

INTRODUCTION TO THE
THEORY OF FOURIER'S
SERIES AND INTEGRALS

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PREFACE

THIS book forms the first volume of the new edition of my book on *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*, published in 1906, and now for some time out of print. Since 1906 so much advance has been made in the Theory of Fourier's Series and Integrals, as well as in the mathematical discussion of Heat Conduction, that it has seemed advisable to write a completely new work, and to issue the same in two volumes. The first volume, which now appears, is concerned with the Theory of Infinite Series and Integrals, with special reference to Fourier's Series and Integrals. The second volume will be devoted to the Mathematical Theory of the Conduction of Heat.

No one can properly understand Fourier's Series and Integrals without a knowledge of what is involved in the convergence of infinite series and integrals. With these questions is bound up the development of the idea of a limit and a function, and both are founded upon the modern theory of real numbers. The first three chapters deal with these matters. In Chapter IV. the Definite Integral is treated from Riemann's point of view, and special attention is given to the question of the convergence of infinite integrals. The theory of series whose terms are functions of a single variable, and the theory of integrals which contain an arbitrary parameter are discussed in Chapters V. and VI. It will be seen that the two theories are closely related, and can be developed on similar lines.

The treatment of Fourier's Series in Chapter VII. depends on Dirichlet's Integrals. There, and elsewhere throughout the book, the Second Theorem of Mean Value will be found an essential part of the argument. In the same chapter the work of Poisson is adapted to modern standards, and a prominent place is given to Fejér's work, both in the proof of the fundamental theorem

and in the discussion of the nature of the convergence of Fourier's Series. Chapter IX. is devoted to Gibbs's Phenomenon, and the last chapter to Fourier's Integrals. In this chapter the work of Pringsheim, who has greatly extended the class of functions to which Fourier's Integral Theorem applies, has been used.

Two appendices are added. The first deals with *Practical Harmonic Analysis and Periodogram Analysis*. In the second a bibliography of the subject is given.

The functions treated in this book are "ordinary" functions. An interval (a, b) for which $f(x)$ is defined can be broken up into a finite number of open partial intervals, in each of which the function is monotonic. If infinities occur in the range, they are isolated and finite in number. Such functions will satisfy most of the demands of the Applied Mathematician.

The modern theory of integration, associated chiefly with the name of Lebesgue, has introduced into the Theory of Fourier's Series and Integrals functions of a far more complicated nature. Various writers, notably W. H. Young, are engaged in building up a theory of these and allied series much more advanced than anything treated in this book. These developments are in the meantime chiefly interesting to the Pure Mathematician specialising in the Theory of Functions of a Real Variable. My purpose has been to remove some of the difficulties of the Applied Mathematician.

The preparation of this book has occupied some time, and much of it has been given as a final course in the Infinitesimal Calculus to my students. To them it owes much. For assistance in the revision of the proofs and for many valuable suggestions, I am much indebted to Mr. E. M. Wellish, Mr. R. J. Lyons and Mr. H. H. Thorne of the Department of Mathematics in the University of Sydney.

H. S. CARSLAW.

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HISTORICAL INTRODUCTION

IN the middle of the eighteenth century there was a prolonged controversy as to the possibility of the expansion of an arbitrary function of a real variable in a series of sines and cosines of multiples of the variable. The question arose in connection with the problem of the Vibrations of Strings. The theory of these vibrations reduces to the solution of the Differential Equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2},$$

and the earliest attempts at its solution were made by D'Alembert,* Euler,† and Bernoulli.‡ Both D'Alembert and Euler obtained the solution in the functional form

$$y = \phi(x + at) + \psi(x - at).$$

The principal difference between them lay in the fact that D'Alembert supposed the initial form of the string to be given by a single analytical expression, while Euler regarded it as lying along any arbitrary continuous curve, different parts of which might be given by different analytical expressions. Bernoulli, on the other hand, gave the solution, when the string starts from rest, in the form of a trigonometrical series

$$y = A_1 \sin x \cos at + A_2 \sin 2x \cos 2at + \dots,$$

and he asserted that this solution, being perfectly general, must contain those given by Euler and D'Alembert. The importance of his discovery was immediately recognised, and Euler pointed out that if this statement of the solution were correct, an arbitrary function of a single variable must be developable in an infinite series of sines of multiples of the variable. This he

* *Mém. de l'Académie de Berlin*, **3**, p. 214, 1747.

† *loc. cit.*, **4**, p. 69, 1748.

‡ *loc. cit.*, **9**, p. 173, 1753.

held to be obviously impossible, since a series of sines is both periodic and odd, and he argued that if the arbitrary function had not both of these properties it could not be expanded in such a series.

While the debate was at this stage a memoir appeared in 1759* by Lagrange, then a young and unknown mathematician, in which the problem was examined from a totally different point of view. While he accepted Euler's solution as the most general, he objected to the mode of demonstration, and he proposed to obtain a satisfactory solution by first considering the case of a finite number of particles stretched on a weightless string. From the solution of this problem he deduced that of a continuous string by making the number of particles infinite.† In this way he showed that when the initial displacement of the string of unit length is given by $f(x)$, and the initial velocity by $F(x)$, the displacement at time t is given by

$$y = 2 \int_0^1 \sum_1^{\infty} (\sin n\pi x' \sin n\pi x \cos n\pi at) f(x') dx' \\ + \frac{2}{a\pi} \int_0^1 \sum_1^{\infty} \frac{1}{n} (\sin n\pi x' \sin n\pi x \sin n\pi at) F(x') dx'.$$

This result, and the discussion of the problem which Lagrange gave in this and other memoirs, have prompted some mathematicians to deny the importance of Fourier's discoveries, and to attribute to Lagrange the priority in the proof of the development of an arbitrary function in trigonometrical series. It is true that in the formula quoted above it is only necessary to change the order of summation and integration, and to put $t=0$, in order that we may obtain the development of the function $f(x)$ in a series of sines, and that the coefficients shall take the definite integral forms with which we are now familiar. Still Lagrange did not take this step, and, as Burkhardt remarks,‡ the fact that he did not do so is a very instructive example of the ease with which an author omits to draw an almost obvious conclusion from his results, when his investigation has been undertaken with another end in view. Lagrange's purpose was to demon-

* Cf. Lagrange, *Œuvres*, T. I., p. 37.

† *loc. cit.*, § 37.

‡ Burkhardt, "Entwicklungen nach oscillirenden Functionen," *Jahresber. D. Math. Ver., Leipzig*, 10, Hft. II., p. 32, 1901.

strate the truth of Euler's solution, and to defend its general conclusions against D'Alembert's attacks. When he had obtained his solution he therefore proceeded to transform it into the functional form given by Euler. Having succeeded in this, he held his demonstration to be complete.

The further development of the theory of these series was due to the astronomical problem of the expansion of the reciprocal of the distance between two planets in a series of cosines of multiples of the angle between the radii. As early as 1749 and 1754 D'Alembert and Euler had published discussions of this question in which the idea of the definite integral expressions for the coefficients in Fourier's Series may be traced, and Clairaut, in 1757,* gave his results in a form which practically contained these coefficients. Again, Euler,† in a paper written in 1777 and published in 1793, actually employed the method of multiplying both sides of the equation

$$f(x) = a_0 + 2a_1 \cos x + 2a_2 \cos 2x + \dots + 2a_n \cos nx + \dots$$

by $\cos nx$ and integrating the series term by term between the limits 0 and π . In this way he found that

$$a_n = \frac{1}{\pi} \int_0^\pi f(x) \cos nx \, dx.$$

It is curious that these papers seem to have had no effect upon the discussion of the problem of the Vibrations of Strings in which, as we have seen, D'Alembert, Euler, Bernoulli, and Lagrange were about the same time engaged. The explanation is probably to be found in the fact that these results were not accepted with confidence, and that they were only used in determining the coefficients of expansions whose existence could be demonstrated by other means. It was left to Fourier to place our knowledge of the theory of trigonometrical series on a firmer foundation, and, owing to the material advance made by him in this subject the most important of these expansions are now generally associated with his name and called Fourier's Series.

His treatment was suggested by the problems he met in the Mathematical Theory of the Conduction of Heat. It is to be

* *Paris, Hist. Acad. sci.*, 1754 [59], Art. iv. (July 1757).

† *Petrop. N. Acta.*, 11, 1793 [98], p. 94 (May 1777).

found in various memoirs, the most important having been presented to the Paris Academy in 1811, although it was not printed till 1824-6. These memoirs are practically contained in his book, *Théorie mathématique de la chaleur* (1822). In this treatise several discussions of the problem of the expansion of a function in trigonometrical series are to be found. Some of them fail in rigour. One is the same as that given by Euler. However, it is a mistake to suppose that Fourier did not establish in a rigorous and conclusive manner that a quite arbitrary function (meaning by this any function capable of being represented by an arc of a continuous curve or by successive portions of different continuous curves), could be represented by the series we now associate with his name, and it is equally wrong to attribute the first rigorous demonstration of this theorem to Dirichlet, whose earliest memoir was published in 1829.* A closer examination of Fourier's work will show that the importance of his investigations merits the fullest recognition, and Darboux, in the latest complete edition of Fourier's mathematical works,† points out that the method he employed in the final discussion of the general case is perfectly sound and practically identical with that used later by Dirichlet in his classical memoir. In this discussion Fourier followed the line of argument which is now customary in dealing with infinite series. He proved that when the values

$$\left. \begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx', \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx', \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx', \end{aligned} \right\} n \geq 1,$$

are inserted in the terms of the series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

the sum of the terms up to $\cos nx$ and $\sin nx$ is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \frac{\sin \frac{1}{2}(2n+1)(x'-x)}{\sin \frac{1}{2}(x'-x)} dx'.$$

He then discussed the limiting value of this sum as n becomes

* Dirichlet, *J. Math., Berlin*, 4, p. 157, 1829.

† *Œuvres de Fourier*, T. I., p. 512, 1888.

infinite, and thus obtained the sum of the series now called Fourier's Series.

Fourier made no claim to the discovery of the values of the coefficients

$$\left. \begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx', \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx', \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx', \end{aligned} \right\} n \equiv 1.$$

We have already seen that they were employed both by Clairaut and Euler before this time. Still there is an important difference between Fourier's interpretation of these integrals and that which was current among the mathematicians of the eighteenth century. The earlier writers by whom they were employed (with the possible exception of Clairaut) applied them to the determination of the coefficients of series whose existence had been demonstrated by other means. Fourier was the first to apply them to the representation of an entirely arbitrary function, in the sense in which we have explained this term. In this he made a distinct advance upon his predecessors. Indeed Riemann* asserts that when Fourier, in his first paper to the Paris Academy in 1807, stated that a completely arbitrary function could be expressed in such a series, his statement so surprised Lagrange that he denied the possibility in the most definite terms. It should also be noted that he was the first to allow that the arbitrary function might be given by different analytical expressions in different parts of the interval; also that he asserted that the sine series could be used for other functions than odd ones, and the cosine series for other functions than even ones. Further, he was the first to see that when a function is defined for a given range of the variable, its value outside that range is in no way determined, and it follows that no one before him can have properly understood the representation of an arbitrary function by a trigonometrical series.

The treatment which his work received from the Paris Academy is evidence of the doubt with which his contemporaries viewed

* Cf. Riemann, "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe," *Göttingen, Abh. Ges. Wiss.*, 13, §2, 1867.

his arguments and results. His first paper upon the Theory of Heat was presented in 1807. The Academy, wishing to encourage the author to extend and improve his theory, made the question of the propagation of heat the subject of the *grand prix de mathématiques* for 1812. Fourier submitted his *Mémoire sur la propagation de la Chaleur* at the end of 1811 as a candidate for the prize. The memoir was referred to Laplace, Lagrange, Legendre, and the other adjudicators; but, while awarding him the prize, they qualified their praise with criticisms of the rigour of his analysis and methods,* and the paper was not published at the time in the *Mémoires de l'Académie des Sciences*. Fourier always resented the treatment he had received. When publishing his treatise in 1822, he incorporated in it, practically without change, the first part of this memoir; and two years later, having become Secretary of the Academy on the death of Delambre, he caused his original paper, in the form in which it had been communicated in 1811, to be published in these *Mémoires*.† Probably this step was taken to secure to himself the priority in his important discoveries, in consequence of the attention the subject was receiving at the hands of other mathematicians. It is also possible that he wished to show the injustice of the criticisms which had been passed upon his work. After the publication of his treatise, when the results of his different memoirs had become known, it was recognised that real advance had been made by him in the treatment of the subject and the substantial accuracy of his reasoning was admitted.‡

* Their report is quoted by Darboux in his Introduction (p. vii) to *Œuvres de Fourier*, T. I. :—" Cette pièce renferme les véritables équations différentielles de la transmission de la chaleur, soit à l'intérieur des corps, soit à leur surface; et la nouveauté du sujet, jointe à son importance, a déterminé la Classe à couronner cet Ouvrage, en observant cependant que la manière dont l'Auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du côté de la rigueur."

† *Mémoires de l'Acad. des Sc.*, 4, p. 185, and 5, p. 153.

‡ It is interesting to note the following references to his work in the writings of modern mathematicians :

Kelvin, *Coll. Works*, Vol. III., p. 192 (Article on "Heat," *Enc. Brit.*, 1878).

"Returning to the question of the Conduction of Heat, we have first of all to say that the theory of it was discovered by Fourier, and given to the world through the French Academy in his *Théorie analytique de la Chaleur*, with

The next writer upon the Theory of Heat was Poisson. He employed an altogether different method in his discussion of the question of the representation of an arbitrary function by a trigonometrical series in his papers from 1820 onwards, which are practically contained in his books, *Traité de Mécanique* (T. I. (2^e éd.) 1833), and *Théorie mathématique de la Chaleur* (1835). He began with the equation

$$\frac{1 - h^2}{1 - 2h \cos(x' - x) + h^2} = 1 + 2 \sum_1^{\infty} h^n \cos n(x' - x),$$

solutions of problems naturally arising from it, of which it is difficult to say whether their uniquely original quality, or their transcendently intense mathematical interest, or their perennially important instructiveness for physical science, is most to be praised."

Darboux, Introduction, *Œuvres de Fourier*, T. I., p. v, 1888.

"Par l'importance de ses découvertes, par l'influence décisive qu'il a exercée sur le développement de la Physique mathématique, Fourier méritait l'hommage qui est rendu aujourd'hui à ses travaux et à sa mémoire. Son nom figurera dignement à côté des noms, illustres entre tous, dont la liste, destinée à s'accroître avec les années, constitue dès à présent un véritable titre d'honneur pour notre pays. La *Théorie analytique de la Chaleur* . . . , que l'on peut placer sans injustice à côté des écrits scientifiques les plus parfaits de tous les temps, se recommande par une exposition intéressante et originale des principes fondamentaux; il éclaire de la lumière la plus vive et la plus pénétrante toutes les idées essentielles que nous devons à Fourier et sur lesquelles doit reposer désormais la Philosophie naturelle; mais il contient, nous devons le reconnaître, beaucoup de négligences, des erreurs de calcul et de détail que Fourier a su éviter dans d'autres écrits."

Poincaré, *Théorie analytique de la propagation de la Chaleur*, p. 1, § 1, 1891.

"La théorie de la chaleur de Fourier est un des premiers exemples de l'application de l'analyse à la physique; en partant d'hypothèses simples qui ne sont autre chose que des faits expérimentaux généralisés, Fourier en a déduit une série de conséquences dont l'ensemble constitue une théorie complète et cohérente. Les résultats qu'il a obtenus sont certes intéressants par eux-mêmes, mais ce qui l'est plus encore est la méthode qu'il a employée pour y parvenir et qui servira toujours de modèle à tous ceux qui voudront cultiver une branche quelconque de la physique mathématique. J'ajouterai que le livre de Fourier a une importance capitale dans l'histoire des mathématiques et que l'analyse pure lui doit peut-être plus encore que l'analyse appliquée."

Boussinesq, *Théorie analytique de la Chaleur*, T. I., p. 4, 1901.

"Les admirables applications qu'il fit de cette méthode (*i. e.* his method of integrating the equations of Conduction of Heat) sont, à la fois, assez simples et assez générales, pour avoir servi de modèle aux géomètres de la première moitié de ce siècle; et elles leur ont été d'autant plus utiles, qu'elles ont pu, avec de légères modifications tout au plus, être transportées dans d'autres branches de la Physique mathématique, notamment dans l'Hydrodynamique et dans la Théorie de l'élasticité."

h being numerically less than unity, and he obtained, by integration,

$$\begin{aligned} & \int_{-\pi}^{\pi} f(x') \frac{1-h^2}{1-2h \cos(x'-x)+h^2} dx' \\ &= \int_{-\pi}^{\pi} f(x') dx' + 2 \sum_1^{\infty} h^n \int_{-\pi}^{\pi} f(x') \cos n(x'-x) dx'. \end{aligned}$$

While it is true that by proceeding to the limit we may deduce that

$$f(x) \text{ or } \frac{1}{2}[f(x-0)+f(x+0)]$$

is equal to

$$\text{Lt}_{h \rightarrow 1} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx' + \frac{1}{\pi} \sum_1^{\infty} h^n \int_{-\pi}^{\pi} f(x') \cos n(x'-x) dx' \right],$$

we are not entitled to assert that this holds *for the value* $h=1$, unless we have already proved that the series converges for this value. This is the real difficulty of Fourier's Series, and this limitation on Poisson's discussion has been lost sight of in some presentations of Fourier's Series. There are, however, other directions in which Poisson's method has led to most notable results. The importance of his work cannot be exaggerated.*

After Poisson, Cauchy attacked the subject in different memoirs published from 1826 onwards,† using his method of residues, but his treatment did not attract so much attention as that given about the same time by Dirichlet, to which we now turn.

Dirichlet's investigation is contained in two memoirs which appeared in 1829 ‡ and 1837.§ The method which he employed we have already referred to in speaking of Fourier's work. He based his proof upon a careful discussion of the limiting values of the integrals

$$\begin{aligned} & \int_0^a f(x) \frac{\sin \mu x}{\sin x} dx \dots a > 0, \\ & \int_a^b f(x) \frac{\sin \mu x}{\sin x} dx \dots b > a > 0, \end{aligned}$$

* For a full treatment of Poisson's method, reference may be made to Bôcher's paper, "Introduction to the Theory of Fourier's Series," *Ann. Math., Princeton, N. J.* (Ser. 2), 7, 1906.

† See Bibliography, p. 303.

‡ *J. Math., Berlin*, 4, 1829.

§ *Dove's Repertorium der Physik*, Bd. I., p. 152, 1837.

as μ increases indefinitely. By this means he showed that the sum of the series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

where the coefficients a_0 , etc., are those given by Fourier, is

$$\begin{aligned} \frac{1}{2} [f(x-0) + f(x+0)] \dots -\pi < x < \pi, \\ \frac{1}{2} [f(-\pi+0) + f(\pi-0)] \dots x = \pm \pi, \end{aligned}$$

provided that, while $-\pi < x < \pi$, $f(x)$ has only a finite number of ordinary discontinuities and turning points, and that it does not become infinite in this range. In a later paper,* in which he discussed the expansion in Spherical Harmonics, he showed that the restriction that $f(x)$ must remain finite is not necessary, provided that $\int_{-\pi}^{\pi} f(x) dx$ converges absolutely.

The work of Dirichlet led in a few years to one of the most important advances not only in the treatment of trigonometrical series, but also in the Theory of Functions of a Real Variable; indeed it may be said to have laid the foundations of that theory. This advance is to be found in the memoir by Riemann already referred to, which formed his *Habilitationsschrift* at Göttingen in 1854, but was not published till 1867, after his death. Riemann's aim was to determine the *necessary* conditions which $f(x)$ must satisfy, if it can be replaced by its Fourier's Series. Dirichlet had shown only that certain conditions were *sufficient*. The question which Riemann set himself to answer, he did not completely solve: indeed it still remains unsolved. But in the consideration of it he perceived that it was necessary to widen the concept of the definite integral as then understood.

Cauchy, in 1823, had defined the integral of a *continuous* function as the limit of a sum, much in the way in which it is still treated in our elementary text-books. The interval of integration (a, b) is first divided into parts by the points

$$a = x_0, x_1, x_2, \dots, x_{n-1}, x_n = b.$$

The sum

$$S = (x_1 - x_0)f(x_1) + (x_2 - x_1)f(x_2) + \dots + (x_n - x_{n-1})f(x_n)$$

is formed. And the integral $\int_a^b f(x) dx$ is defined as the limit of

**J. Math., Berlin*, 17, 1837.

this sum when the number of parts is increased indefinitely and their length diminished indefinitely. On this understanding every continuous function has an integral.

For discontinuous functions, he proceeded as follows:

If a function $f(x)$ is continuous in an interval (a, b) except at the point c , in the neighbourhood of which $f(x)$ may be bounded or not, the integral of $f(x)$ in (a, b) is defined as the sum of the two limits

$$\text{Lt}_{h \rightarrow 0} \int_a^{c-h} f(x) dx, \quad \text{Lt}_{h \rightarrow 0} \int_{c+h}^b f(x) dx,$$

when these limits exist.

Riemann dismisses altogether the requirement of continuity, and in forming the sum S multiplies each interval $(x_r - x_{r-1})$ by the value of $f(x)$, not necessarily at the beginning (or end) of the interval, but at a point ξ_r arbitrarily chosen between these, or by a number intermediate between the lower and upper bounds of $f(x)$ in (x_{r-1}, x_r) . The integral $\int_a^b f(x) dx$ is defined as the limit of this sum, if such exists, when the number of the partial intervals is increased indefinitely and their length tends to zero.

Riemann's treatment, given in the text in a slightly modified form, is now generally adopted in a scientific treatment of the Calculus. It is true that a more general theory of integration has been developed in recent years, chiefly due to the writings of Lebesgue,* de la Vallée Poussin and Young; that theory is mainly for the specialist in certain branches of Pure Mathematics. But no mathematician can neglect the concept of the definite integral which Riemann introduced.

One of the immediate advances it brought was to bring within the integrable functions a class of discontinuous functions whose discontinuities were infinitely numerous in any finite interval. An example, now classical, given by Riemann, was the function defined by the convergent series:

$$1 + \frac{[x]}{1^2} + \frac{[2x]}{2^2} + \dots + \frac{[nx]}{n^2}$$

where $[nx]$ denotes the positive or negative difference between nx and the nearest integer, unless nx falls midway between two consecutive integers, when the value of $[nx]$ is to be taken as

* See footnote, p. 77.

zero. The sum of this series is discontinuous for every rational value of x of the form $p/2n$, where p is an odd integer prime to n .

With Riemann's definition the restrictions which Dirichlet had placed upon the function $f(x)$ were considerably relaxed. To this Riemann contributed much, and the numerous writers who have carried out similar investigations since his time have still further widened the bounds, while the original idea that every continuous function admitted of such an expansion has been shown to be false. Still it remains true that for all practical purposes, and for all ordinary functions, Dirichlet's investigation established the convergence of the expansions. Simplifications have been introduced in his proof by the introduction of the Second Theorem of Mean Value, and the use of a modified form of Dirichlet's Integral, but the method which he employed is still the basis of most rigorous discussions of Fourier's Series.

The nature of the convergence of the series began to be examined after the discovery by Stokes (1847) and Seidel (1848) of the property of Uniform Convergence. It had been known since Dirichlet's time that the series were, in general, only conditionally convergent, their convergence depending upon the presence of both positive and negative terms. It was not till 1870 that Heine showed that, if the function is finite and satisfies Dirichlet's Conditions in the interval $(-\pi, \pi)$, the Fourier's Series converges uniformly in any interval lying within an interval in which $f(x)$ is continuous. This condition has, like the other conditions of that time, since been somewhat modified.

In the last thirty or forty years quite a large literature has arisen dealing with Fourier's Series. The object of many of the investigations has been to determine sufficient conditions to be satisfied by the function $f(x)$, in order that its Fourier's Series may converge, either throughout the interval $(-\pi, \pi)$, or at particular points of the interval. It appears that the convergence or non-convergence of the series for a particular value of x really depends only upon the nature of the function in an arbitrarily small neighbourhood of that point, and is independent of the general character of the function throughout the interval, this general character being limited only by the necessity for the existence of the coefficients of the series. These memoirs—

associated chiefly with the names of Du Bois-Reymond, Lipschitz, Dini, Heine, Cantor, Jordan, Lebesgue, de la Vallée Poussin, Hobson and Young—have resulted in the discovery of sufficient conditions of wide scope, which suffice for the convergence of the series, either at particular points, or, generally, throughout the interval. The necessary and sufficient conditions for the convergence of the series at a point of the interval, or throughout any portion of it, have not been obtained. In view of the general character of the problem, this is not surprising. Indeed it is not improbable that no such necessary and sufficient conditions may be obtainable.

In many of the works referred to above, written after the discovery by Lebesgue (1902) of his general theory of integration, series whose terms did not exist under the old definition of the integral are included in the discussion.

The fact that divergent series may be utilised in various ways in analysis has also widened the field of investigation, and indeed one of the most fruitful advances recently made arises from the discussion of Fourier's Series which diverge. The word "sum," when applied to a divergent series, has, of course, to be defined afresh; but all methods of treatment agree in this, that when the same process is applied to a convergent series the "sum," according to the new definition, is to agree with the "sum" obtained in the ordinary way. One of the most important methods of "summation" is due to Cesàro, and in its simplest form is as follows:

Denote by s_n the sum of the first n terms of the series

$$u_1 + u_2 + u_3 + \dots$$

Let

$$S_n = \frac{s_1 + s_2 + \dots + s_n}{n}.$$

When $\text{Lt}_{n \rightarrow \infty} S_n = S$, we say that the series is "summable," and that its "sum" is S .

