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ON THE POSSIBLE LINE ELEMENTS FOR THE UNIVERSE

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1. *Introduction.*—As is well known, two different line elements, both compatible with general relativity, have been proposed respectively by Einstein and by de Sitter for the treatment of the large-scale behavior of the universe as a whole. Since neither of these line elements appears to account in any very direct or inevitable way for our present observational facts with regard to the distances and Doppler shifts for the extra-galactic nebulae,[1] it is interesting to inquire whether these are the only possible line elements that could be proposed as applicable to cosmological behavior.

The present investigation will show that these two line elements, together with that given by the special theory of relativity, are as a matter of fact the only ones which are compatible with those assumptions as to the physical nature of the universe which it would seem most natural to make, and with the general theory of relativity. The possibility, however, of modifying the physical assumptions so as to permit a different type of line element will be mentioned. Opportunity will also be afforded by the article to make clear the range of circumstances which would agree with these different known line elements.

2. *The Conditions to Be Satisfied.*—As the first condition to be satisfied by the line element, we shall require it to be compatible for a limited region in space and time with the special theory of relativity and hence in polar coördinates to reduce to the form

$$ds^2 = -dr^2 - r^2d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2 \qquad (1)$$

for small values of r, and at least for a limited range in time. The justification for this requirement is afforded by our experimental knowledge of the validity of the special theory of relativity for a limited space-time region, provided we neglect the effect of local gravitational fields, as we must obviously do in obtaining a line element for large-scale cosmological applications.

As the second condition for the line element we shall require the possibility of writing it in a form which is spherically symmetrical in the spatial variables, symmetrical with respect to past and future time, and static with respect to the time. These requirements, as is well known, lead necessarily to the form,

$$ds^2 = -e^{\lambda}dr^2 - r^2d\theta^2 - r^2 \sin^2 \theta + e^{\nu}dt^2, \qquad (2)$$

where λ and ν are functions of r alone. The requirement of spherical symmetry is an obvious one to impose, since otherwise the universe regarded on a large scale would have different properties in different directions. The requirement of symmetry with respect to past and future time means that the large-scale behavior of the universe is reversible, and the static form of the line element means that by and large the universe is in a steady state. These also are natural requirements to make, but they are not inevitable and we shall briefly return to them later.

The final conditions to be imposed on the line element arise from the fundamental relation given by the general theory of relativity connecting the metrical properties of space-time with the distribution of matter. This relation is given by the equation

$$-8\pi T_{\mu\nu} = G_{\mu\nu} - {}^1/_2 G g_{\mu\nu} + \Lambda g_{\mu\nu} \qquad (3)$$

where $T_{\mu\nu}$ is the material energy tensor, $G_{\mu\nu}$ the contracted Riemann-Christoffel tensor, G its associated invariant, $g_{\mu\nu}$ the fundamental metrical tensor and Λ Einstein's cosmological constant. In applying this equation to cosmological considerations, our scale of interest becomes so large compared even with astronomical objects, that we shall regard the matter in the universe as a perfect fluid, and hence can write for the material energy tensor the known expression

$$T^{\mu\nu} = (\rho_{00} + p_0) \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} - g^{\mu\nu} p_0 \qquad (4)$$

where ρ_{00} is the proper macroscopic density of the fluid, p_0 its proper pressure, and the quantities dx_μ/ds are macroscopic "velocities." The further conditions now to be imposed are that ρ_{00} and p_0 are to be constant throughout the universe, and that the spatial velocities are to be zero. These final requirements are also evidently natural ones to impose. The quantities ρ_{00} and p_0 are the macroscopic density and pressure as they would be measured by local observers, so that the first of the above requirements is merely the statement that the material in the universe shall be homogeneous when regarded from a large scale point of view. And the additional requirement that the material in the universe shall be stationary in the coördinates chosen is necessitated by the static nature which has already been assigned to the universe by the form of line element assumed in equation (2).

3. *Dependence of ρ_{00} and p_0 on λ and ν.*—We may now proceed to the detailed calculation of the result of the conditions that we have imposed.

