AN EXTENSION OF THE THEOREM THAT NO COUNTABLE POINT SET IS PERFECT

BY R. L. MOORE

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF TEXAS

Communicated March 10, 1924

In 1918, Sierpinski[1] showed that if a bounded and closed point set is the sum of a countable number of mutually exclusive closed point sets then it is not connected. In the present paper I will prove the following theorem.

THEOREM 1. *If a bounded and closed point set is the sum of a countable number of mutually exclusive closed and connected point sets M_1, M_2, M_3... then not every point set of this sequence contains a limit point of the sum of the remaining ones.*

Proof. Suppose, on the contrary, that there exists a countable sequence of mutually exclusive continua M_1, M_2, M_3,...such that their sum M is closed and bounded and such that, for each n, M_n contains at least one limit point of M-M_n. By the above mentioned theorem of Sierpinski's, for each n, M_n is a maximal connected subset of M. It follows, by an easily established modification of a theorem of Zoretti's,[2] that there exists a domain D_1 containing M_1 but no point of the closed point set M_2 and such that the boundary of D_1 contains no point of M. Let G_1 denote the sequence M_1, M_2, M_3,... Since M_2 contains at least one limit point of M-M_2 it follows that there exists an *infinite* sequence G_2 whose elements are those continua of the sequence G_1 which contain at least one point without D_1 and therefore lie wholly without D_1. Let n_1 denote the smallest positive integer n, greater than 2, such that M_n lies without D_1 and therefore belongs to G_2. There exists a domain D_2 containing M_2, but no point of $D_1' + M_{n_1}$ and such that the boundary of D_2 contains no point of M. Since M_{n_1} contains at least one limit point of M-M_{n_1} there exists an infinite sequence G_3 whose elements are those continua of the sequence G_2 which lie without D_2. Let n_2 denote the smallest integer n, greater than n_1, such that M_n lies without D_2 and therefore belongs to G_3. There exists a domain D_3 containing M_{n_1}, but no point of $D_1' + D_2' + M_{n_2}$, and such that the boundary of D_3 contains no point of M. There exists an infinite sequence G_4 whose elements are those continua of the sequence G_3 which lie without D_3. Let n_3 denote the smallest integer n, greater than n_2, such that M_n lies without D_3. There exists a domain D_4 containing M_{n_2} but no point of $D_1' + D_2' + D_3' + M_{n_3}$ and such that the boundary of D_4 contains no point of M. This process may be continued. Thus there exists a countably *infinite* set of mutually exclusive domains D_1, D_2, D_3,...such that (a) every point set of the sequence G_1 is in some domain of the set K and (b) each domain of the set K contains some point set of the sequence G_1. It follows that M is not covered by any proper subset of the set of domains K. But, since M is closed and bounded, this contradicts the Heine-Borel-Lebesgue Theorem. The truth of Theorem 1 is therefore established.

That the above theorem does not remain true if the stipulation that M be bounded is omitted, may be seen with the aid of the following example.

Example 1. For each positive integer n, let K_n denote the set of all decimal fractions in which there are just n digits equal to 1 and in which all the other digits are equal to 0. Let d_{1n}, d_{2n}, $d_{3n}, \ldots$ denote the fractions of the set K_n. For each n and m, let A_{mn} denote the point $(1/n, d_{mn}/n)$. Let K denote the point set consisting of the point (1, 0) and all points A_{mn} for all values of the positive integers m and n. The points of the countable set K may be designated P_1, P_2, $P_3, \ldots$ For each n let OP_n denote the straight line interval whose end-points are O and P_n and let M_n denote the ray into which the point set $(OP_n - O)$ is thrown by an inversion of the plane about a circle of radius 2 with center at O. Let M denote the sum of all the point sets of the sequence M_1, M_2, $M_3, \ldots$ The point set M is closed and, for each n, M_n contains at least one limit point of $(M\text{-}M_n)$.