It is not difficult to show that if the series

$$u_1 + u_2 + u_3 + \dots$$

is convergent, then $\text{Lt}_{n \rightarrow \infty} S_n = \text{Lt}_{n \rightarrow \infty} s_n$,

so that the "condition of consistency" is satisfied. [Cf. § 102.]

Fejér was the first to consider this sequence of Arithmetic Means,

$$s_1, \quad \frac{s_1 + s_2}{2}, \quad \frac{s_1 + s_2 + s_3}{3}, \quad \dots$$

for the Fourier's Series. He established the remarkable theorem* that the sequence is convergent, and its limit is

$$\frac{1}{2}(f(x+0) + f(x-0))$$

at every point in $(-\pi, \pi)$ where $f(x+0)$ and $f(x-0)$ exist, the only conditions attached to $f(x)$ being that, if bounded, it shall be integrable in $(-\pi, \pi)$, and that, if it is unbounded, $\int_{-\pi}^{\pi} f(x) dx$ shall be absolutely convergent.

Later, Hardy showed that if a series

$$u_1 + u_2 + u_3 + \dots$$

is summable by this method, and the general term u_n is of the order $1/n$, the series is convergent in the ordinary sense, and thus the sum $S (= \text{Lt } S_n)$ and the sum $s (= \text{Lt } s_n)$ will be the same.

[Cf. § 102.]

Hardy's theorem, combined with Fejér's, leads at once to a new proof of the convergence of Fourier's Series, and it can also be applied to the question of its uniform convergence. Many of the results obtained by earlier investigators follow directly from the application to Fourier's Series of the general theory of summable series.†

Recent investigations show that the coefficients in Fourier's Series, now frequently called Fourier's Constants, have important properties, independent of whether the series converges or not. For example, it is now known that if $f(x)$ and $\phi(x)$ are two functions, bounded and integrable in $(-\pi, \pi)$, and a_0, a_n, b_n are Fourier's Constants for $f(x)$, and a'_0, a'_n, b'_n those for $\phi(x)$, the series

$$2a_0a'_0 + \sum_1^{\infty} (a_n a'_n + b_n b'_n)$$

converges, and its sum is $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \phi(x) dx$. To this theorem,

* *Math. Ann.*, Leipzig, **58**, 1904.

† Chapman, *Q. J. Math.*, London, **43**, 1912; Hardy, *London, Proc. Math. Soc.* (Ser. 2), **12**, 1913.

and to the results which follow from it, much attention has recently been given, and it must be regarded as one of the most important in the whole of the theory of Fourier's Series.*

The question of the approximation to an arbitrary function by a finite trigonometrical series was examined by Weierstrass in 1885.† He proved that if $f(x)$ is a continuous and periodic function, given the arbitrary small positive quantity ϵ , a finite Fourier's Series can be found in a variety of ways, such that the absolute value of the difference of its sum and $f(x)$ will be less than ϵ for any value of x in the interval. This theorem was also discussed by Picard, and it has been generalised in recent memoirs by Stekloff and Kneser.

In the same connection, it should be noted that the application of the method of least squares to the determination of the coefficients of a finite trigonometrical series leads to the Fourier coefficients. This result was given by Töpler in 1876.‡ As many applications of Fourier's Series really only deal with a finite number of terms, these results are of special interest.

From the discussion of the Fourier's Series it was a natural step to turn to the theory of the Trigonometrical Series

$$a_0 + (a_1 \cos x + b_1 \sin x) + a_2 \cos 2x + b_2 \sin 2x + \dots,$$

where the coefficients are no longer the Fourier coefficients. The most important question to be answered was whether such an expansion was unique; in other words, whether a function could be represented by more than one such trigonometrical series. This reduces to the question of whether zero can be represented by a trigonometrical series in which the coefficients do not all vanish. The discussion of this and similar problems was carried on chiefly by Heine and Cantor,§ from 1870 onwards, in a series of papers which gave rise to the modern Theory of Sets of Points, another instance of the remarkable influence Fourier's Series have had upon the development of mathematics. || In this place it will

* Cf. Young, *London, Proc. R. Soc. (A)*, **85**, 1911.

† *J. Math., Berlin*, **71**, 1870.

‡ Töpler, *Wien, Anz. Ak. Wiss.*, **13**, 1876.

§ Bibliography, p. 305.

|| Van Vleck, "The Influence of Fourier's Series upon the Development of Mathematics," *American Association for the Advancement of Science (Atlanta)*, 1913.

See also a paper with a similar title by Jourdain, *Scientia*, Bologna (Ser. 2), **22**, 1917.

be sufficient to state that Cantor showed that all the coefficients of the trigonometrical series must vanish, if it is to be zero for all values of x in the interval $(-\pi, \pi)$, with the exception of those which correspond to a set of points constituting, in the language of the Theory of Sets of Points, a set of the n^{th} order, for which points we know nothing about the value of the series.

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CHAPTER I

RATIONAL AND IRRATIONAL NUMBERS THE SYSTEM OF REAL NUMBERS

1. Rational Numbers. The question of the convergence of Infinite Series is only capable of satisfactory treatment when the difficulties underlying the conception of irrational number have been overcome. For this reason we shall first of all give a short discussion of that subject.

The idea of number is formed by a series of generalisations. We begin with the positive integers. The operations of addition and multiplication upon these numbers are always possible; but if a and b are two positive integers, we cannot determine positive integers x and y , so that the equations $a=b+x$ and $a=by$ are satisfied, unless, in the first case, a is greater than b , and, in the second case, a is a multiple of b . To overcome this difficulty fractional and negative numbers are introduced, and the system of *rational numbers* placed at our disposal.*

The system of rational numbers is *ordered*, i.e. if we have two different numbers a and b of this system, one of them is greater than the other. Also, if $a > b$ and $b > c$, then $a > c$, when a , b and c are numbers of the system.

Further, if two different rational numbers a and b are given, we can always find another rational number greater than the one and less than the other. It follows from this that between

*The reader who wishes an extended treatment of the system of rational numbers is referred to Stolz und Gmeiner, *Theoretische Arithmetik*, Leipzig, 1900-1902, and Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre*, Leipzig, 1916.

any two different rational numbers there are an infinite number of rational numbers.*

2. The introduction of fractional and negative rational numbers may be justified from two points of view. The fractional numbers are necessary for the representation of the subdivision of a unit magnitude into several equal parts, and the negative numbers form a valuable instrument for the measurement of magnitudes which may be counted in opposite directions. This may be taken as the argument of the applied mathematician. On the other hand there is the argument of the pure mathematician, with whom the notion of number, positive and negative, integral and fractional, rests upon a foundation independent of measurable magnitude, and in whose eyes analysis is a scheme which deals with numbers only, and has no concern *per se* with measurable quantity. It is possible to found mathematical analysis upon the notion of positive integral number. Thereafter the successive definitions of the different kinds of number, of equality and inequality among these numbers, and of the four fundamental operations, may be presented abstractly.†

3. **Irrational Numbers.** The extension of the idea of number from the rational to the irrational is as natural, if not as easy, as is that from the positive integers to the fractional and negative rational numbers.

Let a and b be any two positive integers. The equation $x^b = a$ cannot be solved in terms of positive integers unless a is a perfect b^{th} power. To make the solution possible in general the irrational numbers are introduced. But it will be seen below that the system of irrational numbers is not confined to numbers which arise as the roots of algebraical equations whose coefficients are integers.

So much for the desirability of the extension from the abstract side. From the concrete the need for the extension is also evident. We have only to consider the measurement of any quantity to

* When we say that a set of things has a finite number of members, we mean that there is a positive integer n , such that the total number of members of the set is less than n .

When we say that it has an infinite number of members, we mean that it has not a finite number. In other words, however large n may be, there are more members of the set than n .

† Cf. Hobson, *London, Proc. Math. Soc.* (Ser. 1), **35**, p. 126, 1913; also the same author's *Theory of Functions of a Real Variable*, p. 13.

which the property of unlimited divisibility is assigned, *e.g.* a straight line L produced indefinitely. Take any segment of this line as unit of length, a definite point of the line as origin or zero point, and the directions of right and left for the positive and negative senses. To every rational number corresponds a

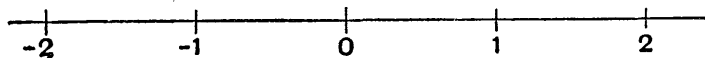


FIG. 1.

definite point on the line. If the number is an integer, the point is obtained by taking the required number of unit segments one after the other in the proper direction. If it is a fraction $\pm p/q$, it is obtained by dividing the unit of length into q equal parts and taking p of these to the right or left according as the sign is positive or negative. These numbers are called the *measures* of the corresponding segments, and the segments are said to be *commensurable* with the unit of length. The points corresponding to rational numbers may be called *rational points*.

There are, however, an infinite number of points on the line L which are not rational points. Although we may approach them as nearly as we please by choosing more and more rational points on the line, we can never quite reach them in this way. The simplest example is the case of the points coinciding with one end of the diagonal of a square, the sides of which are the unit of length, when the diagonal lies along the line L and its other end coincides with any rational point.

Thus, without considering any other case of incommensurability, we see that *the line L is infinitely richer in points than the system of rational numbers in numbers.*

Hence it is clear that if we desire to follow arithmetically all the properties of the straight line, the rational numbers are insufficient, and it will be necessary to extend this system by the creation of other numbers.

4. Returning to the point of view of the pure mathematician, we shall now describe Dedekind's method of introducing the irrational number, in its most general form, into analysis.*

* Dedekind (1831-1916) published his theory in *Stetigkeit und irrationale Zahlen*, Braunschweig, 1872 (English translation in Dedekind's *Essays on Number*, Chicago, 1901).

Let us suppose that by some method or other we have divided all the rational numbers into two classes, a lower class A and an upper class B, such that every number α of the lower class is less than every number β of the upper class.

When this division has been made, if a number α belongs to the class A, every number less than α does so also; and if a number β belongs to the class B, every number greater than β does so also.

Three different cases can arise :

(I) *The lower class can have a greatest number and the upper class no smallest number.*

This would occur if, for example, we put the number 5 and every number less than 5 in the lower class, and if we put in the upper class all the numbers greater than 5.

(II) *The upper class can have a smallest number and the lower class no greatest number.*

This would occur if, for example, we put the number 5 and all the numbers greater than 5 in the upper class, while in the lower class we put all the numbers less than 5.

It is impossible that the lower class can have a greatest number m , and the upper class a smallest number n , in the same division of the rational numbers; for between the rational numbers m and n there are rational numbers, so that our hypothesis that the two classes contain all the rational numbers is contradicted.

But a third case can arise :

(III) *The lower class can have no greatest number and the upper class no smallest number.*

For example, let us arrange the positive integers and their squares in two rows, so that the squares are underneath the numbers to which they correspond. Since the square of a fraction in its lowest terms is a fraction whose numerator and denominator are perfect squares,* we see that there are not rational numbers whose squares are 2, 3, 5, 6, 7, 8, 10, 11, ... ,

1	2	3	4	...												
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...

* If a formal proof of this statement is needed, see Dedekind, *loc. cit.*, English translation, p. 14, or Hardy, *Course of Pure Mathematics* (2nd Ed.), p. 6.

However, there are rational numbers whose squares are as near these numbers as we please. For instance, the numbers

$$\begin{aligned} &2, 1.5, 1.42, 1.415, 1.4143, \dots, \\ &1, 1.4, 1.41, 1.414, 1.4142, \dots, \end{aligned}$$

form an upper and a lower set in which the squares of the terms in the lower are less than 2, and the squares of the terms in the upper are greater than 2. We can find a number in the upper set and a number in the lower set such that their squares differ from 2 by as little as we please.*

Now form a lower class, as described above, containing all negative rational numbers, zero and all the positive rational numbers whose squares are less than 2; and an upper class containing all the positive rational numbers whose squares are greater than 2. Then every rational number belongs to one class or the other. Also every number in the lower class is less than every number in the upper. The lower class has no greatest number and the upper class has no smallest number.

5. When by any means we have obtained a division of all the rational numbers into two classes of this kind, the lower class having no greatest number and the upper class no smallest number, we create a new number defined by this division. We call it an *irrational number*, and we say that it is greater than all the rational numbers of its lower class, and less than all the rational numbers of its upper class.

Such divisions are usually called *sections*.† The irrational number $\sqrt{2}$ is defined by the section of the rational numbers described above. Similar sections would define the irrational numbers $\sqrt[3]{3}$, $\sqrt[4]{5}$, etc. The system of irrational numbers is given by all the possible divisions of the rational numbers into a lower class A and an upper class B, such that every rational number is in one class or the other, the numbers of the lower class being less than the numbers of the upper class, while the lower class has no greatest number, and the upper class no smallest number.

In other words, every irrational number is defined by its section (A, B). It may be said to “correspond” to this section.

* Cf. Hardy, *loc. cit.*, p. 8.

† French, *coupure*; German, *Schnitt*.

The system of *rational* numbers and *irrational* numbers together make up the system of *real* numbers.

The rational numbers themselves “correspond” to divisions of rational numbers.

For instance, take the rational number m . In the lower class A put the rational numbers less than m , and m itself. In the upper class B put all the rational numbers greater than m . Then m corresponds to this division of the rational numbers.

Extending the meaning of the term *section*, as used above in the definition of the irrational number, to divisions in which the lower and upper classes have greatest or smallest numbers, we may say that the rational number m corresponds to a *rational section* (A, B),* and that the irrational numbers correspond to *irrational sections*. When the rational and irrational numbers are defined in this way, and together form the system of real numbers, the real number which corresponds to the rational number m (to save confusion it is sometimes called the *rational-real* number) is conceptually distinct from m . However, the relations of magnitude, and the fundamental operations for the real numbers, are defined in such a way that this rational-real number has no properties distinct from those of m , and it is usually denoted by the same symbol.

6. Relations of Magnitude for Real Numbers. We have extended our conception of number. We must now arrange the system of real numbers in order; *i.e.* we must say when two numbers are equal or unequal to, greater or less than, each other.

In this place we need only deal with cases where at least one of the numbers is irrational.

An irrational number is never *equal* to a rational number. They are always *different* or *unequal*.

Next, in § 5, we have seen that the *irrational* number given by the section (A, B) is said to be *greater* than the *rational* number m , when m is a member of the lower class A, and that the *rational* number m is said to be *greater* than the *irrational* number given by the section (A, B), when m is a member of the upper class B.

Two irrational numbers are *equal*, when they are both given by the same section. They are *different* or *unequal*, when they are given by different sections.

The *irrational* number a given by the section (A, B) is *greater* than the *irrational* number a' given by the section (A', B'), when the class A contains numbers of the class B'. Now the class A has no greatest number. But if a certain number of the class A belongs to the class B', all the numbers of A

*The rational number m could correspond to two sections: the one named in the text, and that in which the lower class A contains all the rational numbers less than m , and the upper class B, m and all the rational numbers greater than m . To save ambiguity, one of these sections only must be chosen.

greater than this number also belong to B' . The class A thus contains an infinite number of members of the class B' , when $a > a'$.

If a real number a is *greater* than another real number a' , then a' is *less* than a .

It will be observed that the notation $>$, $=$, $<$ is used in dealing with real numbers as in dealing with rational numbers.

The real number β is said to *lie between* the real numbers a and γ , when one of them is greater than β and the other less.

With these definitions the system of real numbers is *ordered*. If we have two different real numbers, one of them is greater than the other; and if we have three real numbers such that $\alpha > \beta$ and $\beta > \gamma$, then $\alpha > \gamma$.

These definitions can be simplified when the rational numbers themselves are given by sections, as explained at the end of § 5.

7. Between any two different rational numbers there is an infinite number of rational numbers. A similar property holds for the system of real numbers, as will now be shown :

(I) *Between any two different real numbers a , a' there are an infinite number of rational numbers.*

If a and a' are rational, the property is known.

If a is rational and a' irrational, let us assume $a > a'$. Let a' be given by the section (A', B') . Then the rational number a is a member of the upper class B' , and B' has no least number. Therefore an infinite number of members of the class B' are less than a . It follows from the definitions of § 5 that there are an infinite number of rational numbers greater than a' and less than a .

A similar proof applies to the case when the irrational number a' is greater than the rational number a .

There remains the case when a and a' are both irrational. Let a be given by the section (A, B) and a' by the section (A', B') . Also let $a > a'$.

Then the class A of a contains an infinite number of members of the class B' of a' ; and these numbers are less than a and greater than a' .

A similar proof applies to the case when $a < a'$.

The result which has just been proved can be made more general :—

(II) *Between any two different real numbers there are an infinite number of irrational numbers.*

Let a , a' be the two given numbers, and suppose $a < a'$.

Take any two rational numbers β and β' , such that $a < \beta < \beta' < a'$. If we can show that between β and β' there must be an irrational number, the theorem is established.

Let i be an irrational number. If this does not lie between β and β' , by adding to it a suitable rational number we can make it do so. For we can find two rational numbers m , n , such that $m < i < n$ and $(n - m)$ is less than $(\beta' - \beta)$. The number $\beta - m + i$ is irrational, and lies between β and β' .

8. Dedekind's Theorem. We shall now prove a very important property of the system of *real** numbers, which will be used frequently in the pages which follow.

If the system of real numbers is divided into two classes A and B, in such a way that

- (i) *each class contains at least one number,*
- (ii) *every number belongs to one class or the other,*
- (iii) *every number in the lower class A is less than every number in the upper class B;*

then there is a number a such that

every number less than a belongs to the lower class A, and every number greater than a belongs to the upper class B.

The separating number a itself may belong to either class.

Consider the rational numbers in A and B.

These form two classes—*e.g.* A' and B' —such that every rational number is in one class or the other, and the numbers in the lower class A' are all less than the numbers in the upper class B' .

As we have seen in § 4, three cases, and only three, can arise.

- (i) *The lower class A' can have a greatest number m and the upper class B' no smallest number.*

The rational number m is the number a of the theorem. For it is clear that every real number a less than m belongs to the class A, since m is a member of this class. Also every real number b , greater than m , belongs to the class B. This is evident if b is rational, since b then belongs to the class B' , and B' is part of B. If b is irrational, we can take a rational number n between m and b . Then n belongs to B, and therefore b does so also.

- (ii) *The upper class B' can have a smallest number m and the lower class A' no greatest number.*

It follows, as above, that the rational number m is the number a of our theorem.

- (iii) *The lower class A' can have no greatest number and the upper class B' no smallest number.*

Let m be the irrational number defined by this section (A' , B'). Every rational number less than m belongs to the class A,

* It will be observed that the system of *rational* numbers does not possess this property.

and every rational number greater than m belongs to the class B.

We have yet to show that every irrational number less than m belongs to the class A, and every irrational number greater than m to the class B.

But this follows at once from § 6. For if m' is an irrational number less than m , we know that there are rational numbers between m and m' . These belong to the class A, and therefore m' does so also.

A similar argument applies to the case when $m' > m$.

In the above discussion the *separating* number a belongs to the lower class, and is rational, in case (i); it belongs to the upper class, and is again rational, in case (ii); it is irrational, and may belong to either class, in case (iii).

9. The Linear Continuum. Dedekind's Axiom. We return now to the straight line L of § 3, in which a definite point O has been taken as origin and a definite segment as the unit of length.

We have seen how to effect a correspondence between the rational numbers and the "rational points" of this line. The "rational points" are the ends of segments obtained by marking

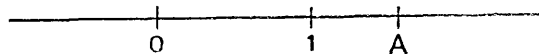


FIG. 2.

off from O on the line lengths equal to multiples or sub-multiples of the unit segment, and the numbers are the measures of the corresponding segments.

Let OA be a segment incommensurable with the unit segment. The point A divides the rational points of the line into two classes, such that all the points of the lower class are to the left of all the points of the upper class. The lower class has no last point, and the upper class no first point.

We then say that A is an *irrational point* of the line, and that the measure of the segment OA is the irrational number defined by this section of the rational numbers.

Thus to any point of the line L corresponds a real number, and to different points of the line correspond different real numbers.

There remains the question—*To every real number does there correspond a point of the line?*

For all rational numbers we can answer the question in the affirmative. When we turn to the irrational numbers, the question amounts to this: *If all the rational points of the line are divided into two classes, a lower and an upper, so that the lower class has no last point and the upper class no first point, is there one, and only one, point on the line which brings about this separation?*

The existence of such a point on the line cannot be proved. The assumption that there is one, and only one, for every section of the rational points is nothing less than an axiom by means of which we assign its continuity to the line.

This assumption is Dedekind's Axiom of Continuity for the line. In adopting it we may now say that *to every point P of the line corresponds a number, rational or irrational, the measure of the segment OP , and that to every real number corresponds a point P of the line, such that the measure of OP is that number.*

The correspondence between the points of the line L (*the linear continuum*) and the system of real numbers (*the arithmetical continuum*) is now perfect. The points can be taken as the images of the numbers, and the numbers as the signs of the points. In consequence of this perfect correspondence, we may, in future, use the terms number and point in this connection as identical.

10. The Development of the System of Real Numbers. It is instructive to see how the idea of the system of real numbers, as we have described it, has grown.* The irrational numbers, belonging as they do in modern arithmetical theory to the realm of arithmetic, arose from the geometrical problems which required their aid. They appeared first as an expression for the ratios of incommensurable pairs of lines. In this sense the Fifth Book of Euclid, in which the general theory of Ratio is developed, and the Tenth Book, which deals with Incommensurable Magnitudes, may be taken as the starting point of the theory. But the irrationalities which Euclid examines are only definite cases of the ratios of incommensurable lines, such

* Cf. Pringsheim, "Irrationalzahlen u. Konvergenz unendlicher Prozesse," *Enc. d. math. Wiss.*, Bd. I., Tl. I., p. 49 *et seq.*, 1898.

as may be obtained with the aid of ruler and compass; that is to say, they depend on square roots alone. The idea that the ratio of any two such incommensurable lines determined a definite (irrational) number did not occur to him, nor to any of the mathematicians of that age.

Although there are traces in the writings of at least one of the mathematicians of the sixteenth century of the idea that every irrational number, just as much as every rational number, possesses a determinate and unique place in the ordered sequence of numbers, these irrational numbers were still considered to arise only from certain cases of evolution, a limitation which is partly due to the commanding position of Euclid's methods in Geometry, and partly to the belief that the problem of finding the n^{th} root of an integer, which lies between the n^{th} powers of two consecutive integers, was the only problem whose solution could not be obtained in terms of rational numbers.

The introduction of the methods of Coordinate Geometry by Descartes in 1637, and the discovery of the Infinitesimal Calculus by Leibnitz and Newton in 1684-7, made mathematicians regard this question in another light, since the applicability of number to spatial magnitude is a fundamental postulate of Coordinate Geometry. "The view now prevailed that number and quantity were *the* objects of mathematical investigation, and that the two were so similar as not to require careful separation. Thus number was applied to quantity without any hesitation, and, conversely, where existing numbers were found inadequate to measurement, new ones were created on the sole ground that every quantity must have a numerical measure." *

It was reserved for the mathematicians of the nineteenth century—notably Weierstrass, Cantor, Dedekind and Heine—to establish the theory on a proper basis. Until their writings appeared, a number was looked upon as an expression for the result of the measurement of a line by another which was regarded as the unit of length. To every segment, or, with the natural modification, to every point, of a line corresponded a definite number, which was either rational or irrational; and by the term irrational number was meant a number defined by an

* Cf. Russell, *Principles of Mathematics*, Ch. XIX., p. 417, 1903.

infinite set of arithmetical operations (*e.g.* infinite decimals or continued fractions). The justification for regarding such an unending sequence of rational numbers as a definite number was considered to be the fact that this system was obtained as the equivalent of a given segment by the aid of the same methods of measurement as those which gave a definite rational number for other segments. However it does not in any way follow from this that, conversely, any arbitrarily given arithmetical representation of this kind can be regarded in the above sense as an irrational number; that is to say, that we can consider as evident the existence of a segment which would produce by suitable measurement the given arithmetical representation. Cantor* has the credit of first pointing out that the assumption that a definite segment must correspond to every such sequence is neither self-evident nor does it admit of proof, but involves an actual axiom of Geometry. Almost at the same time Dedekind showed that the axiom in question (or more exactly one which is equivalent to it) first gave a meaning, which we can comprehend, to that property which, so far without any sufficient definition, had been spoken of as the continuity of the line.

To make the theory of number independent of any geometrical axiom and to place it upon a basis entirely independent of measurable magnitude was the object of the arithmetical theories associated with the names of Weierstrass, Dedekind and Cantor. The theory of Dedekind has been followed in the previous pages. Those of Weierstrass and Cantor, which regard irrational numbers as the limits of convergent sequences, may be deduced from that of Dedekind. In all these theories irrational numbers appear as new numbers, to each of which a definite place in the domain of rational numbers is assigned, and with which we can operate according to definite rules. The ordinary operations of arithmetic for these numbers are defined in such a way as to be in agreement with the ordinary operations upon the rational numbers. They can be used for the representation of definite quantities, and to them can be ascribed definite quantities, according to the axiom of continuity to which we have already referred.

