the result that if the principle of relativity is correct, the principle of constancy of velocity of light must be wrong, and conversely. In other words, logic seems to force us to abandon one of these two principles.

That we cannot deny the law of constancy of velocity of light has been established with a fair degree of certainty by the physicists. Must we abandon the principle of relativity?

The special theory of relativity consists in analyzing the concepts of kinematics and succeeds in retaining both the principle of relativity and constancy of velocity of light by rejecting what had been regarded previously as a necessarily true proposition of kinematics: "If *a* and *b* are simultaneous events with reference to one system, they will be simultaneous with reference to all systems." This was the Newtonian conception of "Absolute Time"; events simultaneous for an observer on the earth would be simultaneous for an observer on Mars. Some definition of simultaneity is required, but the following seems to meet this requirement: Suppose one event happens at A and another at B; an observer is situated at the mid-point M of the distance AB. This observer has an apparatus (e.g., two mirrors inclined at 90°) by which he may observe both places A and B at the same time. If he sees the two events at the same time, then they are simultaneous.

Supposing again the traveling box, we ask: Are two events which are simultaneous relative to K also simultaneous with reference to this box? The answer must be in the negative, for suppose the events at A and B are simultaneous for an observer stationary relative to K; this means that the rays of light from A and B strike his mirror at the same time. But for a scientist observing these events from his box, the events might not be simultaneous. If the box is traveling along the line AB, the events will not be simultaneous, for since the observer is traveling towards the beam from A, say, this will strike his mirror first, and the events will not be judged simultaneous by him.

It follows that the concept of time must be taken in a relative sense (relative to the reference system chosen). In general, the units of time (second, minute, hour) will vary depending on the reference system chosen. Hence these two principles are not really in conflict. The velocity of light *is* constant relative to all uniformly moving systems, i.e., the velocity is always k miles per second, where k is constant, but the duration of a second is not the same in all these systems, and hence there is no conflict with the law of addition of velocities or the principle of relativity.

The special theory of relativity also developed the notion of "relative distances," so that measuring rods may vary depending on the reference system chosen.[10]

III. At the beginning of this chapter a second remedy was suggested for the resolution of conflicts between the laws of a given science and those of logic: the expedient of changing the laws of logic. Singer's article [11] is an example of the contribution of logic (although the difficulty was one of logic's own making) to the long standing controversy in biology between the Mechanism and Vitalism. On one hand, Democritus proposed that a sufficient explanation for all phenomena might be made completely in mechanical terms. Aristotle, on the contrary, held that things must have in addition to non-living matter an active principle, a form, function, or entelechy, to complete them.

Both of these positions bear the weight of tradition and quite as traditionally have been taken to be simultaneously untenable. As may well be imagined the contest was acute when relating specifically to the nature of living organisms. Supporting the Democritean hypothesis were, among others, Epicurus, Booerhaave, probably Freidrich Hoffman, and Jacques Loeb. Following the tradition of Aristotle there are the *spiritus vitae* of

[10] For fuller exposition, see Einstein's *Relativity; The Special and General Theory*. Translated by Robert W. Lawson, Henry Holt and Co., 1920; P. Smith, 1931. A great part of this discussion is taken from this book.

[11] E. A. Singer, Jr., *Beyond Mechanism and Vitalism,* Philosophy of Science, July 1934; vol. 1, no. 3, pp. 273-295. This discussion is largely a summary of the article.

Paracelsus, the *archai* of Van Helmont, G. E. Stahl's theory of phlogiston, Leibnitz' monads, and the entelechies of Dreisch.

It is not, however, the history of these schools that is of interest but rather the delineation of some third and synthetic viewpoint. Although Robert Boyle apparently employed both mechanical and teleological explanation with no concern for their consistency, Immanuel Kant first rigorously stated that while the universe was inconceivable without purpose or function, still there must and can be no instance of the violation of mechanical law. The Kantian solution is not altogether satisfactory since it is dependent upon the nonexperimental concept, the noumenon. We turn now to an exposition of the problem which is consistent with experimental science.

Following Singer's demonstration, though without its elegance, we may set down three propositions:

D. The Democritean Postulate. Everything in Nature is structural (independent of environment) in nature.
A. The Aristotelian Postulate. Some things in Nature are nonstructural (dependent upon environment) in nature.
C. The Classic Logician's Postulate. Nothing nonstructural in Nature is structural in nature.

An examination of these will show that if any two of the propositions are taken to be premises of a formally valid syllogism the conclusion will be the contradictory of the third. The Mechanist holds that the existence of living beings may be accounted for by the same kind of laws which determine nonliving objects, i.e., everything must find its place in a mechanical image. He would affirm the Democritean postulate (D) and deny the Aristotelian postulate (A). The Vitalist proposes that there is a unique factor present in living organisms differentiating them from phenomena comprehended in terms of physical laws, and therefore inexpressible in a mechanical image, but implying a function. He would affirm A and deny D. Believing it to be evident that a thing could not at one time be expressed structurally and nonstructurally, both positions affirm the Classic Logician's postulate (C). There is but one other course [12] open and that alone will enable us to affirm the Democritean

[12] This omits the possibility of a sceptical, "mystical," or "transcendental" alternative.

and the Aristotelian postulates and perforce to deny the Classic Logician's—this may be called the Naturalist's position. In summary: [13]

The Mechanist: D A′ C
The Vitalist: D′ A C
The Naturalist: D A C′

We shall now inquire into the conditions, if there be such, under which the Democritean and the Aristotelian are compatible. It is obvious that the problem is contingent upon the relation which exists between the terms *structural* and *nonstructural*. If it is true that they are contradictories, that D and A are contradictory propositions, then nothing can be accomplished. If, however, the historical formulation is lacking in rigor and some equivocation has been committed, if it should turn out that the Democritean and Aristotelian postulates are not formal contradictories, then the Naturalist's position becomes at least a possibility. This alternative was not available to the eighteenth century, for it depends upon De Morgan's invention of the term universe of discourse.[14]

The first step in Singer's demonstration is the establishment of a universe of discourse in which structural and nonstructural are contradictories. It is proposed that this universe be composed solely of classes within which the structural and nonstructural are to be contradictory Classes or classes such that everything in nature shall belong to one and only one of the contradictories so defined.

The second step establishes the universe of discourse within which the things of the Democritean "everything" and of the Aristotelian something lie. It is proposed that " individual" be substituted for "thing" (for if "class" be used the cause is already lost) ; not any individual, of course, but one belonging to a universe, *u*. Utilizing the discussion of these two steps and conveniently replacing "individual of the *u*-class" by body, we may restate the postulates: D. Every body in Nature is a body included in a structural class. A. Some bodies in Nature are bodies included in a nonstructural class. It is now necessary to show that these are not incompatible.

[13] Where, for example, DA′C means "D is true, A is false, and C is true."
[14] Cf. p. 98.

At first sight it would seem that body so defined belongs to a null class, that is to say it is no body at all. In other words:

If C and C′ are contradictory Classes of classes
p is a class included in C
q is a class included in C′, and
b is a body

then if *b* is included in *p*, and *p* is included in C, then *b* is included in C; also, if *b* is included in *q*, and *q* in C′ (non-C), then *b* is included in C′. Hence, *b* is included in both C and C′, i.e., *b* does not exist. This reasoning is in error; for if it were not we should be obliged to hold: "If a given individual is one of a certain class, and that class is one of a Class of classes, then the individual is one of a Class of classes." This is manifestly absurd.

Stated more generally, it is not a question of the nature of inclusion but simply that the relation of inclusion established in one universe of discourse cannot be assumed to hold in another. Thus if *b* is included in both *p* and *p′*, or if *b* is included in both *q* and *q′*, then *b* does not exist; or if *p* is included in both C and C′, then *p* does not exist, and if *q* is included in both C and C′, then *q* does not exist. But it does not follow that if *b* belongs to *p* and *p* to C, and *b* belongs to *q* and *q* to C′, that *b* does not exist. Hence a body may belong to a class, which class is called structural, i.e., belongs to the structural side of the class of classes. The same body may belong to another class, which class is nonstructural. Without logical inconsistency we may say that such a body exists. That is, we contradict no law of logic when we say that there exist organisms which belong to both a class of things defined mechanically (structurally) and a class of things defined nonmechanically (nonstructurally). The existence of the mechanical and the biological does not conflict with logic, *provided the traditional laws of logic are modified in the light of the concept of the universe of discourse.* It should be noted, as a final word of caution, that this discussion only shows the *consistency* of the naturalist's position; it is not the logician's task to verify this school in any other manner.

IV. Probably no science has been so productive of paradoxical results as that branch of mathematics known as the "Theory of

Sets" (Mengenlehre). The theory of sets in its most general form is merely the theory of classes, and as such is a part of logic. We have examined certain paradoxes which arise in the case of the general theory (cf. page 203). Here, however, we restrict our attention to one property of sets, namely, the number of elements in the set. For every set or class of objects there corresponds a definite (cardinal) number. As an example of the manner in which this theory confines itself to numerical considerations, take the following definition of the equivalence of two sets:

> "We say that two sets A and B are 'equivalent' if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them there corresponds one and only one element of the other."

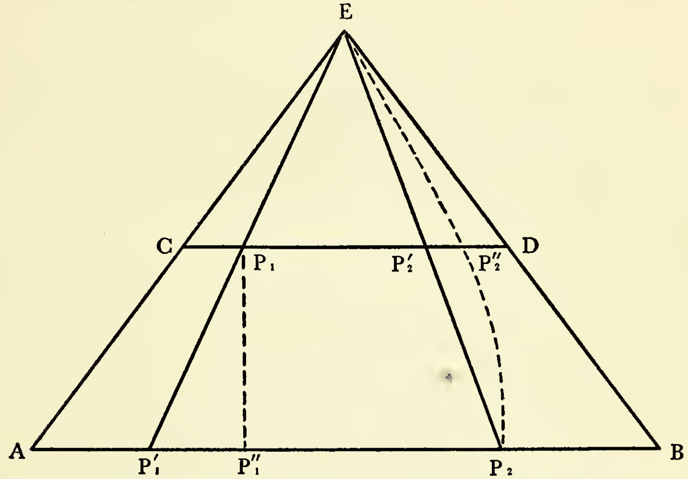
Thus the set of jurors on a jury and the set of a dozen eggs are equivalent, since it is not difficult to find a law defining the correspondence of an element in one set to a unique element in the other. In general, in the case of sets which contain a finite number of objects, two sets will be equivalent if they have the same number of objects, no matter how they may differ in other respects.

Paradoxical results arise in this theory when a given set contains an infinite number of objects. Examples of such sets are plentiful enough: the set of points on a line one inch long, the set of all proper fractions, the set of all whole numbers, the set of all real numbers, the set of all atoms in the universe (matter being infinite), etc.

A result of this theory surprising to the beginner is the following: the set of all points on a line one inch long is equivalent to the set of all points on a line two inches long, and, in general, the set of points on any finite segment of length greater than zero is equivalent to the set of points on any other such segment. This may readily be proved as follows: Let CD and AB be given segments (AB > CD); form the triangle ABE, with AB as base, so that C falls on AE and D on EB, a construction always possible when CD < AB.

We can easily find a law by which every point on CD corresponds to just one point on AB and vice versa. Suppose P_1 is any point on CD. Then draw EP_1, and extend this line till it meets

AB in, say, P_1'. Then P_1' is the only point corresponding to P_1 according to this scheme, for if there were another point P_1'' distinct from P_1', then the line from E through P_1 would cut AB in two distinct points, i.e., two straight lines would have a segment in common. Again, for every point P_2 on AB there corresponds just one point on CD, the point determined by the intersection of P_2E and CD, for if there were two such points P_2' and P_2'' we would have two straight lines enclosing a space.



This demonstration makes use of geometrical principles (in particular, of Euclidean geometry, since it assumes a law, not true in Riemannian geometry, that two lines do not enclose a space) but the theorem may be proved analytically as well.

Another theorem of the same sort shows that the set of all positive whole numbers is equivalent to the set of all whole numbers, positive or negative (excluding 0 for convenience sake). This result seems to lead to an obvious contradiction of an arithmetical law, for it would seem that the set of positive numbers is only one-half the set of all numbers, so that we have the half equal to the whole. In order to prove the required equivalence we must find a rule by which we can make the elements correspond uniquely. There is a wide choice, but the following seems simplest: Suppose A is the set of all positive whole numbers and B the set of all whole numbers; then let the number 1 in A correspond to — 1 in B, the number 2 in A correspond to $+1$ in B, the number 3 in A to -2 in B, the number 4 in A to $+2$ in B, etc. In general, if n is the number in A, then its corresponding element in B will be $+n/2$, if n is even and $-\frac{n+1}{2}$ if n is odd. Thus 84 in A corresponds to 42 in B and 127 in A corresponds to -64 in B.

With this rule, if we are given any element of A, i.e., any positive number, we can determine a unique element of B, either positive or negative. The reverse is also true, for given any element of B, we can readily pass to a unique element of A. If the given element of B is negative, then the required element of A is odd and is equal to $-(2n-1)$, while if the given element, n, of B is positive, the required element of A is even and is equal to $2n$, and these again are unique. Hence the correspondence (called "one to one" correspondence) is established. This theorem may be generalized as follows: Suppose A is a set equivalent to the set of all positive whole numbers; then 2A (i.e., doubling the elements of A), 3A, . . . , nA, are all equivalent to A, where n is any (finite) integer; stronger still, $A.A = A^2$, A^3, . . . , A^n are all equivalent to A, so long as n is a (finite) integer.

A paradoxical result of the same kind appears in Sterne's *Tristram Shandy*. The narrator discovers that it has taken him two years to write the first two days of his history and ponders whether, if providence spared him forever, he would complete his story at that rate. He would, indeed, in the sense that no day would be left untold, if he should live forever. For in order to discover during what years he will write about a certain day, we need only apply a simple mathematical formula; he will write of the nth day n years from the time he began; in infinite time, the set of all days and the set of all years are equivalent.

These results, surprising as they may seem, yield no conflict between the theory of sets and logic or arithmetic. It would, indeed, be contradictory to say that the set of all soldiers in a line one mile long was equivalent to the set of all soldiers (similarly placed) in a line ten miles long, for both of these are finite sets. But when the above definition of equivalence is applied to infinite sets of objects, extraordinary but not contradictory results may follow. To assume that there is a contradiction here is to commit the fallacy of composition. Thus, in the case of finite sets, when we say "all of the elements of the set" we mean the total collection of these elements, but when we say "all the points on a line," we cannot mean the total collection of such points, since there is no such thing as an infinite collection. An infinite collection would be a collection or completion of things which cannot by definition be collected or completed. When I refer to "all the points on a line," I must mean "all" in the distributive sense: "each and every point taken on a line." This distinction is often overlooked by the mathematician himself.

The definition of equivalence between two infinite sets thus becomes:

"Two infinite sets are equivalent if all elements (i.e., each and every element taken) of one can be made to correspond uniquely with all elements (each and every element) of the other; or, whatever element is taken from one, there is but one element of the other to which it corresponds."

Here "all" is not taken in the collective sense, and the definition is perfectly consistent, as are the results proved above, which are not so paradoxical if the distributive sense is borne in mind. Thus, it is not true that a whole is equal to a half if a "whole" refers to some collection (necessarily finite); but for infinite sets there is no such thing as the "whole of the set" since such a set is never completed. Those who use such terms as "all the integers," the "whole class of integers" must be careful to bear in mind that they are using these terms in a sense quite different from ordinary usage, and should expect to obtain startling results. If one defines an elephant as a three-sided figure, he might deduce that "All elephants have angles summing to two right angles," a result paradoxical enough if we forget his idiosyncrasies in defining things, but perfectly consistent if we adhere to his convention. Similarly we might be startled to learn that "The whole class is equal to the half" if we were not aware that the speaker was taking "the whole class" to mean all the class in the distributive sense. In sum, then, a one-to-one correspondence can be set up between the elements of A, the set of all positive integers, and the elements of B, the set of all integers, positive and negative, but we cannot say that the entire set A is equivalent to the entire set B, since "entirety" is not applicable to infinite aggregates.

There are results of the theory of sets, however, which are not merely odd, but also seem to involve a contradiction. The exposition of some of these would require a long explanation concerning the assumptions of the theory, but there are a few which can be stated fairly simply. A good example of the latter type is the paradox arising from the concept of the set of all cardinal numbers. The set of all cardinal numbers includes not only the usual numbers of ordinary arithmetic, 1, 2, 3, but also certain "transfinite" numbers. We have already seen example of a number of the latter type, for the set of all whole numbers is infinite, and hence the number representing its elements must be infinite. This number is usually symbolized by $\aleph_0$ ("aleph null"). Thus the cardinal number of the set of all the days in eternity, the set of all minus numbers, the set of all whole numbers, is aleph null. Is there a cardinal number larger than aleph null? It can apparently be proved that for any given cardinal number there is another cardinal larger than the first. Hence, there are transfinite numbers larger than $\aleph_0$, and ones larger than these, and so on, ad infinitum.

We now introduce the concept of the sum of two or more cardinal numbers; if m is the cardinal number of the set M and n the cardinal number of the set N, the $m + n$ will be the cardinal number of the set composed of all the elements of both M and N. If M and N happen to be finite, $m + n$ will be merely the ordinary arithmetical addition of two whole numbers. But if either M or N is infinite, then $m + n$ may be equal to m or n, for example, if M is the set of whole positive numbers, whose cardinal number is $\aleph_0$, and N the set of whole minus numbers, whose cardinal number is $\aleph_0$, then $m + n$ would be the cardinal number of the set of all whole numbers (excluding 0) and this again, as we have shown, is $\aleph_0$. But one thing seems plain, and that is the fact that the cardinal number of the sets which are parts of another set can never exceed the cardinal number of the whole set. We consider the set K of all cardinal numbers.

The sum of all elements is a cardinal number which cannot be smaller than any cardinal number, for this would make the sum smaller than one of the elements, and this is impossible; but this is a contradiction of the statement made above that for any cardinal number there exists a larger.

Detailed solutions of this paradox are impossible because of the complex nature of the science. But, as a suggestion, one might point out that the argument seems to commit the Fallacy of Composition, in that infinite sets of objects are taken as collections. The proofs usually offered that there exist cardinal numbers greater than $\aleph_0$ seem to assume that infinite sets have some definite form.

V. No science is completely free of such conflicts as have been described; logical difficulties arise even in sciences which, on account of their complexity, seem far removed from the problems of logic. The science of Law is a case in point. One such conflict which occurs between law and logic is the problem of *renvoi.*

Suppose [15] that Smith, an American citizen, dies in France, leaving movable property in New York. At the time of his death, Smith was domiciled in France according to the law of New York but not according to the law of France. The law of New York directs the distribution of movables in accordance with the law of the domicile of the deceased. French law directs such distribution according to the law of the nationality of the deceased. The New York court is seised of the case.

Difficulties would arise if the New York court should refer the matter to the French courts, and the latter should refer the matter back to New York. In this situation, we either have an infinite regress analogous to that arising from a circular definition or we have a contradiction analogous to that of the Epimenides paradox. The contradiction would appear if the matter were considered in this way: assuming that the New York court's decision is final, we are led to the conclusion that it cannot be, since by its own decision, it refers the matter elsewhere. Assuming the French court is final, we are led to the same result; since in this case either the New York or the French court must be final (analogous, in the case of the Epimenides paradox, to the statement that either p is true or p is false), we are led to a contradiction.

An infinite regress would occur if the New York court should refer the matter to the French court without any reference to a final decision, so that the latter might consistently refer the ques-

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[15] This illustration was taken from an article in the Univ. of Penna. Law Review, vol. 87, pp. 341-49, by T. A. Cowan, entitled *Renvoi Does Not Involve a Logical Fallacy;* my thanks are due Prof. Cowan for the privilege of using this material.

tion back to New York. The case of an infinite regress involves no logical paradox (cf. page 207). Hence, to avoid a conflict with logic, the principle of *renvoi* need only be stated in this way, in which case the question of final decision is always left open. Nevertheless, however satisfied the logician might be with this solution, the result would be bad because, as is usual in cases of infinite regress, it would be totally impractical. That is, there would still remain a conflict between law and sociology. Criteria other than consistency are required in determining the value of any law, and one fundamental stipulation is that the courts arrive at a decision on a given matter in a finite time. There are many ways of restating *renvoi* so as to make it more practical, but such solutions, in that they are not solving a conflict between law and logic, are beyond the interests of this book.

Renvoi by no means represents the only logical conflict to be found in legal theory. Most countries solve conflicts of law by establishing a Supreme Court of Appeal which shall make consistent, if necessary, any inconsistencies arising between the laws of different communities. However, there is no Supreme International Court, so that legal theory is more or less helpless in the face of such cases as those in which an individual is declared to be a citizen of one country because he was born there and the citizen of another because his parents were its citizens, despite the fact that both countries insist that allegiance be paid to their flag only.

EXERCISES

GROUP A

In the following examples, determine what sciences are involved in the conflict. Where possible, suggest solutions of the difficulties.