The line element given by equation (2) is of a standard form, and expressions for the quantities $G_{\mu\nu} - {}^1/_2 G g_{\mu\nu}$ corresponding to this line element have already been calculated.[2] By substituting these expressions in equation (3) we obtain as the only surviving components of the energy tensor,

$$\begin{aligned} -8\pi T_{11} &= -\nu'/r - (1 - e^{\lambda})/r^2 - e^{\lambda}\Lambda \\ -8\pi T_{22} &= -r^2 e^{-\lambda}\left\{\nu''/2 - \lambda'\nu'/4 + \nu'^2/4 + (\nu' - \lambda')/2r\right\} - r^2\Lambda \\ -8\pi T_{33} &= -8\pi T_{22}\sin^2\theta \qquad (5) \\ -8\pi T_{44} &= e^{\nu-\lambda}\left\{-\lambda'/r + (1 - e^{\lambda})/r^2\right\} + e^{\nu}\Lambda \end{aligned}$$

where the accents denote differentiation with respect to r.

On the other hand, in our case the components of the energy tensor are also given by equation (4). Making use of the values of the metrical tensor given by our line element, introducing our requirement that the spatial velocities dr/ds, $d\theta/ds$, and $d\phi/ds$ are to be zero, and lowering suffixes twice, we easily find that equation (4) goes over into the expressions.

$$T_{11} = e^{\lambda}p_0, \quad T_{22} = r^2 p_0, \quad T_{33} = T_{22}\sin^2\theta, \quad T_{44} = e^{\nu}\rho_{00} \qquad (6)$$

Combining equations (5) and (6) and noting that the second and third resulting equations are identical, we obtain

$$8\pi p_0 = e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 + \Lambda \qquad (7)$$

$$8\pi p_0 = e^{-\lambda}\left\{\nu''/2 - \lambda'\nu'/4 + \nu'^2/4 + (\nu' - \lambda')/2r\right\} + \Lambda \qquad (8)$$

$$8\pi \rho_{00} = e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 - \Lambda. \qquad (9)$$

We note that there are two independent restrictions placed on the proper pressure by these equations. This results from our assumption of a perfect fluid, which brings with it the necessity for equal stresses parallel and perpendicular to the radius vector.

The second of the two pressure equations gives, however, a very complicated dependence on the metrical variables λ and ν, which makes it desirable to change the form of these equations before proceeding. To obtain the desired transformation, we subtract equation (7) from (8) and write

$$e^{-\lambda}\left\{\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right\} - e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \qquad (10)$$

Multiplying through by $2/r$ and changing the order of terms, this can be rewritten in the form

$$e^{-\lambda}\left\{\frac{\nu''}{r}-\frac{\nu'}{r^2}-\frac{2}{r^3}\right\}-e^{-\lambda}\lambda'\left\{\frac{\nu'}{r}+\frac{1}{r^2}\right\}+\frac{2}{r^3}+e^{-\lambda}\left(\frac{\lambda'}{r}+\frac{\nu'}{r}\right)\frac{\nu'}{2} = 0. \qquad (11)$$

It will be noticed, however, that the first three terms of this new equation can be obtained by differentiating equation (7) with respect to r, while the last term contains the sum of equations (7) and (9). This permits us to rewrite (11) in the simple form

$$8\pi\frac{dp_0}{dr} + 8\pi(\rho_{00} + p_0)\frac{\nu'}{2} = 0. \qquad (12)$$

And this makes it desirable to replace equations (7), (8) and (9) by the equivalent set

$$\rho_{00} = \frac{e^{-\lambda}}{8\pi}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{8\pi r^2} - \frac{\Lambda}{8\pi} \qquad (13)$$

$$p_0 = \frac{e^{-\lambda}}{8\pi}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{8\pi r^2} + \frac{\Lambda}{8\pi} \qquad (14)$$

$$\frac{dp_0}{dr} = -\frac{\rho_{00} + p_0}{2}\nu', \qquad (15)$$

which give in a very simple and convenient form the relation which connects the density and pressure of matter with the space-time metric.

4. *The Result of Imposing the Remaining Conditions.*—In the process of obtaining equations (13), (14) and (15), it will be noted that we have already imposed all the conditions discussed in section 2 except three. These are the requirements that the line element shall reduce for small values of r to the form given by equation (1) valid in the special theory of relativity, and that the proper density and pressure shall be constant throughout the universe. It is now easy to apply these further conditions.