* *Math. Ann.*, Leipzig, 5, p. 127, 1872.

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CHAPTER II

INFINITE SEQUENCES AND SERIES

11. Infinite Aggregates. We are accustomed to speak of the positive integral numbers, the prime numbers, the integers which are perfect squares, etc. These are all examples of infinite sets of numbers or sets which have more than a finite number of terms. In mathematical language they are termed aggregates, and the theory of such infinite aggregates forms an important branch of modern pure mathematics.*

The terms of an aggregate are all different. Their number may be finite or infinite. In the latter case the aggregates are usually called infinite aggregates, but sometimes we shall refer to them simply as aggregates. After the discussion in the previous chapter, there will be no confusion if we speak of an aggregate of points on a line instead of an aggregate of numbers. The two notions are identical. We associate with each number the point of which it is the abscissa. It may happen that, however far we go along the line, there are points of the aggregate further on. In this case we say that it extends to infinity. An aggregate is said to be *bounded on the right*, or *bounded above*, when there is no point of it to the right of some fixed point. It is said to be *bounded on the left*, or *bounded below*, when there is no point of it to the left of some fixed point. The aggregate

*Cantor may be taken as the founder of this theory, which the Germans call *Menge-Lehre*. In a series of papers published from 1870 onward he showed its importance in the Theory of Functions of a Real Variable, and especially in the rigorous discussion of the conditions for the development of an arbitrary function in trigonometric series.

Reference may be made to the standard treatise on the subject by W. H. and Grace Chisholm Young, *Theory of Sets of Points*, Cambridge, 1906.

of rational numbers greater than zero is bounded on the left. The aggregate of rational numbers less than zero is bounded on the right. The aggregate of real positive numbers less than unity is bounded above and below; in such a case we simply say that it is *bounded*. The aggregate of integral numbers is unbounded.

12. The Upper and Lower Bounds of an Aggregate. *When an aggregate $(E)^*$ is bounded on the right, there is a number M which possesses the following properties:*

no number of (E) is greater than M ;

however small the positive number ϵ may be, there is a number of (E) greater than $M - \epsilon$.

We can arrange all the real numbers in two classes, A and B, relative to the aggregate. A number x will be put in the class A if one or more numbers of (E) are greater than x . It will be put in the class B if no number of (E) is greater than x . Since the aggregate is bounded on the right, there are members of both classes, and any number of the class A is smaller than any number of the class B.

By Dedekind's Theorem (§ 8) there is a number M separating the two classes, such that every number less than M belongs to the class A, and every number greater than M to the class B. We shall now show that this is the number M of our theorem.

In the first place, there is no number of (E) greater than M . For suppose there is such a number $M + h$ ($h > 0$). Then the number $M + \frac{1}{2}h$, which is also greater than M , would belong to the class A, and M would not separate the two classes A and B.

In the second place, whatever the positive number ϵ may be, the number $M - \epsilon$ belongs to the class A. It follows from the way in which the class A is defined that there is at least one number of (E) greater than $M - \epsilon$.

This number M is called the *upper bound* of the aggregate (E) . It may belong to the aggregate. This occurs when the aggregate contains a finite number of terms. But when the aggregate contains an infinite number of terms, the upper bound need not belong to it. For example, consider the rational numbers whose

* This notation is convenient, the letter E being the first letter of the French term *ensemble*.

squares are not greater than 2. This aggregate is bounded on the right, its upper bound being the irrational number $\sqrt{2}$, which does not belong to the aggregate. On the other hand, the aggregate of real numbers whose squares are not greater than 2 is also bounded on the right, and has the same upper bound. But $\sqrt{2}$ belongs to this aggregate.

If the upper bound M of the aggregate (E) does not belong to it, there must be an infinite number of terms of the aggregate between M and $M - \epsilon$, however small the positive number ϵ may be. If there were only a finite number of such terms, there would be no term of (E) between the greatest of them and M , which is contrary to our hypothesis.

It can be shown in the same way that *when an aggregate (E) is bounded on the left, there is a number m possessing the following properties:*

no number of (E) is smaller than m ;

however small the positive number ϵ may be, there is a number of (E) less than $m + \epsilon$.

The number m defined in this way is called the *lower bound* of the aggregate (E). As above, it may, or may not, belong to the aggregate when it has an infinite number of terms. But when the aggregate has only a finite number of terms it must belong to it.

13. Limiting Points of an Aggregate. Consider the aggregate

$$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$$

There are an infinite number of points of this aggregate in any interval, however small, extending from the origin to the right. Such a point round which an infinite number of points of an aggregate cluster, is called a *limiting point** of the aggregate. More definitely, *a will be a limiting point of the aggregate (E) if, however small the positive number ϵ may be, there is in (E) a point other than a whose distance from a is less than ϵ .* If there be one such point within the interval $(a - \epsilon, a + \epsilon)$, there will be an infinite number, since, if there were only n of them,

* Sometimes the term *point of condensation* is used; French, *point-limite*, *point d'accumulation*; German, *Häufungspunkt*, *Verdichtungspunkt*.

and α_n were the nearest to α , there would not be in (E) a point other than α whose distance from α was less than $|\alpha - \alpha_n|$.^{*} In that case α would not be a limiting point, contrary to our hypothesis.

An aggregate may have more than one limiting point. The rational numbers between zero and unity form an aggregate with an infinite number of limiting points, since every point of the segment $(0, 1)$ is a limiting point. It will be noticed that some of the limiting points of this aggregate belong to it, and some, namely the irrational points of the segment and its end-points, do not.

In the example at the beginning of this section,

$$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots,$$

the lower bound, zero, is a limiting point, and does not belong to the aggregate. The upper bound, unity, belongs to the aggregate, and is not a limiting point.

The set of real numbers from 0 to 1, inclusive, is an aggregate which is identical with its limiting points.

14. Weierstrass's Theorem. *An infinite aggregate, bounded above and below, has at least one limiting point.*

Let the infinite aggregate (E) be bounded, and have M and m for its upper and lower bounds.

We can arrange all the real numbers in two classes relative to the aggregate (E) . A number x will be said to belong to the class A when an infinite number of terms of (E) are greater than x . It will be said to belong to the class B in the contrary case.

Since m belongs to the class A and M to the class B, there are members of both classes. Also any number in the class A is less than any number in the class B.

By Dedekind's Theorem, there is a number μ separating the two classes. However small the positive number ϵ may be, $\mu - \epsilon$ belongs to the class A, and $\mu + \epsilon$ to the class B. Thus the interval $(\mu - \epsilon, \mu + \epsilon)$ contains an infinite number of terms of the aggregate.

^{*} It is usual to denote the difference between two real numbers a and b , taken positive, by $|a - b|$, and to call it the *absolute value* or *modulus* of $(a - b)$. With this notation $|x + y| \leq |x| + |y|$, and $|xy| = |x||y|$.

Hence μ is a limiting point.

As will be seen from the example of § 13, this point may coincide with M or m .

An infinite aggregate, when unbounded, need not have a limiting point; *e.g.* the set of integers, positive or negative. But if the aggregate has an infinite number of points in an interval of finite length, then it must have at least one limiting point.

15. Convergent Sequences. We speak of an infinite sequence of numbers

$$u_1, u_2, u_3, \dots u_n, \dots$$

when some law is given according to which the general term u_n may be written down.

The sequence u_1, u_2, u_3, \dots

is said to be convergent and to have the limit A , when, by indefinitely increasing n , the difference between A and u_n becomes, and thereafter remains, as small as we please.

This property is so fundamental that it is well to put it more precisely, as follows: *The sequence is said to be convergent and to have the limit A , when, any positive number ϵ having been chosen, as small as we please, there is a positive integer ν such that*

$$|A - u_n| < \epsilon, \text{ provided that } n \geq \nu.$$

For example, the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots$$

has the limit zero, since $1/n$ is less than ϵ for all values of n greater than $1/\epsilon$.

The notation that is employed in this connection is

$$\text{Lt}_{n \rightarrow \infty} u_n = A,$$

and we say that *as n tends to infinity, u_n has the limit A .**

The letter ϵ is usually employed to denote an arbitrarily small positive number, as in the above definition of convergence to a limit as n tends to infinity. Strictly speaking, the words *as small as we please* are unnecessary in the definition, but they are inserted as making clearer the property that is being defined.

We shall very frequently have to employ the form of words which occurs in this definition, or words analogous to them, and

* The phrase " u_n tends to the limit A as n tends to infinity" is also used.

the beginner is advised to make himself familiar with them by formally testing whether the following sequences are convergent or not:

$$(a) \ 1, \ \frac{1}{2}, \ \frac{1}{2^2}, \ \dots \qquad (c) \ 1, \ 1 + \frac{1}{2}, \ 1 + \frac{1}{2} + \frac{1}{2^2}, \ \dots$$

$$(b) \ 1, \ -\frac{1}{2}, \ \frac{1}{3}, \ \dots \qquad (d) \ 1, \ -1, \ 1, \ -1, \ \dots$$

A sequence cannot converge to two distinct limits A and B . If this were possible, let $\epsilon < \frac{|A-B|}{2}$. Then there are only a finite number of terms of the sequence outside the interval $(A - \epsilon, A + \epsilon)$, since the sequence converges to the value A . This contradicts the statement that the sequence has also the limit B , for we would only have a finite number of terms in the interval of the same length with B as centre.

The application of the test of convergency contained in the definition involves the knowledge of the limit A . Thus it will frequently be impossible to use it. The required criterion for the convergence of a sequence, when we are not simply asked to test whether a given number is or is not the limit, is contained in the fundamental *general principle of convergence*:—*

A necessary and sufficient condition for the existence of a limit to the sequence u_1, u_2, u_3, \dots

is that a positive integer ν exists such that $|u_{n+p} - u_n|$ becomes as small as we please when $n \geq \nu$, for every positive integer p .

More exactly:

A necessary and sufficient condition for the existence of a limit to the sequence u_1, u_2, u_3, \dots

is that, if any positive number ϵ has been chosen, as small as we please, there shall be a positive integer ν such that

$$|u_{n+p} - u_n| < \epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p.$$

* This is one of the most important theorems of analysis. In the words of Pringsheim, "Dieser Satz, mit seiner Übertragung auf *beliebige* (z.B. stetige) Zahlenmengen—von du Bois-Reymond als das '*allgemeine Convergenzprinzip*' bezeichnet (Allg. Funct.-Theorie, pp. 6, 260)—ist der eigentliche *Fundamentalsatz der gesamten Analysis* und sollte mit genügender Betonung seines fundamentalen Characters an der Spitze jedes rationellen Lehrbuches der Analysis stehen," *loc. cit.*, *Enc. d. math. Wiss.* p. 66.

We shall first of all show that the condition is *necessary*; *i.e.* if the sequence converges, this condition is satisfied; secondly that, if this condition is satisfied, the sequence converges; in other words, the condition is *sufficient*.

(i) *The condition is necessary.*

Let the sequence converge to the limit A .

Having chosen the arbitrary positive number ϵ , then take $\frac{1}{2}\epsilon$.

We know that there is a positive integer ν such that

$$|A - u_n| < \frac{1}{2}\epsilon, \text{ when } n \geq \nu.$$

But $(u_{n+p} - u_n) = (u_{n+p} - A) + (A - u_n)$.

Therefore $|u_{n+p} - u_n| \leq |u_{n+p} - A| + |A - u_n|$
 $< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon,$
 if $n \geq \nu$, for every positive integer p ,
 $< \epsilon.$

(ii) *The condition is sufficient.*

We must examine two cases; first, when the sequence contains an infinite number of terms equal to one another; second, when it does not.

(a) Let there be an infinite number of terms equal to A .

Then, if

$$|u_{n+p} - u_n| < \epsilon, \text{ when } n \geq \nu, \text{ and } p \text{ is any positive integer,}$$

we may take $u_{n+p} = A$ for some value of p , and we have

$$|A - u_n| < \epsilon, \text{ when } n \geq \nu.$$

Therefore the sequence converges, and has A for its limit.

(b) Let there be only a finite number of terms equal to one another.

Having chosen the arbitrary positive number ϵ , then take $\frac{1}{2}\epsilon$.

We know that there is a positive integer N such that

$$|u_{n+p} - u_n| < \frac{1}{2}\epsilon, \text{ when } n \geq N, \text{ for every positive integer } p.$$

It follows that we have

$$|u_n - u_N| < \frac{1}{2}\epsilon, \text{ when } n \geq N.$$

Therefore all the terms of the sequence

$$u_{N+1}, \quad u_{N+2}, \quad u_{N+3}, \quad \dots$$

lie within the interval whose end-points are $u_N - \frac{1}{2}\epsilon$ and $u_N + \frac{1}{2}\epsilon$.

There must be an infinite number of distinct terms in this sequence. Otherwise we would have an infinite number of terms equal to one another.

Consider the infinite aggregate (E) formed by the distinct terms in

$$u_1, u_2, u_3, \dots$$

This aggregate is bounded and must have at least one limiting point A within, or at an end of, the above interval. (Cf. § 14.)

There cannot be another limiting point A' , for if there were, we could choose ϵ equal to $\frac{1}{4}|A - A'|$ say, and the formula

$$|u_{n+p} - u_n| < \epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p,$$

shows that all the terms of the sequence

$$u_1, u_2, u_3, \dots,$$

except a finite number, would lie within an interval of length $\frac{1}{2}|A - A'|$. This is impossible if A, A' are limiting points of the aggregate.

Thus the aggregate (E) has one and only one limiting point A .

We shall now show that the sequence

$$u_1, u_2, u_3, \dots$$

converges to A as n tends to ∞ .

We have $u_n - A = (u_n - u_N) + (u_N - A)$.

$$\begin{aligned} \text{Therefore } |u_n - A| &\leq |u_n - u_N| + |u_N - A| \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon, & \text{when } n \geq N, \\ &< \epsilon, & \text{when } n \geq N. \end{aligned}$$

Thus the sequence converges, and has A for its limit.

We have therefore proved this theorem :

A necessary and sufficient condition for the convergence of the sequence

$$u_1, u_2, u_3, \dots$$

is that, to the arbitrary positive number ϵ , there shall correspond a positive integer ν such that

$$|u_{n+p} - u_n| < \epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p.$$

It is easy to show that the above condition may be replaced by the following :

In order that the sequence

$$u_1, u_2, u_3, \dots$$

may converge, it is necessary and sufficient that, to the arbitrary positive number ϵ , there shall correspond a positive integer n such that

$$|u_{n+p} - u_n| < \epsilon \text{ for every positive integer } p.$$

It is clear that if the sequence converges, this condition is satisfied by $n = \nu$.

Further, if this condition is satisfied, and ϵ is an arbitrary positive number, to the number $\frac{1}{2}\epsilon$ there corresponds a positive integer n such that

$$|u_{n+p} - u_n| < \frac{1}{2}\epsilon \text{ for every positive integer } p.$$

$$\begin{aligned} \text{But } |u_{n+p'} - u_{n+p''}| &\leq |u_{n+p''} - u_n| + |u_{n+p'} - u_n| \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon, \end{aligned}$$

when p', p'' are any positive integers.

Therefore the condition in the text is also satisfied, and the sequence converges.

16. Divergent and Oscillatory Sequences.* When the sequence

$$u_1, u_2, u_3, \dots$$

does not converge, several different cases arise.

(i) In the first place, the terms may have the property that if any positive number A , however large, is chosen, there is a positive integer ν such that

$$u_n > A, \text{ when } n \geq \nu.$$

In this case we say that the sequence is *divergent*, and that it *diverges to* $+\infty$, and we write this

$$\text{Lt}_{n \rightarrow \infty} u_n = +\infty.$$

(ii) In the second place, the terms may have the property that if any negative number $-A$ is chosen, however large A may be, there is a positive integer ν such that

$$u_n < -A, \text{ when } n \geq \nu.$$

In this case we say that the sequence is *divergent*, and that it *diverges to* $-\infty$, and we write this

$$\text{Lt}_{n \rightarrow \infty} u_n = -\infty.$$

The terms of a sequence may all be very large in absolute value, when n is very large, yet the sequence may not diverge to $+\infty$ or to $-\infty$. A sufficient illustration of this is given by the sequence whose general term is $(-1)^n n$.

* In the first edition of this book, the term *divergent* was used as meaning merely not convergent. In this edition the term is applied only to the case of divergence to $+\infty$ or to $-\infty$, and sequences which oscillate infinitely are placed among the oscillatory sequences.

After some value of n the terms must all have the same sign, if the sequence is to diverge to $+\infty$ or to $-\infty$, the sign being positive in the first alternative, and negative in the second.

(iii) *When the sequence does not converge, and does not diverge to $+\infty$ or to $-\infty$, it is said to oscillate.*

An oscillatory sequence is said to oscillate finitely, if there is a positive number A such that $|u_n| < A$, for all values of n ; and it is said to oscillate infinitely when there is no such number. For example, the sequence whose general term is $(-1)^n$ oscillates finitely; the sequence whose general term is $(-1)^n n$ oscillates infinitely.

We may distinguish between convergent and divergent sequences by saying that *a convergent sequence has a finite limit, i.e. $\text{Lt}_{n \rightarrow \infty} u_n = A$, where A is a definite number; a divergent sequence has an infinite limit, i.e. $\text{Lt}_{n \rightarrow \infty} u_n = +\infty$ or $\text{Lt}_{n \rightarrow \infty} u_n = -\infty$.*

But it must be remembered that the symbol ∞ , and the terms *infinite, infinity* and *tend to infinity*, have purely conventional meanings. **There is no number infinity.** Phrases in which the term is used have only a meaning for us when we have previously, by definition, attached a meaning to them.

When we say that *n tends to infinity*, we are using a short and convenient phrase to express the fact that n assumes an endless series of values which eventually become and remain greater than any arbitrary (large) positive number. So far we have supposed n , in this connection, to advance through integral values only. This restriction will be removed later.

A similar remark applies to the phrases *divergence to $+\infty$ or to $-\infty$* , and *oscillating infinitely*, as well as to our earlier use of the terms *an infinite number, infinite sequence* and *infinite aggregates*. In each case a definite meaning has been attached to the term, and it is employed only with that meaning.

It is true that much of our work might be simplified by the introduction of new numbers $+\infty$, $-\infty$, and by assuming the existence of corresponding points upon the line which we have used as the domain of the numbers. But the creation of these numbers, and the introduction of these points, would be a matter for separate definition.

17. Monotonic Sequences. *If the terms of the sequence*

$$u_1, u_2, u_3, \dots$$

satisfy either of the following relations

$$u_1 \leq u_2 \leq u_3 \dots \leq u_n, \dots$$

or

$$u_1 \geq u_2 \geq u_3 \dots \geq u_n, \dots,$$

*the sequence is said to be **monotonic**.*

In the first case, the terms never decrease, and the sequence may be called *monotonic increasing*; in the second case, the terms never increase, and the sequence may be called *monotonic decreasing*.*

Obviously, when we are concerned with the convergence or divergence of a sequence, the monotonic property, if such exist, need not enter till after a certain stage.

The tests for convergence or divergence are extremely simple in the case of monotonic sequences.

If the sequence u_1, u_2, u_3, \dots

is monotonic increasing, and its terms are all less than some fixed number B , the sequence is convergent and has for its limit a number β such that $u_n \leq \beta \leq B$ for every positive integer n .

Consider the aggregate formed by distinct terms of the sequence. It is bounded by u_1 on the left and by B on the right. Thus it must have an upper bound β (cf. § 12) equal to or less than B , and, however small the positive number ϵ may be there will be a term of the sequence greater than $\beta - \epsilon$.

Let this term be u_ν . Then all the terms after $u_{\nu-1}$ are to the right of $\beta - \epsilon$ and not to the right of β . If any of them coincide with β , from that stage on the terms must be equal.

Thus we have shown that

$$|\beta - u_n| < \epsilon, \text{ when } n \geq \nu,$$

and therefore the sequence is convergent and has β for its limit.

The following test may be proved in the same way:

If the sequence u_1, u_2, u_3, \dots

is monotonic decreasing, and its terms are all greater than some fixed number A , then the sequence is convergent and has for

* The words *steadily increasing* and *steadily decreasing* are sometimes employed in this connection, and when none of the terms of the sequence are equal, the words *in the stricter sense* are added.

its limit a number a such that $u_n \geq a \geq A$ for every positive integer n .

It is an immediate consequence of these theorems that a monotonic sequence either tends to a limit or diverges to $+\infty$ or to $-\infty$.

18. Let A_1, A_2, A_3, \dots be an infinite set of intervals, each lying entirely within the preceding, or lying within it and having with it a common end-point; also let the length of A_n tend to zero as n tends to infinity. Then there is one, and only one, point which belongs to all the intervals, either as an internal point of all, or from and after a definite stage, as a common end-point of all.

Let the representative interval A_n be given by

$$a_n \leq x \leq b_n.$$

Then we have $a_1 \leq a_2 \leq a_3 \dots < b_1$,

and $b_1 \geq b_2 \geq b_3 \dots > a_1$.

Thus the sequence of end-points

$$a_1, a_2, a_3 \dots \dots \dots (1)$$

has a limit, say a , and $a_n \leq a$ for every positive integer n (§ 17).

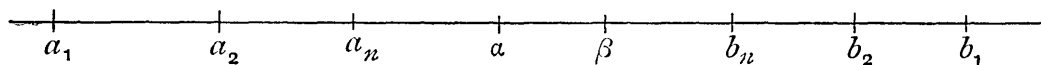


FIG. 3.

Also the sequence of end-points

$$b_1, b_2, b_3 \dots, \dots \dots (2)$$

has a limit, say β , and $b_n \geq \beta$ for every positive integer n (§ 17).

Now it is clear that, under the given conditions, β cannot be less than a .

Therefore, for every value of n ,

$$b_n - a_n > \beta - a \geq 0.$$

But $\text{Lt}_{n \rightarrow \infty} (b_n - a_n) = 0$.

It follows that $a = \beta$.*

* This result also follows at once from the fact that, if $\text{Lt } a_n = a$ and $\text{Lt } b_n = \beta$, then $\text{Lt } (a_n - b_n) = a - \beta$. (Cf. § 26, Theorem I.)

Therefore this common limit of the sequences (1) and (2) satisfies the inequalities

$$a_n \leq a \leq b_n \text{ for every positive integer } n,$$

and thus belongs to all the intervals.

Further, no other point (e.g. γ) can satisfy $a_n \leq \gamma \leq b_n$ for all values of n .

Since we would have at the same time

$$\lim_{n \rightarrow \infty} a_n \leq \gamma \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n \geq \gamma,$$

which is impossible unless $\gamma = a$.

19. The Sum of an Infinite Series.

Let u_1, u_2, u_3, \dots

be an infinite sequence, and let the successive sums

$$\begin{aligned} S_1 &= u_1, \\ S_2 &= u_1 + u_2, \\ &\dots\dots\dots \\ S_n &= u_1 + u_2 + u_3 + \dots + u_n \end{aligned}$$

be formed.

If the sequence S_1, S_2, S_3, \dots

is convergent and has the limit S , then S is called the sum of the infinite series $u_1 + u_2 + u_3 + \dots$

and this series is said to be convergent.

It must be carefully noted that what we call the sum of the infinite series is a limit, the limit of the sum of n terms of

$$u_1 + u_2 + u_3 + \dots,$$

as n tends to infinity. Thus we have no right to assume without proof that familiar properties of finite sums are necessarily true for sums such as S .

When $\lim_{n \rightarrow \infty} S_n = +\infty$ or $\lim_{n \rightarrow \infty} S_n = -\infty$, we shall say that the infinite series is *divergent*, or *diverges to $+\infty$ or $-\infty$* , as the case may be.

If S_n does not tend to a limit, or to $+\infty$ or to $-\infty$, then it oscillates finitely or infinitely according to the definitions of these terms in §16. In this case we shall say that the series *oscillates finitely or infinitely*.*

* Cf. footnote, p. 37.

The conditions obtained in §15 for the convergence of a sequence allow us to state the criteria for the convergence of the series in either of the following ways :

(i) *The series converges and has S for its sum, if, any positive number ϵ having been chosen, as small as we please, there is a positive integer ν such that*

$$|S - S_n| < \epsilon, \text{ when } n \geq \nu.$$

(ii) *A necessary and sufficient condition for the convergence of the series is that, if any positive number ϵ has been chosen, as small as we please, there shall be a positive integer ν such that*

$$|S_{n+p} - S_n| < \epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p.*$$

It is clear that, *if the series converges, $\text{Lt}_{n \rightarrow \infty} u_n = 0$* . This is contained in the second criterion. It is a *necessary* condition for convergence, but it is not a *sufficient* condition; e.g. the series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is divergent, though $\text{Lt}_{n \rightarrow \infty} u_n = 0$.

If we denote

$$u_{n+1} + u_{n+2} + \dots + u_{n+p}, \quad \text{or} \quad S_{n+p} - S_n, \quad \text{by } {}_pR_n$$

the above necessary and sufficient condition for convergence of the series may be written

$$|{}_pR_n| < \epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p.$$

Again, *if the series $u_1 + u_2 + u_3 + \dots$ converges and has S for its sum, the series*

$$u_{n+1} + u_{n+2} + u_{n+3} + \dots$$

converges and has $S - S_n$ for its sum.

For we have $S_{n+p} = S_n + {}_pR_n$.