1. "Something, A, changes, and therefore it cannot be permanent. On the other hand, if A is not permanent, what is it that changes? It will no longer be A, but something else. In other words, let A be free from change in time, and it does not change. But let it contain change, and at once it becomes A^1, A^2, A^3. Then what becomes of A, and of its change, for we are left with something else? Again, we may put the problem thus. The diverse states of

A must exist within one time; and yet they cannot, because they are successive."[16]

2. "The dilemma, I think, can now be made plain. (*a*) *Causation must be continuous.* For suppose that it is not so. You would then be able to take a solid section from the flow of events, solid in the sense of containing no change. I do not merely mean that you could draw a line without breadth across the flow, and could find that this abstraction cut no alteration. I mean that you could take a slice off, and that this slice would have no change in it. But any such slice, being divisible, must have duration. If so, however, you would have your cause, enduring unchanged through a certain number of moments, and then suddenly changing. And this is clearly impossible, for what could have altered it? Not any other thing, for you have taken the whole course of events. And, again, not itself, for you have got itself already without any change. In short, if the cause can endure unchanged for even the very smallest piece of duration, then it must endure forever. It cannot pass into the effect, and it therefore is not a cause at all. On the other hand, (*b*) *Causation cannot be continuous.* This would mean that the cause was entirely without duration. It would never be itself except in the time occupied by a line drawn across the succession. And since this time is not a time, but a mere abstraction, the cause itself will be no better. It is unreal, a nonentity, and the whole succession of the world will consist of these nonentities. But this is much the same as to suppose that solid things are made of points and lines and surfaces. These may be fictions useful for some purposes, but still fictions they remain. The cause must be a real event, and yet there is no fragment of time in which it can be real. Causation is therefore not continuous; and so, unfortunately, it is not causation, but mere appearance."[17]
3. "Every composite substance in the world consists of simple parts; and there exists nothing that is not either itself simple or composed of simple parts, for the term 'composite' means a composition of something, so that if there were no simple parts we would have a composition of nothings, i.e., we would have nothing at all, and the world would not exist. On the other hand, no composite thing in the world consists of simple parts, and there does not exist in the world any simple substance. For suppose there were simple substances; then, like all reality, these must occupy

[16] F. H. Bradley, *Appearance and Reality,* chap. V.
[17] *Ibid.*

a certain space, i.e., they must have a certain extension. If this is the case, they may be divided by planes cutting them, i.e., they are not simple, but made up of extended parts; hence, there can be no simple elements in the world." [18]

4. "If Socrates was born, Socrates became either when Socrates existed not or when Socrates already existed; but if he shall be said to have become when he already existed, he will have become twice; and if when he did not exist, Socrates was both existent and non-existent at the same time—existent through having become, non-existent by hypothesis. And if Socrates died, he died either when he lived or when he died. Now he did not die when he lived, since he would have been at once both dead and alive; nor yet when he died, since he would have been dead twice. Therefore Socrates did not die. And by applying this argument to each of the things said to become or perish it is possible to abolish becoming and perishing." [19]

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[18] Kant's *Second Antimony,* restated.

[19] Sextus Empiricus, *Outlines of Phyrronism.*

Part III

Modern Developments in Logic

Exposition of the Boolean Algebra

12

IT HAS been scarcely a century since the science of logic has taken an enormous step forward through the discovery of what has since become known as Symbolic Logic. The great strides which have been made in this field, especially in the last three decades, make it impossible to present an all-inclusive survey; the succeeding chapters aim merely at an introduction.

The German philosopher Leibnitz (1646-1716), one of the discoverers of the differential and integral calculus, impressed as he was by the rigor of symbols in mathematics, had suggested that the mathematical method might well be carried over to other sciences. His aim was to establish a "universal language," not a verbal language such as Esperanto, but a symbolic one. Note the advantages of the symbols in mathematics. In the first place, these symbols are universally recognizable. Thus, the equation $a + b = b + a$ is as intelligible to a Frenchman, a German, a Swede, as it is to an Englishman. There is no need of translating the sentence into another language as there would be had the equation been written "a plus b equals b plus a." Everyone acquainted with the subject understands the meaning of mathematical symbols. Second, and by far the more important point, symbols avoid the ambiguity which is so often found in words. Thus, *verbally,* both of the following sentences are correct: "Nothing is greater than infinity" and "Nothing is less than every positive number." But if we symbolize these two sentences, we must write (using ∞ for "infinity") "$\infty > x$" and "$0 < x$" (where x is any positive finite number), and here no contradiction is apparent. We shall see later in the chapter what an important role the exactness of symbols plays. Third, symbols are much more concise than words. Try, for example, translating some complicated fractional equation of algebra into words. Another important advantage of symbols, their abstractness, will appear later.

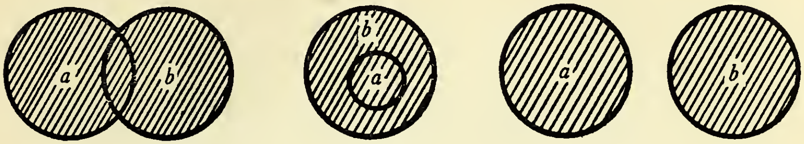
The application of symbols to logic has already been studied in part in this book, for the deductive system of the Aristotelian class logic in chapters IV and V owes much of its exactness and conciseness to the symbolic method.

The dream of Leibnitz was realized in part by George Boole about a century ago. Boole attempted to construct what he called an "algebra" of classes. That is, instead of considering the *a*'s, *b*'s, *c*'s, of ordinary algebra as quantities, he considered them classes. The resulting system has become known as Boolean algebra, though its original form has been considerably developed and simplified through the work of such men as De Morgan, Peirce, Venn, Schroeder, and others.[1]

We note first that the addition of two integral numbers is primarily the process of increasing one quantity by so many units. We shall define the addition of two classes, symbolized in the usual manner by $a + b$, as the class composed of all objects which belong to either *a* or *b*. That is, $a + b$ in this algebra represents another class, just as $a + b$ in ordinary algebra represents another number. This class is always definite if *a* and *b* are given, and represents all the members of both *a* and *b*. "$a + b$" may then be read "what is either *a* or *b*." Note that the words "either, or" are ambiguous, sometimes meaning "either, or (but not both)": "either white or black," and sometimes "either, or (or both)": "either Americans or Pennsylvanians"; the distinction is recognized in the Latin language, the former meaning being translated by "aut," the latter by "vel." That the latter interpretation is what is meant here will be obvious from the diagrams below and from the system itself. As a concrete illustration, the addition of the two classes "white things" and "animals" is the class whose membership comprises all white things and all animals. The addition of the classes "triangles" and "geometric figures" is the class which comprises all triangles and all geometric figures, or, since all triangles are geometric figures, the addition of these two classes is merely the class of geometric figures. Pictorially, addition may be shown in the

[1] For an excellent historical summary, see C. I. Lewis', *Survey of Symbolic Logic,* chap. I.

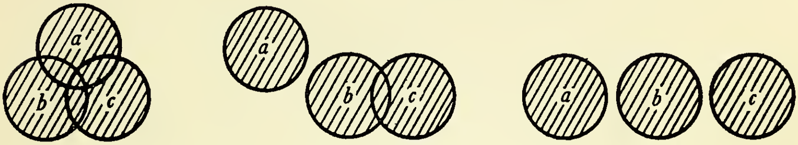
following manner by Euler's diagrams, the shaded portion representing $a + b$:



1 2 3

$a + b$

The addition of three classes, a, b, c, will then be composed of all objects which belong to either a, b, or c:



1 2 3

$a + b + c$

From this, it is apparent that addition in this algebra has many of the properties of addition in ordinary algebra. For instance, addition is "commutative" in both; i.e.,

$$a + b = b + a.$$

(Note that subtraction and division in ordinary algebra are not commutative; i.e., $a - b \neq b - a$, and $a/b \neq b/a$ in general.)

Addition is also "associative" in both:

$$a + (b + c) = (a + b) + c.$$

"What is either a or else (b or c) is the same as what is either (a or b) or else c." (Again, subtraction and division are not associative:

$$(a - (b - c) \neq (a - b) - c, \text{ and } a/(b/c) \neq (a/b)/c \text{ in general.})$$

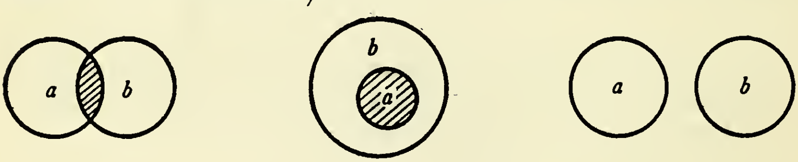
But addition in this algebra has a property which definitely distinguishes it from addition in ordinary algebra, in that, if a and b happen to be the same, we do not have $2a$ as the result of our addition, but rather a:

$$a + a = a$$

"What is either *a* or *a* is *a* itself." This property simplifies the algebra considerably.

Before defining the analogue of subtraction, we must introduce the definition of multiplication. The multiplication of two classes *a* and *b* (symbolized *ab* as in ordinary algebra) is defined as that class composed of objects which belong to both *a* and *b*, i.e., the members which *a* and *b* have in common. For example, the multiplication of the two classes "white things" and "animals" is the class whose members are both white and animals, i.e., the class of white animals. In general, a noun modified by one or more adjectives represents a Boolean multiplication; thus "large, green books" represents the multiplication of the classes "large things," "green things," and "books." The multiplication of the classes "triangles" and "geometric figures" will be the class whose members are both triangles and geometric figures, or the class whose members are triangles. Again, the class constructed from the multiplication of the classes "circles" and "squares" will have as members objects which are both circles and squares ("square-circles"), i.e., it will have no members at all.

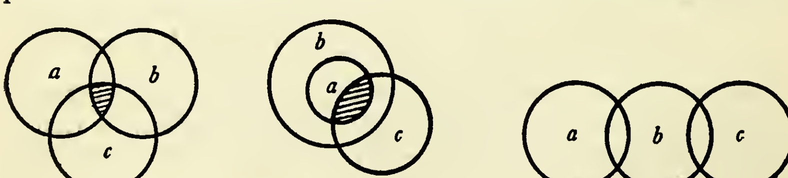
Multiplication is shown in the following Euler's diagrams, the shaded portion being *ab* (where no portion is shaded, the class *ab* has no members):



1 2 3

ab

The multiplication of three classes *a*, *b*, *c*, will be the class composed of the members which *a*, *b*, and *c* have in common:



1 2 3

abc

Multiplication here (like multiplication in ordinary algebra) also has the property of commutativity and associativity:

$$ab = ba$$
$$a\,(bc) = (ab)\,c$$

But again multiplication has a peculiar property analogous to that given above in the case of addition:

$$aa = a$$

"What is both *a* and *a*, i.e., what is common to *a* and *a*, is *a* itself." Hence there are no "powers" (such as a^3) in this algebra.

Multiplication here, as in ordinary algebra, is "distributive" with respect to addition; that is,

$$a\,(b + c) = ab + ac$$

as may easily be verified by Euler's diagrams. Thus the multiplication of two binomials is the same in form in both algebras:

$$(a + b)\ (c + d) = ac + ad + bc + bd.$$

Similarly, the same rules apply for the multiplication of trinomials, and, in general, for any polynomials.

It is a peculiarity of this algebra, however, that addition is distributive with respect to multiplication. We say that an operation "o" between two elements is distributive with respect to another operation "O" when the following law holds:

$$a \mathrm{o} (b \mathrm{O} c) = (a \mathrm{o} b) \mathrm{O} (a \mathrm{o} c)\,.$$

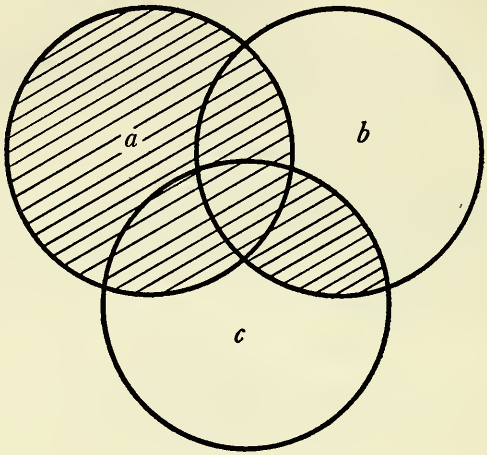
If "o" is the Boolean "plus" and "O" the Boolean "times," we have:

$$a + bc = (a + b)\ (a + c)\,,$$

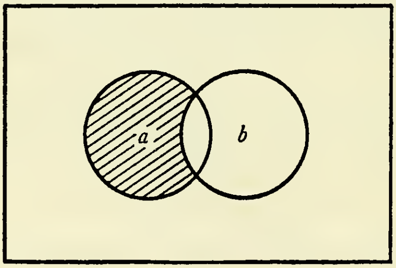
which is a true law of the algebra, as the diagram on page 250 indicates, the shaded portion being both $a + bc$ and $(a + b)$ $(a + c)$.

In order to define "subtraction" in the algebra we must introduce the concept of "non-*a*" or "what isn't a certain class," "what remains in the universe when a certain class is taken out." For a full discussion of this concept, see chapter IV on the

Aristotelian class calculus. If we let a' (or $-a$) symbolize non-a, we can define the subtraction of the class b from the class a as ab'. That is, the subtraction of one class from another is the



class composed of all objects which the latter has in common with the negative of the former: [2]



$a - b$ or ab'

Here ab' is the shaded portion. But note that subtraction cannot be written (as in ordinary algebra) $a + (b')$ or $b' + a$, for these are not equivalent to ab'. Note also the following peculiar property of subtraction:

$$ab' + b = a + b$$

(not just a as in mathematical algebra).

[2] That is, $a - b$ represents all the members of a which are not b's. The class of animals minus the class of things with backbones is the class of invertebrates.

The concept of the "negative" of a class plays a very important role in this algebra. For example, by means of this concept, we can define the operation of multiplication in terms of addition, and vice versa. That is:

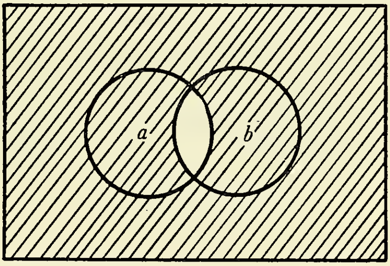
$$ab = (a' + b')'$$

"What is common to *a* and *b* is equivalent to what does not belong to either non-*a* or non-*b*." E.g., "white houses" might be defined as those things which are not either nonwhite or nonhouses. Similarly,

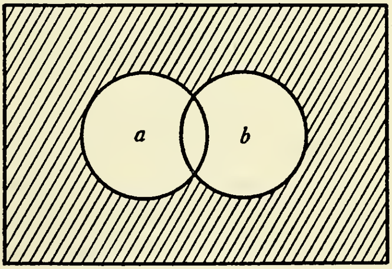
$$a + b = (a'b')'$$

"What is either *a* or *b* is equivalent to what is not both non-*a* and non-*b*." Thus what is either black or white is what is not in the class of colored objects (objects which are both not-black and not-white).

These propositions may be illustrated by Euler's diagrams:



The lined portion represents $a' + b'$. Thus what is not $a' + b'$, that is, $(a' + b')'$ is exactly that portion which is not lined, i.e., ab. Similarly,



here the lined portion is the product of a' and b', and hence the negative of this, $(a'b')'$, is $a + b$.

The above two propositions are known as De Morgan's Law.[3] Note that De Morgan's Law may be extended:

$$(a' + b')' = ab$$
$$(a' + b' + c') = abc$$
$$(a' + b' + c' + d' + \ldots) = abcd \ldots$$
$$(a'b')' = a + b$$
$$(a'b'c')' = a + b + c$$
$$(a'b'c'd' \ldots)' = a + b + c + d + \ldots$$

De Morgan's Law is often expressed in the following two formulas, which follow directly from the above:

$$(a + b)' = a'b'$$

"What is not either a or b is what is common to non-a and non-b."

$$(ab)' = a' + b'$$

"What is not an object which a and b have in common belongs to either non-a or non-b."

Essentially, De Morgan's Law means that in denying a certain expression we change all plus-operations into times-operations and all times- into plus-, and prime all the terms. Thus the negation of

$$a + bc' + a'cd + b'e$$

is

$$a'(b' + c)\ (a + c' + d')\ (b + e'),$$

and the negation of

$$(a + b')\ (c + de)\ (a'b + c'd)$$

is

$$a'b + c(d' + e') + (a + b')\ (c + d')$$

The symbol a' (or $-a$) has properties closely analogous to the symbol $-a$ ("minus a") in mathematical algebra. For example a'' ("what is not non-a") is equivalent to a, just as $--a = a$ in ordinary algebra. Also, if we subtract a from a we derive the first "constant" of our algebra, which, keeping the analogy with ordinary algebra, we symbolize as 0:

---

[3] Cf. p. 34, for De Morgan's law in the logic of propositions.

$$aa' = 0.$$

0 is then a class (for the operation of multiplication on two classes always yields a new class), but it is a class that has no members (hence often called the "null class").[4] The negative of the class 0 is symbolized by 1, and this is the universe, for if we take nothing or 0 from the universe, we have the universe remaining:

$$0' = 1.$$

But, since $0 = aa'$, we might define 1 in terms of a by deriving the negative of aa' by De Morgan's Law (plus the fact that $a'' = a$):

$$1 = 0' = (aa')' = a' + a'' = a' + a = a + a'.$$

That is, the universe may be defined as that class composed of what belongs to a or to non-a (i.e., everything). It is to be noted here, as in the case of the Aristotelian class "calculus" or algebra, that the term "universe" is considered as the "universe of discourse" (cf. page 98).

The following properties of 0 and 1 show their close analogy to the corresponding symbols in mathematical algebra:

$$a0 = 0$$

"What any given class and the null class have in common is nothing, i.e., is the null class."

$$a + 0 = a$$

"The class constructed from the addition of any given class and the null class is merely the given class."

$$a1 = a$$

"What any given class and the universe have in common is the given class."

But peculiar to this algebra is:

$$a + 1 = 1$$

---

[4] What a and non-a have in common is the null class. For example, square circles, talking mutes, bilateral triangles are all null classes.

"The addition of any class to the universe always yields the universe over again."

The most important relation of the Boolean algebra, however, is that of "inclusion," and the closest analogue in ordinary algebra to this is $\leq$, "less than or equal to." We shall symbolize the relation which expresses the inclusion of one class (a) in another (b) by

$$a \supset b$$

"a is included in b."

(It is very important to note that $a \supset b$ is not identical in all its properties with the Aristotelian A (ab), despite the similarity in the words by which the two are interpreted; cf. below.)

The analogies between $\supset$ and $\leq$ are apparent when Euler's diagrams are used, since a is either "less than" b or identical with b when $a \supset b$. The following properties of inclusion are important:

1. With respect to addition, the following relation is always true:

$$a \supset (a + b)$$

"a is included in the addition of itself and any other class." (This is analogous to the mathematical $a \leq (a + b)$, where b is greater than or equal to 0.)

2. With respect to multiplication, we have

$$ab \supset a$$

"What is common to a and any other class is included in a." (Here no apparent analogue exists in ordinary algebra.)

3. Again, the following relation exists between inclusion and addition:

$$\text{If } a \supset b, \text{ then } a + b = b.$$

"If a is included in b, then the addition of a and b is the class b." And conversely,

$$\text{If } a + b = b, \text{ then } a \supset b.$$

These two properties may be partially verified by an examination of Euler's diagrams. Thus inclusion has the property of making addition "absorptive"; this is also true in the case of multiplication, where we have

4. If $a \supset b$, then $ab = a$. "If a is included in b, then what is common to a and b is the same class as a," and conversely,

$$\text{If } ab = a, \text{ then } a \supset b.$$

It is noteworthy that properties 3 and 4 generalize the propositions $a + a = a$ and $aa = a$, since $a = b$ is a special case of the more general relation $a \supset b$.

5. From this follows an important property of multiplication and addition:

$$ab + a = a \quad (\text{since } ab \supset a).$$

That is, if a term in a sum of products appears by itself, then all other products containing this term may be "cancelled out": e.g.,

$$abc + bd + a + bfc + c + ghb + b = a + b + c.$$

This result may also be expressed:

$$a(a + b) = a;$$

the multiplication of a single term by any sum of products in which the term appears at least once "alone" is equal to that term itself. For example:

$$a(bc + d + afg + a + gh) = a.$$

Similarly:

$$(a + b)(a + c) = a(a + c) + b(a + c) = a + ba + bc = a + bc.^{5}$

6. If $a \supset b$ and $b \supset c$, then $a \supset c$. "If a is included in b, and b in c, then a is included in c." This principle is analogous to the Aristotelian syllogism "Barbara" (page 70), to the transitivity of implication in the calculus of propositions (page 28), and to the transitivity of $\leqq$ in ordinary algebra. It may be generalized in a manner similar to the generalization of the analogous principle of implication:

If $b \supset a$ and $ac \supset d$, then $bc \supset d$;
if $b \supset c$ and $ac \supset d$, then $ab \supset d$;
if $d \supset b$ and $ac \supset d$, then $ac \supset b$.

[5] Cf. p. 249; this is the formal reason for the validity of the distributive law $a + bc = (a + b)(a + c)$.

7. We can now apply property 6 to 1 and 2. For $ab \supset a$ and $a \supset a + b$; hence

$$ab \supset (a + b).$$

"What is both a and b is included in what is either a or b (or both)."

8. If $a \supset b$, then $b' \supset a'$. "If a is included in b, then what isn't b (non-b) is included in what isn't a (non-a)." Again, this is analogous to contraposition in the Aristotelian algebra ("A (ab) is equivalent to A ($b'a'$)") (page 100), the Law of Contradiction and Interchange in the calculus of propositions (page 24), and to the principle "If $a \leq b$, then $-b \leq -a$ (where a and b are real numbers)." This principle may also be generalized:

If $ab \supset c$, then $ac' \supset b'$,
and if $ab \supset c$, then $c'b \supset a'$.

9. If $a \supset b$, then $(a + c) \supset (b + c)$, and if $a \supset b$, then $ac \supset bc$, where c in each case may be any class whatsoever. That is, we may add the same class to or multiply the same class by both sides of an inclusion-relation without altering the validity. Hence also:

10. If $a \supset b$, then $a \supset (b + c)$, and $ac \supset b$.

But note that we do *not* have: "If $a \supset b$, then $(a + c) \supset bc$," nor are the converses of the propositions in 9 true; for from the fact that $(a + c) \supset (b + c)$ we cannot necessarily infer that $a \supset b$, as the reader may verify for himself by use of Euler's diagrams. Similarly it does not follow that $a \supset b$ if $ac \supset bc$.

