All things considered I estimate that the experimental results may be in error by four per cent.

The table exhibits the results of the investigation. In it the ratio of the first maximum to the succeeding minima and maxima as computed by the method of Fresnel (Mascart, *Traité d'Optique,* Vol. I, p. 280) are compared with the same ratios as determined by experiment.

The agreement between theory and experiment appears to be within the limit of error.

TABLE 1

	EXPERIMENT					
THEORY	1	2	3	4	5	THEORY
1.762	1.76	1.82	1.70	1.70	1.78	1.762
1.145	1.16	1.16	1.16	1.15	1.14	1.145
1.625	1.61	1.68	1.59	1.57	1.62	1.625
1.191	1.21	1.22	1.21	1.19	1.16	1.191
1.572	1.55	1.59	1.53	1.53	1.57	1.572

I am indebted to my colleague, Dr. Crawford, for the microphotometer records and to my assistant, Mr. H. W. Leighton, for skillful experimental manipulation.

NOTE ON RANGE-FINDING

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1. *The Problem on a Plane.*—The problem is the location of an object such as a gun P by timing a sound or other vibration of known velocity at different stations P_i $(i = 1, 2, \ldots, n)$. The time for a fixed longitude recorded at P_i multiplied by the known velocity is a known distance d_i. Denote the distance between P and P_i by r_i. Then for any two stations P_1 and P_2, we have

$$r_1 - r_2 = d_1 - d_2,$$

so that

$$r_i = d_i + k \ (i = 1, 2, \ldots, n). \qquad (1)$$

For three stations we have, then, in determinant form,

$$\left| r_i^2 - d_i^2, d_i, 1 \right| = 0. \qquad (2)$$

This is the equation of a line on the gun.

If we were restricted to three stations, the problem would be to find the centers of the two circles which touch properly three given directed circles. The three circles here are the circles (P_i, d_i) and from (1) k is the radius of the touching circle. To identify the line, we notice that (2) is true when

$$r_1^2 - d_1^2 = r_2^2 - d_2^2 = r_3^2 - d_3^2,$$

that is, it is true for the radical center. Again it is true when

$$(r_1^2 - d_1^2)/d_1 = (r_2^2 - d_2^2)/d_2 = (r_3^2 - d_3^2)/d_3. \qquad (3)$$

For two directed circles (P_1, d_1) and (P_2, d_2) there is one system of properly tangent circles, all being orthogonal to the circle

$$(r_1^2 - d_1^2)/d_1 = (r_2^2 - d_2^2)/d_2.$$

This then is the circle of an inversion which sends each tangent circle into itself. For the three circles (3) there are three such circles, and the line (2) is their radical axis. It is the line discussed in article 118 of Salmon's *Conic Sections*.

2. *Coördinates of the Gun.*—It is proper to take four stations for the planar problem. Using rectangular coördinates, let P_i be the point (x_i, y_i). Then for three stations, the line (2) is

$$|(x - x_i)^2 + (y - y_i)^2 - d_i^2, d_i, 1| = 0$$

or

$$2x|x_i, d_i, 1| + 2y|y_i, d_i, 1| = |x_i^2 + y_i^2 - d_i^2, d_i, 1|.$$

For four stations we have four such equations. To eliminate y we multiply each by the omitted y_i and so build

$$|y_i, y_i, d_i, 1|$$

which is zero. We have then

$$2x|x_i, y_i, d_i, 1| = |x_i^2 + y_i^2 - d_i^2, y_i, d_i, 1|, \qquad (4)$$

giving the x-coördinate of the gun. For the y-coördinate we interchange x and y.

3. *Elimination of Temperature.*—The velocity of sound is a function of the temperature. If we assume that the function is linear, it is proper to take five stations. We have to get rid of

$$|d_i^2, y_i, d_i, 1|,$$

so that from the five equations (4) we build

$$|d_i^2, d_i^2, y_i, d_i, 1|$$

which is zero. We have then

$$2x|x_i, y_i, d_i^2, d_i, 1| = |x_i^2 + y_i^2, y_i, d_i^2, d_i, 1|, \qquad (5)$$

which being homogeneous in the d_i needs no correction for the temperature.