Also keeping n fixed, it is clear that

$$\text{Lt}_{p \rightarrow \infty} S_{n+p} = S.$$

Therefore

$$\text{Lt}_{p \rightarrow \infty} ({}_pR_n) = S - S_n.$$

* As remarked in §15, this condition can be replaced by: *To the arbitrary positive number ϵ there must correspond a positive integer n such that*

$$|S_{n+p} - S_n| < \epsilon \text{ for every positive integer } p.$$

Thus if we write R_n for the sum of the series

$$u_{n+1} + u_{n+2} + \dots,$$

we have

$$S = S_n + R_n.$$

The first criterion for convergence can now be put in the form

$$|R_n| < \epsilon, \text{ when } n \geq \nu.$$

R_n is usually called the *remainder* of the series after n terms, and ${}_p R_n$ a *partial remainder*.

20. Series whose Terms are all Positive.

Let $u_1 + u_2 + u_3 + \dots$

be a series whose terms are all positive. The sum of n terms of this series either tends to a limit, or it diverges to $+\infty$.

Since the terms are all positive, the successive sums

$$\begin{aligned} S_1 &= u_1, \\ S_2 &= u_1 + u_2, \\ S_3 &= u_1 + u_2 + u_3, \\ &\dots\dots\dots \end{aligned}$$

form a monotonic increasing sequence, and the theorem stated above follows from § 17.

When a series whose terms are all positive is convergent, the series we obtain when we take the terms in any order we please is also convergent and has the same sum.

This change of the order of the terms is to be such that there will be a one-one correspondence between the terms of the old series and the new. The term in any assigned place in the one series is to have a definite place in the other.

Let $S_1 = u_1,$
 $S_2 = u_1 + u_2,$
 $S_3 = u_1 + u_2 + u_3,$
 $\dots\dots\dots$

Then the aggregate (U) , which corresponds to the sequence

$$S_1, S_2, S_3, \dots,$$

is bounded and its upper bound S is the sum of the series.

Let (U') be the corresponding aggregate for the series obtained by taking the terms in any order we please, on the understanding we have explained above. Every number in

(U') is less than S . In addition, if A is any number less than S , there must be a number of (U) greater than A , and *a fortiori* a number of (U') greater than A . The aggregate (U') is thus bounded on the right, and its upper bound is S . The sum of the new series is therefore the same as the sum of the old.

It follows that *if the series*

$$u_1 + u_2 + u_3 + \dots,$$

whose terms are all positive, diverges, the series we obtain by changing the order of the terms must also diverge.

The following theorems may be proved at once by the use of the second condition for convergence (§ 19):

If the series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and all its terms are positive, the series we obtain from this, either

(1) *by keeping only a part of its terms,*

or (2) *by replacing certain of its terms by others, either positive or zero, which are respectively equal or inferior to them,*

or (3) *by changing the signs of some of its terms,*
are also convergent.

21. Absolute and Conditional Convergence. The trigonometrical series, whose properties we shall investigate later, belong to the class of series whose convergence is due to the presence of both positive and negative terms, in the sense that the series would diverge if all the terms were taken with the same sign.

A series with positive and negative terms is said to be absolutely convergent, when the series in which all the terms are taken with the same sign converges.

In other words, the series

$$u_1 + u_2 + u_3 + \dots$$

is absolutely convergent when the series of absolute values

$$|u_1| + |u_2| + |u_3| + \dots$$

is convergent.

It is obvious that an absolutely convergent series is also convergent in the ordinary sense, since the absolute values of the partial remainders of the original series cannot be greater than

those of the second series. There are, however, convergent series which are not absolutely convergent :

e.g. $1 - \frac{1}{2} + \frac{1}{3} \dots$ is convergent.

$1 + \frac{1}{2} + \frac{1}{3} \dots$ is divergent.

Series in which the convergence depends upon the presence of both positive and negative terms are said to be conditionally convergent.

The reason for this name is that, as we shall now prove, an *absolutely convergent series* remains convergent, and has the same sum, even although we alter the order in which its terms are taken; while a *conditionally convergent series* may converge for one arrangement of the terms and diverge for another. Indeed we shall see that we can make a conditionally convergent series have any sum we please, or be greater than any number we care to name, by changing the order of its terms. There is nothing very extraordinary in this statement. The rearrangement of the terms introduces a new function of n , say S'_n , instead of the old function S_n , as the sum of the first n terms. There is no *a priori* reason why this function S'_n should have a limit as n tends to infinity, or, if it has a limit, that this should be the same as the limit of S_n .*

22. Absolutely Convergent Series. *The sum of an absolutely convergent series remains the same when the order of the terms is changed.*

Let (S) be the given absolutely convergent series; (S') the series formed with the *positive* terms of (S) in the order in which they appear; (S'') the series formed with the absolute values of the *negative* terms of (S) , also in the order in which they appear.

If the number of terms either in (S') or (S'') is limited, the theorem requires no proof, since we can change the order of the terms in the finite sum, which includes the terms of (S) up to the last of the class which is limited in number, without altering its sum, and we have just seen that when the terms are of the same sign, as in those which follow, the alteration in the order in the convergent series does not affect its sum.

* Cf. Osgood, *Introduction to Infinite Series*, p. 44, 1897.

Let Σ be the sum of the infinite series formed by the absolute values of the terms of (S) .

Let S_n be the sum of the first n terms of (S) :

In this sum let n' terms be positive and n'' negative.

Let $S_{n'}$ be the sum of these n' terms.

Let $S_{n''}$ be the sum of the absolute values of these n'' terms, taking in each case these terms in the order in which they appear in (S) .

Then
$$S_n = S_{n'} - S_{n''},$$

$$S_{n'} < \Sigma,$$

$$S_{n''} < \Sigma.$$

Now, as n increases $S_{n'}$, $S_{n''}$ never diminish. Thus, as n increases without limit, the successive values of $S_{n'}$, $S_{n''}$ form two infinite monotonic sequences such as we have examined in § 17, whose terms do not exceed the fixed number Σ . These sequences, therefore, tend to fixed limits, say, S' and S'' .

Thus
$$\text{Lt}_{n \rightarrow \infty} (S_n) = S' - S''.$$

Hence the sum of the absolutely convergent series (S) is equal to the difference between the sums of the two infinite series formed one with the positive terms in the order in which they appear, and the other with the absolute values of the negative terms, also in the order in which they appear in (S) .

Now any alteration in the order of the terms of (S) does not change the values of S' and S'' ; since we have seen that in the case of a convergent series whose terms are all positive we do not alter the sum by rearranging the terms. It follows that *(S) remains convergent and has the same sum when the order of its terms is changed in any way we please, provided that a one-one correspondence exists between the terms of the old series and the new.*

We add some other results with regard to absolutely convergent series which admit of simple demonstration:

Any series whose terms are either equal or inferior in absolute value to the corresponding terms of an absolutely convergent series is also absolutely convergent.

An absolutely convergent series remains absolutely convergent when we suppress a certain number of its terms.

$$\begin{aligned} \text{If} \quad & u_1 + u_2 + \dots, \\ & v_1 + v_2 + \dots, \end{aligned}$$

are two absolutely convergent series whose sums are U and V , the series

$$(u_1 + v_1) + (u_2 + v_2) + \dots$$

and

$$(u_1 - v_1) + (u_2 - v_2) + \dots$$

are also absolutely convergent and their sums are equal to $U \pm V$ respectively.

23. Conditionally Convergent Series. *The sum of a conditionally convergent series depends essentially on the order of its terms.*

Let (S) be such a series. The positive and negative terms must both be infinite in number, since otherwise the series would converge absolutely.

Further, the series formed by the positive terms in the order in which they occur in (S) , and the series formed in the same way by the negative terms, must both be *divergent*.

Both could not *converge*, since in that case our series would be equal to the difference of two absolutely convergent series, some of whose terms might be zero, and therefore would be absolutely convergent (§ 22). Also (S) could not converge, if one of these series converged and the other diverged.

We can therefore take sufficient terms from the positive terms to make their sum exceed any positive number we care to name. In the same way we can take sufficient terms from the negative terms to make the sum of their absolute values exceed any number we care to name.

Let a be any positive number.

First take positive numbers from (S) in the order in which they appear, stopping whenever the sum is greater than a . Then take negative terms from (S) , in the order in which they appear, stopping whenever the combined sum is less than a . Then add on as many from the remaining positive terms as will make the sum exceed a , stopping when the sum first exceeds a ; and then proceed to the negative terms; and so on.

In this way we form a new series (S') composed of the same terms as (S) , in which the sum of n terms is sometimes greater than a and sometimes less than a .

Now the series (S) converges. Let its terms be u_1, u_2, u_3, \dots . Then, with the usual notation,

$$|u_n| < \epsilon, \text{ when } n \geq \nu.$$

Let the points B_ν and A_ν (Fig. 4) correspond to the sums obtained in (S') , as described above, when ν groups of positive terms and ν groups of negative terms have been taken.

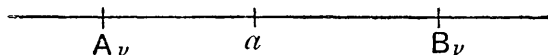


FIG. 4.

Then it is clear that $(a - A_\nu)$ and $(B_\nu - a)$ are each less than $|u_\nu|$, since each of these groups contains at least one term of (S) , and $(a - A_\nu)$, $(B_\nu - a)$ are at most equal to the absolute value of the last term in each group.

Let these 2ν groups contain in all ν' terms.

The term $u'_{n'}$ in (S') , when $n' \geq \nu'$, is less in absolute value than ϵ . Thus, if we proceed from A_ν , the sums $S'_{n'}$ lie within the interval $(a - \epsilon, a + \epsilon)$, when $n' \geq \nu'$.

In other words, $|S'_{n'} - a| < \epsilon$, when $n' \geq \nu'$.

Therefore $\text{Lt}_{n' \rightarrow \infty} S'_{n'} = a$.

A similar argument holds for the case of a negative number, the only difference being that now we begin with the negative terms of the series.

We have thus established the following theorem:

If a conditionally convergent series is given, we can arrange the order of the terms as to make the sum of the n series converge to any value we care to name.

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CHAPTER III

FUNCTIONS OF A SINGLE VARIABLE LIMITS AND CONTINUITY

24. The Idea of a Function. In Elementary Mathematics, when we speak of a function of x , we usually mean a real expression obtained by certain operations, *e.g.* x^2 , \sqrt{x} , $\log x$, $\sin^{-1}x$. In some cases, from the nature of the operations, the range of the variable x is indicated. In the first of the above examples, the range is unlimited; in the second, $x \geq 0$; in the third $x > 0$; and in the last $0 \leq x \leq 1$.

In Higher Mathematics the term "function of x " has a much more general meaning. Let a and b be any two real numbers, where $b > a$. If to every value of x in the interval $a \leq x \leq b$ there corresponds a (real) number y , then we say that y is a function of x in the interval (a, b) , and we write $y = f(x)$.

Sometimes the end-points of the interval are excluded from the domain of x , which is then given by $a < x < b$. In this case the interval is said to be *open* at both ends; when both ends are included (*i.e.* $a \leq x \leq b$) it is said to be *closed*. An interval may be *open* at one end and *closed* at the other (*e.g.* $a < x \leq b$).

Unless otherwise stated, when we speak of an interval in the rest of this work, we shall refer to an interval *closed* at both ends. And when we say that x lies in the interval (a, b) , we mean that $a \leq x \leq b$, but when x is to lie between a and b , and not to coincide with either, we shall say that x lies in the *open interval* (a, b) .*

* In Ch. II., when a point x lies between a and b , and does not coincide with either, we have referred to it as *within* the interval (a, b) . This form of words is convenient, and not likely to give rise to confusion.

Consider the aggregate formed by the values of a function $f(x)$, given in an interval (a, b) . If this aggregate is bounded (cf. § 11), we say that the function $f(x)$ is *bounded* in the interval. The numbers M and m , the *upper* and *lower bounds* of the aggregate (cf. § 12), are called the *upper* and *lower bounds* of the function in the interval. And a function can have an upper bound and no lower bound, and *vice versa*.

The difference $(M - m)$ is called the *oscillation of the function in the interval*.

It should be noticed that a function may be determinate in an interval, and yet not bounded in the interval.

E.g. let $f(0) = 0$, and $f(x) = \frac{1}{x}$ when $x > 0$.

Then $f(x)$ has a definite value for every x in the interval $0 \leq x \leq a$, where a is any given positive number. But $f(x)$ is not bounded in this interval, for we can make $f(x)$ exceed any number we care to name, by letting x approach sufficiently near to zero.

Further, a bounded function need not attain its upper and lower bounds; in other words, M and m need not be members of the aggregate formed by the values of $f(x)$ in the interval.

E.g. let $f(0) = 0$, and $f(x) = 1 - x$ when $0 < x \leq 1$.

This function, given in the interval $(0, 1)$, attains its lower bound zero, but not its upper bound unity.

25. Lt $f(x)$. In the previous chapter we have dealt with the limit when $n \rightarrow \infty$ of a sequence u_1, u_2, u_3, \dots . In other words, we have been dealing with a function $\phi(n)$, where n is a positive integer, and we have considered the limit of this function as $n \rightarrow \infty$.

We pass now to the function of the real variable x and the limit of $f(x)$ when $x \rightarrow a$. The idea is familiar enough. The Differential Calculus rests upon it. But for our purpose we must put the matter on a precise arithmetical footing, and a definition of what exactly is meant by the limit of a function of x , as x tends to a definite value, must be given.

$f(x)$ is said to have the limit b as x tends to a , when, any positive number ϵ having been chosen, as small as we please,

there is a positive number η such that $|f(x) - b| < \epsilon$, for all values of x for which $0 < |x - a| \leq \eta$.

In other words $|f(x) - b|$ must be less than ϵ for all points in the interval $(a - \eta, a + \eta)$ except the point a .

When this condition is satisfied, we employ the notation $\text{Lt}_{x \rightarrow a} f(x) = b$, for the phrase *the limit of $f(x)$, as x tends to a , is b* , and we say that $f(x)$ converges to b as x tends to a .

One advantage of this notation, as opposed to $\text{Lt}_{x=a} f(x) = b$, is that it brings out the fact that we say nothing about what happens when x is equal to a . In the definition it will be observed that a statement is made about the behaviour of $f(x)$ for all values of x such that $0 < |x - a| \leq \eta$. The first of these inequalities is inserted expressly to exclude $x = a$.

Sometimes x tends to a from the right hand only (*i.e.* $x > a$), or from the left hand only (*i.e.* $x < a$).

In these cases, instead of $0 < |x - a| \leq \eta$, we have $0 < (x - a) \leq \eta$ (right hand) and $0 < (a - x) \leq \eta$ (left hand), in the definition.

The notation adopted for these right-hand and left-hand limits is

$$\text{Lt}_{x \rightarrow a+0} f(x) \quad \text{and} \quad \text{Lt}_{x \rightarrow a-0} f(x).$$

The assertion that $\text{Lt}_{x \rightarrow a} f(x) = b$ thus includes

$$\text{Lt}_{x \rightarrow a+0} f(x) = \text{Lt}_{x \rightarrow a-0} f(x) = b.$$

It is convenient to use $f(a+0)$ for $\text{Lt}_{x \rightarrow a+0} f(x)$ when this limit exists, and similarly $f(a-0)$ for $\text{Lt}_{x \rightarrow a-0} f(x)$ when this limit exists.

When $f(x)$ has not a limit as $x \rightarrow a$, it may happen that it diverges to $+\infty$, or to $-\infty$, in the sense in which these terms were used in §16. Or, more precisely, it may happen that *if any positive number A , however large, is chosen, there corresponds to it a positive number η such that*

$$f(x) > A, \text{ when } 0 < |x - a| \leq \eta.$$

In this case we say that $\text{Lt}_{x \rightarrow a} f(x) = +\infty$.

Again, it may happen that *if any negative number $-A$ is chosen, however large A may be, there corresponds to it a positive number η such that $f(x) < -A$, when $0 < |x - a| \leq \eta$.*

In this case we say that $\text{Lt}_{x \rightarrow a} f(x) = -\infty$.

The modifications when $f(a \pm 0) = \pm \infty$ are obvious.

When $\text{Lt}_{x \rightarrow a} f(x)$ does not exist, and when $f(x)$ does not *diverge* to $+\infty$, or to $-\infty$, as $x \rightarrow a$, it is said to *oscillate* as $x \rightarrow a$. It *oscillates finitely* if $f(x)$ is bounded in some neighbourhood of that point.* It *oscillates infinitely* if there is no neighbourhood of a in which $f(x)$ is bounded. (Cf. § 16.)

The modifications to be made in these definitions when $x \rightarrow a$ only from the right, or only from the left, are obvious.

26. Some General Theorems on Limits. I. The Limit of a Sum.

If $\text{Lt}_{x \rightarrow a} f(x) = a$ and $\text{Lt}_{x \rightarrow a} g(x) = \beta$, then $\text{Lt}_{x \rightarrow a} [f(x) + g(x)] = a + \beta$.†

Let the positive number ϵ be chosen, as small as we please. Then to $\epsilon/2$ there correspond the positive numbers η_1, η_2 such that

$$|f(x) - a| < \frac{\epsilon}{2}, \text{ when } 0 < |x - a| \leq \eta_1,$$

$$|g(x) - \beta| < \frac{\epsilon}{2}, \text{ when } 0 < |x - a| \leq \eta_2.$$

Thus, if η is not greater than η_1 or η_2 ,

$$|f(x) + g(x) - a - \beta| \leq |f(x) - a| + |g(x) - \beta|,$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}, \text{ when } 0 < |x - a| \leq \eta,$$

$$< \epsilon, \text{ when } 0 < |x - a| \leq \eta.$$

Therefore $\text{Lt}_{x \rightarrow a} [f(x) + g(x)] = a + \beta$.

This result can be extended to the sum of any number of functions. The *Limit of a Sum* is equal to the *Sum of the Limits*.

* $f(x)$ is said to satisfy a certain condition *in the neighbourhood of* $x = a$ when there is a positive number h such that the condition is satisfied when $0 < |x - a| \leq h$.

Sometimes the *neighbourhood* is meant to include the point $x = a$ itself. In this case it is defined by $|x - a| \leq h$.

† The corresponding theorem for functions of the positive integer n , as $n \rightarrow \infty$, is proved in the same way, and is useful in the argument of certain sections of the previous chapter.

II. **The Limit of a Product.** If $\text{Lt}_{x \rightarrow a} f(x) = a$ and $\text{Lt}_{x \rightarrow a} g(x) = \beta$,
then $\text{Lt}_{x \rightarrow a} [f(x)g(x)] = a\beta$.

Let $f(x) = a + \phi(x)$ and $g(x) = \beta + \psi(x)$.

Then $\text{Lt}_{x \rightarrow a} \phi(x) = 0$ and $\text{Lt}_{x \rightarrow a} \psi(x) = 0$.

Also $f(x)g(x) = a\beta + \beta\phi(x) + a\psi(x) + \phi(x)\psi(x)$.

From Theorem I. our result follows if $\text{Lt}_{x \rightarrow a} [\phi(x)\psi(x)] = 0$.

Since $\phi(x)$ tends to zero as $x \rightarrow a$ and $\psi(x)$ tends to zero as $x \rightarrow a$, a proof of this might appear unnecessary. But if a formal proof is required, it could run as follows:

Given the arbitrary position number ϵ , we have, as in I.,

$$\begin{aligned} |\phi(x)| &< \sqrt{\epsilon}, \text{ when } 0 < |x-a| \leq \eta_1, \\ |\psi(x)| &< \sqrt{\epsilon}, \text{ when } 0 < |x-a| \leq \eta_2. \end{aligned}$$

Thus, if η is not greater than η_1 or η_2 ,

$$|\phi(x)\psi(x)| < \epsilon, \text{ when } 0 < |x-a| \leq \eta;$$

Therefore $\text{Lt}_{x \rightarrow a} [\phi(x)\psi(x)] = 0$.

This result can be extended to any number of functions. The *Limit of a Product* is equal to the *Product of the Limits*.

III. **The Limit of a Quotient.**

(i) If $\text{Lt}_{x \rightarrow a} f(x) = a \neq 0$, then $\text{Lt}_{x \rightarrow a} \frac{1}{f(x)} = \frac{1}{a}$.

This follows easily on putting $f(x) = \phi(x) + a$ and examining the expression

$$\frac{1}{a} - \frac{1}{\phi(x) + a}.$$

(ii) If $\text{Lt}_{x \rightarrow a} f(x) = a$, and $\text{Lt}_{x \rightarrow a} g(x) = \beta \neq 0$, then $\text{Lt}_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{a}{\beta}$.

This follows from II. and III. (i).

This result can obviously be generalised as above.

IV. **The Limit of a Function of a Function.** $\text{Lt}_{x \rightarrow a} f[\phi(x)]$.

Let $\text{Lt}_{x \rightarrow a} \phi(x) = b$ and $\text{Lt}_{u \rightarrow b} f(u) = f(b)$.

Then $\text{Lt}_{x \rightarrow a} f[\phi(x)] = f[\text{Lt}_{x \rightarrow a} \phi(x)]$.

We are given that $\text{Lt}_{u \rightarrow b} f(u) = f(b)$.

Therefore to the arbitrary positive number ϵ there corresponds a positive number η_1 such that

$$|f[\phi(x)] - f(b)| < \epsilon, \text{ when } |\phi(x) - b| \leq \eta_1. \dots\dots\dots(1)$$

Also we are given that $\text{Lt}_{x \rightarrow a} \phi(x) = b$.

Therefore to this positive number η_1 there corresponds a positive number η such that

$$|\phi(x) - b| < \eta_1, \text{ when } 0 < |x - a| \leq \eta. \dots\dots\dots(2)$$

Combining (1) and (2), to the arbitrary positive number ϵ there corresponds a positive number η such that

$$|f[\phi(x)] - f(b)| < \epsilon, \text{ when } 0 < |x - a| \leq \eta.$$

Thus
$$\text{Lt}_{x \rightarrow a} f[\phi(x)] = f(b) = f[\text{Lt}_{x \rightarrow a} \phi(x)].$$

EXAMPLES.

1. If n is a positive integer, $\text{Lt}_{x \rightarrow 0} x^n = 0$.
 2. If n is a negative integer, $\text{Lt}_{x \rightarrow +0} x^n = +\infty$; and $\text{Lt}_{x \rightarrow -0} x^n = -\infty$ or $+\infty$ according as n is odd or even.
[If $n=0$, then $x^n = 1$ and $\text{Lt}_{x \rightarrow 0} x^n = 1$.]
 3. $\text{Lt}_{x \rightarrow 0} (a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n) = a_n$.
 4. $\text{Lt}_{x \rightarrow 0} \left(\frac{a_0 x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \dots + b_{n-1} x + b_n} \right) = \frac{a_m}{b_n}$, unless $b_n = 0$.
 5. $\text{Lt}_{x \rightarrow a} x^n = a^n$, if n is any positive or negative integer.
 6. If
$$P(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m,$$
 then
$$\text{Lt}_{x \rightarrow a} P(x) = P(a).$$
 7. Let
$$P(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m,$$
 and
$$Q(x) = b_0 x^n + b_1 x^{n-1} + \dots + b_{n-1} x + b_n.$$
 Then
$$\text{Lt}_{x \rightarrow a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}, \text{ if } Q(a) \neq 0.$$
 8. If $\text{Lt}_{x \rightarrow a} f(x)$ exists, it is the same as $\text{Lt}_{x \rightarrow 0} f(x+a)$.
 9. If
$$f(x) < g(x) \text{ for } a-h < x < a+h,$$
 and
$$\text{Lt}_{x \rightarrow a} f(x) = \alpha, \quad \text{Lt}_{x \rightarrow a} g(x) = \beta,$$
 then
$$\alpha \leq \beta.$$
 10. If $\text{Lt}_{x \rightarrow a} f(x) = 0$, then $\text{Lt}_{x \rightarrow a} |f(x)| = 0$, and conversely.
 11. If $\text{Lt}_{x \rightarrow a} f(x) = l \geq 0$, then $\text{Lt}_{x \rightarrow a} |f(x)| = |l|$.
- The converse does not hold.

12. Let $f(x)$ be defined as follows :

$$\left. \begin{aligned} f(x) &= x \sin 1/x, \text{ when } x \geq 0 \\ f(0) &= 0 \end{aligned} \right\} f.$$

Then $\text{Lt}_{x \rightarrow a} f(x) = f(a)$ for all values of a .

27. **Lt f(x).** A precise definition of the meaning of the term “the limit of $f(x)$ when x tends to $+\infty$ (or to $-\infty$)” is also needed.

$f(x)$ is said to have the limit b as x tends to $+\infty$, if, any positive number ϵ having been chosen, as small as we please, there is a positive number X such that

$$|f(x) - b| < \epsilon, \text{ when } x \geq X.$$

When this condition is satisfied, we write

$$\text{Lt}_{x \rightarrow +\infty} f(x) = b.$$

A similar notation, $\text{Lt}_{x \rightarrow -\infty} f(x) = b$,

is used when $f(x)$ has the limit b as x tends to $-\infty$, and the precise definition of the term can be obtained by substituting “a negative number $-X$ ” and “ $x \leq -X$ ” in the corresponding places in the above.

When it is clear that only positive values of x are in question the notation $\text{Lt}_{x \rightarrow \infty} f(x)$ is used instead of $\text{Lt}_{x \rightarrow +\infty} f(x)$.