11. From property 2, $ab \supset a$, we derive an important proposition of the algebra by letting b have the value a' or non-a:

$$aa' \supset a.$$

But aa' is 0, and hence we have

$$0 \supset a$$

no matter what a may be. That is, the null class is included in every class. Of course, we must establish the fact that the null class is always the same, that it is really a constant. We might suspect that this is the case, since classes here are taken in extension, i.e., are defined by their membership, and every null class has exactly the same members, namely, none at all. But formally we can show the same thing by proving that $aa' = bb'$; that is, no matter what class is used to define the null class, the resulting class is always the same: square circles, veracious liars, opaque transparencies, are all equivalent classes. Thus we can say not only that the class of square circles is included in the class squares and the class circles, but also in the class men, the class houses. More significant still, the null class has the *unique* property of being included in its own negative: $0 \supset 0'$ or $0 \supset 1$. Hence whenever a given class is such that it satisfies (makes true) the proposition $a \supset a'$, we can say that the class is 0.

Common usage recognizes the truth of the proposition $0 \supset a$ in some cases; for instance, if we have "Everyone robbing money from this bank will be sent to jail," and no one actually robs, then the given statement is still considered to be true: "Everyone who has robbed the bank (no one has—i.e., this is a null class) has been sent to jail." Such confirmations are by no means proofs of the validity of the proposition. In fact, common usage is so ambiguous that it often denies the validity of the statement (cf., page 299).

By property 1, $a \supset (a + b)$, we derive the following by letting $b = a'$:

$$a \supset a + a'.$$

But $a + a'$ is 1, the universe:

$$a \supset 1.$$

"Every class is included in the universe." Note that, analogous to the case of 0, 1 has the unique property of including its own negative; i.e., if $a' \supset a$, then a is 1.

The fundamental laws of the Boolean algebra given above display a property which for the early algebraists was the most astonishing of the calculus. If we write a "plus" for every "times" and a "times" for every "plus" and interchange 0 and 1 in these laws, the resulting expressions will also be laws of the calculus. For example, the distributive law in one form was $a(b + c) = ab + ac$, and in the other was $a + bc = (a + b)(a + c)$, and these two expressions may be changed into one another in the manner suggested. This principle is known as the Law of Duality, and one law is said to be the "dual" of another (the second is, of course, the dual of the first as well). The dual of $a + a' = 1$ is $aa' = 0$, the dual of $a + ab = a$ is $a(a + b) = a$, the dual of $a1 = a$ is $a + 0 = a$, etc. The truth of the Law of Duality is dependent principally on De Morgan's Law, though it also depends on other principles.

Before going on to the formation of the deductive system of the Boolean algebra, it is important to point out a significant aspect of this system. For we must now note that the interpretation which we have suggested is by no means forced on us. In fact, we may regard the symbols in a purely abstract manner, without any interpretation whatsoever. Or, we may give them some other interpretation for which they express a perfectly coherent and consistent system. To illustrate this point, we shall show how the Boolean algebra may be consistently interpreted as a logic of propositions.

We now think of the terms of the system as propositions, not as classes. In this case it is customary (but of course not obligatory) to write $p, q, r, \ldots$ instead of $a, b, c, \ldots$ We now seek new interpretations for each of the relations of the system: [6]

ADDITION: $p + q$: "Either p is true or q is true (or both are true)." Addition is usually referred to as "disjunction" under this interpretation. Thus the statement "Either today is Monday or it is raining" is a disjunction.

MULTIPLICATION: pq : "Both p and q are true." Multiplication here is usually termed "conjunction." Thus the conjunction of the above two propositions is the statement "Today is Monday and it is raining."

NEGATION: p' or $\sim p$: "p is false." Here we shall use the symbol p' for negation to keep the analogy with the class calculus, though $\sim p$ is slightly more common.

IMPLICATION: $p \supset q$: "p implies q" or "If p is true, then q is true." [7]

Thus, translating these properties of the Boolean algebra under this new interpretation, we have

$$p + p = p,$$

[6] Cf. p. 37.
[7] Symbolized "p $\angle$ q" above, but p $\supset$ q is used here to keep the analogy with the Boolean class algebra.

"To say that either p is true or p is true is to say (merely) that p is true." Similarly,

$$pp = p,$$

"To say that p is true and p is true is to say (merely) that p is true." Also

$$p + q = q + p,$$

" 'Either p or q is true' is equivalent to 'Either q or p is true.' " And

$$pq = qp,$$

" 'p and q are both true' is equivalent to 'q and p are both true.' " Again,

$$p(q + r) = pq + pr,$$

" 'p is true and either q or r is true' is equivalent to 'Either (p and q) or (p and r) are true.' "

De Morgan's Law, $p + q = (p'q')'$, now becomes "To say that either p or q is true is equivalent to saying that it is false that both p and q are false," and $pq = (p' + q')'$ becomes "To say that p and q are both true is to say that it is false that either p or q is false." These laws are usually phrased:

$$(p + q)' = p'q'$$
$$(pq)' = p' + q' \quad \text{(cf. page 35.)}$$

If now we examine the characteristics of the symbol $\supset$ given above under this new interpretation (where it means "implies" instead of "is included in"), we derive the following properties of the calculus of propositions: [8]

1. $p \supset p + q,$

"If p is true, then the statement 'Either p is true or some other given statement is true' is a true statement."

2. $pq \supset p,$

"If p and q are both true, then p is true."

3. $p \supset q \,.=.\, p + q = q,$

"If p implies q, then to say that either p or q is true is to say that q is true, and conversely;" or, "A necessary and sufficient condi-

[8] Not all these properties are true of the most general calculus of propositions (cf. p. 310).

tion that p imply q is that the expression 'Either p or q is true' be equivalent to 'q is true.' " The truth of this proposition will appear from the following considerations: when p implies q, p is false if q is false; hence the truth or falsity of $p + q$ depends solely on q alone. If q is true, then "Either p or q" is true, regardless of what p is; but if q is false, p must also be false, and hence $p + q$ is false. Conversely, if the truth of $p + q$ depends on q, then $p \supset q$. Similarly:

4. $p \supset q \; . = . \; pq = p.$

"p implies q if and only if the conjunction 'Both p and q are true' is equivalent to 'p is true.' " That is, when p implies q, the truth or falsity of pq depends solely on p. If p is true, then q is also true, and hence pq is true; but if p is false, then pq ("Both p and q are true") is false. Also, if $pq = p$, then $p \supset q$.

Properties 1-4 show clearly the close relation between the term "class" and the term "proposition" as we have used the words. For we considered a class extensionally, i.e., from the point of view of membership, and not from the point of view of its definition (cf. page 49). Similarly, we consider a proposition extensionally; we are not interested in its *meaning* but in the number of times it is true, in its "truth-membership." Thus, we call two propositions "equivalent" (symbolized here by $p = q$) when they are true in exactly the same cases and (hence) false in exactly the same cases. Hence, two propositions having quite distinct meanings are equivalent, e.g., "This triangle has three equal sides" and "This triangle has three equal angles." Thus $p + q$ may be considered as the class of cases where either one or the other, p or q, (or both) is true; pq becomes the class of the cases where both p and q are true; p' is the class of cases where p is not true; $p \supset q$ means that the class of cases where p is true is included in the class of cases where q is true. The remaining properties of the Boolean algebra under this interpretation will substantiate the foregoing; e.g.,

5. $pq + p = p.$

"To say that p and q are both true or else p is true is to say that p is true." The two sides of 5 may readily be seen to be equivalent under the above conception of equivalence, for they are both true in exactly the same cases. Whenever p is true, $pq + p$ will be true, no matter what q may be, for the second part of the alternative will be verified, and when p is false, $pq + p$ will be false, since both alternatives, pq and p, will be false.

6. $(p \supset q)\ (q \supset r) \supset (p \supset r)$.
Note that the expression "if . . . then" may be replaced by an implication sign here. This is the Principle of strengthening and weakening (cf. page 29).

7. $pq \supset p + q$,
"If p and q are both true, then either p or q is true."

8. $(p \supset q) \supset (q' \supset p')$.
This (and its generalization) is the Principle of Contradiction and Interchange (page 24).

9. $(p \supset q) \supset [(p + r) \supset (q + r)]$.
"If p implies q, then 'p or r' implies 'q or r.' " Similarly,

10. $(p \supset q) \supset (pr \supset qr)$.

11. $(p \supset q) \supset (p \supset q + r)$.

But perhaps the most interesting results of this interpretation of the Boolean algebra occur with respect to the "constants" 0 and 1. If we retain the definition of 0, we have:

$$0 = pp',$$

"The null proposition is that proposition which asserts that p is both true and false at the same time"; i.e., the null proposition is a contradiction. Similarly,

$$1 = p + p',$$

"The universal proposition is that proposition which asserts that either p is true or p is false"; i.e., 1 is a necessarily true proposition.

The rules governing 0 and 1 with respect to multiplication and addition are obvious enough:

$$p + 0 = p,$$

"To say that either a given proposition p is true or a contradiction is true is no more than to say that p is true" (since the second half of the disjunction is immediately excluded).

$$p + 1 = 1.$$

"To say that either p is true or a true proposition is true is to assert a true proposition (no matter what p may be)."

$$p \cdot 0 = 0,$$

"The expression 'Both p and a contradiction are true' is itself a contradiction."

$$p \cdot 1 = p$$

" 'Both p and a true proposition are true' is equivalent to 'p is true.' " Note that both sides have the same truth-value. This proposition expresses (among other things) the right to suppress a true premise (cf. page 103).

More significant still are the following:

$$0 \supset p$$

$$p \supset 1.$$

"If a contradiction is true, then anything follows," and "A necessarily true proposition is implied by any proposition." The former is not so strange, perhaps, for we might argue that if a contradiction were true, then all logic would collapse and implication has no meaning: anything both follows and does not follow from anything else. (Note that the second of these propositions follows from the first by Contradiction and Interchange.) These two propositions may be expressed: "A contradiction is the strongest proposition possible," and "A necessary proposition is the weakest proposition possible," (cf. exercise 9, page 44), statements which seem to substantiate the use of the words "strong" and "weak," since the strongest statements are usually the ones we consider false (at first), while the weakest are the trite, everyday truths.

EXERCISES

GROUP A

1. Multiply the following:

   EXAMPLE: $(a + b)\ (a + bc)$.

   We first multiply as in ordinary algebra. That is, we multiply the first term of the first sum (a) by each of the terms of the second sum, and then repeat the process for the second term:

$$(a + b)\ (a + bc) = aa + abc + ba + bbc.$$

But $aa = a$ and $(bb)c = bc$;
also $a + abc = a$, and $a + ab = a$.
Hence $(a + b)(a + bc) = a + bc$.

Note that certain expressions are reducible further, even though they do not appear so at first sight. Thus $ax + ax'$ is $a(x + x')$ or $a \cdot 1$, and this is a.

a) $(a + cd + b')(a + c)$
b) $(a + a'b' + b)(a' + b')$
c) $(ab + ac + bc)(a' + b' + c')$
d) $(a + b)(a + c)(b + c)$
e) $(a + b)(a + b')(a' + b)(a' + b')$
f) $(ab' + b'a)(ab + a'b')$

2. Give the equivalent of the following expressions by means of De Morgan's Law and the principle $a'' = a$:

EXAMPLE: $(a' + b)'$.

Now $(a + b)' = a'b'$; hence if we replace a by a' we have $(a' + b)' = a''b'$. But $a'' = a$. Hence $(a' + b)' = ab'$.

a) $(ab')'$
b) $(a'b)'$
c) $(a + b')'$
d) $(a' + b + c')'$
e) $(abc)'$
f) $(a + b' + c)'$
g) $(ab'c')'$

3. Reduce the following expressions by De Morgan's Law:[1]

EXAMPLE: $(ab' + c'd)'$.

First we consider ab' and $c'd$ as two single terms, and reduce the expression by the principle $(a + b)' = a'b'$:

$$(ab' + c'd)' = (ab')'(c'd)'$$

Now $(ab')' = a' + b$, and $(c'd)' = c + d'$. Hence

$$(ab')'(c'd)' = (a' + b)(c + d').$$

We now multiply the expression on the right as in ordinary algebra:

$$(a' + b)(c + d') = a'c + a'd' + bc + bd'$$

a) $(ab + c)'$
b) $(a + bc)'$

[1] Cf. p. 252.

c) $(a + bc + d)'$
d) $[(ab' + a'b)'(a'b' + ab)']'$
e) $[(ab + a'b')(ab)']'$
f) $[(ab + ca' + b'c')(a + bc)(c + ab)]'$
g) $[(a + ca' + c'a')(a' + c)]'$
h) Given that $a \supset c$ and $b \supset d$, reduce the following:

$$(a + b + cd)'$$

(Note that if $a \supset b$, then $ab = a$ and, since $b' \supset a'$, $b'a' = b'$.)

i) Given that $a \supset c$, $b \supset d$, $e \supset b$, and $f \supset a$, reduce the following:

$$(ab + a'b'cd + ce + fd)'$$

j) Reduce the following expressions of the calculus of propositions:
(1) $(p + p'q + q)'$
(2) $[(p \supset q)(p \supset q') + (p' \supset q')]'$
(3) $[(a \supset b)(b \supset a) + (a \supset b)(a \supset b')'(b' \supset a)']'$

4. a) The class A is defined as the set of all white cats plus the set of all white dogs. What is its negative? Let a = "white (things)," b = "cats," c = "dogs." Then $A = ab + ac$. Hence $A' = (a' + b')(a' + d') = a' + b'd'$, i.e., A' is the class of things which are not white plus the class of things which are neither cats nor dogs.

b) Give the negatives of the following classes:

(1) A = the class of all unhappy people plus the class of all happy angels.
(2) A = the class of all impractical theories plus the class of all practical problems.
(3) A = what the class of featherless bipeds and hairless animals have in common.
(4) A = all pigs that are either fat or lazy.
(5) A = class of dissatisfied Republicans plus the class of satisfied but ignorant Democrats plus the class of wise Socialists. (Assume that everyone is either a Republican, a Democrat, or a Socialist.)
(6) A = what the class of people who are either happy, poor, or lonely have in common with the class of people who are either happy or rich or not lonely.

5. Are there any operations, other than plus and times, in ordinary algebra which are such that the first is distributive with respect to the second?

6. Determine whether multiplication is distributive with respect to subtraction in Boolean algebra. Is subtraction distributive with respect to either addition or multiplication? Is addition distributive with respect to subtraction?
7. Given that $a \triangle b = ab + a'b'$ and $a \circ b = a'b + ab'$;
   a) What is the relation between $a \triangle b$ and $a \circ b$?
   b) (1) $a \triangle a = ?$ $a \circ a = ?$ $a \triangle a' = ?$ $a \circ a' = ?$
   (2) If $a \supset b$, $a \triangle b = ?$, $a \circ b = ?$
   (3) If $a = 0$, $a \triangle b = ?$, $a \circ b = ?$; is the converse true?
   (4) If $a = 1$, $a \triangle b = ?$ $a \circ b = ?$ is the converse of this also true?
   (5) Are "$\triangle$" and "o" commutative?
   (6) Are "$\triangle$" and "o" associative?
   (7) Is "$\triangle$" distributive with respect to "o"; is "o" distributive with respect to "$\triangle$"? Are either of these operations distributive with respect to "plus," "times," or "minus"? Are the latter three distributive with respect to "$\triangle$" or "o"?
8. Which of the following pairs of propositions are equivalent (i.e., which are true in exactly the same cases)?

   a) $a = 2b + c$
   $3a - 6b = 3c$

   b) The Romans conquer the Carthaginians.
   The Carthaginians are conquered by the Romans.
   c) The speed of light is greater than the speed of any other moving object.
   If the speed of light is greater than that of any other moving object, then Einstein is right.
   d) The triangle A has a side and an angle equal respectively to a side and an angle of the triangle B.
   The triangle A has two sides which are equal respectively to two sides of the triangle B.

   e) $a \supset bc$
   $b'c' \supset a'$

   f) $ab + c = c$
   $abc = ab$ (in Boolean algebra)

   g) $a \supset c$ and $c \supset b$ and $b \supset a'$.
   $a = 0$ and $bc = c$.

   h) $a = b$
   $ab' + a'b = 0$

i) $a + b = 0$
$a = 0$ and $b = 0$

j) $a - b = b - a$
$a = b$

k) $ab = a + b$
$a = b$

l) $a + b = a - b$
$b = 0$

m) $ab = a - b$
$a = 0$

Deductive System of the Boolean Algebra

13

WE PROCEED now to "formalize" the Boolean algebra, by way of constructing a set of definitions and assumptions which will be sufficient to allow us to deduce all the properties mentioned in the preceding chapter. The general characteristics of a deductive science were dealt with in chapter I, and chapter IV presented an example of the methods employed. The deductive system here differs from that of chapter IV in its emphasis on the axiom of Substitution (axioms 4 and 5 below).

Concerning this deductive system of the Boolean algebra, the following points are noteworthy:

1. It is not necessary to assume as indefinables all the relations and operations of the system. For example, in De Morgan's Law we can define the operation $a + b$ (addition) in terms of multiplication and negation, or multiplication in terms of addition and negation. It is, then, more or less arbitrary whether we take multiplication or addition as indefinable. We define addition in terms of multiplication and negation.

2. The null class (0) is defined by aa', that is, by means of multiplication and negation. To define this concept, however, we must assume that it is unique or "constant." This we do in Postulate 4 by assuming the proposition $aa' = bb'$. That is, no matter what class we take for a in the definition $0 = aa'$, we always have the same result: $0 = aa' = bb' = cc'$, etc.

3. We may define the relation of inclusion in terms of 0, negation and multiplication; or, since 0 may be defined by multiplication and negation, we may define the relation of inclusion in terms of the last two only; $a \supset b$ is to mean the same thing as the statement that a and non-b have no members in

common. That is, $a \supset b$ means $ab' = 0$. Hence, by an axiom which asserts that when two statements are definitionally equivalent, they imply each other, if $a \supset b$, then $ab' = 0$, and if $ab' = 0$, then $a \supset b$. However, on account of the considerations mentioned concerning the uniqueness of 0, the definition of inclusion will first read: "$a \supset b$ means $ab' = aa'$. We shall postulate that aa' is unique, and as a theorem deduce "$a \supset b$ means $ab' = 0$." In the case of the calculus of propositions, implication, $p \supset q$, is then equivalent to the expression "p is true and q false is a contradiction." That is, if whenever p is true, q is also true, then it is a contradiction to say that p is true and q is false, and, vice versa, if it is impossible that p be true and q be false at the same time, then p implies q.

4. The symbol "$=$" will be taken as indefinable in a certain sense only, for we shall assume an axiom which will describe its essential property, namely, that two equal terms, a and b, may be substituted for each other in any expression.

Deductive System of the Boolean Algebra [1]

Indefinables

1. A class K of elements $a, b, c \ldots$
2. ab ("multiplication")
3. a' ("negation")
4. $a = b$ ("equality")

Definitions

1. $(a + b) = (a'b')'$ ("addition")
2. $a \supset b$ means $ab' = aa'$ ("inclusion")

Postulates [2]

1. $ab = ba$.
2. $a(bc) = (ab)c$.
3. If $a \supset b$, then $ab = a$.
4. $aa' = bb'$.

---

[1] Cf. chap. V, p. 102.

[2] Most sets of postulates for Boolean algebra assume also the following or their equivalent:

0.1 K has at least 2 elements.
0.2 If a and b belong to K, then ab belongs to K.
0.3 If a is in K, a' is in K.

The following definitions may now be added:

3. $0 = kk'$, where k is some fixed element of K.
4. $1 = 0'$

AXIOMS [3]

1. If p and q are definitionally equivalent (i.e., if p and q "mean the same thing"), then p implies q and q implies p.
2. If p is implied by a postulate or theorem, then p is true (i.e., p is a theorem).
3. The terms of the postulates, definitions, and theorems (a, b, c, etc.) may be replaced throughout by any other terms or expressions involving multiplication (ab), addition ($a + b$), or negation (a') (i.e., any other element of K), without altering the validity of the given expression. (In the case of the calculus of propositions, this may be extended to the sign "=" and to $\supset$. Note that the substitution must take place *throughout* the given expression; i.e., if we replace the term a in a certain place by another term, we must replace it wherever it appears by the same term.)
4. If $a = b$, then a may replace b in any expression without altering its truth or falsity and b may similarly replace a; "=" is also taken to be reflexive; i.e., $a = a$. We can thus deduce the following properties of equivalence, which will be important in the sequel (some of these statements depend on theorems to be proved):

   a) $a = a$ ("=" is reflexive; cf. page 22).
   b) If $a = b$, then $b = a$ ("=" is symmetrical).
   c) If $a = b$, and $b = c$, then $a = c$ ("=" is transitive).
   d) If $a = b$, then $ac = bc$ and $a + c = b + c$.
   e) If $a = b$, then $a' = b'$.
   f) If $a = b$, then $a \supset b$ and $b \supset a$.

5. If p implies q and q implies r, then p implies r (The Principle of Strengthening and Weakening).
6. The statement "p is true and a given theorem or postulate is true" is equivalent to the statement "p is true." (Cf. Axiom 12, page 103; axioms 5 and 6 will be used tacitly.)

[3] These axioms, with the exception of 4, are all drawn from the logic of propositions. Axiom 4, according to the principles for a formally correct deductive system set down in chap. VII, should really be a postulate, since it includes an indefinable of the system. However, this method of assuming it as an axiom (often implicitly), though formally incorrect, is almost invariably followed in postulate sets for Boolean algebra.

The following theorems lay no claim to a thorough survey of all the Boolean algebra. The principal purpose here is to indicate the method of proof and to demonstrate some of the important properties named above. We first prove that $a \supset b$ may be defined as $ab' = 0$:

THEOREM 0.1 $0 = aa' = bb' = cc'$, etc.

