to their inverses and if one of the largest subgroups composed of operators which correspond to their inverses under this automorphism is of odd index, then G contains an abelian subgroup of index 2 which is the direct product of an abelian group whose order is this odd index and another abelian group.

From this theorem it results that when the index of H under G is an odd number then G admits a number of conjugate automorphisms in which more than one-half of the operators correspond to their inverses which is equal to this index. In each of these all the operators of the abelian subgroup of index 2 correspond to their inverses and the number of additional operators which correspond to their inverses is equal to the order of the central of G. It therefore results that whenever H is of index $2k + 1$, k being a positive integer, then G admits exactly $2k + 2$ automorphisms in which more than half of its operators correspond to their inverses. One of these is characteristic while the rest form a single set of conjugates under G. A necessary and sufficient condition that the characteristic automorphism is the identical automorphism is that each of the operators in H besides the identity is of order 2. The simplest illustration of this general theorem is furnished by the automorphism of the symmetric group of order 6. It is not difficult to determine the number of distinct groups of order g which belong to the given category. This is clearly the product of the number of abelian groups of order $2k + 1$ and the sum of the numbers of the sets of subgroups of index 2 contained in the abelian groups of order $g/(2k + 1)$ such that each set is composed of all those subgroups which are conjugate under the groups of isomorphisms of the corresponding abelian groups.

LINEAR TRANSFORMATIONS IN HILBERT SPACE. III. OPERATIONAL METHODS AND GROUP THEORY

BY M. H. STONE

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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In the present note, we wish to outline certain further developments of the theory of transformations in Hilbert space sketched in previous notes.[1] The results which we shall give have a somewhat special and fragmentary character, due to the fact that they are here isolated as the most interesting and novel aspects of the theory in which they appear.

On the basis of the results already outlined, we can lay the foundations of an operational calculus: if T is an arbitrary self-adjoint transformation and E_λ is the corresponding canonical resolution of the identity, we define $F(T)$ as the transformation whose field comprises those and only those elements f such that $\int_{-\infty}^{+\infty} | F(\lambda) |^2 dQ(E_\lambda f)$ converges and which takes f into the element $F(T)f$ such that

$$Q(F(T)f, g) = \int_{-\infty}^{+\infty} F(\lambda)\, dQ\,(E_\lambda f, g)]$$

for every g in $\mathfrak{H}$. In this definition, the integrals are to be treated as Lebesgue-Stieltjes integrals, and the function $F(\lambda)$ is required to have such attributes as to ensure the possibility of this interpretation; in particular, the definition is always significant for Borel measurable functions. By means of a systematic development of the requisite fundamental theorems, we justify the use of the symbol $F(T)$, which reflects the isomorphism between the algebra of the transformations and that of the functions defining them, and obtain a full-fledged operational calculus suitable for application to other problems. We may mention that a particular case of great interest arises in connection with the differential operator $i \frac{d}{dx}$ in the Hilbert space consisting of all complex-valued Lebesgue measurable functions $f(x)$, $-\infty < x < +\infty$, for which the integral $\int_{-\infty}^{+\infty} |f(x)|^2 dx$ exists: the differential operator defines a self-adjoint transformation, and the related operational calculus is the Heaviside calculus. In this instance, our definitions coincide with those given by Wiener.[2]

A first important application of the operational calculus yields a beautiful discussion of the theory of the unitary equivalence of self-adjoint transformations, a theory elaborated for bounded transformations in an essentially identical but less perspicuous form by Hellinger[3] and Hahn.[4] We shall not give details here.

A second application, to a field in which few results have been published hitherto, concerns group-theoretic questions of fundamental importance in the quantum mechanics. Among these questions is the study of the unitary representations of the group of translations of a straight line into itself. The solution of this problem is embodied in the following statement:

THEOREM. If U_τ, $-\infty < \tau < +\infty$ is a group of unitary transformations in $\mathfrak{H}$ such that

$$U_0 = I, \quad U_{-\tau} = U_\tau^{-1}, \quad U_{\sigma+\tau} = U_\sigma U_\tau,$$

then there exists a unique self-adjoint transformation T with the canonical resolution of the identity E_λ, such that $Q(U_\tau f, g) = \int_{-\infty}^{+\infty} e^{i\tau\lambda} dQ(E_\lambda f, g)$

while $\frac{U_\tau - U_0}{\tau} f \longrightarrow iTf$ when $\tau \longrightarrow 0$ and f is in the field of T; the transformation iT may be called the generating infinitesimal transformation of the group. Conversely, when T is a given self-adjoint transformation, there exists a one-parameter group of unitary transformations of the type described above which has iT as its generating infinitesimal transformation.[5]

The device upon which the demonstration of this theorem depends is a transition from the group U_τ to the resolvent R_l of T, by means of Fourier analysis: we form the function $\psi(\tau; l) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\lambda\tau}\frac{1}{\lambda - l}\,d\lambda$, and then study the integral $\int_{-\infty}^{+\infty} Q(U_\tau f, g)\psi(\tau; l)d\tau$, showing that it defines a transformation R_l which is the resolvent of a self-adjoint transformation T.