From the definition of the limit of $f(x)$ as x tends to $\pm\infty$, it follows that

$$\text{Lt}_{x \rightarrow +\infty} f(x) = b$$

carries with it

$$\text{Lt}_{x \rightarrow +0} f\left(\frac{1}{x}\right) = b.$$

And, conversely, if

$$\text{Lt}_{x \rightarrow +0} f(x) = b,$$

then

$$\text{Lt}_{x \rightarrow +\infty} f\left(\frac{1}{x}\right) = b.$$

Similarly we have

$$\text{Lt}_{x \rightarrow -\infty} f(x) = \text{Lt}_{x \rightarrow -0} f\left(\frac{1}{x}\right).$$

The modifications in the above definitions when

$$(i) \quad \text{Lt}_{x \rightarrow +\infty} f(x) = +\infty \quad \text{or} \quad -\infty,$$

$$\text{and (ii) } \quad \text{Lt}_{x \rightarrow -\infty} f(x) = +\infty \quad \text{or} \quad -\infty,$$

will be obvious, on referring to § 25.

And oscillation, finite or infinite, as x tends to $+\infty$ or to $-\infty$, is treated as before.

28. A necessary and sufficient condition for the existence of a limit to $f(x)$ as x tends to a . The general principle of convergence.*

A necessary and sufficient condition for the existence of a limit to $f(x)$ as x tends to a , is that, when any positive number ϵ has been chosen, as small as we please, there shall be a positive number η such that $|f(x'') - f(x')| < \epsilon$ for all values of x', x'' for which $0 < |x'' - a| < |x' - a| \leq \eta$.

(i) *The condition is necessary.*

Let $\text{Lt}_{x \rightarrow a} f(x) = b$.

Let ϵ be a positive number, as small as we please,

Then to $\epsilon/2$ there corresponds a positive number η such that

$$|f(x) - b| < \frac{\epsilon}{2}, \text{ when } 0 < |x - a| \leq \eta.$$

Now let x', x'' be any two values of x satisfying

$$0 < |x'' - a| < |x' - a| \leq \eta.$$

Then $|f(x'') - f(x')| \leq |f(x'') - b| + |f(x') - b|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$< \epsilon.$$

(ii) *The condition is sufficient.*

Let $\epsilon_1, \epsilon_2, \epsilon_3, \dots$

be a sequence of positive numbers such that

$$\epsilon_{n+1} < \epsilon_n \quad \text{and} \quad \text{Lt}_{n \rightarrow \infty} \epsilon_n = 0.$$

Let $\eta_1, \eta_2, \eta_3, \dots$

be corresponding positive numbers such that

$$\left. \begin{aligned} |f(x'') - f(x')| < \epsilon_n, \text{ when } 0 < |x'' - a| < |x' - a| \leq \eta_n \\ (n = 1, 2, 3, \dots). \end{aligned} \right\} \dots\dots(1)$$

Then, since $\epsilon_{n+1} < \epsilon_n$, we can obviously assume that $\eta_n \geq \eta_{n+1}$.

Now take ϵ_1 and the corresponding η_1 .

In the inequalities (1) put $x' = a + \eta_1$ and $x'' = x$.

Then we have

$$0 < |f(x) - f(a + \eta_1)| < \epsilon_1, \text{ when } 0 < |x - a| < \eta_1.$$

* See footnote, p. 34.

Therefore

$$f(a+\eta_1) - \epsilon_1 < f(x) < f(a+\eta_1) + \epsilon_1, \text{ when } 0 < |x-a| < \eta_1. \quad (2)$$

In Fig. 5 $f(x)$ lies within the interval A_1 of length $2\epsilon_1$, with centre at $f(a+\eta_1)$, when $0 < |x-a| < \eta_1$.

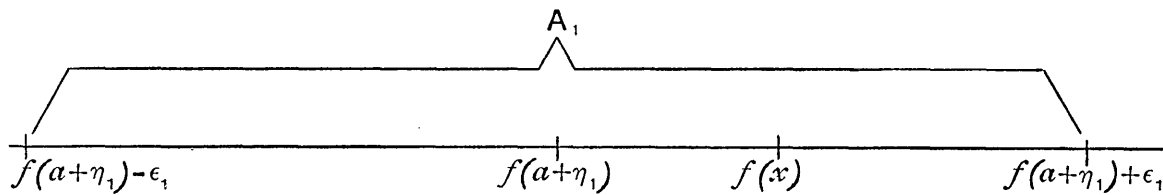


FIG. 5.

Now take ϵ_2 and the corresponding η_2 , remembering that $\eta_2 \leq \eta_1$.

We have, as above,

$$f(a+\eta_2) - \epsilon_2 < f(x) < f(a+\eta_2) + \epsilon_2, \text{ when } 0 < |x-a| < \eta_2. \quad (3)$$

Since $\eta_2 \leq \eta_1$, the interval for x in (3) cannot extend beyond the interval for x in (2), and $f(a+\eta_2)$ is in the open interval A_1 .

Therefore, in Fig. 6, $f(x)$ now lies within the interval A_2 , which lies entirely within A_1 , or lies within it and has with it a common end-point. An overlapping part of

$$\{f(a+\eta_2) - \epsilon_2, f(a+\eta_2) + \epsilon_2\}$$

could be cut off, in virtue of (2).

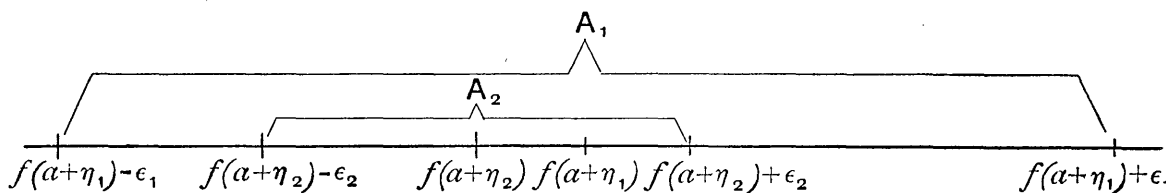


FIG. 6.

In this way we obtain a series of intervals

$$A_1, A_2, A_3, \dots,$$

each lying entirely within the preceding, or lying within it and having with it a common end-point; and, since the length of $A_n \leq 2\epsilon_n$, we have $\text{Lt}_{n \rightarrow \infty} A_n = 0$, for we are given that $\text{Lt}_{n \rightarrow \infty} \epsilon_n = 0$.

If we denote the end-points of these intervals by a_1, a_2, a_3, \dots and $\beta_1, \beta_2, \beta_3, \dots$, where $\beta_n > a_n$, then we know from 18 that

$$\text{Lt}_{n \rightarrow \infty} a_n = \text{Lt}_{n \rightarrow \infty} \beta_n.$$

Denote this common limit by α .

We shall now show that α is the limit of $f(x)$ as $x \rightarrow a$.

We can choose ϵ_n in the sequence $\epsilon_1, \epsilon_2, \epsilon_3, \dots$ so that $2\epsilon_n < \epsilon$, where ϵ is any given positive number.

Then we have, as above in (2) and (3),

$$a_n < f(x) < \beta_n, \text{ when } 0 < |x - a| < \eta_n.$$

But
$$a_n \leq a \leq \beta_n.$$

Therefore
$$\begin{aligned} |f(x) - a| &< \beta_n - a_n \\ &< 2\epsilon_n \\ &< \epsilon, \end{aligned} \quad \text{when } 0 < |x - a| < \eta_n.$$

It follows that
$$\text{Lt}_{x \rightarrow a} f(x) = a.$$

As a matter of fact, we have not obtained

$$|f(x) - a| < \epsilon, \text{ when } 0 < |x - a| \leq \eta_n$$

in the above, but when $0 < |x - a| < \eta_n$.

However, we need only take η smaller than this η_n , and we obtain the inequalities used in our definition of a limit.

29. In the previous section we have supposed that x tends to a from both sides. The slight modification in the condition for convergence when it tends to a from one side only can easily be made.

Similarly, a necessary and sufficient condition for the existence of a limit to $f(x)$ as x tends to $+\infty$, is that, if the positive number ϵ has been chosen, as small as we please, there shall be a positive number X such that

$$|f(x'') - f(x')| < \epsilon, \text{ when } x'' > x' \geq X.$$

In the case of $\text{Lt}_{x \rightarrow -\infty} f(x)$, we have, in the same way, the condition

$$|f(x'') - f(x')| < \epsilon, \text{ when } x'' < x' \leq -X.$$

The conditions for the existence of a limit to $f(x)$ as x tends to $+\infty$ or to $-\infty$ can, of course, be deduced from those for the existence of a limit as x tends to $+0$ or to -0 .

Actually the argument given in the preceding section is simpler when we deal with $+\infty$ or $-\infty$,* and the case when the variable tends to zero from the right or left can be deduced from these two, by substituting $x = \frac{1}{u}$; when it tends to a , we must substitute $x = a + \frac{1}{u}$.

* Cf. Osgood, *Lehrbuch der Funktionentheorie*, Bd. I., p. 27, Leipzig, 1907.

The general principle of convergence of § 15 can also be established in this way.

30. Continuous Functions. The function $f(x)$ is said to be continuous when $x = x_0$, if $f(x)$ has a limit as x tends to x_0 from either side, and each of these limits is equal to $f(x_0)$.

Thus $f(x)$ is continuous when $x = x_0$, if, to the arbitrary positive number ϵ , there corresponds a positive number η such that

$$|f(x) - f(x_0)| < \epsilon, \text{ when } |x - x_0| \leq \eta.$$

When $f(x)$ is defined in an interval (a, b) , we shall say that it is continuous in the interval (a, b) , if it is continuous for every value of x between a and b ($a < x < b$), and if $f(a+0)$ exists and is equal to $f(a)$, and $f(b-0)$ exists and is equal to $f(b)$.

In such cases it is convenient to make a slight change in our definition of continuity at a point, and to say that $f(x)$ is continuous at the end-points a and b when these conditions are satisfied.

It follows from the definition of continuity that the sum or product of any number of functions, which are continuous at a point, is also continuous at that point. The same holds for the quotient of two functions, continuous at a point, unless the denominator vanishes at that point (cf. § 26). A continuous function of a continuous function is also a continuous function (cf. § 26 (IV.)).

The polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

is continuous for all values of x .

The rational function

$$R(x) = P(x)/Q(x)$$

is continuous in any interval which does not include values of x making the polynomial $Q(x)$ zero.

The functions $\sin x$, $\cos x$, $\tan x$, etc. and the corresponding functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, etc. are continuous except, in certain cases, at particular points.

e^x is continuous everywhere; $\log x$ is continuous for the interval $x > 0$.

31. Properties of Continuous Functions.* We shall now prove several important theorems on continuous functions, to

* This section follows closely the treatment given by Goursat (*loc. cit.*), T. I. (3^e éd.), § 8.

which reference will frequently be made later. It will be seen that in these proofs we rely only on the definition of continuity and the results obtained in the previous pages.

THEOREM I. *Let $f(x)$ be continuous in the interval $(a, b)^*$, and let the positive number ϵ be chosen, as small as we please. Then the interval (a, b) can always be broken up into a finite number of partial intervals, such that $|f(x') - f(x'')| < \epsilon$, when x' and x'' are any two points in the same partial interval.*

Let us suppose that this is not true. Then let $c = (a + b)/2$. At least one of the intervals (a, c) , (c, b) must be such that it is impossible to break it up into a finite number of partial intervals which satisfy the condition named in the theorem. Denote by (a_1, b_1) this new interval, which is half of (a, b) . Operating on (a_1, b_1) in the same way as we have done with (a, b) , and then proceeding as before, we obtain an infinite set of intervals such as we have met in the theorem of § 18. The sequence of end-points a, a_1, a_2, \dots converges, and the sequence of end-points b, b_1, b_2, \dots also converges, the limit of each being the same, say α . Also each of the intervals (a_n, b_n) has the property we have ascribed to the original interval (a, b) . It is impossible to break it up into a finite number of partial intervals which satisfy the condition named in the theorem.

Let us suppose that α does not coincide with a or b . Since the function $f(x)$ is continuous when $x = \alpha$, we know that there is a positive number η such that $|f(x) - f(\alpha)| < \epsilon/2$ when $|x - \alpha| \leq \eta$. Let us choose n so large that $(b_n - a_n)$ is less than η . Then the interval (a_n, b_n) is contained entirely within $(\alpha - \eta, \alpha + \eta)$, for we know that $a_n \leq \alpha \leq b_n$. Therefore, if x' and x'' are any two points in the interval (a_n, b_n) , it follows from the above that

$$|f(x') - f(\alpha)| < \frac{\epsilon}{2} \quad \text{and} \quad |f(x'') - f(\alpha)| < \frac{\epsilon}{2}.$$

$$\text{But} \quad |f(x') - f(x'')| \leq |f(x') - f(\alpha)| + |f(x'') - f(\alpha)|.$$

$$\text{Thus we have} \quad |f(x') - f(x'')| < \epsilon,$$

and our hypothesis leads to a contradiction.

* In these theorems the continuity of $f(x)$ is supposed given in the closed interval $(a \leq x \leq b)$, as explained in § 30.

There remains the possibility that a might coincide with either a or b . The slight modification required in the above argument is obvious.

Hence the assumption that the theorem is untrue leads in every case to a contradiction, and its truth is established.

COROLLARY I. Let $a, x_1, x_2, \dots, x_{n-1}, b$ be a mode of subdivision of (a, b) into partial intervals satisfying the conditions of Theorem I.

Then

$$\begin{aligned} |f(x)| &\leq |f(a)| + |f(x) - f(a)| \\ &< |f(a)| + \epsilon, \quad \text{when } 0 < (x - a) \leq (x_1 - a). \end{aligned}$$

Therefore

$$|f(x_1)| < |f(a)| + \epsilon.$$

In the same way

$$\begin{aligned} |f(x)| &\leq |f(x_1)| + |f(x) - f(x_1)| \\ &< |f(x_1)| + \epsilon, \quad \text{when } 0 < (x - x_1) \leq (x_2 - x_1) \\ &< |f(a)| + 2\epsilon, \quad \text{when } 0 < (x - x_1) \leq (x_2 - x_1). \end{aligned}$$

Therefore

$$|f(x_2)| < |f(a)| + 2\epsilon.$$

Proceeding in the same way for each successive partial interval, we obtain from the n^{th} interval

$$|f(x)| < |f(a)| + n\epsilon, \quad \text{when } 0 < (x - x_{n-1}) \leq (b - x_{n-1}).$$

Thus we see that in the whole interval (a, b)

$$|f(x)| < |f(a)| + n\epsilon.$$

It follows that *a function which is continuous in a given interval is bounded in that interval.*

COROLLARY II. Let us suppose the interval (a, b) divided up into n partial intervals $(a, x_1), (x_1, x_2), \dots, (x_{n-1}, b)$, such that $|f(x') - f(x'')| < \epsilon/2$ for any two points in the same partial interval. Let η be a positive number smaller than the least of the numbers $(x_1 - a), (x_2 - x_1), \dots, (b - x_{n-1})$. Now take any two points x' and x'' in the interval (a, b) , such that $|x' - x''| \leq \eta$. If these two points belong to the same partial interval, we have

$$|f(x') - f(x'')| < \epsilon/2.$$

On the other hand, if they do not belong to the same partial interval, they must lie in two consecutive partial intervals. In this case it is clear that $|f(x') - f(x'')| < \epsilon/2 + \epsilon/2 = \epsilon$.

Hence, *the positive number ϵ having been chosen, as small as we please, there is a positive number η such that $|f(x') - f(x'')| < \epsilon$, when x', x'' are any two values of x in the interval (a, b) for which $|x' - x''| \leq \eta$.*

We started with the assumption that $f(x)$ was continuous in (a, b) . It follows from this assumption that if x is any point in this interval, and ϵ any arbitrary positive number, then there is a positive number η such that

$$|f(x') - f(x)| < \epsilon, \text{ when } |x' - x| \leq \eta.$$

To begin with, we have no justification for supposing that the same η could do for all values of x in the interval. But the theorem proved in this corollary establishes that this is the case. This result is usually expressed by saying that $f(x)$ is *uniformly continuous* in the interval (a, b) .

We have thus shown that *a function which is continuous in an interval is also uniformly continuous in the interval.*

THEOREM II. *If $f(a)$ and $f(b)$ are unequal and $f(x)$ is continuous in the interval (a, b) , as x passes from a to b , $f(x)$ takes at least once every value between $f(a)$ and $f(b)$.*

First, let us suppose that $f(a)$ and $f(b)$ have different signs, e.g. $f(a) < 0$ and $f(b) > 0$. We shall show that for at least one value of x between a and b , $f(x) = 0$.

From the continuity of $f(x)$, we see that it is negative in the neighbourhood of a and positive in the neighbourhood of b . Consider the set of values of x between a and b which make $f(x)$ positive. Let λ be the lower bound of this aggregate. Then $a < \lambda < b$. From the definition of the lower bound $f(x)$ is negative or zero in $a \leq x < \lambda$. But $\lim_{x \rightarrow \lambda - 0} f(x)$ exists and is equal to $f(\lambda)$. Therefore $f(\lambda)$ is also negative or zero. But $f(\lambda)$ cannot be negative. For if $f(\lambda) = -m$, m being a positive number, then there is a positive number η such that

$$|f(x) - f(\lambda)| < m, \text{ when } |x - \lambda| \leq \eta,$$

since $f(x)$ is continuous when $x = \lambda$. The function $f(x)$ would then be negative for the values of x in (a, b) between λ and $\lambda + \eta$

and λ would not be the lower bound of the above aggregate. We must therefore have $f(\lambda) = 0$.

Now let N be any number between $f(a)$ and $f(b)$, which may be of the same or different signs. The continuous function $\phi(x) = f(x) - N$ has opposite signs when $x = a$ and $x = b$. By the case we have just discussed, $\phi(x)$ vanishes for at least one value of x between a and b , *i.e.* in the *open* interval (a, b) .

Thus our theorem is established.

Again, if $f(x)$ is continuous in (a, b) , we know from Corollary I. above that it is bounded in that interval. In the next theorem we show that it attains these bounds.

THEOREM III. *If $f(x)$ is continuous in the interval (a, b) , and M, m are its upper and lower bounds, then $f(x)$ takes the value M and the value m at least once in the interval.*

We shall show first that $f(x) = M$ at least once in the interval.

Let $c = (a + b)/2$; the upper bound of $f(x)$ is equal to M , for at least one of the intervals $(a, c), (c, b)$. Replacing (a, b) by this interval, we bisect it, and proceed as before. In this way, as in Theorem I., we obtain an infinite set of intervals $(a, b), (a_1, b_1), (a_2, b_2), \dots$ tending to zero in the limit, each lying entirely within the preceding, or lying within it and having with it a common end-point, the upper bound of $f(x)$ in each being M .

Let λ be the common limit of the sequences a, a_1, a_2, \dots and b, b_1, b_2, \dots . We shall show that $f(\lambda) = M$.

For suppose $f(\lambda) = M - h$, where $h > 0$. Since $f(x)$ is continuous at $x = \lambda$, there is a positive number η such that

$$|f(x) - f(\lambda)| < \frac{h}{2}, \text{ when } |x - \lambda| \leq \eta.$$

Thus $f(x) < M - \frac{h}{2}$, when $|x - \lambda| \leq \eta$.

Now take n so large that $(b_n - a_n)$ will be less than η . The interval (a_n, b_n) will be contained wholly within $(\lambda - \eta, \lambda + \eta)$. The upper bound of $f(x)$ in the interval (a_n, b_n) would then be different from M , contrary to our hypothesis.

Combining this theorem with the preceding we obtain the following additional result:

THEOREM IV. *If $f(x)$ is continuous in the interval (a, b) , and M, m are its upper and lower bounds, then it takes at least*

once in this interval the values M , m , and every value between M and m .

Also, since the *oscillation* of a function in an interval was defined as the difference between its upper and lower bounds (cf. § 24), and since the function attains its bounds at least once in the interval, we can state Theorem I. afresh as follows:

If $f(x)$ is continuous in the interval (a, b) , then we can divide (a, b) into a finite number of partial intervals

$$(a, x_1), (x_1, x_2), \dots (x_{n-1}, b),$$

in each of which the oscillation of $f(x)$ is less than any given positive number.

And a similar change can be made in the statement of the property known as uniform continuity.

32. Continuity in an Infinite Interval. Some of the results of the last section can be extended to the case when $f(x)$ is continuous in $x \geq a$, where a is some definite positive number, and $\text{Lt}_{x \rightarrow \infty} f(x)$ exists.

Let $u = a/x$. When $x \geq a$, we have $0 < u \leq 1$.

With the values of u in $0 < u \leq 1$, associate the values of $f(x)$ at the corresponding points in $x \geq a$, and to $u = 0$ assign $\text{Lt}_{x \rightarrow \infty} f(x)$.

We thus obtain a function of u , which is continuous in the closed interval $(0, 1)$.

Therefore it is bounded in this interval, and attains its bounds M and m . Also it takes at least once every value between M and m , as u passes over the interval $(0, 1)$.

Thus we may say that $f(x)$ is bounded in the range* given by $x \geq a$ and the new "point" $x = \infty$, at which $f(x)$ is given the value $\text{Lt}_{x \rightarrow \infty} f(x)$.

Also $f(x)$ takes at least once in this range its upper and lower bounds and every value between these bounds.

For example, the function $\frac{x^2}{a^2 + x^2}$ is continuous in $(0, \infty)$. It does not attain its upper bound—unity—when $x \geq 0$, but it takes this value when $x = \infty$, as defined above.

33. Discontinuous Functions. When $f(x)$ is defined for x_0 and the neighbourhood of x_0 (e.g. $0 < |x - x_0| \leq h$), and $f(x_0 + 0) = f(x_0 - 0) = f(x_0)$, then $f(x)$ is continuous at x_0 .

On the other hand, when $f(x)$ is defined for the neighbourhood of x_0 , and it may be also for x_0 , while $f(x)$ is not continuous at

*It is convenient to speak of this range as the interval (a, ∞) , and to write $f(\infty)$ for $\text{Lt}_{x \rightarrow \infty} f(x)$.

x_0 , it is natural to say that $f(x)$ is *discontinuous* at x_0 , and to call x_0 a *point of discontinuity* of $f(x)$.

Points of discontinuity may be classified as follows :

I. $f(x_0+0)$ and $f(x_0-0)$ may exist and be equal. If their common value is different from $f(x_0)$, or if $f(x)$ is not defined for x_0 , then we have a point of discontinuity there.

Ex. $f(x) = (x - x_0) \sin 1/(x - x_0)$, when $x \geq x_0$.

Here $f(x_0+0) = f(x_0-0) = 0$, and if we give $f(x_0)$ any value other than zero, or if we leave $f(x_0)$ undefined, x_0 is a point of discontinuity of $f(x)$.

II. $f(x_0+0)$ and $f(x_0-0)$ may exist and be unequal. Then x_0 is a point of discontinuity of $f(x)$, whether $f(x_0)$ is defined or not.

Ex. $f(x) = \frac{1}{1 - e^{1/(x - x_0)}}$, when $x \geq x_0$.

Here $f(x_0+0) = 0$ and $f(x_0-0) = 1$.

In both these cases $f(x)$ is said to have an *ordinary* or *simple* discontinuity at x_0 . And the same term is applied when the point x_0 is an end-point of the interval in which $f(x)$ is given, and $f(x_0+0)$, or $f(x_0-0)$, exists and is different from $f(x_0)$, if $f(x)$ is defined for x_0 .

III. $f(x)$ may have the limit $+\infty$, or $-\infty$, as $x \rightarrow x_0$ on either side and it may oscillate on one side or the other. Take in this section the cases in which there is no oscillation. These may be arranged as follows :

(i) $f(x_0+0) = f(x_0-0) = +\infty$ (or $-\infty$).

Ex. $f(x) = 1/(x - x_0)^2$, when $x \geq x_0$.

(ii) $f(x_0+0) = +\infty$ (or $-\infty$) and $f(x_0-0) = -\infty$ (or $+\infty$).

Ex. $f(x) = 1/(x - x_0)$, when $x \geq x_0$.

(iii) $f(x_0+0) = +\infty$ (or $-\infty$) } or $f(x_0-0) = +\infty$ (or $-\infty$) }
 $f(x_0-0)$ exists } $f(x_0+0)$ exists }

Ex. $f(x) = 1/(x - x_0)$, when $x > x_0$ }
 $f(x) = x - x_0$, when $x \leq x_0$ }

In these cases we say that the point x_0 is an *infinity* of $f(x)$, and the same term is used when x_0 is an end-point of the interval in which $f(x)$ is given, and $f(x_0+0)$, or $f(x_0-0)$, is $+\infty$ or $-\infty$.

It is usual to say that $f(x)$ becomes *infinite* at a point x_0 of the kind given in (i), and that $f(x_0) = +\infty$ (or $-\infty$). But this

must be regarded as simply a short way of expressing the fact that $f(x)$ diverges to $+\infty$ (or to $-\infty$) as $x \rightarrow x_0$.

It will be noticed that $\tan x$ has an infinity at $\frac{1}{2}\pi$, but that $\tan \frac{1}{2}\pi$ is not defined. On the other hand,

$$\tan\left(\frac{1}{2}\pi - 0\right) = +\infty \quad \text{and} \quad \tan\left(\frac{1}{2}\pi + 0\right) = -\infty.$$

IV. When $f(x)$ oscillates at x_0 on one side or the other, x_0 is said to be a *point of oscillatory discontinuity*. The oscillation is *finite* when $f(x)$ is bounded in some neighbourhood of x_0 ; it is *infinite* when there is no neighbourhood of x_0 in which $f(x)$ is bounded (cf. § 25).