This theorem follows from postulate 4, where k is written for a, and a for b, and definition 3. This theorem will be used implicitly in the proofs of the remaining theorems.

THEOREM 1. $a \supset b$ means $ab' = 0$.

For $a \supset b$ means $ab' = aa'$ (Definition 2)
But $aa' = 0$
Hence $a \supset b$ means $ab' = 0$ (Axiom 4)

THEOREM 2. $a \supset a$

$a \supset b$ means $ab' = 0$ (Theorem 1)
Now replace b by a:
$a \supset a$ means $aa' = 0$.
But $aa' = 0$; hence
$a \supset a$ is true (Axioms 1 and 2)

THEOREM 3. $aa = a$

In Postulate 3 replace b by a:
If $a \supset a$, then $aa = a$.
But $a \supset a$ is true (Theorem 2). Hence (Axiom 2)
$aa = a$ is true.

THEOREM 4. $a0 = 0$

Now $0 = aa'$
Hence $a0 = a(aa')$ (Axiom 4).
But $a(aa') = (aa)a'$ (Postulate 2).
But $aa = a$ (Theorem 3).
Hence $(aa)a' = aa'$ (Axiom 4).
But $aa' = 0$
Hence $a0 = 0$ (Axiom 4). Q.E.D.

THEOREM 5. If $ab = a$, then $a \supset b$. (This is the converse of Postulate 3).

If $ab = a$, then $ab' = (ab)b'$ (Axiom 4).
But $(ab)b' = a(bb')$ (Postulate 2)
But $bb' = 0$
Hence $a(bb') = a0$ (Axiom 4).
But $a0 = 0$ (Theorem 4).
Hence $ab' = 0$ (Axiom 4).
Hence $a \supset b$ (Theorem 1, and Axioms 1 and 5). Q.E.D.

THEOREM 6. $0 \supset a$

Now $a0 = 0$ (Theorem 4).
But $a0 = 0a$ (Postulate 1, where 0 is written for b).
Hence $0a = 0$ (Axiom 4).
Now in Theorem 5 put 0 for a and a for b:
If $0a = 0$, then $0 \supset a$.
But $0a = 0$; hence (Axiom 2)
$0 \supset a$ is true. Q.E.D.

THEOREM 7. If $(a \supset b)$ and $(b \supset a)$, then $a = b$. (This is the converse of f under Axiom 4.)

If $a \supset b$, then $ab = a$ (Postulate 3),
and if $b \supset a$, then $ba = b$ (Postulate 3).
But $ab = ba$ (Postulate 1).
Hence $a = ab = ba = b$, and $a = b$ (Axiom 4). Q.E.D.

THEOREM 8. If $a \supset b$, then $ac \supset bc$ (or $ca \supset cb$).

We show that $ac \supset bc$ by showing that $(ac)(bc) = (ac)$. For if the latter is the case, then by Theorem 5 (where a is replaced by ac and b by bc), the former must also be the case. This method of proof is very common in the proofs of the theorems of the Boolean algebra and consists in showing that the multiplication of what precedes the inclusion sign by what comes after reduces simply to the former.

$$\text{Now } (ac)(bc) = (ab)(cc).$$

This is a consequence of Postulates 1 and 2. The actual proof is left as an exercise.

But $ab = a$ (Postulate 3, since $a \supset b$ by hypothesis),
and $cc = c$ (Theorem 3, where a is replaced by c).
Hence $(ab)(cc) = ac$ (Axiom 4).
Hence $(ac)(bc) = (ac)$, and $ac \supset bc$. Q.E.D.

THEOREM 9. If $a \supset 0$, then $a = 0$.

For $0 \supset a$ (Theorem 6)
Hence, if $a \supset 0$, we have $a = 0$ by Theorem 7 and Axiom 6.

THEOREM 10. $a'' = a$

$aa' = a'a = 0$ (Postulate 1 and Definition 3)
Now replace a by a'.
$(a'')\,a' = 0$.
In Theorem 1 replace a by a'' and b by a
$a'' \supset a$ means $[(a'')\,a'] = 0$.
Hence (Axioms 1 and 2)
$a'' \supset a$.

Now replace a by a':
$a''' \supset a'$
Hence, by Theorem 8, where a is written for c:
$aa''' \supset aa'$,
or $a(a'')' \supset 0$, or (Theorem 9)
$a(a'')' = 0$, and hence (Theorem 1)
$a \supset a''$.
Thus $a'' \supset a$ and $a \supset a''$, and hence (Theorem 7)
$a = a''$ Q.E.D.

(Note that in the proof of this theorem we seem to assume the fact that $(a')' = a''$. But actually there is no assumption here at all. a'' is merely shorthand for $(a')'$, a convenient method for writing the longer expression.)

Theorem 11. $a \supset b$ means $b' \supset a'$.
$a \supset b$ means $ab' = 0$ (Theorem 1)
But $ab' = b'a$ (Postulate 1)
Hence, $a \supset b$ means $b'a = 0$.
But $b'a = b'a'' = b'(a')'$ (Theorem 10)
Hence, $a \supset b$ means $b'(a')' = 0$.
But $(b')(a')' = 0$ means $b' \supset a'$ (Theorem 1)
Hence, $a \supset b$ means $b' \supset a'$.
Hence $a \supset b$ implies $b' \supset a'$, and conversely. (Axiom 1).

Theorem 12. If $a \supset b$, and $b \supset c$, then $a \supset c$.
We show that, under the hypothesis, $ac = a$ and hence that $a \supset c$. (By Theorem 5).
Since $a \supset b$, we have $ab = a$ (Postulate 3). Hence
$ac = (ab)c = a(bc)$ (Postulate 2)
But since $b \supset c$, $bc = b$. (Postulate 3)
Hence $a(bc) = ab = a$.
Thus $ac = a$, and $a \supset c$. Q.E.D.

The proofs of the following theorems are greatly abridged, all references being omitted:

Theorem 13. $ab \supset a$.
We show that $(ab)a' = 0$, and hence, by Definition 2 and Axioms 1 and 2, that $ab \supset a$. The formal proof is left to be done by the student.

Theorem 14. $a + b = b + a$.
$a'b' = b'a'$.
Hence $(a'b')' = (b'a')'$.
Hence $a + b = b + a$.

THEOREM 15. $a + a = a$.
In Definition 1 put a for b:
$(a'a')' = a + a$.
But $a'a' = a'$; hence
$(a')' = a + a$, or
$a + a = a$. Q.E.D.

THEOREM 16. $a + (b + c) = (a + b) + c$

THEOREM 17. $1 = a + a'$
For $0 = a'a$; hence
$0' = (a'a)'$; but
$0' = 1$, and $(a'a)' = a'' + a' = a + a'$.
Hence $1 = a + a'$ Q.E.D.

THEOREM 18. $a \supset (a + b)$.
Hint: use Theorems 13 and 11.

THEOREM 19. $ab \supset a + b$.

THEOREM 20. If $a \supset b$, then $a + b = b$, and conversely.
If $a \supset b$, then $b' \supset a'$ or $b'a' = b'$.
Hence $(b'a')' = (a'b')' = b'' = b$.
But $(a'b')' = a + b$.
Hence, if $a \supset b$, then $a + b = b$.

The converse follows similarly.

(Proofs of the following theorems are to be worked out by the student.)

21. $ab + a = a$.
22. If $a \supset b$ and $a \supset b'$, then $a = 0$.
23. If $a \supset a'$, then $a = 0$.
24. If $b \supset a$ and $b' \supset a$, then $a = 1$.
25. If $a' \supset a$, then $a = 1$.
26. $(a'b'c')' = a + b + c$.
    Hint: take $a'b'c'$ as $a'(b'c')$.
27. $(a' + b')' = ab$.
28. $ab \supset c$ means $ac' \supset b'$, or $ab \supset c$ means $c'b \supset a'$.
29. If $a \supset b$, then $a' + b = 1$, and conversely,
30. If $a' + b = 1$, then $a \supset b$.
31. If $a \supset b'$, then $b \supset a'$.
32. If $a' \supset b$, then $b' \supset a$.
33. $a1 = a$.
34. $a \supset 1$.
35. $a + 1 = 1$.
36. $a + 0 = a$.

37. If $a + b = 0$, then $a = 0$ and $b = 0$.
38. If $ab = 1$, then $a = 1$ and $b = 1$.
39. If $a \supset b$, and $c \supset d$, then
    a) $ac \supset bd$,
    b) $ac \supset b + d$
    c) $(a + c) \supset (b + d)$
40. *a*) If $a \supset b$, and $c \supset b$, then $ac \supset b$ and $(a + c) \supset b$.
    b) If $a \supset b$, and $a \supset c$, then $a \supset bc$ and $a \supset b + c$.
41. $a(b + c) = ab + ac$ (Distributive Law)

This must be shown in two parts:

a) $(ab + ac) \supset a(b + c)$

To prove this, show that $ab \supset a(b + c)$ and $ac \supset a(b + c)$ (Theorem 39) and apply Theorem 40.

b) $a(b + c) \supset (ab + ac)$
Now $a(ab)'b = (ab)(ab)' = 0$.
Hence $a(ab)' \supset b'$
Similarly $a(ac)' \supset c'$
Hence (39) $a(ab)'(ac)' \supset (b'c')$
Hence $a(b'c')' \supset [(ab)'(ac)']'$ (Theorem 28),
which, by definition, gives:
$a(b + c) \supset (ab + ac)$.[4]

42. $(b + c)a = ba + ca = ab + ac$
43. $a + bc = (a + b)(a + c)$
44. $a(b + c + d) = ab + ac + ad$
45. $(a + b)(c + d) = ac + ad + bc + bd$
46. $(ab + cd)' = a'c' + a'd' + b'c' + b'd'$
47. $ab' + b = a + b$
48. $a = ax + ax'$ (i.e., any element may be "expanded" by any other element)
49. $1 = a + a' = ab + ab' + a'b + a'b'$ (These last four terms are called the "quadrants" of Boolean algebra.)
    In general, 1 may be expanded by any set of elements a_1, a_2, a_3, . . ., a_n, by adding the products of all the terms with the primes permuted in all possible ways:

$$1 = a_1a_2 \ldots a_n + a_1a_2 \ldots a'_n + a_1a_2 \ldots a'_{n-1}a_n + \ldots + a_1a_2 \ldots a'_{n-1}a'_n + \ldots + a'_1a'_2 \ldots a'_n.$$

This can be shown by mathematical induction (cf. page 31).

[4] This simple proof of the theorem, which has often been thought extremely complex for postulate sets like the above, is due to Mr. S. I. Askovitz. Substantially the same proof appears in Peano's *Formulaire de mathematique.*

50. $0 = aa' = (a + b)\ (a' + b)\ (a + b')\ (a' + b')$
$= (a_1 + \ . \ . \ . \ + a_n)\ (a'_1 + a_2 \ . \ . \ . \ + a_n) \ . \ . \ .$
$(a'_1 + a'_2 + \ . \ . \ . \ + a'_n)$
51. If $ac = bc$ and $a + c = b + c$, then $a = b$.
52. (Cf. Exercise 7, page 265)
a) $(a \circ b) = (a' \circ b')$; $(a \bigtriangleup b) = (a' \bigtriangleup b')$; $(a \bigtriangleup b) = (a \circ b')$;
$(a \circ b) = (a \bigtriangleup b')$
b) $a \circ (b \bigtriangleup c) = b \circ (a \bigtriangleup c) = c \circ (b \bigtriangleup a)$
c) $a \bigtriangleup (b \circ c) = b \bigtriangleup (a \circ c) = c \bigtriangleup (a \circ b)$
d) $a \circ (b \bigtriangleup c) = a \bigtriangleup (b \circ c)$
e) If $a \bigtriangleup c = b \bigtriangleup c$, then $a = b$; if $a \circ c = b \circ c$, then $a = b$

As we have pointed out already, the Boolean algebra may be considered as a calculus of propositions. We will now prove certain well-known laws of this calculus. For example:

1. If $(p = 1)$ (i.e., if p is true) and $(p \supset q)$, then $q = 1$ (q is true).

We can show this if we can show that $1 \supset q$ is the same thing as $q = 1$. But if $1 \supset q$, then, since $q \supset 1$ always (34), we have $1 = q$ (Theorem 7). Hence, if $p = 1$, then $p \supset q$ is $1 \supset q$, and hence $q = 1$.

2. If $p \supset q$, then $pr \supset qr$.

"If when p is true, q is true, then when p and any other proposition r are true, q and r are true." (Cf. Theorem 8).

3. We can now prove the dilemmas given on page 35. To complete their proof from the above postulates, we must add the additional assumption:

If $p \supset q$, then $p' + q$ ("either p is false or q is true"), and, conversely, if $p' + q$, then $p \supset q$.

Constructive Dilemma:

If $(p \supset q)\ (r \supset s)\ (p + r)$, then $(q + s)$.
Now if $(p \supset q)$, then $(q' \supset p')$ (Theorem 11)

But, according to 2 above, we may multiply both sides of this implication by any expression whatsoever and the resulting form will still be true; let this expression be $p' \supset r$:

If $(p \supset q)\ (p' \supset r)$, then $(q' \supset p')\ (p' \supset r)$.

But the conclusion of this proposition may be weakened by virtue of Theorem 12 to $(q' \supset r)$:

If $(p \supset q)\ (p' \supset r)$, then $(q' \supset r)$.

Again, multiply both sides of this expression by $r \supset s$:

If $(p \supset q)\ (r \supset s)\ (p' \supset r)$, then $(q' \supset r)\ (r \supset s)$.

Again, the conclusion may be weakened to $q' \supset s$:

If $(p \supset q)\ (r \supset s)\ (p' \supset r)$, then $(q' \supset s)$.

But, by the additional postulate, $(p' \supset r)$ may be strengthened to $p'' + r$ or $p + r$, and $(q' \supset s)$ may be weakened to $q'' + s$ or $q + s$. Hence, substituting these values in the last expression above, we derive the constructive dilemma.

Destructive Dilemma:

If $(p \supset q)\ (r \supset s)\ (q' + s')$, then $(p' + r')$.

In this case we start with Theorem 12:

If $(p \supset q)\ (q \supset s')$, then $(p \supset s')$.

Now multiply both sides by $(s' \supset r')$:

If $(p \supset q)\ (q \supset s')\ (s' \supset r')$, then $(p \supset s')\ (s' \supset r')$.

But the premise $(s' \supset r')$ may be strengthened to $(r \supset s)$, since if $(r \supset s)$ is true, $(s' \supset r')$ is true. Also the conclusion may be weakened to $(p \supset r')$:

If $(p \supset q)\ (r \supset s)\ (q \supset s')$, then $(p \supset r')$.

But $(q \supset s')$ may be strengthened to $(q' + s')$, and $(p \supset r')$ may be weakened to $(p' + r')$:

If $(p \supset q)\ (r \supset s)\ (q' + s')$, then $(p' + r')$.

The remaining "mixed" forms are to be done by the student:

If $(p \supset q)\ (r \supset s)\ (q' + r)$, then $(p' + s)$, and
If $(p \supset q)\ (r \supset s)\ (p + s')$, then $(q + r')$.

The following theorems are also left to be proved by the student:

a) If p, q, r, and s are four propositions such that $p(q + r) = 0$ and $s \supset p$, then
$s \supset q'$ and $s \supset r'$.

b) If p, q, r, and s are four mutually exclusive and mutually inclusive propositions (cf. page 128), and t, u, v, and w are also four mutually exclusive and inclusive propositions, and $p \supset t$, $q \supset u$, $r \supset v$, $s \supset w$, then $t \supset p$, $u \supset q$, $v \supset r$, and $w \supset s$.

c) $(p \supset q)\ (r \supset s)\ (t \supset u)\ (p + r + t) \supset (q + s + u)$,

d) $(p \supset q)\ (r \supset s)\ (t \supset u)\ (v \supset w)\ (p + r + t + v) \supset (q + s + u + w)$.

Elementary Mathematics of the Boolean Algebra

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THE BOOLEAN algebra may be extended in much the same manner as arithmetical algebra to considerations of functions, equations, and solutions of equations.

Expansions of Boolean Elements

Any element of the Boolean algebra may be "expanded" to as many terms as we please in the following manner:

$a = a(b + b') = ab + ab' = ab(c + c') + ab'(d + d') =$
$abc + abc' + ab'd + ab'd'$, etc., etc. (Theorem 48, page 274)

Such expansions are often valuable in determining the values of certain expressions. Thus if the complex expression

$$a + a'b + b'$$

is expanded in its two extreme terms, we have

$$a(b + b') + a'b + b'(a + a') = ab + ab' + a'b + b'a + b'a',$$

the last expression being 1 (cf. page 274).

If $f(x)$ signifies some expression containing x and the usual Boolean operations of $+$, $\times$, and $'$, then $f(x)$ may always be expressed in the "normal form":

$$\mathrm{A}x + \mathrm{B}x',$$

where A and B are expressions not containing x. For example $ab' + ax + c$ may be written

$$ab'x + ab'x' + ax + cx + cx',$$

by expanding the terms not containing x by means of x. Collecting and factoring, we have

$$(ab' + a + c)\,x + (ab' + c)\,x'.$$

Here A is $(ab' + a + c)$ and B is $(ab' + c)$.

The normal form of a function containing two variables is

$$f(x, y) = (Ay + By')\,x + (Cy + Dy')\,x', \text{ or}$$
$$f(x, y) = Axy + Bxy' + Cx'y + Dx'y'.$$

An expression containing any number of variables, x, y, z, etc., may be written in a normal form in a similar manner.

(See Exercises, Group A, at end of chapter.)

Theory of Boolean Equations

The normal form of any Boolean equation containing one unknown is

$$Ax + Bx' = 0.$$

We require some method for transferring terms from one side to another in order to reduce equations to this normal form. In ordinary algebra, if $a = b$, then we transpose by subtracting b from both sides and deriving the *equivalent* expression $a - b = 0$. This cannot be done in Boolean algebra, since $ab' = 0$ is not equivalent to, but is weaker than, $a = b$. Rather, our method here is to multiply each side of the equation by the negation of the other, add results and equate to 0. For example, $a = x$ is equivalent to $ax' + a'x = 0$. For when $a + b = 0$, $a = 0$ *and* $b = 0$ in this algebra. Hence $ax' = 0$ and $a'x = 0$, or $a \supset x$ and $x \supset a$, which gives $a = x$.

(See Exercises, Group B, at end of chapter.)

Solution of Boolean Equations

By the solution of the equation $Ax + Bx' = 0$, we mean determining the "value" of x in terms of A and B. More specifically, we mean determining the range of x. Now the solution of this equation may be given as

$$B \supset x \supset A'.$$

For if $Ax + Bx' = 0$, then $Bx' = 0$ and $Ax = 0$, which give $B \supset x$ and $x \supset A'$ respectively. This solution is unique, on the one hand, when B and A' are the same, and, on the other hand, it is completely indeterminate, when B is 0 and A' is 1, for in the latter case any element whatsoever would satisfy the conditions. Again, if $AB \neq 0$, there is no solution (cf. Exercise 2, Group C). If the equation gives an ambiguous answer, we may have another (simultaneous) equation $Cx + Dx' = 0$. The solution of two simultaneous equations

$$Ax + Bx' = 0$$
$$Cx + Dx' = 0$$

is given by the expression $B + D \supset x \supset A'C'$. (Why?) This result may be extended for any number of equations.

A peculiarity of the Boolean algebra is that if we have two unknowns we do not necessarily need two equations (as we do in ordinary algebra) for a complete solution. In general, in this algebra there is no necessary relation between the number of unknowns and the number of equations required for a unique solution. If the given equation has two unknowns, we may always eliminate one of them. For if $Ax + Bx' = 0$, then $AB = 0$, for if $Ax + Bx' = 0$, then $B \supset x \supset A'$, or $B \supset A'$, and hence $AB = 0$. This provides a method for eliminating x. Suppose the given equation is $ay + bx' + c = 0$. Writing this in the normal form with respect to x, we have

$$(ay + c)\,x + (ay + b + c)\,x' = 0.$$

Since the product of the coefficients of x is equal to 0, we have the following equation, with x eliminated:

$$(ay + c)\;(ay + b + c) = 0,$$
or
$$(a + c)\,y + cy' = 0.$$

Hence $c \supset y \supset (a + c)'$, or $c \supset y \supset c'a'$. Since this gives $c \supset c'$ or $c = 0$, y is indeterminate as regards its lower limit. Its upper limit is a'. (Note that we could have eliminated c in the beginning, by virtue of the original equation. Why?)

The normal form of an equation in two unknowns is

$$f(x, y) = Axy + Bxy' + Cx'y + Dx'y' = 0.$$

Any equation with two unknowns may be reduced to this normal form, where A, B, C, and D do not contain x and y. A, B, C, and D are called the "discriminants" of the equation, just as A and B are the discriminants of the equation

$$Ax + Bx' = 0.$$

For equations with three unknowns there will also be a normal form with eight discriminants:

$$f(x, y, z) = Axyz + Bxyz' + Cxy'z + Dx'yz + Exy'z' + Fx'yz' + Gx'y'z + Hx'y'z'.$$

In general, the normal form for equations of n variables will contain a sum of products, the products representing all possible combinations of primed and unprimed elements. The general solution of an equation $f(x, y)$ in two unknowns will be

$$CD \supset x \supset A' + B', \text{ and}$$
$$BD \supset y \supset A' + C',$$

as verified by the elimination method suggested here.

(See Exercises, Group C, at end of chapter.)

The Boolean algebra, like ordinary algebra, may be used in the solution of "word" problems:

EXAMPLE: x is a certain class of animals. There are no members of x which are not white and there are no members of x which are neither cats nor dogs; on the other hand, every animal is either not white, or an x, or neither a cat nor a dog; determine the membership of x.