An examination of equation (15) immediately shows that the requirement of uniform pressure can only be met by setting either $(\rho_{00} + p_0)$, or ν', or both, equal to zero. This leads directly to three alternative cases to which we can give separate treatment.

Case I. Let us first consider the case given by

$$\rho_{00} + p_0 = 0. \qquad (16)$$

In accordance with equations (13) and (14) this immediately leads to

$$\lambda' = -\nu', \qquad (17)$$

or since λ and ν must both become zero at $r = 0$ in order for the line element to reduce in the neighborhood of the origin to the special relativity form given by equation (1), we may write at once

$$\lambda = -\nu. \tag{18}$$

On the other hand applying the remaining condition that ρ_{00} is a constant, we can immediately integrate equation (13) and easily obtain, as can be verified by differentiation,

$$e^{-\lambda} = 1 - \frac{\Lambda + 8\pi\rho_{00}}{3} r^2 + \frac{A}{r}, \tag{19}$$

where A is a constant of integration, and this constant must evidently be equal to zero if the line element is to reduce to the special relativity form for $r = 0$.

Hence writing for convenience

$$\frac{\Lambda + 8\pi\rho_{00}}{3} = \frac{1}{R^2}, \tag{20}$$

we obtain for the line element from equations (2), (18) and (19) the expression

$$ds^2 = -\frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + (1 - r^2/R^2)dt^2, \tag{21}$$

which will be immediately recognized as one of the known forms for the de Sitter line element.[3]

Case II. Let us now turn to the second possible case which is given by

$$\nu' = 0. \tag{22}$$

Integrating and remembering that the line element must reduce to the form (1) for small values of r we at once obtain

$$\nu = \text{const.} = 0. \tag{23}$$

On the other hand substituting (22) in (14) and solving we at once obtain

$$e^{-\lambda} = 1 - (\Lambda - 8\pi\rho_0)\, r^2. \tag{24}$$

Hence writing for convenience

$$\Lambda - 8\pi p_0 = \frac{1}{R^2}, \tag{25}$$

we obtain for the line element from equations (2), (23) and (24)

$$ds^2 = -\frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + dt^2, \tag{26}$$

which will be immediately recognized as one of the known forms for the Einstein line element.[4]

Case III. Finally, let us consider the case in which we have both

$$\rho_{00} + p_0 = 0 \qquad \text{and} \qquad \nu' = 0. \tag{27}$$

Under these circumstances we shall have equation (18) from Case I and equation (23) from Case II both true, which gives at once

$$\lambda = \nu = 0. \tag{28}$$

And the line element reduces completely to the special relativity form

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2, \tag{29}$$

which is also a special case of the other two line elements obtained by taking $R = \infty$.

We thus see, as stated at the beginning of the article, that the only cosmological line elements which are compatible with the conditions that were imposed, are those of de Sitter, of Einstein, and of the special theory of relativity. Let us now investigate somewhat further the relation of these three line elements to the quantities ρ_{00}, p_0, and Λ.

5. *The de Sitter Line Element.*—To obtain the de Sitter line element we had to place the sum, $\rho_{00} + p_0$, equal to zero. The proper density ρ_{00} is, however, from its nature necessarily either zero or a positive quantity. Hence, unless we are willing to assume that the idealized fluid which fills the universe is endowed with sufficient cohesion to support a relatively enormous negative pressure, we can obtain the above result only by setting density p_{00} and pressure p_0 each individually equal to zero. We are thus led to the conclusion that the de Sitter universe would be a substantially empty one, any matter actually in it merely producing a local disturbance away from the general form of line element.

Introducing the condition $\rho_{00} = 0$ into equation (20), we now obtain the usual de Sitter expression

$$\Lambda = \frac{3}{R^2} \tag{30}$$

connecting the cosmological constant Λ with the "radius" of the universe R.

Since the field equations (3) provide a correct description of planetary orbits, without the introduction of the cosmological term, the quantity Λ must be small enough so that the term has no appreciable effect until we consider distances greater than the radii of these orbits. Furthermore Λ must be a quantity which is independent of the coördinates, since the divergence of the left-hand side of (3) is taken as equal to zero because of the known facts as to the behavior of energy and momentum, and the divergence of the right side is then to be taken as vanishing identically. There seems to be no *a priori* reason in the case of the de Sitter universe, however, which would make Λ necessarily a positive quantity. If Λ is positive, R is real and we have the usual closed de Sitter universe. If Λ were negative, on the other hand, R would be imaginary and the universe would not be closed.