- Ex.** (i) $f(x) = \sin 1/(x - x_0)$, when $x \geq x_0$.
(ii) $f(x) = 1/(x - x_0) \sin 1/(x - x_0)$, when $x \geq x_0$.

In both these examples x_0 is a point of oscillatory discontinuity. The first oscillates finitely at x_0 , the second oscillates infinitely. The same remark would apply if the function had been given only for one side of x_0 .

The *infinities* defined in III. and the points at which $f(x)$ oscillates infinitely are said to be *points of infinite discontinuity*.

34. Monotonic Functions. *The function $f(x)$, given in the interval (a, b) , is said to be monotonic in that interval if*

- either (i) $f(x') \leq f(x'')$, when $a \leq x' < x'' \leq b$;
or (ii) $f(x') \geq f(x'')$, when $a \leq x' < x'' \leq b$.*

In the first case, the function never decreases as x increases and it is said to be *monotonic increasing*; in the second case, it never increases as x increases, and it is said to be *monotonic decreasing*.*

The monotonic character of the function may fail at the end-points of the interval, and in this case it is said to be *monotonic in the open interval*.

The properties of monotonic functions are very similar to those of monotonic sequences, treated in § 17, and they may be established in precisely the same way:

- (i) *If $f(x)$ is monotonic increasing when $x \geq X$, and $f(x)$ is less than some fixed number A when $x \geq X$, then $\text{Lt}_{x \rightarrow +\infty} f(x)$ exists and is less than or equal to A .*

* The footnote, p. 39, also applies here.

(ii) If $f(x)$ is monotonic increasing when $x \leq X$, and $f(x)$ is greater than some fixed number A when $x \leq X$, then $\lim_{x \rightarrow -\infty} f(x)$ exists and is greater than or equal to A .

(iii) If $f(x)$ is monotonic increasing in an open interval (a, b) , and $f(x)$ is greater than some fixed number A in that open interval, then $f(a+0)$ exists and is greater than or equal to A .

(iv) If $f(x)$ is monotonic increasing in an open interval (a, b) , and $f(x)$ is less than some fixed number A in that open interval, then $f(b-0)$ exists and is less than or equal to A .

These results can be readily adapted to the case of monotonic decreasing functions, and it follows at once from (iii) and (iv) that if $f(x)$ is bounded and monotonic in an open interval, it can only have ordinary discontinuities in that interval, or at its ends.

It may be worth observing that if $f(x)$ is monotonic in a closed interval, the same result follows, but that if we are only given that it is monotonic in an open interval, and not told that it is bounded, the function may have an infinity at either end.

E.g. $f(x) = 1/x$ is monotonic in the open interval $(0, 1)$, but not bounded.

At first one might be inclined to think that a function which is bounded and monotonic in an interval can have only a finite number of points of discontinuity in that interval.

The following example shows that this is not the case:

$$\text{Let } f(x) = 1, \quad \text{when } \frac{1}{2} < x \leq 1;$$

$$\text{let } f(x) = \frac{1}{2}, \quad \text{when } \frac{1}{2^2} < x \leq \frac{1}{2};$$

and, in general,

$$\text{let } f(x) = \frac{1}{2^n}, \quad \text{when } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n},$$

(n being any positive integer).

Also let $f(0) = 0$.

Then $f(x)$ is monotonic in the interval $(0, 1)$.

This function has an infinite number of points of discontinuity, namely at $x = \frac{1}{2^n}$ (n being any positive integer).

Obviously there can only be a finite number of points of discontinuity at which the jump would be greater than or equal to k , where k is any fixed positive number, if the function is monotonic (and bounded) in an interval.

35. Inverse Functions. Let the function $f(x)$, defined in the interval (a, b) , be continuous and monotonic in the *stricter sense** in (a, b) .

For example, let $y=f(x)$ be continuous and continually increase from A to B as x passes from a to b .

Then to every value of y in (A, B) there corresponds one and only one value of x in (a, b) . [§ 31, Theorem II.]

This value of x is a function of y —say $\phi(y)$, which is itself continually increasing in the interval (A, B) .

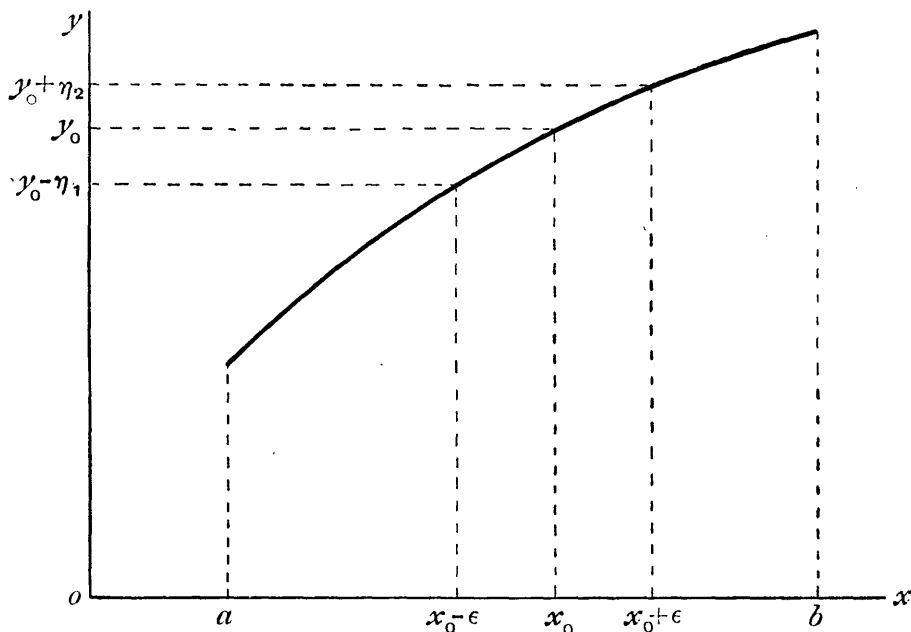


FIG. 7.

The function $\phi(y)$ is called the *inverse* of the function $f(x)$.

We shall now show that $\phi(y)$ is a *continuous function* of y in the interval in which it is defined.

For let y_0 be any number between A and B , and x_0 the corresponding value of x . Also let ϵ be an arbitrary positive number such that $x_0 - \epsilon$ and $x_0 + \epsilon$ lie in (a, b) (Fig. 7).

Let $y_0 - \eta_1$ and $y_0 + \eta_2$ be the corresponding values of y .

Then, if the positive number η is less than the smaller of η_1 and η_2 , it is clear that

$$|x - x_0| < \epsilon, \text{ when } |y - y_0| \leq \eta.$$

Therefore $|\phi(y) - \phi(y_0)| < \epsilon$, when $|y - y_0| \leq \eta$.

Thus $\phi(y)$ is continuous at y_0 .

A similar proof applies to the end-points A and B , and it is obvious that the same argument applies to a function which is monotonic in the *stricter sense* and decreasing.

* Cf footnote, p. 39.

The functions $\sin^{-1}x$, $\cos^{-1}x$, etc., thus arise as the inverses of the functions $\sin x$ and $\cos x$, where $0 \leq x \leq \frac{1}{2}\pi$, and so on.

In the first place these appear as functions of y , namely $\sin^{-1}y$, where $0 \leq y \leq 1$, $\cos^{-1}y$, where $0 \leq y \leq 1$, etc. The symbol y is then replaced by the usual symbol x for the independent variable.

In the same way $\log x$ appears as the inverse of the function e^x .

There is a simple rule for obtaining the graph of the inverse function $f^{-1}(x)$ when the graph of $f(x)$ is known. $f^{-1}(x)$ is the image of $f(x)$ in the line $y=x$. The proof of this may be left to the reader.

The following theorem may be compared with that of § 26 (IV.):

Let $f(u)$ be a continuous function, monotonic in the stricter sense, and let

$$\text{Lt}_{x \rightarrow a} f[\phi(x)] = f(b).$$

Then $\text{Lt}_{x \rightarrow a} \phi(x)$ exists and is equal to b .

A strict proof of this result may be obtained, relying on the property proved above, that the inverse of the function $f(u)$ is a continuous function.

The theorem is almost intuitive, if we are permitted to use the graph of $f(u)$. The reader is familiar with its application in finding certain limiting values, where logarithms are taken. In these cases it is shown that $\text{Lt} \log u = \log b$, and it is inferred that $\text{Lt} u = b$.*

36. Let the bounded function $f(x)$, given in the interval (a, b) , be such that this interval can be divided up into a finite number of open partial intervals, in each of which $f(x)$ is monotonic.

Suppose that the points x_1, x_2, \dots, x_{n-1} divide this interval into the open intervals $(a, x_1), (x_1, x_2), \dots, (x_{n-1}, b)$, in which $f(x)$ is monotonic. Then we know that $f(x)$ can only have ordinary discontinuities, which can occur at the points $a, x_1, x_2, \dots, x_{n-1}, b$, and also at any number of points within the partial intervals. (§ 34.)

I. Let us take first the case where $f(x)$ is continuous at $a, x_1, \dots, x_{n-1}, b$, and alternately monotonic increasing and decreasing. To make matters clearer, we shall assume that there are only three of these points of section, namely x_1, x_2, x_3 , $f(x)$ being monotonic increasing in the first interval (a, x_1) , decreasing in the second, (x_1, x_2) , and so on (Fig. 8).

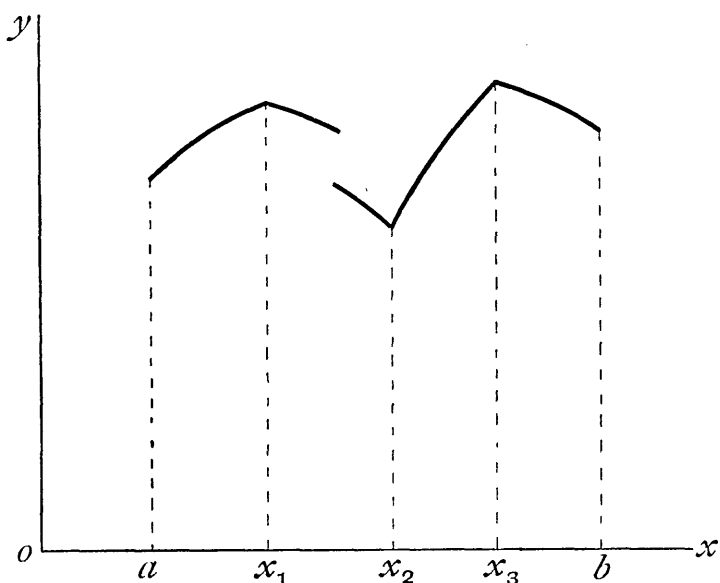


FIG. 8.

* Cf Hobson, *Plane Trigonometry*, (3rd Ed.), p. 130.

It is obvious that the intervals may in this case be regarded as closed, the monotonic character of $f(x)$ extending to the ends of each.

Consider the functions $F(x)$, $G(x)$ given by the following scheme :

$F(x)$	$G(x)$	
$f(x)$	0	$a \leq x \leq x_1$
$f(x_1)$	$f(x_1) - f(x)$	$x_1 \leq x \leq x_2$
$f(x_1) - f(x_2) + f(x)$	$f(x_1) - f(x_2)$	$x_2 \leq x \leq x_3$
$f(x_1) - f(x_2) + f(x_3)$	$f(x_1) - f(x_2) + f(x_3) - f(x)$	$x_3 \leq x \leq b$

It is clear that $F(x)$ and $G(x)$ are *monotonic increasing* in the closed interval (a, b) , and that $f(x) = F(x) - G(x)$ in (a, b) .

If $f(x)$ is decreasing in the first partial interval, we start with

$$F(x) = f(a), \quad G(x) = f(a) - f(x), \quad \text{when } a \leq x \leq x_1,$$

and proceed as before, *i.e.* we begin with the second line of the above diagram, and substitute a for x_1 , etc.

Also, since the function $f(x)$ is bounded in (a, b) , by adding some number to both $F(x)$ and $G(x)$, we can make both these functions positive if, in the original discussion, one or both were negative.

It is clear that the process outlined above applies equally well to n partial intervals.

We have thus shown that *when the bounded function $f(x)$, given in the interval (a, b) , is such that this interval can be divided up into a finite number of partial intervals, $f(x)$ being alternately monotonic increasing and monotonic decreasing in these intervals, and continuous at their ends, then we can*

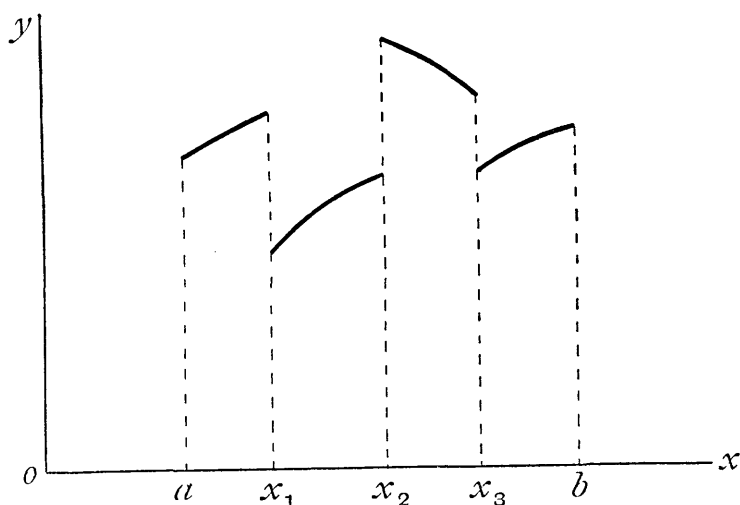


FIG. 9.

express $f(x)$ as the difference of two (bounded) functions, which are positive and monotonic increasing in the interval (a, b) .

II. There remains the case when some or all of the points $a, x_1, x_2, \dots, x_{n-1}, b$ are points of discontinuity of $f(x)$, and the proviso that the function is alternately monotonic increasing and decreasing is dropped.

We can obtain from $f(x)$ a new function $\phi(x)$, with the same monotonic properties as $f(x)$ in the open partial intervals $(a, x_1), (x_1, x_2), \dots, (x_{n-1}, b)$, but continuous at their ends.

The process is obvious from Fig. 9. We need only keep the first part of the curve fixed, move the second up or down till its end-point $(x_1, f(x_1+0))$

coincides with $(x_1, f(x_1 - 0))$, then proceed to the third curve and move it up or down to the new position of the second, and so on.

If the values of $f(x)$ at $a, x_1, x_2, \dots, x_{n-1}, b$ are not the same as

$$f(a+0), f(x_1+0) \quad \text{or} \quad f(x_1-0), \text{ etc.},$$

we must treat these points separately.

In this way, but arithmetically,* we obtain the function $\phi(x)$ defined as follows :

In $a \leqq x < x_1$, $\phi(x) = f(x)$, supposing for clearness $f(a) = f(a+0)$.

At x_1 , $\phi(x) = f(x) + \alpha_1$.

In $x_1 < x < x_2$, $\phi(x) = f(x) + \alpha_1 + \alpha_2$.

At x_2 , $\phi(x) = f(x) + \alpha_1 + \alpha_2 + \alpha_3$.

And so on,

$\alpha_1, \alpha_2, \alpha_3, \dots$ being definite numbers depending on $f(x_1 \pm 0), f(x_1)$, etc.

We can now apply the theorem proved above to the function $\phi(x)$ and write

$$\phi(x) = \Phi(x) - \Psi(x) \quad \text{in } (a, b),$$

$\Phi(x)$ and $\Psi(x)$ being positive and monotonic increasing in this interval.

It follows that :

In $a \leqq x < x_1$, $f(x) = \Phi(x) - \Psi(x)$.

At x_1 , $f(x) = \Phi(x) - \Psi(x) - \alpha_1$.

In $x_1 < x < x_2$, $f(x) = \Phi(x) - \Psi(x) - \alpha_1 - \alpha_2$.

And so on.

If any of the terms $\alpha_1, \alpha_2, \dots$ are negative we put them with $\Phi(x)$: the positive terms we leave with $\Psi(x)$. Thus finally we obtain, as before, that

$$f(x) = F(x) - G(x) \quad \text{in } (a, b),$$

where $F(x)$ and $G(x)$ are positive and monotonic increasing in this interval.

We have thus established the important theorem :

If the bounded function $f(x)$, given in the interval (a, b) , is such that this interval can be divided up into a finite number of open partial intervals, in each of which $f(x)$ is monotonic, then we can express $f(x)$ as the difference of two (bounded) functions, which are positive and monotonic increasing in the interval (a, b) .

Also it will be seen from the above discussion that the discontinuities of $F(x)$ and $G(x)$, which can, of course, only be ordinary discontinuities, can occur only at the points where $f(x)$ is discontinuous.

It should, perhaps, also be added that, while the monotonic properties ascribed to $f(x)$ allow it to have only ordinary discontinuities, the number of points of discontinuity may be infinite (§ 34).

37. Functions of Several Variables. So far we have dealt only with functions of a single variable. If to every value of x in the interval $a \leqq x \leqq b$ there corresponds a number y , then we

* It will be noticed that in the proof the curves and diagrams are used simply as illustrations.

have said that y is a function of x in the interval (a, b) , and we have written $y=f(x)$.

The extension to functions of two variables is immediate:—

To every pair of values of x and y , such that

$$a \leq x \leq a', \quad b \leq y \leq b',$$

let there correspond a number z . Then z is said to be a function of x and y in this domain, and we write $z=f(x, y)$.

If we consider x and y as the coordinates of a point in a plane, to every pair of values of x and y there corresponds a point in the plane, and the region defined by $a \leq x \leq a'$, $b \leq y \leq b'$ will be a rectangle.

In the case of the single variable, it is necessary to distinguish between the *open* interval $(a < x < b)$ and the *closed* interval $(a \leq x \leq b)$. So, in the case of two dimensions, it is well to distinguish between *open* and *closed* domains. In the former the boundary of the region is not included in the domain; in the latter it is included.

In the above definition we have taken a rectangle for the domain of the variables. A function of two variables may be defined in the same way for a domain of which the boundary is a curve C : or again, the domain may have a curve C for its external boundary, and other curves, C' , C'' , etc., for its internal boundary.

A function of three variables, or any number of variables, will be defined as above. For three variables, we can still draw upon the language of geometry, and refer to the domain as contained within a surface S , etc.

We shall now refer briefly to some properties of functions of two variables.

A function is said to be *bounded* in the domain in which it is defined, if the set of values of z , for all the points of this domain, forms a bounded aggregate. The *upper* and *lower bounds*, M and m , and the *oscillation*, are defined as in § 24.

$f(x, y)$ is said to have the limit l as (x, y) tends to (x_0, y_0) , when, any positive number ϵ having been chosen, as small as we please, there is a positive number η such that $|f(x, y) - l| < \epsilon$ for all values of (x, y) for which

$$|x - x_0| \leq \eta, \quad |y - y_0| \leq \eta \quad \text{and} \quad 0 < |x - x_0| + |y - y_0|.$$

In other words, $|f(x, y) - l|$ must be less than ϵ for all points in the square, centre (x_0, y_0) , whose sides are parallel to the coordinate axes and of length 2η , the centre itself being excluded from the domain.

A necessary and sufficient condition for the existence of a limit to $f(x, y)$ as (x, y) tends to (x_0, y_0) is that, to the arbitrary positive number ϵ , there shall correspond a positive number η such that $|f(x', y') - f(x'', y'')| < \epsilon$, where (x', y') and (x'', y'') are any two points other than (x_0, y_0) in the square, centre (x_0, y_0) , whose sides are parallel to the coordinate axes and of length 2η .

The proof of this theorem can be obtained in exactly the same way as in the one-dimensional case, squares taking the place of the intervals in the preceding proof.

A function $f(x, y)$ is said to be continuous when $x = x_0$ and $y = y_0$, if $f(x, y)$ has the limit $f(x_0, y_0)$ as (x, y) tends to (x_0, y_0) .

Thus, $f(x, y)$ is continuous when $x = x_0$ and $y = y_0$, if, to the arbitrary positive number ϵ , there corresponds a positive number η such that $|f(x, y) - f(x_0, y_0)| < \epsilon$ for all values of (x, y) for which $|x - x_0| \leq \eta$ and $|y - y_0| \leq \eta$.

In other words, $|f(x, y) - f(x_0, y_0)|$ must be less than ϵ for all points in the square, centre (x_0, y_0) , whose sides are parallel to the coordinate axes and of length 2η .*

It is convenient to speak of a function as continuous *at a point* (x_0, y_0) instead of *when* $x = x_0$ and $y = y_0$. Also when a function of two variables is continuous at (x, y) , as defined above, for every point of a domain, we shall say that it is a continuous function of (x, y) in the domain.

It is easy to see that we can substitute for the square, with centre at (x_0, y_0) , referred to above, a circle with the same centre. The definition of a limit would then read as follows:

$f(x, y)$ is said to have the limit l as (x, y) tend to (x_0, y_0) , if, to the arbitrary positive number ϵ , there corresponds a positive number η such that $|f(x, y) - l| < \epsilon$ for all values of (x, y) for which $0 < |(x - x_0)^2 + (y - y_0)^2| \leq \eta$.

If a function converges at (x_0, y_0) according to this definition (based on the circle), it converges according to the former defini-

* There are obvious changes to be made in these statements when we are dealing with a point (x_0, y_0) on one of the boundaries of the domain in which the function is defined.

tion (based on the square); and conversely. And the limits in both cases are the same.

Also continuity at (x_0, y_0) would now be defined as follows:

$f(x, y)$ is continuous at (x_0, y_0) , if, to the arbitrary positive number ϵ , there corresponds a positive number η such that $|f(x, y) - f(x_0, y_0)| < \epsilon$ for all values of (x, y) for which

$$0 < |(x - x_0)^2 + (y - y_0)^2| \leq \eta.$$

Every function, which is continuous at (x_0, y_0) under this definition, is continuous at (x_0, y_0) under the former definition, and conversely.

It is important to notice that if a function of x and y is continuous with respect to the two variables, as defined above, it is also continuous when considered as a function of x alone, or of y alone.

For example, let $f(x, y)$ be defined as follows:

$$\begin{cases} f(x, y) = \frac{2xy}{x^2 + y^2}, & \text{when at least one of the variables is not zero,} \\ f(0, 0) = 0. \end{cases}$$

Then $f(x, y)$ is a continuous function of x , for all values of x , when y has any fixed value, zero or not; and it is a continuous function of y , for all values of y , when x has any fixed value, zero or not.

But it is not a continuous function of (x, y) in any domain which includes the origin, since $f(x, y)$ is not continuous when $x=0$ and $y=0$.

For, if we put $x=r \cos \theta$, $y=r \sin \theta$, we have $f(x, y) = \sin 2\theta$, which is independent of r , and varies from -1 to $+1$.

However, it is a continuous function of (x, y) in any domain which does not include the origin.

On the other hand, the function defined by

$$\begin{cases} f(x, y) = \frac{2xy}{\sqrt{(x^2 + y^2)}}, & \text{when at least one of the variables is not zero,} \\ f(0, 0) = 0, \end{cases}$$

is a continuous function of (x, y) in any domain which includes the origin.

The theorems as to the continuity of the sum, product and, in certain cases, quotient of two or more continuous functions, given in § 30, can be readily extended to the case of functions of two or more variables. A continuous function of one or more continuous functions is also continuous.

In particular we have the theorem:

Let $u = \phi(x, y)$, $v = \psi(x, y)$ be continuous at (x_0, y_0) , and let $u_0 = \phi(x_0, y_0)$, $v_0 = \psi(x_0, y_0)$.

Let $z = f(u, v)$ be continuous in (u, v) at (u_0, v_0) .

Then $z = f[\phi(x, y), \psi(x, y)]$ is continuous in (x, y) at (x_0, y_0) .

Further, the general theorems on continuous functions, proved in § 31, hold, with only verbal changes, for functions of two or more variables.

For example :

If a function of two variables is continuous at every point of a closed domain, it is uniformly continuous in the domain.

In other words, when the positive number ϵ has been chosen, as small as we please, there is a positive number η such that $|f(x', y') - f(x'', y'')| < \epsilon$, when (x', y') and (x'', y'') are any two points in the domain for which

$$\sqrt{\{(x' - x'')^2 + (y' - y'')^2\}} \leq \eta.$$

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CHAPTER IV

THE DEFINITE INTEGRAL

38. In the usual elementary treatment of the Definite Integral, defined as the limit of a sum, it is assumed that the function of x considered may be represented by a curve. The limit is the area between the curve, the axis of x and the two bounding ordinates.

For long this demonstration was accepted as sufficient. To-day, however, analysis is founded on a more solid basis. No appeal is made to the intuitions of geometry. Further, even among the continuous functions of analysis, there are many which cannot be represented graphically.

E.g. let $f(x) = x \sin \frac{1}{x}$, when $x \geq 0$,
and $f(0) = 0$.

Then $f(x)$ is continuous for every value of x , but it has not a differential coefficient when $x = 0$.

It is continuous at $x = 0$, because

$$|f(x) - f(0)| = |f(x)| \leq |x|;$$

and $|f(x) - f(0)| < \epsilon$, when $0 < |x| \leq \eta$, if $\eta < \epsilon$.