SOLUTION: If x is the class, then the problem may be put in the following symbolic form, where $a =$ "white things," $b =$ "cats," $d =$ "dogs," the equations becoming:

$$(a' + b'd')\,x = 0$$
$$a' + b'd' + x = 1$$

(where "animals" represents the universe of discourse)

The solution of these two equations is $x = ab + ad$, i.e., x is the class of white cats plus the class of white dogs.

(See Exercises, Group D, at end of chapter.)

EXERCISES

GROUP A

1. Express in the normal form the following:

   a) $ax' + b + cdx$ (As in ordinary algebra, take x, y, z, etc., as "unknowns" and a, b, c, etc. as "knowns.")

b) $a'b + cx + d'b + e$
c) $ax + a'x' + a'x$
d) $a + b$ (How may we introduce x?)
e) $ax' + b + cy'$ (2 variables)
f) $a'x + b'y + c + ay + x'$
g) $ax + by + cz$ (What is the normal form for three variables?)
h) $abx' + c'dy + ez' + fx + a'$

2. If $f(x) = Ax + Bx'$, determine the values of $f(1)$, of $f(0)$, of $f(A)$, of $f(B)$, of $f(AB)$, of $f(A + B)$, where, for example, $f(1)$ means the result of substituting 1 for x throughout the expression. Hence show that

$$f(x) = f(1)\,x + f(0)\,x'$$ [1]

3. Given the normal form for $f(x, y)$ as above, determine the values of $f(0, 0)$, $f(0, 1)$, $f(1, 0)$, $f(1, 1)$. Rewrite the normal form using these results.
4. Prove that $AB \supset Ax + Bx' \supset A + B$.
5. Prove that $[f(x)]' = A'x + B'x'$.
6. *a*) Prove that if $f(x) = Ax + Bx'$ and $\phi(x) = Cx + Dx'$, then $f(x) + \phi(x) = (A + B)\,x + (C + D)\,x'$. Prove a corresponding theorem for functions with two variables.
   b) Prove that $f(x) \cdot \phi(x) = ACx + BDx'$, and prove a corresponding theorem for two variables.

GROUP B

1. Reduce the following to normal form equations:

a) $ax = bx'$
b) $a + cx = d$
c) $a'x = (c + d)\,x + x'$
d) $ay + bx = ax + by$
e) $axy = bx'y'$
f) $ax + by' = cx' + dy$
g) $ax + bxy = cx' + dy' + ex'y'$

GROUP C

1. Solve the following equations:

EXAMPLE 1) $abx + a'b' = 0$

In the normal form this is

$$(ab + a'b')\,x + a'b'x' = 0.$$

[1] This proposition can be said to "characterize" the Boolean algebra, since with this as the only formal assumption, the entire algebra follows.

Hence the solution is

$$a'b' \supset x \supset (ab + a'b')' \text{ or}$$
$$a'b' \supset x \supset a'b + b'a.$$

But, as generally happens, we may determine more concerning these "limits." For if $Ax + Bx' = 0$, then $AB = 0$. Hence $(ab + a'b') \cdot (a'b') = 0$, or $a'b' = 0$. Hence, the lower limit of x is undetermined. The upper limit becomes $a' + b'$, since $a' \supset b$ and $b' \supset a$.

EXAMPLE 2) $x + b'x' + b = 0$

As in ordinary algebra, an equation or a set of equations may be inconsistent, in which case, of course, there is no solution. Thus in ordinary algebra, the equation $x + 3 = x + 5$ and the simultaneous set $3x + 9y = 8$ and $x + 3y = 2$ are inconsistent. Example 2 is inconsistent in Boolean algebra, for its normal form is $x + x' = 0$, which is a contradiction.[2] In general, if the product of the coefficients or discriminants, AB, is not 0, then the equation is inconsistent. When the coefficients are undetermined, i.e., are a, b, c, etc., then this restriction places values upon them, as in Example 1. What conditions must be satisfied in order that the equation $Axy + Bxy' + Cx'y + Dx'y' = 0$ be consistent? Generalize this for equations with any number of unknowns.

a) $ax + b' = 0$
b) $a'x + ax' = 0$
c) $ax + x' = a'$
d) $ax + b = 1$
e) $a'x + ax' = 1$
f) $xy + x'y + x'y' = 0$
g) $xy + ax + ay + x'y' = 0$
h) $ax + by' = cx + dy'$
i) $a'x'y + x + ax'y' = 0$
j) $xy + ax' + by' + x'y' = 0$
k) $xy + xb' + ay + b'y + a'x + bx'y' = 0$
l) $xy + x'y' + a'x + aby + b'x = 0$
m) $[(a + b + x' + y')(a'b' + x + y)]' + [(ab'c + x' + y)(a'bc' + x + y')]' + [(a + c + x' + y')(b + c' + x' + y')]' = 0$
n) $a'x'y' + x'y + ad + bx'y' + a'y + b'xy' + by = 0$

[2] I.e., a contradiction in a Boolean algebra having more than one element.

o) $xz + x'z + ayz' + a'xz' + x'y'z' = 0$
p) $ax + bx' = 0$
$ax + cx' = 0$
q) $a'xy + axy' + b'xy' + bx'y' = 0$
$a'b'xy + a'xy' + bxy' + ax'y' = 0$
r) $xy + a'x + x'y + ax'y' = 0$
$a'xy + xy' + a'x' + x'y' = 0$
s) $ax + b'y' + bx + xy' = 0$
$abx'y + a'b'xy + b'x'y' + a' = 0$

2. Prove that if $A = 0$ in an equation of two unknowns then the upper limit of x and y is 1; if $B = 0$, the upper limit of x is 1, the lower limit of y is 0; if $C = 0$, the lower limit of x is 0, the upper limit of y is 1; if $D = 0$, the lower limit of x and y is 0. What follows when $A = 1$, or $B = 1$, or $C = 1$, or $D = 1$?

3. Solve the general equation: $Ax + Bx' = Cx + Dx'$ for x in terms of A, B, C, and D, and the general equation: $Axy + Bxy' + Cx'y + Dx'y' = Exy + Fxy' + Gx'y + Hx'y'$ for x and y in terms of A, B, C, etc. With the results obtained solve the following equations:

a) $ax + bx' = a'x + dx'$
b) $a'b'x + ax' = a'b'x'$
c) $ax + bx' = ax + b'x'$
d) $a'y + x'y + ay' + x'y' + by' = a'xy' + ab'xy$
e) $xy + xy' + x'y + ax'y' + bcx'y' = (a'b' + a'c' + x + y)'$

4. Show that a necessary and sufficient condition that the general equation $Axyz \ldots st + Bxyz \ldots st' + Cxyz \ldots s't + \ldots + Lx'y'z' \ldots s't' = 0$ have a unique solution: $x = a,\ y = b,\ z = c, \ldots, s = k,\ t = m$, is that 1) The product of the discriminants "vanish," i.e., $ABC \ldots L = 0$ (the *Condition of Consistency*), and 2) $A'B' = 0,\ A'C' = 0, \ldots, K'L' = 0$, i.e., the products of the negatives of the discriminants taken two at a time vanish (the *Condition of Uniqueness*).

(*Hint:* Show: *a*) That an equation having the unique solution mentioned will be $\phi(x, y, z, \ldots, t) = a'x + ax' + b'y + by' + c'z + cz' + \ldots + m't + mt' = 0$.
b) Two equations are equivalent if and only if their discriminants are the same.
c) The discriminants of any equation of n unknowns are given by the equation
$f(1, 1, 1, \ldots, 1) = A$
$f(1, 1, \ldots, 1, 0) = B$
$f(0, 0, 0, \ldots, 0) = L$

Hence *d*) In order that the original equation and $\phi(x, y, z, \ldots t) = 0$ be equivalent we must have:

$$A = a' + b' + c' + \ldots + m'$$
$$B = a' + b' + c' + \ldots + m$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$L = a + b + c + \ldots + m$$

which gives the required theorem.)

GROUP D

1. Solve the following problems:

a) The "Liberserfs" is a club. The members of this club that belong to either the Elks or the Kiwanis plus those that belong to neither are none. Determine the club's membership.

b) From a certain class of objects A picks out the small round things and the large white things and B picks out from the remainder the small white things and round things that are not white. All that is left comprises the class of small things that are not round. What can be determined about the class originally?

c) x and y are two classes which have the following properties: Their product vanishes, their sum is the universe; the class composed of things that are neither white nor y is empty as are the classes of things that are x but not men and things that are y and are white men. Determine the values of x and y.

d) "Three persons A, B, C are assigned to sort a heap of books in a library. A is told to collect all the English historical works, and the bound foreign novels: B is to take bound historical works and the English novels, provided they are not historical: to C is assigned the bound English works and the unbound historical novels. What works will be claimed by two of them? Will any be claimed by all three?"[3]

e) "It is found that when all the books in a library except philosophy and divinity are rejected they are reduced to philosophy and protestant divinity, but include all the works on those subjects. What is the widest and narrowest extent, so far as expressible in these class terms, which the library could have possessed under the given conditions?" (Venn)

f) "At a certain town where an examination is held, it is known that,

(1) Every candidate is either a junior who does not take Latin, or a senior who takes composition.

[3] Venn, *Symbolic Logic.*

(2) Every junior candidate takes either Latin or composition.
(3) All candidates who take composition also take Latin, and are juniors.
Determine the membership of the class of candidates." (Venn)

g) Determine all relations in the Boolean algebra which are transitive (cf. page 27) by giving $a\mathrm{R}b$ the general form $\mathrm{A}ab + \mathrm{B}ab' + \mathrm{C}a'b + \mathrm{D}a'b' = 0$, and determining the values of the discriminants A, B, C, and D which will make true the implication "$(a\mathrm{R}b)$ $(b\mathrm{R}c)$ implies $(a\mathrm{R}c)$." Determine all relations which are reflexive; all which are symmetrical.[4] Determine all relations which are "contra-symmetrical": if $a\mathrm{R}b$, then $b'\mathrm{R}a'$; all which are "contra-reflexive": $a\mathrm{R}a'$.

[4] B. A. Bernstein, *A Generalization of the Syllogism,* Bulletin Am. Math. Soc., vol. 30, pp. 125-127.

Abstract Nature of the Boolean Algebra

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WE HAVE shown that the symbols of the Boolean algebra are not confined to one interpretation, for we may read $a + b$ as "What belongs to either a or b," or "Either a or b is true"; that is, the Boolean algebra may be considered as an algebra of classes or as an algebra of propositions. But it must seem evident enough that these two interpretations do not exhaust the possibilities. That is, there is nothing to prevent our considering the symbols of the algebra in an entirely different manner. The following are some of the interpretations which have been suggested:

I. Let us consider two "things," which we shall call A and B. We define the "multiplication" of A and B (AB or BA) as equal to A, while AA = A, and BB = B. We say that the "negation" of A, A′, is B, and B′ = A. The following tables represent these conventions:

$\times$	A	B
A	A	A
B	A	B

1.

	′
A	B
B	A

2.

(Here, to determine the result of the operation in the upper left-hand column in table 1, start with the element in the vertical column on the left, pass to the element in the horizontal column and complete the rectangle. Thus to determine BA, start with the B in the bottom of the vertical column on the left, draw a line through the $\times$-relationship at the top to the A on the horizontal line. If we complete this partial rectangle, the remaining vertex will give the required value, A.)

Then it will appear that, when $+$, $\times$, 0, and 1 have been defined for this system according to the definitions given above, the resulting system is perfectly consistent with the Boolean algebra. Thus A + B will be equal to $(A'B')'$. But A' is B and B' is A. Hence $(A'B')'$ is $(BA)'$. But BA is A. Hence, $A + B = (A)' = B$. In a similar manner, we can find the value of B + A, A + A, and B + B. The elements of the Boolean algebra, *a, b, c*, etc., are thought of as having one of the two values A or B. This being the case, the four postulates will be verified for "all" values of *a, b,* and *c*, i.e., no matter which of the two values A and B these terms may have. It will be evident that A is identical with the 0-element and B with the 1-element. Hence, $a \supset b$ will be true except when *a* is B and *b* is A ($1 \supset 0$ is false).

When this restriction is placed on the terms of the Boolean algebra, the latter is usually called the "two-valued" algebra, and has had a special application to the calculus of propositions. Let 0 represent a logically false proposition, and 1 a logically true proposition. Then this interpretation expresses the relations between these two types of expression. There are certain laws which are peculiar to this two-valued algebra. Thus we can here assert that $p = 0$ is the same thing as p' ("*p* is false"), and $p = 1$ is the same thing as p ("*p* is true"). Hence, we could define implication, $p \supset q$, as $p' + q$ ("Either *p* is false or *q* is true"). For, by Definition 4, $p \supset q$ means $pq' = 0$. But here $pq' = 0$ is $(pq')' = p' + q$ by De Morgan's Law. The two-valued algebra represents a very simple system. For example, the truth or falsity of any expression may be tested simply by changing all implication signs into disjunctions $(p + q)$ and multiplying out. Thus $p \supset (q \supset pq)$ would become $p \supset (q' + pq)$ or $p' + q' + pq$. Since this result is 1, we can infer that $p \supset (q \supset pq)$ is true.

With respect to the two-valued logic, the proofs of the following propositions will be helpful. These more or less characterize the system:

1. $p \supset (q \supset p)$

"If *p* is true, then *p* is implied by any proposition."

2. $p' \supset (p \supset q)$

"If *p* is false, then *p* implies any proposition."

3. $(p \supset q)' \supset (p = 0)'$

"If *p* does not imply a given proposition *q*, then p is not false (or, *p* is true)."

4. $(p = 0)' = p$
5. $(p \supset q)' \supset q'$

"If q is not implied by a given proposition p, then q is false (i.e., q is not true)."

6. $(p \supset q) + (p \supset q')$

"Any proposition implies another or the contradictory of another."

7. $(pq \supset r) \supset [(p \supset r) + (q \supset r)]$

"If the propositions p and q together imply r, then either p implies r or q implies r."

II. An even simpler (but far less significant) interpretation of the Boolean algebra is a system which has but one element, A, where AA = A, and A′ = A. Thus A = 1 and A = 0. The four postulates will then be true for "all" values (actually only one value) of a, b, and c. $a = b$ will be true always, and hence also $a \supset b$. (This system denies Postulate 0.1, footnote, page 268.)

III. This interpretation is drawn from arithmetic. Let the objects under consideration (i.e., the objects which a, b, c, etc., are to represent) be the number 30 and its factors, namely, 1, 2, 3, 5, 6, 10, and 15. Then let ab represent the highest common factor of the numbers a and b, and $a + b$ the lowest common multiple of a and b. Thus $2 \cdot 3$ is 1 and $2 + 3$ is 6, $10 \cdot 6$ is 2 and $10 + 6$ is 30. The following table represents "negation":

	′
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1

De Morgan's Law can then be seen to be true by testing. (Note that $a + b$ and ab always represent some one of the elements of the system; that is, if a and b are one of the numbers given, then ab and $a + b$ are.) The 0-element will be the *number* 1, and the 1-element 30. "$\supset$" will have the meaning "is a factor of," for $a \supset b$ is the same as $ab = a$, i.e., in this case, it states that the highest common factor of the two numbers a and b is a, and hence a must be a factor of b. Also, if a is a factor of b, then the highest common factor of a and b will be a. Thus $0 \supset a$ means that the 0-element is a factor of every number, and this is a property of the number 1. Also $a \supset 1$ will mean that the 1-element has as a factor every number of the system, and this is true here of the number 30. The four postulates will all be seen to be verified.

These interpretations, which do not by any means exhaust the possibilities, point to the abstract nature of the symbols of the Boolean algebra. These symbols may be considered as pure abstract relations apart from any of the concrete interpretations whatsoever. They express undefined relations between objects. Now these relations do have a certain "meaning" since we cannot interpret them as we please. Thus $a + b$ of the Boolean algebra cannot be the $a + b$ of ordinary algebra, since certain properties of the former are not true in the latter; e.g., $a + a = a$ is not true in ordinary algebra. They are thus pure abstract concepts.

The Boolean algebra is not the only abstract system. Indeed, any branch of theoretical mathematics may be so considered. Geometry, for instance, is generally taken as an abstract deductive system; the term "point" does not necessarily refer to some position in the physical universe; "dimension" does not refer necessarily to some world. In Cartesian three-dimensional geometry, a "point" is merely an array of three numbers, as, for example, $(1, 3, -2)$. A "line" is a mathematical equation, a "circle" is another equation. A "triangle" is defined by three mathematical equations or "lines" and the "vertices" of the triangle are again arrays of numbers, the solutions of two of the equations. Thus, in a sense, the terms of geometry, which originally referred to certain concepts and objects of our visual experience, may be considered in a purely abstract manner and the propositions of geometry may be open to a number of possible interpretations; in the case of the Boolean algebra these interpretations did not seem so strange, but here they do, perhaps, since we are so accustomed to associate such geometrical terms as "line," "circle," "sphere," etc., with visible objects.

When geometry is considered thus abstractly, it can readily be seen that it is nonsense to ask whether Euclid's Fifth Postulate is "true" or not (cf. page 12). In Euclid's abstract system, "parallel lines" have the property he assigns them; but the abstract concept of parallel lines may have different properties, as non-Euclidean geometry has shown. Both Euclidean and non-Euclidean geometries are "consistent"; they merely represent different abstract systems. Of course, once the terms of geometry are interpreted to refer to physical objects and positions, then the question arises whether a given geometry is "true" when so interpreted, and here we are out of the field of deductive sciences and logic and in the field of experimental science.[1]

The abstract nature of the algebra we have been considering is important since it has pointed the way to the solution of a problem that has puzzled the ages, namely, that of the consistency and independence of the postulates of a system. A set of postulates is said to be consistent if the postulates do not contradict each other, and independent if no postulate can be proved as a theorem from any other (cf. page 11). Now the question arises: how can we determine when a set of expressions is consistent and independent? That this problem has not been an easy one to solve, especially with respect to independence, the history of geometry has plainly shown.[2]

The answer to this problem can be phrased in the following manner. We say that an abstract (or "symbolic") set of postulates is "consistent" if an interpretation can be found for the symbols of the system (in the case of the Boolean algebra, $+$, $\times$, $\supset$, $'$; in the case of geometry, "point," "line," etc.) such that all the postulates thus interpreted become true. Thus, any of the three interpretations given above would show the postulates of the Boolean algebra consistent. It is assumed, of course, that the interpretation chosen is a consistent system; thus, in the case of the third interpretation above (the factors of 30) it is assumed that arithmetic is a consistent science. Hence this

[1] Cf. chap. VII.
[2] P. 12.

method could not be applied to all systems, for we must assume at least one consistent system to exist at the beginning, just as in a deductive system we assume certain postulates.

More significant still is the answer to the second question: when can we say that a set of expressions is independent? The answer to this depends on the concept of implication. For if a given postulate were actually a theorem, and could actually be proved from the rest of the consistent set of postulates, then it would always be true when the other postulates were true, no matter what the interpretation. Hence, if an interpretation of the postulates can be found such that one postulate is false while the remainder are true, then we say that this postulate is independent of the rest, since it does not have the necessary qualifications for a theorem. That is, we show that the set of assumptions comprising the assertions of all the postulates but one together with the denial of this one is consistent. Hence, the problem of independence is reduced to one of consistency. To illustrate this, we will show that Postulate 3 for the Boolean algebra is independent of Postulates 1, 2, and 4, i.e., that we cannot prove

"$a \supset b$ implies $ab = a$" from $ab = ba$, $a(bc) = (ab)c$, and $aa' = bb'$.

To do this, we must interpret the undefined ideas, $\times$, $'$, and a, b, c, etc., in such a manner that the latter three propositions are true but Postulate 3 is false. We now consider a, b, c, etc., as the integral numbers of ordinary arithmetic. The Boolean addition of two terms a and b will merely be the arithmetical addition of these numbers. Thus $2 + 5$ will be 7, $3 + (-5)$ will be -2, etc. The "negation" of a term, a', will be the same number with the sign changed. Thus $3'$ is -3, $(-5)'$ is 5, etc. Then the following properties will follow from this interpretation: the Boolean multiplication of two elements will again be the arithmetical sum. For $ab = (a' + b')'$. But since a' is $-a$ and $a + b$ is the same as the arithmetical $a + b$, the right side becomes

$$-(-a + -b),$$

which is $a + b$ by a well-known law of algebra. Hence the Boolean 0-element and 1-element are the same, both being the arithmetical 0. (0 in this case is included among the integers.) $a \supset b$ means that $a - b = 0$, i.e., that $a = b$. Now it will be plain that Postulate 1 and Postulate 2 are true; for $a + b = b + a$ and $a + (b + c) = (a + b) + c$ are both true in ordinary arithmetic. Also, Postulate 4, $aa' = bb'$, will become $a - a = b - b$ and will be true in general. But from the fact that $a \supset b$ (here $a = b$) we cannot infer that the Boolean multiplication of a and b (here the arithmetical sum) will be equal to a. In fact, in most cases this will not be so (it will only be true when a and b are 0). Thus if $a = 3$ and $b = 3$, then $a = b$ is true, and hence $a \supset b$ is true, but $ab = a$ is not, for ab is $3 + 3$ or 6.

Note that when Postulate 3 fails, certain characteristics of the Boolean algebra drop out. Thus, $0 \supset a$, under the interpretation given, is not true for every value of a (it is only true when $a = 0$); the sum of any element and the 1-element (here arithmetical 0) is not the 1-element in general; there is a case where $a = a'$, etc.