4. *The Problem on a Space.*—For three stations on a space (3) is

$$2x \mid x_i, d_i, 1 \mid + 2y \mid y_i, d_i, 1 \mid + 2z \mid z_i, d_i, 1 \mid = \mid x_i^2 + y_i^2 + z_i^2 - d_i^2, d_i, 1 \mid.$$

For four stations on a space we eliminate y, as above, and have

$$2x \mid x_i, y_i, d_i, 1 \mid + 2z \mid z_i, y_i, d_i, 1 \mid = \mid z_i^2 + y_i^2 + z_i^2 - d_i^2, y_i, d_i^2, 1 \mid.$$

For five stations on a space we have five such equations. Multiplying by the omitted z_i and forming

$$\mid z_i, z_i, y_i, d_i, 1 \mid,$$

we have

$$2x \mid x_i, y_i, z_i, d_i, 1 \mid = \mid x_i^2 + y_i^2 + z_i^2 - d_i^2, y_i^2, z_i^2, d_i^2, 1 \mid, \qquad (6)$$

giving a typical coördinate of the gun. As in Article 3, this may be made independent of the temperature by the use of six stations.

5. *The Problem on a Sphere.*—When the vibration travels along the surface of a sphere (as in a tidal wave to a first approximation) the equations (1) refer to the lengths of arcs of great circles. We have then

$$\cos r_i = \cos d_i \cos k - \sin d_i \sin k. \qquad (7)$$

The cosine is necessary, because for constants m_i a circle on the sphere is given by

$$\Sigma m_i \cos r_i = m_0.$$

In Salmon's *Geometry of Three Dimensions*, chapter X, the sine of the arcual distance from a variable point to a great circle is employed. But here it is more convenient to think of this as the cosine of the arc from a variable to a fixed point. We have from (7) for three stations

$$\mid \cos r_i, \cos d_i, \sin d_i \mid = 0, \qquad (8)$$

a great circle on the source of the wave.

It is worth while to mention that the circles of inversion (3) of the planar case are here replaced by

$$(\cos r_i - \cos d_i)/\sin d_i = (\cos r_2 - \cos d_2)/\sin d_2 = (\cos r_3 - \cos d_3)/\sin d_3.$$

Let now the longitude and colatitude of the point P_i be a_i and b_i, those of the source a and b. Then

$$\cos r_i = \cos b \cos b_i + \sin b \sin b_i \cos (a - a_i),$$

so that (8) becomes

$$\begin{aligned} &\cos b \mid \cos b_i, \cos d_i, \sin d_i \mid \\ &\quad + \sin b \cos a \mid \sin b_i \cos a_i, \cos d_i, \sin d_i \mid \\ &\quad + \sin b \sin a \mid \sin b_i \sin a_i, \cos d_i, \sin a_i \mid = 0. \end{aligned}$$

We have then for four observations

$$\cot a \left| \cos b_i, \sin b_i \cos a_i, \cos d_i, \sin d_i \right| + \left| \cos b_i, \sin b_i \sin a_i, \cos d_i, \sin d_i \right| = 0, \qquad (9)$$

and

$$\cot b \left| \cos b_i, \sin b_i \cos a_i, \cos d_i, \sin d_i \right| + \cos a \left| \sin b_i \cos a_i, \sin b_i \sin a_i, \cos d_i, \sin d_i \right| = 0, \qquad (10)$$

giving the longitude and the colatitude of the source.

APPLICATIONS OF AN EXPANSION THEOREM TO THE DEVELOPMENT OF THE DISTURBING FUNCTION

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1. The simplest form of the theorem is stated as follows. If $f(a + \alpha p)$ be a function developable in powers of α and if $D \equiv \alpha\partial/\partial\alpha$, then

$$f(a + \alpha p) = p^D f(a + \alpha). \qquad (1.1)$$

It is assumed that p^D is developed in integral powers of D and, after development, is applied to $f(a + \alpha)$.

The proof is simple. If n be an integer,

$$D^n \alpha^m = m^n \alpha^m.$$

Hence, if $\Psi(D)$ be a function of D developable in integral powers of D,

$$\Psi(D)\alpha^m = \Psi(m)\alpha^m. \qquad (1.2)$$

Suppose $\Psi(m) = p^m$. Then

$$p^m \alpha^m = p^D \alpha^m. \qquad (1.3)$$

Hence, by Taylor's theorem,

$$f(a + \alpha p) = \Sigma_m \frac{p^m \alpha^m}{m!} f^{(m)}(a) = p^D \frac{\Sigma \alpha^m}{m!} f^{(m)}(a) = p^D f(a + \alpha).$$

2. Examples. Let

$$p = (1 + q)^j,$$

where $|q| < 1$ and j is any quantity. Then

$$p^D = (1 + q)^{jD} = 1 + jD.q + \frac{jD(jD - 1)}{1.2} q^2 + \ldots$$