Also it is continuous when $x \geq 0$, since it is the product of two continuous functions [cf. § 30].

It has not a differential coefficient at $x = 0$, because

$$\frac{f(h) - f(0)}{h} = \sin \frac{1}{h},$$

and $\sin 1/h$ has not a limit as $h \rightarrow 0$.

It has a differential coefficient at every point where $x \geq 0$, and at such points

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.$$

More curious still, Weierstrass discovered a function, which is continuous for every value of x , while it has not a differential coefficient anywhere.* This function is defined by the sum of the infinite series

$$\sum_0^{\infty} a^n \cos b^n \pi x,$$

a being a positive odd integer and b a positive number less than unity, connected with a by the inequality $ab > 1 + \frac{3}{2}\pi$.†

Other examples of such extraordinary functions have been given since Weierstrass's time. And for this reason alone it would have been necessary to substitute an exact arithmetical treatment for the traditional discussion of the Definite Integral.

Riemann ‡ was the first to give such a rigorous arithmetical treatment. The definition adopted in this chapter is due to him. The limitations imposed upon the integrand $f(x)$ will be indicated as we proceed.

In recent years a more general definition of the integral has been given by Lebesgue, § and extended by others, notably de la Vallée Poussin and W. H. Young, the chief object of Lebesgue's work being to remove the limitations on the integrand required in Riemann's treatment.

39. The Sums S and s . || Let $f(x)$ be a bounded function, given in the interval (a, b) .

* It seems impossible to assign an exact date to this discovery. Weierstrass himself did not at once publish it, but communicated it privately, as was his habit, to his pupils and friends. Du Bois-Reymond quotes it in a paper published in 1874.

† Hardy has shown that this relation can be replaced by $0 < a < 1$, $b > 1$ and $ab \geq 1$ [cf. *Trans. Amer. Math. Soc.*, **17**, 1916]. An interesting discussion of Weierstrass's function is to be found in a paper, "Infinite Derivates," *Q. J. Math.*, London, **47**, p. 127, 1916, by Grace Chisholm Young.

‡ In his classical paper, "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe." See above, p. 9.

But the earlier work of Cauchy and Dirichlet must not be forgotten.

§ Cf. Lebesgue, *Leçons sur l'Intégration*, Paris, 1904; de la Vallée Poussin, *Intégrales de Lebesgue*, Paris, 1916. And papers by Bliss and Hildebrandt in *Bull. Amer. Math. Soc.*, **24**, 1917.

|| The argument which follows is taken, with slight modifications, from Goursat's *Cours d'Analyse*, T. I. (3^e éd.), pp. 171 et seq.

Suppose this interval broken up into n partial intervals

$$(a, x_1), (x_1, x_2), \dots (x_{n-1}, b),$$

where

$$a < x_1 < x_2 \dots < x_{n-1} < b.$$

Let M, m be the upper and lower bounds of $f(x)$ in the whole interval, and M_r, m_r those in the interval (x_{r-1}, x_r) , writing $a = x_0$ and $b = x_n$.

$$\begin{aligned} \text{Let } S &= M_1(x_1 - a) + M_2(x_2 - x_1) + \dots + M_n(b - x_{n-1}), \\ \text{and } s &= m_1(x_1 - a) + m_2(x_2 - x_1) + \dots + m_n(b - x_{n-1}). \end{aligned}$$

To every mode of subdivision of (a, b) into such partial intervals, there corresponds a sum S and a sum s such that $s \leq S$.

The sums S have a lower bound, since they are all greater than $m(b - a)$, and the sums s have an upper bound, since they are all less than $M(b - a)$.

Let the lower bound of the sums S be J , and the upper bound of the sums s be I .

We shall now show that $I \leq J$.

$$\text{Let } a, x_1, x_2, \dots, x_{n-1}, b$$

be the set of points to which a certain S and s correspond.

Suppose some or all of the intervals $(a, x_1), (x_1, x_2), \dots (x_{n-1}, b)$ to be divided into smaller intervals, and let

$$a, y_1, y_2, \dots, y_{k-1}, x_1, y_k, y_{k+1}, \dots, y_{l-1}, x_2, y_l, \dots, b$$

be the set of points thus obtained.

The second mode of division will be called *consecutive* to the first, when it is obtained from it in this way.

Let Σ, σ be the sums for the new division.

Compare, for example, the parts of S and Σ which come from the interval (a, x_1) .

Let M'_1, m'_1 be the upper and lower bounds of $f(x)$ in (a, y_1) , M'_2, m'_2 in (y_1, y_2) , and so on.

The part of Σ which comes from (a, x_1) is then

$$M'_1(y_1 - a) + M'_2(y_2 - y_1) \dots + M'_k(x_1 - y_{k-1}).$$

But the numbers M'_1, M'_2, \dots cannot exceed M_1 .

Thus the part of Σ which we are considering is at most equal to $M_1(x_1 - a)$.

Similarly the part of Σ which comes from (x_1, x_2) is at most equal to $M_2(x_2 - x_1)$, and so on.

Adding these results we have $\Sigma \leq S$.

Similarly we obtain $\sigma \leq s$.

Consider now any two modes of division of (a, b) .

Denote them by

$$a, x_1, x_2, \dots, x_{m-1}, b, \text{ with sums } S \text{ and } s, \dots\dots(1)$$

and $a, y_1, y_2, \dots, y_{n-1}, b, \text{ with sums } S' \text{ and } s'. \dots\dots(2)$

On superposing these two, we obtain a third mode of division (3), consecutive to both (1) and (2).

Let the sums for (3) be Σ and σ .

Then, since (3) is consecutive to (1),

$$S \geq \Sigma \quad \text{and} \quad \sigma \geq s.$$

Also, since (3) is consecutive to (2),

$$S' \geq \Sigma \quad \text{and} \quad \sigma \geq s'.$$

But $\Sigma \geq \sigma$.

Therefore $S \geq s' \quad \text{and} \quad S' \geq s$.

Thus the sum S arising from any mode of division of (a, b) is not less than the sum s arising from the same, or any other, mode of division.

It follows at once that $I \leq J$.

For we can find a sum s as near I as we please, and a sum S (not necessarily from the same mode of division) as near J as we please. If $I > J$, this would involve the existence of an s and an S for which $s > S$.

The argument of this section will offer less difficulty, if the reader follow it for an ordinary function represented by a curve, when the sums S and s will refer to certain rectangles associated with the curve.

40. Darboux's Theorem. *The sums S and s tend respectively to J and I , when the points of division are multiplied indefinitely, in such a way that all the partial intervals tend to zero.*

Stated more precisely, the theorem reads as follows:

If the positive number ϵ is chosen, as small as we please, there is a positive number η such that, for all modes of division in which all the partial intervals are less than or equal to η , the sum S is greater than J by less than ϵ , and the sum s is smaller than I by less than ϵ .

Let ϵ be any positive number as small as we please.

Since the sums S and s have J and I for lower and upper bounds respectively, there is a mode of division such that the sum S for it exceeds J by less than $\frac{1}{2}\epsilon$.

Let this mode of division be

$$a, \quad a_1, \quad a_2, \quad \dots \quad a_{p-1}, \quad b, \quad \text{with sums } S_1 \text{ and } s_1. \quad \dots\dots(1)$$

Then
$$S_1 < J + \frac{1}{2}\epsilon.$$

Let η be a positive number such that all the partial interval of (1) are greater than η .

Let

$$a = x_0, \quad x_1, \quad x_2, \quad \dots \quad x_{n-1}, \quad b = x_n, \quad \text{with sums } S_2 \text{ and } s_2, \dots(2)$$

be any mode of division such that

$$(x_r - x_{r-1}) \leq \eta, \quad \text{when } r = 1, 2, \dots n.$$

The mode of division obtained by superposing (1) and (2), *e.g.*
 $a, x_1, x_2, a_1, x_3, a_2, x_4, \dots x_{n-1}, b$, with sums S_3 and $s_3, \dots(3)$
 is *consecutive* to (1) and (2).

Then, by § 39, we have
$$S_1 \geq S_3.$$

But
$$S_1 < J + \frac{1}{2}\epsilon.$$

Therefore
$$S_3 < J + \frac{1}{2}\epsilon.$$

Further,

$$S_2 - S_3 = \sum [M(x_{r-1}, x_r)(x_r - x_{r-1}) - M(x_{r-1}, a_k)(a_k - x_{r-1}) \\ - M(a_k, x_r)(x_r - a_k)],$$

$M(x', x'')$ denoting the upper bound of $f(x)$ in the interval (x', x'') , and the symbol Σ standing as usual for a summation, extending in this case to all the intervals (x_{r-1}, x_r) of (2) which have one of the points $a_1, a_2, \dots a_{p-1}$ as an internal point, and not an end-point. From the fact that each of the partial intervals of (1) is greater than η , and that each of those of (2) does not exceed η , we see that no two of the a 's can lie between two consecutive x 's of (2).

There are at most $(p-1)$ terms in the summation denoted by Σ . Let $|f(x)|$ have A for its upper bound in (a, b) .

We can rewrite $S_2 - S_3$ above in the form

$$S_2 - S_3 = \sum [\{M(x_{r-1}, x_r) - M(x_{r-1}, a_k)\}(a_k - x_{r-1}) \\ + \{M(x_{r-1}, x_r) - M(a_k, x_r)\}(x_r - a_k)].$$

But $\{M(x_{r-1}, x_r) - M(x_{r-1}, a_k)\}$ and $\{M(x_{r-1}, x_r) - M(a_k, x_r)\}$ are both positive or zero, and they cannot exceed $2A$.

Therefore $S_2 - S_3 \leq 2A \Sigma(x_r - x_{r-1})$,

the summation having at most $(p-1)$ terms, and $(x_r - x_{r-1})$ being at most equal to η .

Thus $S_2 - S_3 \leq 2(p-1)A\eta$.

Therefore $S_2 < J + \frac{1}{2}\epsilon + 2(p-1)A\eta$,

since we have seen that $S_3 < J + \frac{1}{2}\epsilon$.

So far the only restriction placed upon the positive number η has been that the partial intervals of (1) are each greater than η .

We can thus choose η so that

$$\eta < \frac{\epsilon}{4(p-1)A}.$$

With such a choice of η , $S_2 < J + \epsilon$.

Thus we have shown that for any mode of division such that the greatest of the partial intervals is less than or equal to a certain positive number η , dependent on ϵ , the sum S exceeds J by less than ϵ .

Similarly for s and I ; and it is obvious that we can make the same η satisfy both S and s , by taking the smaller of the two to which we are led in this argument.

41. The Definite Integral of a Bounded Function. We now come to the definition of the definite integral of a bounded function $f(x)$, given in an interval (a, b) .

A bounded function $f(x)$, given in the interval (a, b) , is said to be integrable in that interval, when the lower bound J of the sums S and the upper bound I of the sums s of § 39 are equal.

The common value of these bounds I and J is called the definite integral of $f(x)$ between the limits a and b , and is written

$$\int_a^b f(x) dx. *$$

It follows from the definition that $\int_a^b f(x) dx$ cannot be greater than the sum S or less than the sum s corresponding to any

* The bound J of the sums S is usually called the *upper integral* of $f(x)$ and the bound I of the sums s the *lower integral*.

mode of division of (a, b) . These form approximations by excess and defect to the integral.

We can replace the sums

$$\begin{aligned} S &= M_1(x_1 - a) + M_2(x_2 - x_1) + \dots + M_n(b - x_{n-1}), \\ s &= m_1(x_1 - a) + m_2(x_2 - x_1) + \dots + m_n(b - x_{n-1}), \end{aligned}$$

by more general expressions, as follows:

Let $\xi_1, \xi_2, \dots, \xi_r, \dots, \xi_n$ be any values of x in the partial intervals $(a, x_1), (x_1, x_2), \dots, (x_{r-1}, x_r), \dots, (x_{n-1}, b)$ respectively.

The sum

$$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + \dots + f(\xi_n)(b - x_{n-1}) \dots \dots (1)$$

obviously lies between the sums S and s for this mode of division, since we have $m_r \leq f(\xi_r) \leq M_r$ for each of the partial intervals.

But, when the number of points of division (x_r) increases indefinitely in such a way that all the partial intervals tend to zero, the sums S and s have a common limit, namely $\int_a^b f(x) dx$.

Therefore the sum (1) has the same limit.

Thus we have shown that, for an integrable function $f(x)$, the sum $f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + \dots + f(\xi_n)(b - x_{n-1})$

has the definite integral $\int_a^b f(x) dx$ for its limit, when the number of points of division (x_r) increases indefinitely in such a way that all the partial intervals tend to zero, $\xi_1, \xi_2, \dots, \xi_n$ being any values of x in these partial intervals.*

In particular, we may take $a, x_1, x_2, \dots, x_{n-1}$, or $x_1, x_2, \dots, x_{n-1}, b$, for the values of $\xi_1, \xi_2, \dots, \xi_n$.

42. Necessary and Sufficient Conditions for Integrability.

Any one of the following is a necessary and sufficient condition for the integrability of the bounded function $f(x)$ given in the interval (a, b) :

I. When any positive number ϵ has been chosen, as small as we please, there shall be a positive number η such that $S - s < \epsilon$ for every mode of division of (a, b) in which all the partial intervals are less than or equal to η .

* We may substitute in the above, for $f(\xi_1), f(\xi_2), \dots, f(\xi_n)$, any values $\mu_1, \mu_2, \dots, \mu_n$ intermediate between $(M_1, m_1), (M_2, m_2)$, etc., the upper and lower bounds of $f(x)$ in the partial intervals.

We have $S - s < \epsilon$, as stated above.

But $S \geq J$ and $s \leq I$.

Therefore $J - I < \epsilon$.

And J must be equal to I .

Thus the condition is *sufficient*.

Further, if $I = J$, the condition is satisfied.

For, given ϵ , by Darboux's Theorem, there is a positive number η such that $S - J < \frac{1}{2}\epsilon$ and $I - s < \frac{1}{2}\epsilon$ for every mode of division in which all the partial intervals are less than or equal to η .

But $S - s = (S - J) + (I - s)$, since $I = J$.

Therefore $S - s < \epsilon$.

II. When any positive number ϵ has been chosen, as small as we please, there shall be a mode of division of (a, b) such that $S - s < \epsilon$.

It has been proved in I. that this condition is *sufficient*. Also it is *necessary*. For we are given $I = J$, as $f(x)$ is integrable, and we have shown that in this case there are any number of modes of division, such that $S - s < \epsilon$.

III. Let ω, σ be any pair of positive numbers. There shall be a mode of division of (a, b) such that the sum of the lengths of the partial intervals in which the oscillation is greater than or equal to ω shall be less than σ .*

This condition is *sufficient*. For, having chosen the arbitrary positive number ϵ , take

$$\sigma = \frac{\epsilon}{2(M - m)} \quad \text{and} \quad \omega = \frac{\epsilon}{2(b - a)},$$

where M, m are the upper and lower bounds respectively of $f(x)$ in (a, b) .

Then there is a mode of division such that the sum of the lengths of the partial intervals in which the oscillation is greater than or equal to ω shall be less than σ . Let the intervals (x_{r-1}, x_r) in which the oscillation is greater than or equal to ω be denoted by D_r , and those in which it is less than ω by d_r , and let the oscillation $(M_r - m_r)$ in (x_{r-1}, x_r) be denoted by ω_r .

Then we have, for this mode of division,

$$\begin{aligned} S - s &= \sum \omega_r D_r + \sum \omega_r d_r \\ &< (M - m) \frac{\epsilon}{2(M - m)} + \frac{\epsilon}{2(b - a)} (b - a) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon, \end{aligned}$$

and, by II., $f(x)$ is integrable in (a, b) .

* Cf. Pierpont, *Theory of Functions of Real Variables*, Vol. I., § 498.

Also the condition is *necessary*. For, by II., if $f(x)$ is integrable in (a, b) , there is a mode of division such that $S - s < \omega\sigma$. Using D_r, d_r as above,

$$\begin{aligned} S - s &= \sum \omega_r D_r + \sum \omega_r d_r \\ &\geq \sum \omega_r D_r \\ &\geq \omega \sum D_r. \end{aligned}$$

Therefore $\omega\sigma > \omega \sum D_r$,
and $\sum D_r < \sigma$.

43. Integrable Functions.

I. *If $f(x)$ is continuous in (a, b) , it is integrable in (a, b) .*

In the first place, we know that $f(x)$ is bounded in the interval, since it is continuous in (a, b) [cf. § 31].

Next, we know that, to the arbitrary positive number ϵ , there corresponds a positive number η such that the oscillation of $f(x)$ is less than ϵ in all partial intervals less than or equal to η [cf. § 31].

Now we wish to show that, given the arbitrary positive number ϵ , there is a mode of division such that $S - s < \epsilon$ [§ 42, II.]. Starting with the given ϵ , we know that for $\epsilon/(b-a)$ there is a positive number η such that the oscillation of $f(x)$ is less than $\epsilon/(b-a)$ in all partial intervals less than or equal to η .

If we take a mode of division in which the partial intervals are less than or equal to this η , then for it we have

$$S - s < (b - a) \frac{\epsilon}{b - a} = \epsilon.$$

Therefore $f(x)$ is integrable in (a, b) .

II. *If $f(x)$ is monotonic in (a, b) , it is integrable in (a, b) .*

In the first place, we note that the function, being given in the *closed* interval (a, b) , and being monotonic, is also bounded. We shall take the case of a monotonic increasing function, so that we have

$$f(a) \leq f(x_1) \leq f(x_2) \dots \leq f(x_{n-1}) \leq f(b)$$

for the mode of division given by

$$a, \quad x_1, \quad x_2, \quad \dots \quad x_{n-1}, \quad b.$$

Thus we have

$$\left. \begin{aligned} S &= f(x_1)(x_1 - a) + f(x_2)(x_2 - x_1) \dots + f(b)(b - x_{n-1}), \\ s &= f(a)(x_1 - a) + f(x_1)(x_2 - x_1) \dots + f(x_{n-1})(b - x_{n-1}). \end{aligned} \right\}$$

Therefore, if all the partial intervals are less than or equal to η ,

$$S - s \leq \eta[f(b) - f(a)],$$

since $f(x_1) - f(a), f(x_2) - f(x_1), \dots, f(b) - f(x_{n-1})$
are none of them negative.

If we take
$$\eta < \frac{\epsilon}{f(b) - f(a)},$$

it follows that
$$S - s < \epsilon.$$

Thus $f(x)$ is integrable in (a, b) .

The same proof applies to a monotonic decreasing function.

We have seen that a monotonic function, given in (a, b) , can only have ordinary discontinuities, but these need not be finite in number (cf. § 34). We are thus led to consider other cases in which a bounded function is integrable, when discontinuities of the function occur in the given interval. A simple test of integrability is contained in the following theorem :

III. *A bounded function is integrable in (a, b) , when all its points of discontinuity in (a, b) can be enclosed in a finite number of intervals the sum of which is less than any arbitrary positive number.*

Let ϵ be any positive number, as small as we please, and let the upper bound of $|f(x)|$ in (a, b) be A .

By our hypothesis we can enclose all the points of discontinuity of $f(x)$ in a finite number of intervals, the sum of which is less than $\epsilon/4A$.

The part of $S - s$ coming from these intervals is, at most, $2A$ multiplied by their sum.

On the other hand, $f(x)$ is continuous in all the remaining (closed) intervals.

We can, therefore, break up this part of (a, b) into a finite number of partial intervals such that the corresponding portion of $S - s < \frac{1}{2}\epsilon$ (cf. I.).

Thus the combined mode of division for the whole of (a, b) is such that for it $S - s < \epsilon$.

Hence $f(x)$ is integrable in (a, b) .

In particular, *a bounded function, with only a finite number of discontinuities in (a, b) , is integrable in this interval.*

The discontinuities referred to in this theorem III. need not be ordinary discontinuities, but, as the function is bounded, they cannot be infinite discontinuities.

IV. *If a bounded function is integrable in each of the partial intervals (a, a_1) , (a_1, a_2) , ... (a_{p-1}, b) , it is integrable in the whole interval (a, b) .*

Since the function is integrable in each of these p intervals, there is a mode of division for each (e.g. a_{r-1}, a_r), such that $S-s$ for it is less than ϵ/p , where ϵ is any given positive number.

Then $S-s$ for the combined mode of division of the whole interval (a, b) is less than ϵ .

Therefore the function is integrable.

From the above results it is clear that *if a bounded function is such that the interval (a, b) can be broken up into a finite number of open partial intervals, in each of which the function is monotonic or continuous, then it is integrable in (a, b) .*

V. *If the bounded function $f(x)$ is integrable in (a, b) , then $|f(x)|$ is also integrable in (a, b) .*

This follows at once, since $S-s$ for $|f(x)|$ is not greater than $S-s$ for $f(x)$ for the same mode of division.

It may be remarked that the converse does not hold.

E.g. let $f(x)=1$ for rational values of x in $(0, 1)$,

and $f(x)=-1$ for irrational values of x in $(0, 1)$.

Then $|f(x)|$ is integrable, but $f(x)$ is not integrable, for it is obvious that the condition II. of § 42 is not satisfied, as the oscillation is 2 in any interval, however small.

44. If the bounded function $f(x)$ is integrable in (a, b) , there are an infinite number of points in any partial interval of (a, b) at which $f(x)$ is continuous.*

Let $\omega_1 > \omega_2 > \omega_3 \dots$ be an infinite sequence of positive numbers, such that

$$\lim_{n \rightarrow \infty} \omega_n = 0.$$

Let (a, β) be any interval contained in (a, b) such that $a \leq a < \beta < b$.

Then, by § 42, III., there is a mode of division of (a, b) such that the sum of the partial intervals in which the oscillation of $f(x)$ is greater than or equal to ω_1 is less than $(\beta - a)$.

If we remove from (a, b) these partial intervals, the remainder must cover at least part of (a, β) . We can thus choose within (a, β) a new interval (α_1, β_1) such that $(\beta_1 - \alpha_1) < \frac{1}{2}(\beta - a)$ and the oscillation in (α_1, β_1) is less than ω_1 .

Proceeding in the same way, we obtain within (α_1, β_1) a new interval

* Cf. Pierpont, *loc. cit.*, Vol. I., § 508.

(a_2, β_2) such that $(\beta_2 - a_2) < \frac{1}{2}(\beta_1 - a_1)$ and the oscillation in (a_2, β_2) is less than ω_2 . And so on.

Thus we find an infinite set of intervals A_1, A_2, \dots , each contained *entirely within* the preceding, while the length of A_n tends to zero as $n \rightarrow \infty$, and the oscillation of $f(x)$ in A_n also tends to zero.

By the theorem of § 18, the set of intervals defines a point (*e.g.* c) which lies within all the intervals.

Let ϵ be any positive number, as small as we please.

Then we can choose in the sequence $\omega_1, \omega_2, \dots$ a number ω_r less than ϵ . Let A_r be the corresponding interval (a_r, β_r) , and η a positive number smaller than $(c - a_r)$ and $(\beta_r - c)$.

Then $|f(x) - f(c)| < \epsilon$, when $|x - c| \leq \eta$,

and therefore we have shown that $f(x)$ is continuous at c .

Since this proof applies to any interval in (a, b) , the interval (a, β) contains an infinite number of points at which $f(x)$ is continuous, for any part of (a, β) , however small, contains a point of continuity.

45. Some Properties of the Definite Integral. We shall now establish some of the properties of $\int_a^b f(x) dx$, the integrand being bounded in (a, b) and integrable.

I. *If $f(x)$ is integrable in (a, b) , it is also integrable in any interval (a, β) contained in (a, b) .*

From § 42, I. we know that to the arbitrary positive number ϵ there corresponds a positive number η such that the difference $S - s < \epsilon$ for every mode of division of (a, b) in which all the partial intervals are less than or equal to η .

We can choose a mode of division of this kind with (a, β) as ends of partial intervals.

Let Σ, σ be the sums for the mode of division of (a, β) included in the above.

Then we have $0 \leq \Sigma - \sigma \leq S - s < \epsilon$.

Thus $f(x)$ is integrable in (a, β) [§ 42, II.].

II. *If the value of the integrable function $f(x)$ is altered at a finite number of points of (a, b) , the function $\phi(x)$ thus obtained is integrable in (a, b) , and its integral is the same as that of $f(x)$.*

We can enclose the points to which reference is made in a finite number of intervals, the sum of which is less than $\epsilon/4A$, where ϵ is any given positive number, and A is the upper bound of $|\phi(x)|$ in (a, b) .

The part of $S-s$ for $\phi(x)$, arising from these intervals, is at most $2A$ multiplied by their sum, *i.e.* it is less than $\frac{1}{2}\epsilon$.

On the other hand, $f(x)$ and $\phi(x)$, which is identical with $f(x)$ in the parts of (a, b) which are left, are integrable in each of these parts.

Thus we can obtain a mode of division for the whole of them which will contribute less than $\frac{1}{2}\epsilon$ to $S-s$, and, finally, we have a mode of division of (a, b) for which $S-s < \epsilon$.

Therefore $\phi(x)$ is integrable in (a, b) .

Further,
$$\int_a^b \phi(x) dx = \int_a^b f(x) dx.$$

For we have seen in § 41 that $\int_a^b \phi(x) dx$ is the limit of

$$\phi(\xi_1)(x_1 - a) + \phi(\xi_2)(x_2 - x_1) + \dots + \phi(\xi_n)(b - x_{n-1})$$

when the intervals $(a, x_1), (x_1, x_2), \dots, (x_{n-1}, b)$ tend to zero, and $\xi_1, \xi_2, \dots, \xi_n$ are any values of x in these intervals.