The independence of Postulate 1 can be established in a similar manner. The system is to have four elements, where multiplication and negation are defined by the following tables:

×	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	2	2	2
3	0	3	3	3

	′
0	3
1	0
2	0
3	0

As an example of the manner in which Postulate 1 fails, let $a = 1$ and $b = 2$. Then $1 \cdot 2 \neq 2 \cdot 1$, for the former is 1 while the latter is 2. That Postulate 2 is always true may be verified by inspection. In the case of Postulate 3, it is to be noted that $a \supset b$ is true in every case except where $b = 0$ and $a \neq 0$. But $ab = a$ is also true in every case except where $b = 0$ and $a \neq 0$. Hence Postulate 3 will be verified in every case. Postulate 4 will be true always, since $aa' = 0$ for every element.

The following matrix [3] shows that Postulate 2 is independent of the rest. Let the system have four elements, multiplication and negation being defined as follows:

[3] A matrix is simply a set of tables showing independence or consistency.

×	0	1	2	3
0	0	2	1	0
1	2	1	0	1
2	1	0	2	2
3	0	1	2	3

	′
0	3
1	2
2	1
3	0

As a consequence, the following table will represent the properties of implication, T signifying that the given element in the vertical column is included in one in the horizontal, F that it is not. Thus $0 \supset 3$ is true, but $2 \supset 1$ is false (the null element being 0):

$\supset$	0	1	2	3
0	T	F	F	T
1	F	T	F	F
2	F	F	T	F
3	F	F	F	T

Postulates 1, 3, and 4 can be seen to be true by inspection; Postulate 2 will be false, for example, when $a = 0$, $b = 2$, and $c = 1$, for $0\,(21) \neq (02)\,1$ in this system.

The independence matrix of Postulate 4 may be given in the following simple manner: the system has two elements, 0 and 1:

×	0	1
0	0	0
1	0	1

	′
0	0
1	1

It will then appear that aa' is not unique, for $00'$ is 00 and this is 0, while $11'$ is $11 = 1$. Now $a \supset b$ being defined, as above, as $ab' = aa'$, it follows that $0 \supset 1$, $1 \supset 1$, $0 \supset 0$ are true relations. But it also follows that $01 = 0$, $11 = 1$, $00 = 0$, and hence Postulate 3 will be true always. The truth of Postulates 1 and 2 will appear from inspection.

Further examples of independence and consistency proofs are given in chapter XVIII.

EXERCISES

GROUP A

1. The following examples demonstrate the consistency and independence of the postulates for the Aristotelian class algebra given on page 103, including Postulate 7 mentioned in the footnote, but are not necessarily in the correct order. Determine in each example which postulate is shown independent, or whether the entire set is shown consistent:

a) There are eight elements, 0-7. A (ab) is true only for the cases A (14), A (15), A (24), A (26), A (35), A (36). a' is determined by the following table:

	′
0	7
1	6
2	5
3	4
4	3
5	2
6	1
7	0

b) A (ab) is the same as the Boolean inclusion-relation, $a \supset b$. (a' is the Boolean a').

c) $a, b, c, \ldots$ represent integral numbers $(\neq 0)$. a' is $-a$ ("minus a"). A (ab) means $a = b$.

d) $a, b, c, \ldots$ represent positive numbers. a' means $a/2$. A (ab) means $a \leqq b$ ("a is less than or equal to b").

e) $a, b, c, \ldots$ represent any of eight elements, 0-7. If $a = b$, then A (ab) is true. If $a \neq b$, then A (ab) is only true in the cases enumerated in example *a*). a' is defined as in example *a*).

f) $a, b, c, \ldots$ represent any of eight elements, 0-7; A (ab) and a' are defined as in example *e*), except that A (03) and A (47) are also true.

g) $a, b, c, \ldots$ represent any of the eight elements 0-7. a' is defined as in example *e*). A (ab) is true in all the cases enumerated in *e*), and A (01), A (04), A (05), A (10), A (27), A (37), A (67), and A (76) are also true.

h) *a, b, c,* . . . represent any of eight elements 0-7. A (*ab*) is true in all the cases enumerated in example *e*), and also A (04), A (05), A (10), A (27), A (37), A (76) are true. a' is defined as follows: $0' = 7$, $1' = 7$, $2' = 5$, $3' = 4$, $4' = 3$, $5' = 2$, $6' = 0$, $7' = 0$.

2. Construct a consistency matrix for the Aristotelian algebra analogous to the "factors-of-thirty" example given above for the Boolean algebra. Here there is no element 0 such that A (0*a*) always holds (cf. next chapter).

The Aristotelian and the Boolean Algebras

16

A CERTAIN PECULIARITY which arises from the comparison of the Aristotelian and the Boolean system already may have made itself apparent. For suppose that in the former we define the "null class" in a manner similar to that of the latter. It will then appear that A (ab) and $a \supset b$ do not have the same properties. For suppose that A (ab) and $a \supset b$ were the same. Now for the relation of inclusion we always have

$0 \supset b$, "The null class is included in every class."

Hence, if A (ab) and $a \supset b$ were identical, we should also have

$$A\,(0b).$$

Put otherwise, if the two relations are identical, then A (ab, a), "All of what is common to a and any other class, belongs to a," must be true even when a and b have nothing in common. Thus, "All square-circles are squares" and "All square-circles are circles," etc. But now let us see whether I $(0b)$, "Some of the null class belongs to every class," is true or not. Since E (ab) is definitionally equivalent to A (ab'), or "all a is non-b," it must be the case that E $(0b)$, "all the null class belongs to non-b," is true, since non-b is a class and the null class supposedly belongs to any class whatsoever. I (ab) is the contradictory of E (ab), and hence if E $(0b)$ is true, I $(0b)$ must be false, since two contradictory statements cannot both be true. That is, on the supposition that A $(0b)$ is always true, we derive the fact that I $(0b)$ is false. But this result is contradictory to the Aristotelian class calculus, for there it was postulated that A (ab) implies I (ab), that I (ab) is always true when A (ab) is true. Hence, *if* A (0*b*) were true, then one of the postulates of the Aristotelian system would be false. Hence A (0*b*) is not true in this deductive system, and it follows that A (*ab*) and $a \supset b$ are not the same, no matter how closely we may interpret them in words.

This result may be seen in other ways. For A (*ab*) and E (*ab*) were called contraries. That is, they could not be true at the same time. But if A (0*b*) were true, then A (*ab*) and E (*ab*) would both be true together, in the case where *a* was 0. Again, we deduced as a theorem: [A (*aa′*)]′ ("For no values of *a* is it true that all *a* is non-*a*.") But if A (0*a*) were true, then this theorem would be false for one value of *a*, namely, 0. Similarly, E (*aa*) would be true for one value, and not always false as it is in the Aristotelian calculus.

Again, take the syllogism Darapti:

A (*ba*) A (*bc*) implies A (*ca*).

Here, if *b* is the null class, and if A (0*a*) were always true, we would have both premises verified *independently* of the values of *a* and *c*. But A (*ca*) is not true independently of what *a* and *c* are, and hence the syllogism must become false. As an example, let $b =$ square-circles, $a =$ circles, and $c =$ squares. Many of the other syllogisms will prove false by similar arguments.

This discussion might seem to indicate an inherent inconsistency in the Aristotelian system. But the necessity of asserting A (0*b*) is not obvious; we are under no logical obligation to assume that A (*ab*) and $a \supset b$ are the same relations. Appeals to "common sense" are in the wrong direction from logical rigor and in the end usually depend on personal feeling. There are, indeed, many cases where it would appear that common sense sanctions the proposition "All the null class is in any class." Thus (the illustration is Professor C. I. Lewis') suppose a man hires a watchman and agrees to double his salary if all trespassers on his property are successfully prosecuted. At the end of a year it so happens that there have been no trespassers. Should the man pay the double salary or not? I.e., is it true that all trespassers were prosecuted when there were none? Probably, if a census were taken on the question, a majority would vote "yes" to this latter question and hence substantiate in this instance the claim that A (0*b*) was true.

On the other hand, consider the following proposition (due to Professor H. B. Smith): "Every aged Achilles who is mentioned in the Iliad is there celebrated as the fleet of foot." Though the class of "aged Achilles" is null, the consensus of opinion would probably be in favor of the falsity of the above. All this merely illustrates the futility of appealing to "common sense" in the postulates of a deductive system.[1]

As the foregoing remarks show, both the Boolean algebra and the Aristotelian algebra are abstract systems, the validity of whose postulates does not depend on some particular concrete interpretation. This discussion has meaning only when we are determining which algebra can best be applied to a class calculus, i.e., which algebra best expresses the relations between classes. In sum, then, the only result that such a discussion can claim to have is a criticism of the words by which we interpret the symbol $A(ab)$. Perhaps "All a is b" is not the most accurate rendering of the symbol. But the words, after all, are merely an aid to the imagination. The true definition of $A(ab)$ lies in its properties as expressed by the Aristotelian system, and these properties are perfectly consistent.[2]

The fact that the Aristotelian and the Boolean algebra are both consistent suggests the possibility of "translating" the relation $a \supset b$ in terms of $A(ab)$ and vice versa. This translation, as we have seen, must be very much more complicated than the simple $A(ab) = (a \supset b)$. Professor H. B. Smith found one such translation. He let

$$A(ab) = (a \supset b)\ (b \supset a) + (a \supset b)\ (a \supset b')'\ (b' \supset a)',$$

where $p + q$ means "either, or." Thus, if $a =$ dogs and $b =$ animals, then the Aristotelian "All dogs are animals" is equivalent to the Boolean:

"The class dogs is included in the class animals, and vice versa, or dogs are included in animals but (and) dogs are not included in what are not animals, and non-animals are not included in dogs," the latter part of the disjunction being true.

---

[1] Even though it were "common sense" to say that the null class is included in every class, it is certainly not common sense to assert that when all a is b it does not follow that some a is b, an assertion the Boolean algebra must make if $(a \supset b) = A(ab)$.

[2] The proof that the Aristotelian algebra is consistent will be found in the exercises at the end of the last chapter.

Now, since

$$E(ab) = A(ab'),$$

we can derive the translation of $E(ab)$ in the Boolean algebra by replacing b by b' in the formula for $A(ab)$:

$$E(ab) = (a \supset b')(b' \supset a) + (a \supset b')(a \supset b)'(b \supset a)'.$$

We can then derive the formulas for O and I by contradicting the right side of the formulas for A and E:

$$[A(ab)]' = O(ab) = [(a \supset b)(b \supset a) + (a \supset b)(a \supset b')'(b' \supset a)']'$$

By successive applications of De Morgan's Law (page 259), it follows that

$$O(ab) = (a \supset b)' + (b \supset a)'(a \supset b') + (b \supset a)'(b' \supset a).$$

Similarly,

$$I(ab) = (a \supset b')' + (b' \supset a)'(a \supset b) + (b' \supset a)'(b \supset a).$$

By means of these formulas we can prove on the basis of the validity of Boolean algebra all the postulates of the Aristotelian calculus. Thus we can prove $A(aa)$; for

$$A(aa) = (a \supset a)(a \supset a) + (a \supset a)(a \supset a')'(a' \supset a)',$$

and since the first part of the disjunction, $a \supset a$, is true, the whole disjunction is true. To prove $A(ab)$ implies $I(ab)$ we prove first that the conjunction of $A(ab)$ and $E(ab)$ is an impossibility; i.e., we show that $A(ab) \cdot E(ab) = 0$; we do this by multiplying the formulas for $A(ab)$ and $E(ab)$ together term by term and showing that the result "vanishes." Then, since if $xy' = 0$, we can infer that x implies y, we can deduce from $A(ab)\ E(ab) = 0$ that $A(ab)$ implies $I(ab)$. Similarly, we can prove the syllogism Barbara, $A(ba)\ A(cb)$ implies $A(ca)$, by proving first that $A(ba)\ A(cb)\ O(ca) = 0$.

As an example of the manner in which this last product vanishes take the following term which is one of the results of the multiplication:

$$(b \supset a)(b \supset a')'(a' \supset b)'(c \supset b)(c \supset b')'(b \supset c')'(a \supset c)'(c \supset a').$$

Two of the factors of this product, $(b \supset a)$ and $(c \supset b)$, imply that $(c \supset a)$, and this result, together with $(c \supset a')$ give $c = 0$;

but this result contradicts the assertion of another factor of the product, namely $(c \supset b')'$, which asserts that $c \neq 0$, since $(0 \supset b')'$ is a false statement. Hence the whole product implies a contradiction and hence "vanishes"; there will be a sum of such products which will all vanish in a similar manner.

Note how A $(0a)$ fails. For A $(0a)$ translated becomes:

$$(0 \supset a)\ (a \supset 0) + (0 \supset a)\ (0 \supset a')'(a' \supset 0)'.$$

But $(a \supset 0)\ (0 \supset a)$ is $a = 0$, and since $(0 \supset a')'$ is a false statement (0 is included in every class), the second part of the disjunction is false $(= 0)$, and hence A $(0a)$ reduces to $a = 0$. That is, *A (0a) is true if and only if a is 0.* Hence E $(a0)$ is true if and only if a is 1. And I $(0a)$ is true if and only if a is not 1 (some of the null class belongs to every class but the universe). O $(0a)$ is true if and only if a is not 0.

EXERCISES

GROUP A

1. Given the properties of the Boolean algebra, by means of the above transformation formulas, prove the following propositions of the Aristotelian algebra:

   a) E (ab) implies O (ba). (Hint: show that E (ab) A $(ba) = 0$.)
   b) A $(ab) =$ A $(b'a')$.
   c) I $(ab) =$ I (ba).
   d) A (ba) A (bc) implies I (ca).
   e) A (ab) A (cb) does not necessarily imply I (ca).
   f) Derive the formulas for U (ab) and V (ab) (page 109).

2. Given the properties of the Aristotelian class calculus, prove the following propositions of the Boolean algebra by means of the transformation formula:

   $$(a \supset b) = [\mathrm{A}\,(ab) + \mathrm{A}\,(a0) + \mathrm{A}\,(b'0)],$$

   where 0 is aa'. (Hint: remember that $1 + p = 1$, i.e., that the disjunction of a true proposition and any other proposition always yields a true proposition.)

   a) If $(a \supset b)$, then $(b' \supset a')$
   b) $(a \supset a)$
   c) $(0 \supset a)$
   d) $(a \supset 1)$

e) If $(a \supset b)$ and $(a \supset b')$, then $a \supset 0$ (make use of the fact that if A $(a0)$, then $a = 0$.)

f) If $(a \supset a')$, then $a = 0$.

g) If $(a \supset b)$ and $(b \supset c)$, then $(a \supset c)$ (Remember that if $p + q$ is true, and (p implies r) and (q implies s) are true, then $r + s$ is true (constructive dilemma), and that this may be generalized for a disjunction of any number of variables, $p + q + r + \ldots$.)

h) If $(a \supset b)$ and $(b \supset a)$, then $a = b$.

i) $(a \supset b)' = \mathrm{O}\,(ab)\ \mathrm{O}\,(a0)\ \mathrm{O}\,(b'0)$

j) $(a \supset b') = \mathrm{E}\,(ab) + \mathrm{E}\,(a1) + \mathrm{E}\,(b1)$

k) $(a' \supset b) = \mathrm{U}\,(ab) + \mathrm{U}\,(a0) + \mathrm{U}\,(b0)$

Problems of Symbolic Logic

17

THE FOLLOWING are seven problems with which the science of Symbolic Logic is concerned:

I. A certain difficulty arises with respect to the deductive system of the Boolean algebra when this algebra is interpreted as a calculus of propositions. Among our axioms we assumed certain verbal statements which later we pretended to prove. Thus, it was presupposed that if p implies q and q implies r, then p implies r; but this fact is also stated by Theorem 12. We seem indeed to have committed the fallacy of *petitio principi* in the proof of the latter *if we interpret the algebra as a calculus of propositions.* No such fallacious reasoning results when the algebra is not so interpreted. Another objection to this procedure with respect to this interpretation lies in the fact that we have not completely symbolized the expressions under consideration. Theorem 11, for instance, reads "If $(p \supset q)$, then $(q' \supset p')$." But the words "if . . . then" of this expression themselves represent an implication. Hence the whole expression should be symbolized:

$$(p \supset q) \supset (q' \supset p');$$

similarly, Theorem 12 should be symbolized

$$(p \supset q)\ (q \supset r) \supset (p \supset r)$$

and this would be a postulate or theorem (not an axiom) of the system. A satisfactory algebra of propositions, then, would be one in which the presuppositions or axioms of the Boolean algebra were a part of the system, that is, were theorems or postulates, and all theorems and postulates were completely symbolized. Can these requirements be carried out? The answer to this question, which has puzzled logicians ever since the founding of Symbolic Logic a century ago, is "no." In other words, there must be certain nonsymbolic or "verbal" statements for any deductive system of the propositional calculus. For suppose we are given a completely symbolic set of postulates; how can we deduce anything from them unless we have some rule of procedure, some law which will describe the process of passing from one form to another? And this law must be expressed verbally, at least in part. E.g., suppose we are given

1. $p \supset p$
2. $pq \supset p$

Without some method of changing these, we can but sit and look at them. No theorems will follow without the necessary "gasoline." The need for verbal rules is but an illustration of the abstract nature of symbolic expressions. These two postulates given have no specific meaning, and we must make some sort of presupposition in assuming that "$\supset$" means "implies" and "p" means "a proposition." The verbal rules, on the other hand, avoid this abstractness, and supposedly have unique meaning.

Two of these verbal rules or Rules of Procedure may be evident at once, for (1) we wish to express the right to substitute for the terms of a given expression any other terms of the system, and (2) we wish to express the right to assert a proved theorem independently of its hypothesis, i.e., to assert that q can be inferred from the proposition that "p is true and 'p implies q' is true." The first presupposition is the core of logic; for logic is that science whose expressions are true independently of the meaning of the terms, and hence true no matter what we substitute for the terms. It might seem, indeed, that the second Rule of Procedure could be symbolized: $[p\,(p \supset q)] \supset q$. But such a symbolic proposition would not express the right to assert q *independently of p*. Suppose, as an example, that we know that $p \supset p$ is true and we have also proved the proposition

$$(p \supset p) \supset (p + p').$$

Can we assert that $p + p'$ is an independently true theorem? If the symbolic expression $[p\,(p \supset q)] \supset q$ is all that is given, then we can only assert the following:

$$(p \supset p)\,[(p \supset p) \supset (p + p')] \supset (p + p'),$$

where we have substituted $p \supset p$ for p and $p + p'$ for q in the expression $[p(p \supset q)] \supset q$. But actually what is required is a verbal statement which will assert the right to consider $(p + p')$ as true by itself.[1]

This is one problem, then, which confronts the logician: to construct a symbolic set of postulates adequate for the calculus of propositions.

REFERENCES

B. Russell, and A. N. Whitehead, *Principia Mathematica* (Part I, chapters 1-5).

C. I. Lewis and C. H. Langford, *Symbolic Logic* (chapter VI).

II. *Propositional functions:* Suppose that we have a sentence such as "Jones is a man" and substitute for the subject the variable "x," deriving the expression "x is a man." Is this latter true or is it false? If we take "true" to mean "always true" and false to mean "always false," then we can say that the statement is neither true nor false, but is true some times and false other times. The expression "x is a man" is called a propositional function. The reader is probably familiar in part with a certain group of propositional functions, namely, the mathematical ones such as $x + 2y = 3$, or $3^x = 6$, or $\int_1^2 x\, dx = 0$. But these do not by any means exhaust the possibilities, as our first example here plainly shows. Certain assertions can be made about the truth of a given propositional function. Let us represent any given propositional function by $\phi(x)$, where ϕ represents the form of the given expression and may be "____is a man," "____$+ 2 = 4$," etc. Then we can assert one of four things: (1) $\phi(x)$ is true for all values of x, and this is symbolized $(x) . \phi(x)$; (2) $\phi(x)$ is true for at least one value of x, i.e., there is a value of x such that $\phi(x)$ is true, and this is symbolized $(\exists x) . \phi(x)$ (the symbol $\exists$ being used to denote existence); (3) $\phi(x)$ is sometimes false, i.e., there is at least one value of x such that $\phi(x)$ is false, $(\exists x) . \sim \phi(x)$ (where $\sim \phi(x)$ means

[1] Cf. p. 208 (discussion of Carroll's paradox).

"$\phi(x)$ is false")[2]; (4) $\phi(x)$ is never true, i.e., $\phi(x)$ is false for every value of x: $(x) . \sim \phi(x)$.

It is clear that the form of certain functions cannot be applied to all possible objects. Thus if ϕ is "exceeds $+ 3$" we derive a meaningless expression if the x in $\phi(x)$ is some such object as "triangle" or "beauty." The set of objects to which a given function may be applied meaningfully is called the *domain* of the function. In the case of the propositional function A(ab) the domain consists of all objects whatsoever, but the domain of the function "$x > y$" would be the set of all numbers. The set of propositions which result from substituting for the variable or variables the objects of the domain is called the *range* of the function. The above assertions about the truth or falsity of a given propositional function must be thought of as applying only to the domain: "$\phi(x)$ is always true" means that $\phi(x)$ is true for every element of the domain, i.e., the entire range is true.

Within the domain of a propositional function there may be some objects which make the function true, some which make it false. We symbolize the expression "all the objects of the domain which make $\phi(x)$ true" by $\phi(\hat{x})$ or $\ni \phi(x)$. Thus, if ϕ is the function "x is a rational featherless biped," then $\phi(\hat{x})$ will be the class of human beings in the domain of physical objects. If $\phi(x)$ is "is included in itself" then $\phi(\hat{x})$ is the universe of classes, while if $\phi(x)$ reads "x is a square-circle," then $\phi(\hat{x})$ is the null class.

The Theory of Types, explained in part previously (page 202), imposes certain conditions on the domain of a function, one of these being that the function itself cannot be a member of its own domain. Hence $\phi[\phi(x)]$ is a meaningless symbol, so that if $\phi(x)$ reads "x is false," the expression "'x is false' is false" is meaningless.