6. *The Einstein Line Element.*—Turning now to the Einstein line element, and making use of equations (13), (24) and (25), we easily obtain the following convenient expressions for pressure and density

$$8\pi p_0 = \Lambda - \frac{1}{R^2} \qquad (31)$$

$$8\pi \rho_{00} = -\Lambda + \frac{3}{R^2} \qquad (32)$$

and by intercombination of these equations we can also write,

$$4\pi(\rho_{00} + p_0) = \frac{1}{R^2} \qquad (33)$$

$$4\pi(\rho_{00} + 3p_0) = \Lambda. \qquad (34)$$

Einstein's original deduction of his line element was based on the idea that the universe could be regarded on a large scale as filled with stars having negligible relative velocities. Hence to obtain the original Einstein result we may now put the pressure p_0 of our idealized fluid equal to zero, and write equations (33) and (34) in the well-known form

$$\Lambda = \frac{1}{R^2} = 4\pi\rho_0. \qquad (35)$$

There appears to be no *a priori* reason, however, compelling us to restrict our considerations to a universe filled with material having a negligible pressure. Indeed, if the density of radiation in the universe should even be of the same order as the density of ordinary matter, the neglect of the pressure would not be permissible. For more general considerations, in which the pressure cannot be neglected, equations (31) and (32) should be used in their complete form, as has already been done by the present writer in investigating the final state of thermodynamic equilibrium in the Einstein universe.[5]

As a special case of possible importance,[6] it is interesting to consider an Einstein universe filled primarily with radiation alone, and containing, if any, only a negligible density of ordinary matter. In accordance with the relation between radiation pressure and energy density we then have

$$\rho_{00} = 3p_0 \qquad (36)$$

and equations (33) and (34) become

$$\Lambda = \frac{3}{2R^2} = 8\pi\rho_{00}. \quad (37)$$

In the Einstein universe the cosmological constant Λ must in any case be regarded as a sort of parameter connected with the density or density and pressure of the universe as a whole. Its value, of course, must be small enough so that the older calculations of planetary orbits will not be affected by the introduction of the cosmological term into equation (3), and it must be independent of the coördinates in order that the divergence of the right hand side of equation (3) shall vanish identically. In accordance with equations (33) and (34), Λ and R^2 are both of them quantities which can physically hardly be other than positive. Hence the radius R of the Einstein universe is necessarily real and the universe is closed. Finally the possible physical interpretation of Λ as given by Einstein's later work[7] should be noted.

7. *The Special Relativity Line Element.*—In the case of the special relativity line element we had λ and ν both equal to zero in accordance with equation (28), which substituted in equations (13) and (14) gives us

$$\Lambda = 8\pi p_0 = -8\pi\rho_{00}. \quad (38)$$

Physically, however, this condition can be reasonably met only by taking all three quantities equal to zero. Hence the special relativity line element could be the cosmological line element only in the case of a completely empty universe with the cosmological constant Λ equal to zero.

8. *Possibility of Other Line Elements.*—In conclusion the possibility must be emphasized of obtaining cosmological line elements other than the above three, by imposing conditions different from those laid down in section 2. In particular it should be noted that our assumption of a static line element takes no explicit recognition of any universal evolutionary process which may be going on. The investigation of non-static line elements would be very interesting.

[1] See Tolman, R. C., an article to be published soon in *The Astrophysical Journal,* entitled "On the Astronomical Implications of the de Sitter Line Element for the Universe."

[2] See Eddington, A. S., *The Mathematical Theory of Relativity,* Cambridge, **1923,** Equation (46.9).

[3] See Eddington, loc. cit., Equation (70.1).

[4] This form can be obtained for example by substituting $R \sin \chi = r$ in Eddington's equation (67.12), loc. cit.

[5] Tolman, R. C., THIS JOURNAL, **14,** pp. 268, 343, 348 (1928).

[6] See Silberstein, L., Program for the New York meeting of the American Physical Society, February, 1929, Abstract 10.

[7] Einstein, A., *Ber. Berl.*, p. 349 (1919).