A second question of group theory, to which we can apply the operational calculus, is raised by the Heisenberg permutation relations connecting the self-adjoint transformations P_k, Q_k, $k = 1, \ldots, n$. For convenience, we write these relations in the form

$$P_kQ_l - Q_lP_k = i\delta_{kl}I,\ P_kP_l - P_lP_k = 0,\ Q_kQ_l - Q_lQ_k = 0,\ k, l = 1, \ldots, n.$$

In quantum mechanics, these transformations refer essentially to the coördinates and momenta of a dynamical system of n degrees of freedom. The content of these permutation relations must be made precise by expressing them in terms of the one parameter groups of unitary transformations $U_\tau^{(k)}$ and $V_\tau^{(k)}$ generated by iP_k and iQ_k, respectively. We have $U_\sigma^{(k)} V_\tau^{(k)} = e^{-i\sigma\tau} V_\tau^{(k)} U_\sigma^{(k)}$, and $U_\sigma^{(k)} V_\sigma^{(l)} = V_\tau^{(l)} U_\sigma^{(k)}$ when k and l are different, for $k, l = 1, \ldots, n$; we have also $U_\sigma^{(k)} U_\tau^{(l)} = U_\tau^{(l)} U_\sigma^{(k)}$ and $V_\sigma^{(k)} V_\tau^{(l)} = V_\tau^{(l)} V_\sigma^{(k)}$ for $k, l = 1, \ldots, n$. We prove the following theorem:

THEOREM. If the family of transformations $U_{\sigma_1}^{(1)} \ldots U_{\sigma_n}^{(n)} V_{\tau_1}^{(1)} \ldots V_{\tau_n}^{(n)}$ is irreducible in $\mathfrak{H}$, then there exists a one-to-one linear isometric correspondence or transformation S which takes $\mathfrak{H}$ into $\mathfrak{H}_n$, the space of all complex-valued Lebesgue-measurable functions $f(x_1, \ldots, x_n)$, $-\infty < x_1 < +\infty, \ldots, -\infty < x_n < +\infty$, for which the integral $\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} |f(x_1, \ldots, x_n)|^2 dx_1 \ldots dx_n$ exists, such that

$$SP_kS^{-1}f(x_1, \ldots, x_n) = i\frac{\partial}{\partial x_k}f,\ SQ_kS^{-1}f(x_1, \ldots, x_n) = x_kf.$$

The proof of this theorem depends upon a consideration of the transformations

$$T_k = Q_k P_k Q_k - P_k, \quad S_k = (Q_k - i)/(Q_k + i)$$

for $k = 1, \ldots, n$, and their mutual relations. The principal difficulty lies in showing that the transformations T_k are self-adjoint. The determination of their spectra, under the hypothesis of irreducibility, can then be effected and leads easily to the construction of the desired transformation S by means of a device previously employed by J. v. Neumann in a rather different connection.[6]

The significance of the last two theorems for the quantum mechanics has been pointed out by Weyl, who made no attempt to prove them.[7] We shall add one or two remarks concerning the last theorem. If $\mathfrak{H}$ is the representation space for the states of a dynamical system of n degrees of freedom, the postulate of irreducibility imposed in addition to the Heisenberg permutation relations has the following physical significance: if the family of transformations considered were reducible, it would be reduced by each of two mutually perpendicular or orthogonal closed linear manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$, both proper subsets of $\mathfrak{H}$; these manifolds would reduce any transformation expressible as a function of the transformations P_d, Q_k, $k = 1, \ldots, n$; hence no physically realizable perturbation could take a state representable in $\mathfrak{M}_1$ into a state representable in $\mathfrak{M}_2$, or vice versa; our postulate of irreducibility avoids such a degeneracy into two physically distinct systems. We may also remark that the result of the theorem leads to the following important fact: if we have two sets of canonical coördinates P_k, Q_k and P'_k, Q'_k, satisfying the Heisenberg relations and the postulate of irreducibility, we can obtain a unitary transformation of $\mathfrak{H}$ into itself, which we may call U, with the property that $P'_k = UP_kU^{-1}$, $Q'_k = UQ_kU^{-1}$, for $k = 1, \ldots, n$. The converse of this assertion is trivial.

[1] *Proc. Nat. Acad. Sci.*, **15** (1929), 198–200, 423–425.

[2] Wiener, *Mathematische Annalen*, **95** (1925), 557–584.

[3] Hellinger, *J. Mathematik*, **136** (1909), 210–271.

[4] Hahn, *Monatshefte Mathematik und Physik*, **23** (1912),161–224.

[5] Presented to the American Mathematical Society, December 27, 1928; see *Bull. Am. Math. Soc.*, **35** (1929), 167.

[6] J. v. Neumann, *Mathematische Annalen*, **102** (1929), 49–131, particularly Anhang 4; this is the paper of v. Neumann referred to in footnote 6 of our preceding note.

[7] Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, **1928**, §§. 14, 15, 18, 45, 46.