That Theorem 1 becomes false if the stipulation that M be closed is omitted may be seen with the help of an example given by Miss Anna M. Mullikin in her Doctor's dissertation.[3] That it becomes false if the stipulation that each point set of the sequence M_1, M_2, $M_3, \ldots$ be connected is omitted may be seen with the aid of the following example

Example 2. For each positive integer n, let P_n^n denote the point on OX whose abscissa is $1 - 1/n$. For each pair of positive integers m and n, such that $n > 1$ and $m > n$, let P_m^n denote the point on OX whose abscissa is $[1 - 1/(n-1)] + [1/(n-1) - 1/n]/m$. Let M_1 denote the point set composed of the points P_1 and $(1, O)$. For each n greater than 1, let M_n denote the point set composed of the $n - 1$ points P_n^2, $P_n^3, \ldots P_n^n$.

If M_1, M_2, $M_3, \ldots$ is a sequence of point sets let B denote the statement that their sum is not connected and let A denote the statement that not every point set of this sequence contains at least one limit point of the sum of the remaining ones. Clearly A implies B, but B does not imply A. In other words Statement B is weaker than Statement A. Sometime last year I found that if the word "bounded" is omitted from the statement of Theorem 1, the theorem remains true provided the conclusion (Statement A) is replaced by the weaker statement B. This result was announced at the Summer meeting of the American Mathematical Society, September 6, 1923. The same result has been established by S. Mazurkiewicz in an article in Volume V of *Fundamenta Mathematicae.* This volume bears the date 1924, but a reprint of the article left the press before the appearance of the entire volume, just how long before, I do not know.[4]

[1] Sierpinski, W., "Un theoreme sur les ensembles continus," *Tohoku Math. J.*, **13**, 1918 (300).

[2] Zoretti, L., "Sur les fonctions analytiques uniformes," *J. Math. pures appl.*, **1**, 1905 (9–11).

[3] Certain theorems relating to plane connected point sets, *Trans. Amer. Math. Soc.*, **24**, 1922 (145), Fig. 1.

[4] I submitted my proof for publication in *Fundamenta Mathematicae*, the manuscript being mailed Sept. 28. Sometime in November I received the reprint of the article by Professor Mazurkiewicz.

CONCERNING THE PRIME PARTS OF CERTAIN CONTINUA WHICH SEPARATE THE PLANE

BY R. L. MOORE

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF TEXAS

Communicated March 10, 1924

Hans Hahn[1] has recently introduced the notion of *prime parts* of a continuum. If P is a point of a continuum M, the prime part of M which contains P is the set of all points $[X]$ belonging to M such that for every positive number e there exists a finite set of irregular[2] points, $P_1, P_2, P_3 \ldots P_n$, of M such that the distances $PP_1, P_1P_2, P_2P_3, \ldots P_nX$ are all less than e. Among other things, Hahn shows that if M is *irreducibly* continuous between the points A and B and it has more than one prime part then its prime parts may be ordered in a certain manner and there exists, between them and the points of a straight line interval, a one to one correspondence which preserves order and which is, in a certain sense, continuous.

In the present paper the following theorem will be established.

THEOREM 1. *If, in a plane S,M is a bounded continuum which has more than one prime part and no prime part of M separates S, then, in order that S-M shall be the sum of two mutually exclusive domains such that each prime part of M contains at least one limit point of each of these domains, it is necessary and sufficient that the set whose elements are the prime parts of M shall be a simple closed curve of prime parts in the sense that it is disconnected by the omission of any two of its elements which are not identical.*

In my proof of this theorem I will make use of a number of lemmas.

LEMMA 1. *The outer*[3] *boundary of a bounded domain is a continuum which is not disconnected by the omission of any one of its points.*

That the boundary of a complementary domain of a bounded continuum is itself a continuum has been proved by Brouwer.[4] The truth of Lemma 1 is established in a paper which I have recently submitted for publication in *Fundamenta Mathematicae*.[5]