For the sake of convenience, we introduce a convention which avoids the awkwardness of parenthesis signs, that of using dots as brackets. This use of dots is as follows: the number of dots indicates the range of application of one part of an expression over the rest; thus, if in the calculus of propositions we wish to symbolize $p \supset (q \supset r)$[3], we write $p . \supset : q \supset r$, where the two

[2] The symbol $\sim p$ for "p is false" is more frequently used here than p'.

[3] As is usual, the symbol "$\supset$" is used here for "implies" and the symbol "$\equiv$" for equivalence, i.e., for mutual implication.

dots indicate that the implication sign applies to everything thereafter; if we wish to have $(p \supset q) \supset r$ we write $p \supset q . \supset : r$. Again, in the case of propositional functions, if we wish to say that $(\exists x) . \phi(x)$ and $\sim (x) . \phi(x)$ are equivalent, we write $(\exists x) . \phi(x) : \equiv : \sim (x) . \phi(x)$, where the double dot indicates that the equivalence sign is the principal relation. On the other hand, the expression $(\exists x) : \phi(x) . \equiv . \psi(x)$ would read "For some values of x, the functions $\phi(x)$ and $\psi(x)$ are equivalent."

The following laws may now be stated:

1. $(x) . \phi(x) : \supset : (\exists x) . \phi(x)$

"If $\phi(x)$ is true for every value of x, then $\phi(x)$ is true for at least one value."

2. $(x) . \sim \phi(x) : \supset : (\exists x) . \sim \phi(x)$

"If $\phi(x)$ is false for all values, then it is false for at least one value."

3. $(x) . \sim \phi(x) : \equiv : \sim (\exists x) . \phi(x)$

"A necessary and sufficient condition that $\phi(x)$ be false for all values is that it be false that any x exists for which $\phi(x)$ is true."

4. $(x) . \phi(x) : \equiv : \sim (\exists x) . \sim \phi(x)$

"$\phi(x)$ is true for every x if and only if there is no value of x such that $\phi(x)$ is false."

For more or less obvious reasons, the symbols (x) and $(\exists x)$ are called "quantifiers."

It is evident that we are not restricted to functions having but one variable. Indeed, the propositional functions in most sciences contain many more than one. Examples of such are: "a travels at the speed v during the time t," "$x + y = 3$," "Line a is parallel to b and perpendicular to c." We may symbolize the propositional function containing two variables by $\phi(x, y)$. Then if ϕ is a fixed function, there are a number of ways of asserting the truth of $\phi(x, y)$: (1) $(x)(y) . \phi(x, y)$, "For every x, $\phi(x, y)$ is true for every y," which is equivalent to $(y)(x) . \phi(x, y)$; (2) $(\exists x)(y) . \phi(x, y)$, "There exists an x such that $\phi(x, y)$ is true for every y"; (3) $(x)(\exists y) . \phi(x, y)$, "For every x there exists a value of y such that $\phi(x, y)$ is true":

(4) $(\exists x)\ (\exists y) . \phi(x, y)$, "There exists a value of x such that $\phi(x, y)$ is true for some value of y," which, again, is equivalent to $(\exists y)\ (\exists x) . \phi(x, y)$. There will be similar ways of denying $\phi(x, y)$. Note that (2) and (3) are not equivalent, as may be shown by pointing out an example which can be asserted in the manner of (3) but not of (2). Thus the mathematical function $x + y = 3$ is such that for every value of x there is a value of y which makes the equation true. But we cannot say of $x + y = 3$ that there exists an x such that the equation will be satisfied for every value of y.

As in the case of one variable, there is also a domain of a function of two variables, the set of pairs of terms which yield a meaningful proposition when substituted for the variables. It should be noted that the domain of x may be entirely different from the domain of y; in the propositional function "x was in China at time y" the domain of x is the set, say, of physical objects, while y is some number expressing a measure of time. The class of couples satisfying $\phi(x, y)$ may be symbolized $\phi(\hat{x}, \hat{y})$. If $\phi(x, y)$ reads "x is the husband of y," then $\phi(\hat{x}, \hat{y})$ is the class of all married couples in the domain of humans; these pairs are more often than not "ordered," i.e., we cannot reverse them; if $\phi(x, y)$ is "all x is y," then $\phi(\hat{x}, \hat{y})$ will be the class of all pairs of classes such that the first is included in the second; the pair "horses, animals," then, would be a member of the class, but we could not say that the pair "animals, horses" is a member. If ϕ is a function containing three variables, $\phi(\hat{x}, \hat{y}, \hat{z})$ will represent a class of ordered triads; the concept may be extended to functions of any number of variables and has played an important role in attempts to generalize the calculus of propositional functions.

The reader will be able to discover for himself laws of functions of two variables analogous to those given above for the case of one variable (see the exercise at end of chapter).

Closely related to the theory of propositional functions is the concept of "formal implication." We say that a given propositional function $\phi(x)$ "formally implies" another function $\psi(x)$ if for every value of x, $\phi(x) \supset \psi(x)$ is a true implication, i.e., if $(x) : \phi(x) . \supset . \psi(x)$. This is sometimes symbolized

$$\phi(x) . \supset_x . \psi(x) .$$

The concept may evidently be generalized to propositional functions of any number of variables.

So far we have treated the symbols $\phi(x)$ and $\phi(x, y)$ as though the form of the function were fixed while the terms involved varied. Thus ϕ might represent "is a man" and $\phi(x)$ read "x is a man." But it also possible to regard ϕ as a variable as well. For example, suppose we wish to state that the objects a and b were identical in all respects; this would mean that whatever was said of a could be said of b as well, and vice versa, that is, that any function containing a would be true if and only if the same function were true when a was replaced by b. This can be symbolized:

$$(\phi) : \phi(a) \equiv \phi(b)$$

"For every value of ϕ, $\phi(a)$ is equivalent to $\phi(b)$ (where a and b are not variables)." In general, $\phi(x)$ is an expression containing two variables, ϕ and x; $\phi(x, y)$ is one containing three variables.

The calculus of propositional functions may be considered as a general calculus of classes and relations. Its central problem may be phrased thus: "To find all universal properties of classes and relations, or to find all statements about classes and relations which hold, regardless of what these may be, i.e., independent of any specific meaning one may assign to the classes or relations." Such a calculus considers not only relations holding between two objects ("a is less than b") but relations holding between any number of objects ("a is larger than b by x units, where x is larger than y"). The propositional function

$$\phi(x, y, z, \ldots)$$

represents this general relation.

(See Exercises, Group A, at end of chapter.)

REFERENCES

G. Peano, *Formulaire de mathematique* (especially vol. 2) (The best elementary exposition of Peano's work is in A. Padoa, *La Logique déductive dans sa dernière phase de développement,* Revue de métaphysique et de morale, vols. 19 and 20.)

D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik.*
D. Hilbert and P. Bernays, *Grundlagen der Mathematik.*
B. Russell, *Principles of Mathematics.*
B. Russell and A. N. Whitehead, *Principia Mathematica* (Second Edition).
W. V. Quine, *A System of Logistic.*
A. Tarski, *Einführing in die mathematische Logik.*

III. That the words "true" and "false" do not exhaust the possible ways or "modes" of asserting a given expression is evident. For example, one may assert that a given proposition is "necessarily" true, or perhaps merely "possibly" true. It might be thought, indeed, that the discussion in II covers these modes, for "sometimes true" and "possibly true" seem to assert the same things. But the history of the case has shown that this is not so, and there do appear to be obvious examples which substantiate this; e.g., suppose the given expression has no variable (i.e., is a proposition): to use a classic example, "The moon is made of green cheese." It seems nonsense to talk of this proposition's being "sometimes true," but one can say that it is "possible" (i.e., it is not self-contradictory). If the modes "possible" and "sometimes true" are distinct, then there arises the problem of constructing a logic which takes care of the relations between the modes such as "possible," "necessary," "possibly false." This has become known as "intensional" logic, since here apparently the meaning of a given statement is considered and not merely its truth-value (cf. page 288).

REFERENCES

H. MacColl, *Symbolic Logic and its Applications.*
C. I. Lewis and C. H. Langford, *Symbolic Logic.* (For the problem of the relation of intensional logic to "language," and the relation of logic in general to grammar, see R. Carnap, *Logical Syntax of Language.*)

IV. Closely related to III, though in many respects quite a different problem, is the question of a "complete" calculus of propositions. It is to be remembered that in discussing the two-valued logic it was pointed out that this logic assumed as equivalent the statements $(p = 0)$ and p'. This equivalence seems to hold only among a rather restricted group of propositions (those which are actually true or actually false). Suppose, then, we do not restrict ourselves in this manner, but allow p to represent any proposition (or propositional expression, such as a propositional function). That is, we are to allow our logic to be applied to any case whatsoever. Indeed, logic is sometimes defined as that science whose propositions hold no matter what the terms in them may be. Thus, under this definition, such an expression as $(p = 1) = p$ would not be a law of logic, since it does not hold for every p (e.g., let p be "all a is b"). The general relations of logic turn out to be the simple ones first enumerated in the case of the Boolean algebra ("$\supset$," inclusion, "$+$," disjunction, "$\times$," conjunction, and $'$, negation). Thus, the symbol $\exists$ is not a general relation, since it cannot be applied to all propositional expressions but merely to propositional functions.

A general theory of propositions will consider all the expressions which arise from these general relations. This problem has had its difficulties; for, though it is true that $(p = 0)$ and p' are not the same, we can say that $(p = 0)$ implies p'. Similarly, $(p = 1)$ implies p, and p implies $(p = 0)'$. But suppose we take some more complicated form, such as $(p = 0) = 0$. What is the relation of this to the other forms? There arises, then, the problem of constructing a "table" of relations. The problem becomes much more complex when we consider two variables and attempt to simplify such expressions as $(pq = 0)'$. It is to be noted that this problem really deals with the generalized theory of implication, since every implication is expressible in the form $(p = 0)$ (cf. Definition 2 in the Boolean algebra).

The generalized logic has practical value in that it seems to offer a very simple solution of the Epimenides paradox (cf. page 197). The solution may be given in an abbreviated form in the following manner. The more restricted two-valued calculus of propositions considers all its elements as having either the value 1 (the true) or 0 (the false), and any law holds universally in this logic if its value is always 1 when its elements take on either the value 1 or 0. Thus the proposition "$p \supset (q \supset p)$" is "universally" true, since it is true when $p = 0$ and $q = 1$ or 0, and when $p = 1$ and $q = 1$ or 0. But the general logic does not consider its propositions in this "truth-table" sense. Propositions are not universally true even though they hold for all "truth-values" (0 and 1) of the elements.

In this logic, elements may be considered as "ambiguously true or false." Thus in the assertion "$p \supset (p \supset q)$," where no restrictions are placed on p and q, p and q are not necessarily either 0 or 1. A similar remark applies to the statement "p is false," which is the troublesome proposition in the Epimenides paradox. Here again p is ambiguously true or false: it is *both* true and false in the *distributive* sense; it cannot be both true and false in the *collective* sense (cf. page 169), since this would involve a contradiction; but if no meaning or value is attached to the proposition p, then it may be thought of as containing both the true and the false, much as the symbol $\sqrt{4}$ contains both the values -2 and $+2$; but $\sqrt{4}$ is not both $+2$ and -2 collectively, since this is a contradiction, but rather distributively. This being the case, the assertion "p is false" is also both false and true, but again in the distributive sense, just as the symbol $-\sqrt{4}$ is both -2 and $+2$. The paradox turns out to be a case of the Fallacy of Composition.

REFERENCE

H. B. Smith, *Abstract Logic or the Science of Modality,* Philosophy of Science, vol. 1, 1934.

V. One of the chief problems of what has become known as "metalogic" is concerned with the question of consistency. We have set down as criteria of the postulates of a formal science the stipulations that they must be consistent and independent; the latter criterion is really dependent on the first, since a given postulate is shown to be independent of the rest by showing that the set of propositions composed of the denial of this one and the assertions of the rest is consistent. We have also indicated a method whereby a given set can be proved consistent simply by showing that there is an interpretation of the undefined terms of the system which makes all the propositions true. The following questions now arise:

1. Can every set be proved consistent? We have said that the interpretation of the symbols gives us a set of "true" propositions; if

the term "true" involves the term "consistent," as it surely must, then we still have the problem of showing that this new set is consistent, and such a demonstration would require another consistent set, etc.

2. Since the science of logic sets down the criteria for consistency, how can we ever show that a set of postulates for logic is consistent without involving ourselves in a piece of circular reasoning?

3. How can we say that any postulate is independent of the rest, if it is a law of logic that a true proposition is implied by any other? The postulates are certainly all true, and hence each one must be implied by all the rest and no one can be independent of any other.

Adequate answers of these problems would probably require another volume. The following remarks suggest possible replies:

1. Some set or sets of propositions must be assumed consistent, but the consistency of a set of propositions involving an infinite range of application often can be shown to depend on the consistency of a set involving only a finite range,[4] and in the case of the latter the assumptions necessary to establish consistency are rather simple. Thus the postulates for Boolean algebra may have an infinite range of application, but their consistency may be shown to depend on a set of propositions (the two-valued algebra) whose range of application is restricted to two objects.

2. Presumably the science of logic cannot be proved consistent without a *petitio principi;* but a set of symbolic expressions which are supposed to represent laws of logic may be shown consistent by the usual methods, for here our interest lies in whether we have chosen a proper set for our science.

3. It is important to notice that when we attempt to discover whether a given postulate is independent of the rest of the set or not, we are interested in knowing whether it can be deduced from the remaining members of the set by a proper manipulation of the symbols in the given expression. For this purpose, the question of the actual validity of the propositions which we are considering is ignored and these propositions are treated as propositional functions, whose truth-values remain undetermined. Thus, when we ask whether $aa' = bb'$ is independent of

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[4] No finite consistency matrix for the postulates of arithmetic exists.

$ab = ba$ and $a(bc) = (ab)c$, we merely ask whether the implication

$$[ab = ba]\,[a(bc) = (ab)c] \text{ implies } [aa' = bb']$$

is true or not. If we can show that the proposition

$$[ab = ba][a(bc) = (ab)c][aa' = bb']'$$

is not an absurdity, then we have demonstrated the required independence and this independence-proof does not involve the actual truth or falsity of the constituents.

REFERENCE

C. I. Lewis, "Emch's Calculus and Strict Implication, *Journal of Symbolic Logic,* vol. 1, pp. 85, 86.

VI. Another criterion of a set of postulates was that they be "sufficient." Sufficiency is a relative term and consequently involves a certain amount of vagueness. This vagueness may be overcome by substituting the criterion of "absolute" sufficiency in place of the relative. A set of postulates is said to be absolutely sufficient if, given any meaningful expression involving only the indefinables assumed by the system, we can demonstrate by means of the assumptions either that this expression is a truth of the system or that it is not a truth of the system.

The problem of showing absolute sufficiency [5] is in general quite difficult. Certain simple systems, as, for example, the two-valued algebra, have been shown to be absolutely sufficient,[6] while apparently in the case of certain other systems it has been shown that they cannot be made so.[7]

VII. The Law of Substitution (page 269) as given in almost all systems of logic is much more complicated than any of the axioms. It involves such unanalyzed ideas as "replacing throughout," "expression," etc., which do not appear in any of the other laws of logic. Since this law is so fundamental in all our proofs, it would be desirable to analyze it into one or more simpler propositions. This problem of the analysis of logical substitution

[5] The so-called *Entscheidungsproblem.*

[6] See Hilbert and Bernays, *Grundzüge der theoretischen Logik.*

[7] See B. Rosser, "An Informal Exposition of Godel's Theorems and Church's Theorems," *Journal of Symbolic Logic,* vol. 4, pp. 53-60.

has been studied by several authors and has been solved in a certain sense by H. B. Curry.

Curry found it necessary to reformulate symbolic logic in terms of operations alone, and to eliminate, as far as possible, the use of variable symbols. For example, the result of substituting $g(x)$ for x in $f(x)$ is symbolized by

$$Bfgx,$$

and the new function of x by

$$Bfg,$$

i.e., B operates on f and g to produce a new function. Other types of substitution are similarly treated.[8]

EXERCISES

GROUP A

1. Symbolize the following expressions:

EXAMPLE: "(For every value of x and y) if x and y are not the same, then either x precedes y or y precedes x (cf. page 317)." Let $P(x, y)$ represent the propositional function "x precedes y" and $x = y$ represent "x equals y." Then the given sentence becomes:

$$(x)\ (y) : (x \neq y) \,.\, \supset \,.\, P(x, y) + P(y, x)\,.$$

a) If x precedes y and y precedes z, then x precedes z.
b) If x precedes y, then x and y are not the same.
c) There are numbers which are not equal.
d) For every value of x, $x + y = 0$ can be made a true statement if we choose the correct value for y.
e) There is no largest number.
f) If x and y are not parallel, then there is a point at which they meet.
g) Every man has a father.
h) Not every man has a son.
i) a and b are the same in some respects.

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[8] For detailed development see H. B. Curry, "An Analysis of Logical Substitution," *American Journal of Mathematics,* vol. 51, pp. 363-384, and *Grundlagen der kombinatorischen Logik, ibid.* vol. 52, pp. 509-536, and 789-834.

Note: the reading suggestions made in this chapter are not complete, of course. A complete bibliography of symbolic logic is given in the *Journal of Symbolic Logic,* vol. 1, No. 4. This bibliography is indexed according to subjects in vol. 3, No. 4.

j) If *a* and *b* are not the same in all respects, then they are not identical.
k) Two distinct points determine one and only one straight line.
l) Principle of Mathematical Induction, page 321.
m) The postulates for "betweenness," page 318.

2. In what ways can the following functions be asserted:

EXAMPLE: "*x* is better than *y*." If we let $\phi(x, y)$ represent this function, then we can make the following assertions:

a) There are values of *x* and *y* which make the function true: $(\exists x)\ (\exists y) . \phi(x, y)$.
b) There is no value of *x* such that $\phi(x, y)$ is true for all values of *y*: $\sim (\exists x)\ (y) . \phi(x, y)$.
c) There is a value of *y* such that for every value of *x*, $\phi(x, y)$ is false (if a perfect being exists): $(\exists y)\ (x) . \sim \phi(x, y)$.

(1) *x* is an animal with one horn.
(2) *x* is a square-circle.
(3) $4x = y$.
(4) A is similar but not equal to B (the domain is the set of geometrical figures).
(5) *x* loves *y*.
(6) *x* is to *y* as *y* is to *z*.
(7) *x* is married to *y*.
(8) *x* and *y* have a son *z*.

3. Determine whether the following are true or false: (if false, give examples which show them to be so):

a) $\sim (x)\ (\exists y) . \phi(x, y) : \supset : (x)\ (\exists y) . \sim \phi(x, y)$.
b) $(x)\ (\exists y) . \sim \phi(x, y) : \supset : \sim (x)\ (\exists y) . \phi(x, y)$.
c) $(x) \sim (\exists y) . \phi(x, y) : \supset : (x)\ (y) . \sim \phi(x, y)$.
d) $\sim (\exists x) \sim (\exists y) . \phi(x, y) : \equiv : (\exists x)\ (\exists y) . \phi(x, y)$.
e) $\sim (x)\ (y) . \sim \phi(x, y) : \equiv : (\exists x)\ (\exists y) . \phi(x, y)$.
f) $\sim (x)\ (y)\ (z) . \sim \phi(x, y, z) : \equiv : (\exists x)\ (\exists y)\ (\exists z) . \phi(x, y, z)$.
g) $(y)\ (\exists x) . \phi(x, y) : \equiv : (\exists x)\ (y) . \phi(x, y)$.

Examples of Deductive Systems

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POSTULATES FOR a deductive system are of two sorts: some act merely as definitions, some do not. Often a set of postulates may be considered in either way; thus, Euclid's set of postulates for plane geometry might be taken as definitions of a straight line, a point, etc. That is, we might state the whole Euclidean geometry as follows: "The concepts 'point,' 'line,' 'straight line,' and the like, are all things which satisfy these postulates." On the other hand, if the Euclidean indefinables are taken as definite physical concepts, e.g., if "straight line" means "ray of light," then the postulates do not define, but assert statements about physical concepts (cf. page 143).

The following are examples of deductive systems which are merely definitions.

1. The first presents the definition of a "series" of objects: [1]

DEFINITION: A set of objects K, with elements a, b, c, etc., is said to form a series with respect to a relation which can be applied meaningfully to any two of the elements (the relation symbolized "$<$" and read "precedes") if the following postulates are satisfied: (for all elements of K):

a) If $a \neq b$ (i.e., if a and b are distinct), then either $a < b$ or $b < a$.
b) If $a < b$, then $a \neq b$. ("$<$" is not reflexive)
c) If $a < b$ and $b < c$, then $a < c$. ("$<$" is transitive.)

Examples of series are plentiful enough. The group of Kings of England forms a series, where the relation "$<$" means "comes before in time," for with this meaning all the postulates become

[1] This set is due to E. V. Huntington, in his *Continuum*.

true. Again, the set of all whole numbers is a series, the relation "<" being "is less than." (Can you find another meaning of "<" which will make the set of all whole numbers a series?) These two examples show that "<" may be interpreted in many ways so as to make the given group a series.

The set of three postulates for a series is independent, i.e., no one postulate follows as a theorem from the other two (cf. page 292); we show this by exhibiting groups of objects for which a relation may be found which satisfies two of the postulates but not the third. This could not be done were one of the postulates a theorem; for example, any relation which satisfies postulates 1-3 would also satisfy the proposition "If $a < b$, then $b < a$ is false" since the latter is a theorem of the system.

Postulate 1 is shown independent if K is taken as the class of all human beings and $a < b$ means "*a* is the ancestor of *b*."

Postulate 2 is independent since K may be any class, where "$a < b$" means "*a* and *b* are both in the class K."

For a demonstration of the independence of Postulate 3 let K be the class of all whole numbers, where "$a < b$" means "*a* and *b* are distinct."

2. The following set of postulates, similar to those defining a series, defines the concept of "betweenness," which is important in the development of geometry.[2]

Definition: Given any set of objects K, if a triadic relation (i.e., a relation between three elements X, Y, Z), symbolized XYZ, can be found which will satisfy the following postulates for all elements of K, then this relation is called "betweenness," and XYZ means "Y lies between X and Z":

a) If AXB, then BXA.
b) If A, B, and C are distinct ($A \neq B$, $B \neq C$, $C \neq A$), then either BAC, or CAB, or ABC, or CBA, or ACB or BCA is true. (If Postulate 1 holds, the conclusion of the postulate may be reduced to BAC, or ABC, or ACB.)
c) If A, X, Y are distinct, then AXY and AYX cannot be true together (i.e., AXY implies "AYX is false").
d) If ABC, then $A \neq B$, $B \neq C$, and $C \neq A$.

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[2] The set is due to E. V. Huntington and J. R. Kline, *Postulates for Betweenness*, Transactions American Mathematical Society, vol. 18, pp. 301-325.

e) If XAB and ABY, then XAY, where A, B, X, and Y are all distinct.

f) If XAB and AYB, then XAY, where A; B, X, and Y are all distinct.

3. A very important concept of higher algebra is that of a *group*. A set of objects K is called a "group" with respect to a certain operation between two elements, symbolized $a \text{ o } b$, when the following propositions hold:

a) If a and b belong to the set of objects, then $a \text{ o } b$ belongs to the set.[8] For example, if the set of objects were the whole numbers and 0, and addition were the rule of combination, then this condition would be satisfied, since the addition of two whole numbers is a whole number.

b) If a, b, and c are members of the set, then $a \text{ o } (b \text{ o } c) = (a \text{ o } b) \text{ o } c$. (I.e., the associative law holds.) This condition is again true of the set of whole numbers, for $a + (b + c) = (a + b) + c$.

c) There is at least one element, i, of the set (often called the "unit element") such that when i is combined with any element, the result is merely the latter element:

$$i \text{ o } a = a$$

In the case of the set of whole numbers, 0 is the unit element, since $0 + a = a$. But if multiplication were the rule of combination, 1 would be the unit element.

d) For every element a of the set there is at least one element a', such that

$$a \text{ o } a' = i$$

Thus, in the case of the whole numbers and "+," to find the required element we merely change the sign; for $a + (-a) = 0$. (a' is called the "inverse" of a.)

Thus the set of whole numbers is a "group" with respect to addition, and this shows that the four conditions are consistent. But the elements of the Boolean algebra do not form a group with respect to either multiplication or addition. For if multiplication is the rule of combination, then 4 is not true. For i must be the 1-element in order that 3 be true, and $aa' \neq 1$. Similarly, addition cannot be the rule of combination, for then

[8] The "closure" postulate.

i would be the 0-element, and $a + a' \neq 0$, and hence 4 would again fail.

(See Exercises, Group A, at end of chapter.)

In the case of many deductive systems, it is very difficult to determine whether the postulates are merely definitions of the indefinables or not. Thus, the postulates of the algebra of classes might be taken as definitions of classes, so that, for example, if $(a \supset b)$ does not imply $(b' \supset a')$, then either a or b is not a class. Similarly, the logic of propositions might seem to define what is meant by the term "proposition," so that anything not fulfilling the law of excluded middle, $p + p'$, is not a proposition. (This is often the position taken by the proponents of the Theory of Types (page 202). Since the expression, "I am now lying," is apparently neither true or false, or is both, it cannot be a proposition, i.e., a meaningful statement.)

Leibnitz thought of mathematics as a science built solely on definition. According to him, such a science was the most perfect possible, for it depends only on the Law of Contradiction for the validity of its postulates; that is, were one to attempt to deny one of the postulates, he would contradict himself, since the postulates are merely the definitions of the concepts involved. Just as it is contradictory to assert that a triangle does not have three sides, since the definition of triangle implies three-sidedness, so it is contradictory to assert that $2 + 2 \neq 4$, for the definitions of 2, 4, and "+" imply $2 + 2 = 4$.

There is no unanimity of opinion today as to whether Leibnitz was right about mathematics or whether anyone is right in asserting that logic is altogether a science of "tautologies." From our point of view here, it is only important to point out that not all deductive systems have this character, and such systems are not dependent solely on the criterion of consistency alone for their verification (cf. chapter VIII). Postulate sets are given here for parts of arithmetic, geometry, and mechanics; the latter two are certainly nondefinitional in character. This order follows the scheme of the science suggested in chapter VII, but omits certain sciences. The calculus of propositional functions would presumably fall between logic, as we have defined it, and arithmetic.

The following set of postulates for arithmetic is derived from the set of Peano.[4] They are a set of postulates for whole, positive numbers. Actually, Peano, by suitable definitions, developed an arithmetic for all rational numbers.

Here the only presupposed science is logic. As indefinables, the following concepts are taken:

1. A class of objects, symbolized N_0, one interpretation of which is the set of positive, integral numbers and 0.
2. The element (or "number") 0.
3. The element (or "number") 1.
4. The operation of "adding" two elements, symbolized $a + b$.
5. The operation of "multiplying" two elements, symbolized $a \cdot b$, or ab.
6. The sign "=," which obeys the usual formal properties of the equality relation; cf. Axiom 4, page 269.

Some definitions may now be made:

1. $2 = 1 + 1$
2. $3 = 2 + 1$
3. $4 = 3 + 1$, etc., etc.

Our postulates are:

1. 0 belongs to N_0.
2. If a belongs to N_0, then $a + 1$ belongs to N_0.
3. $a + 0 = a$ (for all elements of N_0)
4. $a + (b + 1) = (a + b) + 1$.
5. If $a + 1 = b + 1$, then $a = b$ (for all elements of N_0)
6. If a belongs to N_0, then $a + 1 \neq 0$.
7. $a \cdot 0 = 0$ (for all elements of N_0)
8. $a \cdot (b + 1) = (a \cdot b) + a$ (for all elements of N_0)
9. Suppose a certain proposition containing a (e.g., $a + b = b + a$) holds in general if a is 0; suppose also that if when the proposition is true for a certain element x of N_0, it must also be true for the next element $x + 1$. Then we say that the proposition is true in general. (This is the Principle of Mathematical Induction; cf. page 31. Despite its logical character, the assumption as here stated is an arithmetical one, since, as is shown in the exercises at the end of the chapter, there are perfectly consistent, i.e., logically correct, systems for which this postulate does not hold. Mathemati-

[4] *Formulaire de mathematique,* 1894.

cal induction is an example of a "nonlogical" method of proof which a given science can use because of the peculiar structure of the system; geometry also introduces a new method through its constructions; cf. page 126.)

Several illustrations of the method of proving theorems by means of this set will suffice:

THEOREM 1. If a and b belong to N_0, then their addition, $a + b$, belongs to N_0.

This statement may be considered as a statement about the element b, and is true when b is 0, for then $a + b = a + 0 = a$ and the result belongs by hypothesis to N_0. Suppose that the statement is true for a certain element of N_0, x; that is, suppose that if a and x belong to N_0, then $a + x$ belongs. We now show that on this assumption $a + (x + 1)$ must belong to N_0. For if $a + x$ belongs to N_0, then $(a + x) + 1$ belongs by Postulate 2. But $(a + x) + 1 = a + (x + 1)$ (Postulate 4), and hence the latter belongs to N_0. The premises of Postulate 9 are now satisfied, and we can infer that the statement is true for all elements of N_0. This theorem, when N_0 is interpreted as suggested above, becomes: "If a and b are whole positive numbers, or 0, then $a + b$ is a whole positive number, or 0."

THEOREM 2. $a + (b + c) = (a + b) + c$. (The "associative law" for addition.)

This is true if $c = 0$ (Postulate 3). Suppose, now, that for a certain x, $a + (b + x) = (a + b) + x$. Then the statement will be true for $x + 1$: $a + [b + (x + 1)] = (a + b) + (x + 1)$. For $(a + b) + (x + 1) = [(a + b) + x] + 1 = [a + (b + x)] + 1 = a + [(b + x) + 1)] = a + [b + (x + 1)]$.

THEOREM 3. $0 + a = a$.

The verification of this for $a = 0$ follows from Postulate 3. If $0 + x = x$, then $0 + (x + 1) = (0 + x) + 1 = (x + 1)$.

(See Exercises, Group B, at end of chapter.)

The deductive system of geometry has already been given. An excellent set of postulates for Euclidean Geometry is found in D. Hilbert's *Foundations of Geometry;* this set avoids the ambiguities and omissions which are to be found in Euclid's set, and is of historical interest in that it is one of the first sets of postulates for which independence proofs were devised.

Between the sciences of geometry and physics should come kinematics, which introduces the concept of motion (or the concept of time). Among the important problems of this science is the question as to whether motion is continuous or not; two distinct formal systems arise depending on whether continuity of path is postulated or lack of continuity is postulated. As in the case of different formal systems of geometry, and arithmetic, the fact that there are at least two formal systems of kinematics will have its effects on all later sciences such as physics, chemistry, astronomy, so that two systems of physics may have the same postulates for physics but differ in that the one chooses the postulate of continuity, the other does not.

Few are inclined to consider the science of physics as a formal science, for physicists seem to spend the greater part of their time in experimentation. This is not strictly the case. Among those working in the field are the "theoretical physicists"; these scientists are interested in developing certain hypotheses, suggested by experiments, in order to see what consequences may be drawn from them. But this is exactly the procedure of the formal scientist. Granting these assumptions, which are taken to be consistent and independent, the formal scientist asks what must follow.

It is true that the physicist does not give way to the mathematician's temptation to investigate systems which appear to have no application. If experiment apparently refutes his theory, his interest in the formal science he has developed ceases; that is, he does feel obliged to restrain his theory within the domain prescribed by experiment. But these restrictions, though they may prevent his investigating all possible formal systems of physics, do not prevent his becoming a formal scientist, since his primary interest still lies in formulating postulates and deducing theorems. Further, since experiment by its very nature can never be conclusive, there is always open more than one possible formal science of physics which conforms with experimental data.

Historically, physicists have developed many formal systems of physics, the most famous of which, perhaps, are those of Democritus, Aristotle, Newton, and Einstein.

The science of physics embraces so much that it would require volumes to develop any complete formal system, if there be any such. The discussion here is confined to the science of mechanics, and in particular, to the mechanics of a single particle; though this science is usually classified as a part of physics, it might be better in a general classification to make the former a presupposition, and not a part of the latter (cf. page 136).

As usual, we must indicate what sciences are presupposed, i.e., what laws and terms we may use. The science of logic, arithmetic (number), and geometry (distance) are all presupposed; in the case of geometry, we assume the Euclidean laws in this treatment. Further, we assume the additional concept, time, introduced by kinematics, and also the formal science of kinematics which asserts that motion is continuous. (The concepts of velocity, acceleration, etc., are all kinematical terms.)

The mechanics of a single particle considers the problem of the laws governing the motion of a particle or point in space. Just as geometry introduces the concept of a line and attributes to every line a certain number (its measure, or distance), so we apply here to the geometrical concept of a point either a number or a quality. The latter method leads to a non-quantitative mechanics, the best known example of which is Aristotle's physics. If we call these new points introduced by mechanics "particles of matter," then Aristotle may be said to have added four "indefinables." To every particle of matter we can attribute two of the following four qualities: "hot," "cold," "wet," and "dry." We can define what is meant by a "particle of fire" in terms of these indefinables: "A particle of matter is said to be 'fire' if it is hot and dry." The particle is water if it is cold and wet; earth if cold and dry; air if hot and wet; the remaining possibilities are self-contradictory: no particle can be both hot and cold, or both wet and dry. Aristotle takes the earth to be the ordinate of his (spherical) system of coordinates. Then, excluding the study of celestial motion which for Aristotle forms a separate science, we have the following postulate:

1. Every particle of matter tends to a certain location (its "proper place"), depending on its kind, unless otherwise impeded by an external force.

2. The proper place of dry and cold particles (earth) is the center of the system.

3. The proper place of water is a position adjacent to earth.

4. The proper place of air is a position adjacent to water.

5. The proper place of fire is a position adjacent to air (i.e., the "outer edge" of the world).

(Note: These assumptions introduce other physical terms not mentioned among the indefinables, namely, "tends," "impeded," "force.")

The Newtonian system of mechanics is a quantitative one. It adds to every particle, not a quality, but a certain quantity, and calls this quantity "mass." The fundamental concepts of mechanics can now be defined in terms of three numbers: length (l), a concept of geometry, time (t), a concept of kinematics, and mass (m), the new concept of Newtonian mechanics. The following scheme gives the required definitions: [5]

1. Velocity $(v) = l/t$
2. Acceleration $(a) = l/t^2 = v/t$
3. Force $(F) = ml/t^2 = ma$
4. Momentum $(u) = ml/t = mv$
5. Work $(W) = ml^2/t^2 = Fl$
6. Kinetic Energy $(K) = ml^2/2t^2 = mv^2/2$

One of the postulates of Newton's original system is included among these definitions, for (3) is Newton's second law; it is always arbitrary in a formal system whether or not we choose to make a certain concept an indefinable and call its definition a postulate. If the physicist finds that force and mass may be considered as independent of one another in some sense, then he may find it more convenient to take F as indefinable and assert (3), or something like it, as a postulate.

The fundamental postulates of Newtonian mechanics follow:

1. (Newton's First Law) Every body tends to remain in a state of rest or of uniform motion in a straight line, unless acted upon by external force to change that state.

2. (Universal Law of Gravitation) The force acting between any two particles of matter is proportional to the product of

[5] Cf. E. Mach, *The Science of Mechanics.* Mach, following Fourier, takes these indefinables as "dimensions."

their masses[6] divided by the square of the distance between them:

$$F = G\frac{m_1\ m_2}{r^2}$$

Newton's Second and Third Laws follow from these postulates and the above definitions:

THEOREM 1. The force acting on any particle is equal to the change of momentum per second:

$$F = u/t$$

THEOREM 2. The force acting between particle A and particle B is equal to the force acting between particle B and particle A.

This follows from Postulate 2, since

$$G\frac{m_1 m_2}{r^2} = G\frac{m_2 m_1}{r^2}$$

Among the most remarkable theorems in Newtonian mechanics were Kepler's laws of planetary motion. The deduction of these illustrates the powerful instrument mathematics becomes in the hands of the physicist. Sciences not presupposing mathematics are necessarily somewhat simple in their method of deducing theorems, but if the whole of mathematics can be presupposed, the method of deduction becomes much more complicated and at the same time many more theorems follow from a given number of assumptions. Kepler's Law that the paths of the planets traveling about the sun are ellipses follows from Newton's second postulate.

Newton's laws actually correct Kepler's, in that the planets do not really travel in ellipses since the forces of attraction of other planets cause perturbations from this path. The Newtonian postulate allows one to calculate these perturbations, given the planets, and, conversely, given the perturbations, to calculate the position and mass of the planets causing them. The latter fact has been an invaluable aid in discovering new planets.

[6] Whether these "masses" are to be identified with the mass taken above as an indefinable and used to define "force," is undetermined in Newtonian mechanics; the assumption that the two are the same is a fundamental postulate of Relativistic mechanics.

The formal system of Newton is not regarded as universally valid by most physicists today; new postulates have been devised, new formal systems constructed. But the general method of constructing formal or deductive systems necessarily follows the scheme outlined in this text.

EXERCISES

GROUP A

1. The following sets of objects are all series; determine a relation which makes them so:

   a) the class of all points on a finite line segment
   b) the class of all points in a square, or the class of all points in space
   c) the class of all proper fractions; find a relation "<" other than "is less than"
   d) the class of all living men

2. Prove that every series must have the following properties (i.e., prove the following theorems from Postulates 1-3 for a series):

   a) If $a < b$, then $b > a$ is false.
   b) Either $a = b$ or $a < b$ or $b < a$.

3. What is the definition of the "first" element of a series in terms of "<"; of the "last"; define "x lies between a and b"; "a immediately precedes b."
4. Not every series has a first element. Give examples to show that this is true. What postulate must be added to the postulates for series which will define "first element" series. Why will this postulate be "independent" of the rest?
5. In some series no element has an "immediate predecessor." Give examples. Such series are called "dense" series. Give the additional postulate necessary to define dense series.
6. Are the following examples series or not?

   a) K is the class of the Presidents of the U. S.; "<" means "precedes in time of taking office."
   b) K is the class of all points (x, y) in a given square. $(x_1, y_1) < (x_2, y_2)$ if and only if x_1 is less than x_2, and y_1 is less than y_2.
   c) K is a family of brothers; $a < b$ means "a is the brother of b."
   d) K is a Boolean algebra. "$a < b$" is any relation of the form $Aab + Bab' + Ca'b + Da'b' = 0$. (Cf. Exercise 1 *g*), page 286.)

7. *a*) Prove that for every series there exists a relation of betweenness; i.e., given the three postulates for series, we can define the relation of betweenness in terms of "precedes" so that all the postulates for betweenness can be deduced. (When K has less than three members, the postulates for betweenness become true "vacuously," i.e., the premises in each case are false and hence imply anything.)
   b) Prove that every set of objects for which the relation of betweenness can be defined is a series.

8. Demonstrate the consistency and independence of the postulates for betweenness given above by means of the following examples:

   a) K is the set of rational numbers; AXY means "A is less than X and X is less than Y."
   b) K contains three elements, the numbers 1, 2, 3. ABC holds only in the following three cases: 111, 123, 321.
   c) K contains 4 elements, 1, 2, 3, 4. ABC is true for 124, 134, 213, 243, 312, 342, 421, 431.
   d) K is the class of real numbers. ABC means $A < B$, $B < C$, and $C < A$. (Remember that a false proposition implies any proposition.)
   e) K consists of four elements, 1, 2, 3, 4. ABC is true for 123, 142, 241, 243, 321, 341, 342.
   f) K is a class having but one member. ABC means that A, B, and C are all distinct.
   g) K is a class containing one member. ABC means $A = B = C$.
   h) K is a class containing three members. ABC means that A, B, and C are all distinct.

9. Give three other examples which demonstrate the consistency of the postulates for betweenness.
10. Prove the following theorems from the postulates for betweenness (A, X, Y, B are all distinct):

    a) If AXB and AYB, then either AXY or YXB.
    b) If XAB and AYB, then XYB.

11. Show the independence and consistency of the postulates for a group.
12. Prove that the following set of postulates for a group are equivalent to those given above:

    a) If a and b are in K, then aob is in K.
    b) $ao\,(boc) = (aob)\,oc$.
    c) The equations $aox = b$ and $yoa = b$ have solutions in K.

13. Prove that the commutative law, $aob = boa$, is not a necessary property of a group by showing that it is independent of the above postulates. A group for which the commutative law holds is called an "abelian" group.
14. Is the Boolean algebra a group with respect to "$\triangle$"? (Cf. page 265.) With respect to "o"?
15. Give a set of postulates defining the relation "is the cause of"; one defining the relation "is a proper part of," i.e., is a part not equal to the whole.

GROUP B

1. Prove the following theorems from Peano's set:

   a) $1 + a = a + 1$
   b) $a + b = b + a$ ("commutative law" for addition)
   c) If $a + c = b + c$, then $a = b$.
   d) If a and b belong to N_0, then ab belongs to N_0.

   (Note that we cannot prove a corresponding theorem for $a - b$, if this concept is introduced, or for a/b, for if $a = 4$ and $b = 5$, then neither $a - b$ or a/b belong to N_0.)

   e) $0a = 0$
   f) $a1 = a$
   g) $a(b + c) = ab + ac$
   $(a + b)c = ac + bc$ (The "distributive laws" for multiplication with respect to addition.)
   h) $ab = ba$ ("commutative law" for multiplication)

2. Determine which of the above postulates for arithmetic are proved independent by the following examples:

   a) N_0 is the class of rational positive numbers and 0; 0, 1, $a + b$, ab are defined as in ordinary algebra.
   b) N_0 represents a class of two elements, 0 and 1. $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, but $1 + 1 = 2$, where 2 is an element not in N_0. $00 = 10 = 01 = 0$, and $1 \cdot 1 = 1$.
   c) N_0 is the class of all whole numbers less than or equal to 0. $a + b$ now means $a - b$. 0, 1, ab defined as usual.
   d) N_0 is the class containing only the two numbers, 1 and 0. ab defined as usual; $0 + 0 = 0$; $0 + 1 = 1 + 0 = 1 + 0 = 1 + 1 = 1$. (That is, N_0 is a two-valued Boolean algebra.)
   e) This example is called a "periodic arithmetic"; we use this arithmetic in calculating dates on a calendar. N_0 is a "peri-

odic" system, that is, a class of elements, $0, 1 \ldots, n$, where $n + 1 = 0$. (The days of the month, the hours of the day, form such systems, for in counting dates we return to 1 again after reaching 30 or 31.) $0, 1, ab, a + b$ (except for the property named) have their usual properties. Thus, if the periodic system has 30 elements, $23 + 10 = 3$, and $5 \cdot 8 = 10$, etc.

f) N_0 is the class of positive, whole numbers plus 0. $a + b$ is defined as usual; $ab = b$.

g) N_0 is the class of all whole numbers greater than 0. Here the symbols 0 and 1 are equal and both represent the number 1. ab represents the addition of 1 and the quantity $(b - 1)$ times a, where "times" means the usual multiplication.

h) N_0 is a null class, i.e., a class with no members. The symbols 0,1, represent existent things; the definition of ab and $a + b$ is arbitrary. Or, N_0 is the class of all whole numbers greater than 0; $0, 1, ab$, $a + b$ defined as usual.

i) N_0 is the class of all whole, positive numbers and 0; $a + b$, 0,1, are defined as usual. $a0 = 1$ (and hence $a1 = 1 + a$, $a2 = 1 + a + a$, etc.; that is, ab represents the sum of 1 and b times a.)

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