We may put $f(\xi_1), f(\xi_2), \dots, f(\xi_n)$ for $\phi(\xi_1), \phi(\xi_2), \dots, \phi(\xi_n)$ in this sum, since in each interval there are points at which $\phi(x)$ and $f(x)$ are equal.

In this way we obtain a sum of the form $\text{Lt } \Sigma f(\xi_r)(x_r - x_{r-1})$, which is identical with $\int_a^b f(x) dx$.

III. It follows immediately from the definition of the integral, that *if $f(x)$ is integrable in (a, b) , so also is $Cf(x)$, where C is any constant.*

Again, *if $f_1(x)$ and $f_2(x)$ are integrable in (a, b) , their sum is also integrable.*

For, let $S, s; S', s'$; and Σ, σ be the sums corresponding to the same mode of division for $f_1(x), f_2(x)$ and $f_1(x) + f_2(x)$.

Then it is clear that

$$\Sigma - \sigma \leq (S - s) + (S' - s'),$$

and the result follows.

Also it is easy to show that

$$\int_a^b Cf(x) dx = C \int_a^b f(x) dx,$$

and
$$\int_a^b \{f_1(x) + f_2(x)\} dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx.$$

IV. *The product of two integrable functions $f_1(x)$, $f_2(x)$ is integrable.*

To begin with, let the functions $f_1(x)$, $f_2(x)$ be positive in (a, b) .

Let M_r, m_r ; M'_r, m'_r ; $\mathbf{M}_r, \mathbf{m}_r$ be the upper and lower bounds of $f_1(x)$, $f_2(x)$ and $f_1(x)f_2(x)$ in the partial interval (x_{r-1}, x_r) .

Let S, s ; S', s' and Σ, σ be the corresponding sums for a certain mode of division in which (x_{r-1}, x_r) is a partial interval.

Then it is clear that

$$\mathbf{M}_r - \mathbf{m}_r \leq M_r M'_r - m_r m'_r = M_r(M'_r - m'_r) + m'_r(M_r - m_r).$$

A fortiori, $\mathbf{M}_r - \mathbf{m}_r \leq M(M'_r - m'_r) + M'(M_r - m_r)$,

where M, M' are the upper bounds of $f_1(x)$, $f_2(x)$ in (a, b) .

Multiplying this inequality by $(x_r - x_{r-1})$ and adding the corresponding results, we have

$$\Sigma - \sigma \leq M(S' - s') + M'(S - s).$$

It follows that $\Sigma - \sigma$ tends to zero, and the product of $f_1(x)$, $f_2(x)$ is integrable in (a, b) .

If the two functions are not both positive throughout the interval, we can always add constants c_1 and c_2 , so that $f_1(x) + c_1$, $f_2(x) + c_2$ remain positive in (a, b) .

The product

$$(f_1(x) + c_1)(f_2(x) + c_2) = f_1(x)f_2(x) + c_1f_2(x) + c_2f_1(x) + c_1c_2$$

is then integrable.

But $c_1f_2(x) + c_2f_1(x) + c_1c_2$ is integrable.

It follows that $f_1(x)f_2(x)$ is integrable.

On combining these results, we see that *if $f_1(x)$, $f_2(x)$... $f_n(x)$ are integrable functions, every polynomial in*

$$f_1(x), f_2(x) \dots f_n(x)$$

*is also an integrable function.**

46. Properties of the Definite Integral (continued).

I.
$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

In the definition of the sums S and s , and of the definite integral $\int_a^b f(x) dx$, we assumed that a was less than b . This restriction is, however, unnecessary, and will now be removed.

* This result can be extended to any continuous function of the n functions [cf. Hobson, *loc. cit.*, p. 345].

If $a > b$, we take as before the set of points

$$a, \quad x_1, \quad x_2, \quad \dots \quad x_{n-1}, \quad b,$$

and we deal with the sums

$$\left. \begin{aligned} S &= M_1(x_1 - a) + M_2(x_2 - x_1) + \dots + M_n(b - x_{n-1}), \\ s &= m_1(x_1 - a) + m_2(x_2 - x_1) + \dots + m_n(b - x_{n-1}). \end{aligned} \right\} \dots\dots(1)$$

The new sum S is equal in absolute value, but opposite in sign, to the sum obtained from

$$b, \quad x_{n-1}, \quad x_{n-2}, \quad \dots \quad x_1, \quad a.$$

The existence of the bounds of S and s in (1) follows, and the definite integral is defined as the common value of these bounds, when they have a common value.

It is thus clear that, with this extension of the definition of §41, we have

$$\int_a^b f(x) dx = - \int_b^a f(x) dx, *$$

a, b being any points of an interval in which $f(x)$ is bounded and integrable.

II. *Let c be any point of an interval (a, b) in which $f(x)$ is bounded and integrable.*

$$\text{Then} \quad \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Consider a mode of division of (a, b) which has not c for a point of section. If we now introduce c as an additional point of section, the sum S is certainly not increased.

But the sums S for (a, c) and (c, b) , given by this mode of division, are respectively not less than $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$.

Thus every mode of division of (a, b) gives a sum S not less than

$$\int_a^c f(x) dx + \int_c^b f(x) dx.$$

It follows that

$$\int_a^b f(x) dx \geq \int_a^c f(x) dx + \int_c^b f(x) dx.$$

But the modes of division of (a, c) and (c, b) together form a possible mode of division of (a, b) . And we can obtain modes

* The results proved in §§ 42-45 are also applicable, in some cases with slight verbal alterations, to the Definite Integral thus generalised.

of division of (a, c) and (c, b) , the sums for which differ from $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$, respectively, by as little as we please.

It follows that the sign of inequality in the above relation must be deleted, so that we have

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

If c lies on (a, b) produced in either direction, it is easy to show, as above, that this result remains true, provided that $f(x)$ is integrable in (a, c) in the one case, and (c, b) in the other.

47. If $f(x) \geq g(x)$, and both functions are integrable in (a, b) , then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$.

Let $\phi(x) = f(x) - g(x) \geq 0$.

Then $\phi(x)$ is integrable in (a, b) , and obviously, from the sum s ,

$$\int_a^b \phi(x) dx \geq 0.$$

Therefore $\int_a^b f(x) dx - \int_a^b g(x) dx \geq 0$.

COROLLARY I. *If $f(x)$ is integrable in (a, b) , then*

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

We have seen in § 43 that if $f(x)$ is integrable in (a, b) , so also is $|f(x)|$.

And $-|f(x)| \leq f(x) \leq |f(x)|$.

The result follows from the above theorem.

COROLLARY II. *Let $f(x)$ be integrable and never negative in (a, b) . If $f(x)$ is continuous at c in (a, b) and $f(c) > 0$, then $\int_a^b f(x) dx > 0$.*

We have seen in § 44 that if $f(x)$ is integrable in (a, b) , it must have points of continuity in the interval. What is assumed here is that at one of these points of continuity $f(x)$ is positive.

Let this point c be an internal point of the interval (a, b) , and not an end-point. Then there is an interval (c', c'') , where

$a < c' < c < c'' < b$, such that $f(x) > k$ for every point of (c', c'') , k being some positive number.

Thus, since $f(x) \geq 0$ in (a, c') , $\int_a^{c'} f(x) dx \geq 0$.

And, since $f(x) > k$ in (c', c'') , $\int_{c'}^{c''} f(x) dx \geq k(c'' - c') > 0$.

Also, since $f(x) \geq 0$ in (c'', b) , $\int_{c''}^b f(x) dx \geq 0$.

Adding these results, we have $\int_a^b f(x) dx > 0$.

The changes in the argument when c is an end-point of (a, b) are slight.

COROLLARY III. *Let $f(x) \geq g(x)$, and both be integrable in (a, b) . At a point c in (a, b) , let $f(x)$ and $g(x)$ both be continuous, and $f(c) > g(c)$. Then $\int_a^b f(x) dx > \int_a^b g(x) dx$.*

This follows at once from Corollary II. by writing

$$\phi(x) = f(x) - g(x).$$

By the aid of the theorem proved in § 44, the following simpler result may be obtained :

If $f(x) > g(x)$, and both are integrable in (a, b) , then

$$\int_a^b f(x) dx > \int_a^b g(x) dx.$$

For, if $f(x)$ and $g(x)$ are integrable in (a, b) , we know that $f(x) - g(x)$ is integrable and has an infinite number of points of continuity in (a, b) .

At any one of these points $f(x) - g(x)$ is positive, and the result follows from Corollary II.

48. The First Theorem of Mean Value. Let $\phi(x)$, $\psi(x)$ be two bounded functions, integrable in (a, b) , and let $\psi(x)$ keep the same sign in this interval; *e.g.* let $\psi(x) \geq 0$ in (a, b) .

Also let M , m be the upper and lower bounds of $\phi(x)$ in (a, b) .

Then we have, in (a, b) ,

$$m \leq \phi(x) \leq M,$$

and multiplying by the factor $\psi(x)$, which is not negative,

$$m\psi(x) \leq \phi(x)\psi(x) \leq M\psi(x).$$

It follows from § 47 that

$$m \int_a^b \psi(x) dx \leq \int_a^b \phi(x)\psi(x) dx \leq M \int_a^b \psi(x) dx,$$

since $\phi(x)\psi(x)$ is also integrable in (a, b) .

Therefore
$$\int_a^b \phi(x) \psi(x) dx = \mu \int_a^b \psi(x) dx$$

where μ is some number satisfying the relation $m \leq \mu \leq M$.

It is clear that the argument applies also to the case when

$$\psi(x) \leq 0 \text{ in } (a, b).$$

If $\phi(x)$ is continuous in (a, b) , we know that it takes the value μ for some value of x in the interval (cf. § 31).

We have thus established the important theorem :

If $\phi(x)$, $\psi(x)$ are two bounded functions, integrable in (a, b) , $\phi(x)$ being continuous and $\psi(x)$ keeping the same sign in the interval, then

$$\int_a^b \phi(x) \psi(x) dx = \phi(\xi) \int_a^b \psi(x) dx,$$

where ξ is some definite value of x in $a \leq x \leq b$.

Further, if $\phi(x)$ is not continuous in (a, b) , we replace $\phi(\xi)$ by μ , where μ satisfies the relation $m \leq \mu \leq M$, m, M being the bounds of $\phi(x)$ in (a, b) .

This is usually called the *First Theorem of Mean Value*.

As a particular case, when $\phi(x)$ is continuous,

$$\int_a^b \phi(x) dx = (b-a) \phi(\xi), \quad \text{where } a \leq \xi \leq b.$$

It will be seen from the corollaries to the theorem in § 47 that in certain cases we can replace $a \leq \xi \leq b$ by $a < \xi < b$.*

However, for most applications of the theorem, the more general statement in the text is sufficient.

49. The Integral considered as a Function of its Upper Limit.

Let $f(x)$ be bounded and integrable in (a, b) , and let

$$F(x) = \int_a^x f(x) dx,$$

where x is any point in (a, b) .

Then if $(x+h)$ is also in the interval,

$$F(x+h) - F(x) = \int_x^{x+h} f(x) dx.$$

Thus
$$F(x+h) - F(x) = \mu h,$$

where $m \leq \mu \leq M$, the numbers M, m being the upper and lower bounds of $f(x)$ in $(x, x+h)$.

It follows that $F(x)$ is a continuous function of x in (a, b) .

* Cf. Pierpont, *loc. cit.*, Vol. I., pp. 367-8.

Further, if $f(x)$ is continuous in (a, b) ,

$$F(x+h) - F(x) = hf(\xi), \quad \text{where } x \leq \xi \leq x+h.$$

When h tends to zero, $f(\xi)$ has the limit $f(x)$.

Therefore
$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

Thus when $f(x)$ is continuous in (a, b) , $\int_a^x f(x) dx$ is continuous in (a, b) , and has a differential coefficient for every value of x in (a, b) , this differential coefficient being equal to $f(x)$.

This is one of the most important theorems of the Calculus. It shows that every continuous function is the differential coefficient of a continuous function, usually called its primitive, or indefinite integral.

It also gives a means of evaluating definite integrals of continuous functions. If $f(x)$ is continuous in (a, b) and

$$F(x) = \int_a^x f(x) dx,$$

then we know that $\frac{d}{dx}F(x) = f(x)$. Suppose that, by some means or other, we have obtained a continuous function $\phi(x)$ such that

$$\frac{d}{dx}\phi(x) = f(x).$$

We must then have $F(x) = \phi(x) + C$, since $\frac{d}{dx}(F(x) - \phi(x)) = 0$ in (a, b) .*

To determine the constant C , we use the fact that $F(x)$ vanishes at $x = a$.

Thus we have
$$\int_a^x f(x) dx = \phi(x) - \phi(a).$$

50. The Second Theorem of Mean Value. We now come to a theorem of which frequent use will be made.

I. Let $\phi(x)$ be monotonic, and therefore integrable, in (a, b) and let $\psi(x)$ be bounded and integrable, and not change sign more than a finite number of times in (a, b) .†

* Cf. Hardy, *loc. cit.*, p. 228.

† This restriction on $\psi(x)$ can be removed. For a proof in which the only condition imposed upon $\psi(x)$ is that it is bounded and integrable in (a, b) , see Hobson, *loc. cit.*, p. 360, or Goursat, *loc. cit.*, T. I., p. 182.

Then
$$\int_a^b \phi(x) \psi(x) dx = \phi(a) \int_a^{\xi} \psi(x) dx + \phi(b) \int_{\xi}^b \psi(x) dx,$$

where ξ is some definite value of x in $a \leq x \leq b$.

For clearness we shall take $\phi(x)$ monotonic *increasing* in (a, b) . The modifications in the proof for a monotonic *decreasing* function will be obvious.

Since we assume that $\psi(x)$ does not change sign more than a finite number of times in (a, b) , we can take

$$a = a_0, \quad a_1, \quad a_2, \quad \dots \quad a_{n-1}, \quad a_n = b,$$

such that $\psi(x)$ keeps the same sign in the partial intervals

$$(a_0, a_1), \quad (a_1, a_2), \quad \dots \quad (a_{n-1}, a_n).$$

Then
$$\int_a^b \phi(x) \psi(x) dx = \sum_1^n \int_{a_{r-1}}^{a_r} \phi(x) \psi(x) dx.$$

Now, by the First Theorem of Mean Value,

$$\int_{a_{r-1}}^{a_r} \phi(x) \psi(x) dx = \mu_r \int_{a_{r-1}}^{a_r} \psi(x) dx,$$

where

$$\phi(a_{r-1}) \leq \mu_r \leq \phi(a_r).$$

Therefore
$$\int_{a_{r-1}}^{a_r} \phi(x) \psi(x) dx = \mu_r [F(a_r) - F(a_{r-1})],$$

where we have written $F(x) = \int_a^x \psi(x) dx$.

Thus we have

$$\int_a^b \phi(x) \psi(x) dx = \sum_1^n \mu_r [F(a_r) - F(a_{r-1})]. \quad \dots \dots (1)$$

Since $F(a) = 0$, we may add on the term $\phi(a) F(a)$, and rewrite (1) in the form :

$$\left. \begin{aligned} \int_a^b \phi(x) \psi(x) dx = & F(a) [\phi(a) - \mu_1] \\ & + F(a_1) [\mu_1 - \mu_2] \\ & + \dots \dots \dots \\ & + F(b) [\mu_n - \phi(b)] \\ & + F(b) \phi(b). \end{aligned} \right\} \dots \dots \dots (2)$$

The multipliers of $F(a), F(a_1), \dots F(b)$ in all but the last line must be negative or zero.

We may therefore replace the sum of these $(n+1)$ terms by

$$\mathbf{M}\{[\phi(a) - \mu_1] + [\mu_1 - \mu_2] + \dots + [\mu_n - \phi(b)]\},$$

i.e. by

$$\mathbf{M}[\phi(a) - \phi(b)],$$

where \mathbf{M} is some definite number intermediate between the greatest and least of $F(a)$, $F(a_1)$, \dots , $F(a_{n-1})$, $F(b)$, or coinciding with one or other.

But, since $F(x) = \int_a^x \psi(x) dx$, we know that $F(x)$ is continuous in (a, b) .

Therefore \mathbf{M} may be taken as $F(\xi)$, where ξ is some definite value of x in $a \leq x \leq b$ (cf. § 31).

It follows from (2) that

$$\begin{aligned} \int_a^b \phi(x) \psi(x) dx &= F(\xi)[\phi(a) - \phi(b)] + F(b) \phi(b) \\ &= \phi(a) F(\xi) + \phi(b)[F(b) - F(\xi)] \\ &= \phi(a) \int_a^\xi \psi(x) dx + \phi(b) \int_\xi^b \psi(x) dx, \end{aligned}$$

where ξ is some definite value of x in $a \leq x \leq b$.

Thus the theorem stated above is proved.

We have seen in § 45 that $\int_a^b f(x) dx$ is unaltered if we change the value of $f(x)$ at a finite number of points in (a, b) .

Now $\phi(x)$ is monotonic in (a, b) , and therefore $\phi(a+0)$, $\phi(b-0)$ exist. Also we may give $\phi(x)$ these values at $x=a$ and $x=b$ respectively, without altering its monotonic character or changing

$$\int_a^b \phi(x) \psi(x) dx.$$

We thus* obtain the result:

II. *Let $\phi(x)$ be monotonic, and therefore integrable, in (a, b) , and let $\psi(x)$ be bounded and integrable, and not change its sign more than a finite number of times in (a, b) .*

Then

$$\int_a^b \phi(x) \psi(x) dx = \phi(a+0) \int_a^\xi \psi(x) dx + \phi(b-0) \int_\xi^b \psi(x) dx,$$

where ξ is some definite value of x in $a \leq x \leq b$.

* This and the other theorems which follow could be obtained at once by making suitable changes in the terms $\phi(a)$ and $\phi(b)$ on the right-hand side of (2).

Also it is clear that we could in the same way replace $\phi(a+0)$ and $\phi(b-0)$, respectively, by any numbers A and B , provided that $A \leq \phi(a+0)$ and $B \geq \phi(b-0)$ in the case of the monotonic increasing function; and $A \geq \phi(a+0)$, $B \leq \phi(b-0)$ in the case of the monotonic decreasing function.

We thus obtain, *with the same limitation on $\phi(x)$ and $\psi(x)$ as before,*

$$\text{III.} \quad \int_a^b \phi(x) \psi(x) dx = A \int_a^{\xi} \psi(x) dx + B \int_{\xi}^b \psi(x) dx,$$

where $A \leq \phi(a+0)$ and $B \geq \phi(b-0)$, if $\phi(x)$ is monotonic increasing, and $A \geq \phi(a+0)$, $B \leq \phi(b-0)$, if $\phi(x)$ is monotonic decreasing, ξ being some definite value of x in $a \leq x \leq b$.

The value of ξ in I., II., III. need not, of course, be the same, and in III. it will depend on the values chosen for A and B .

Finally, as in III., we may take $A=0$ and $B=\phi(b)$, when $\phi(x) \geq 0$ and is monotonic increasing in (a, b) .

Thus, *with the same limitations on $\psi(x)$ as before, when $\phi(x) \geq 0$ and is monotonic increasing in (a, b) , we have*

$$\int_a^b \phi(x) \psi(x) dx = \phi(b) \int_{\xi}^b \psi(x) dx,$$

where ξ is some definite value of x in $a \leq x \leq b$.

Again, when $\phi(x) \geq 0$ and is monotonic decreasing in (a, b) , we may take $A=\phi(a)$ and $B=0$, obtaining in this way, *with the same limitations on $\psi(x)$ as before,*

$$\text{V.} \quad \int_a^b \phi(x) \psi(x) dx = \phi(a) \int_a^{\xi} \psi(x) dx,$$

where ξ is some definite value of x in $a \leq x \leq b$.

Theorems IV. and V. are the earliest form of the Second Theorem of Mean Value, and are due to Bonnet,* by whom they were employed in the discussion of the Theory of Fourier's Series. The other Theorems I., II., III. can be deduced from Bonnet's results.

Theorem I. was given by Weierstrass in his lectures and Du Bois-Reymond,† independently of Bonnet.

Theorem II. is the form in which we shall most frequently use the Second Theorem of Mean Value.

* *Bruxelles, Mém. cour. Acad. roy.*, 23, p. 8, 1850; also *J. Math., Paris*, 14 p. 249, 1849.

† *J. Math., Berlin*, 69, p. 81, 1869; and 79, p. 42, 1875.

INFINITE INTEGRALS. INTEGRAND BOUNDED.
INTERVAL INFINITE.

51. In the definition of the ordinary integral $\int_a^b f(x) dx$, and in the preceding sections of this chapter, we have supposed that the integrand is bounded in the interval of integration which extends from one given point a to another given point b . We proceed to extend this definition so as to include cases in which

- (i) the interval increases without limit,
- (ii) the integrand has a finite number of infinite discontinuities.*

I. Integrals to $+\infty$. $\int_a^\infty f(x) dx$.

Let $f(x)$ be bounded and integrable in the interval (a, b) , where a is fixed and b is any number greater than a . We define the integral $\int_a^\infty f(x) dx$ as $\text{Lt}_{x \rightarrow \infty} \int_a^x f(x) dx$, when this limit exists.†

We speak of $\int_a^\infty f(x) dx$ in this case as an *infinite integral*, and say that it converges.

On the other hand, when $\int_a^x f(x) dx$ tends to ∞ as $x \rightarrow \infty$, we say that the infinite integral $\int_a^\infty f(x) dx$ diverges to ∞ , and there is a similar definition of divergence to $-\infty$ of $\int_a^\infty f(x) dx$.

Ex. 1. $\int_0^\infty e^{-x} dx = 1$; $\int_1^\infty \frac{dx}{x^{\frac{3}{2}}} = 2$.

For $\int_0^\infty e^{-x} dx = \text{Lt}_{x \rightarrow \infty} \int_0^x e^{-x} dx = \text{Lt}_{x \rightarrow \infty} (1 - e^{-x}) = 1$.

And $\int_1^\infty \frac{dx}{x^{\frac{3}{2}}} = \text{Lt}_{x \rightarrow \infty} \int_1^x \frac{dx}{x^{\frac{3}{2}}} = \text{Lt}_{x \rightarrow \infty} 2 \left(1 - \frac{1}{x^{\frac{1}{2}}} \right) = 2$.

Ex. 2. $\int_0^\infty e^x dx = \infty$; $\int_1^\infty \frac{dx}{\sqrt{x}} = \infty$.

For $\int_0^\infty e^x dx = \text{Lt}_{x \rightarrow \infty} \int_0^x e^x dx = \text{Lt}_{x \rightarrow \infty} (e^x - 1) = \infty$.

* For the definition of the term "infinite discontinuities," see § 33.

† It is more convenient to use this notation, but, if the presence of the variable x in the integrand offers difficulty, we may replace these integrals by $\int_a^\infty f(t) dt$ and $\text{Lt}_{x \rightarrow \infty} \int_a^x f(t) dt$.

And
$$\int_1^{\infty} \frac{dx}{\sqrt{x}} = \text{Lt}_{x \rightarrow \infty} \int_1^x \frac{dx}{\sqrt{x}} = \text{Lt}_{x \rightarrow \infty} 2(\sqrt{x} - 1) = \infty.$$

Similarly
$$\int_1^{\infty} \log \frac{1}{x} dx = -\infty; \quad \int_2^{\infty} \frac{dx}{1-x} = -\infty.$$

These integrals diverge to ∞ or $-\infty$, as the case may be.

Finally, when none of these alternatives occur, we say that the infinite integral $\int_a^{\infty} f(x) dx$ oscillates finitely or infinitely, as in §§ 16 and 25.

Ex. 3.
$$\int_a^{\infty} \sin x dx$$
 oscillates finitely.

$$\int_a^{\infty} x \sin x dx$$
 oscillates infinitely.

II. Integrals to $-\infty$.
$$\int_{-\infty}^b f(x) dx.$$

When $f(x)$ is bounded and integrable in the interval (a, b) , where b is fixed and a is any number less than b , we define the integral $\int_{-\infty}^b f(x) dx$ as $\text{Lt}_{x \rightarrow -\infty} \int_x^b f(x) dx$, when this limit exists.

We speak of $\int_{-\infty}^b f(x) dx$ as an infinite integral, and say that it converges.

The cases in which $\int_{-\infty}^b f(x) dx$ is said to diverge to ∞ or to $-\infty$, or to oscillate finitely or infinitely, are treated as before.

Ex. 1.
$$\int_{-\infty}^0 e^x dx = 1, \quad \int_{-\infty}^0 \frac{dx}{(1-2x)^2} = \frac{1}{2}.$$

Ex. 2.
$$\int_{-\infty}^0 e^{-x} dx$$
 diverges to ∞ .

$$\int_{-\infty}^0 \sinh x dx$$
 diverges to $-\infty$.

$$\int_{-\infty}^0 \sin x dx$$
 oscillates finitely.

$$\int_{-\infty}^0 x \sin x dx$$
 oscillates infinitely.

III. Integrals from $-\infty$ to ∞ .
$$\int_{-\infty}^{\infty} f(x) dx.$$

If the infinite integrals $\int_{-\infty}^a f(x) dx$ and $\int_a^{\infty} f(x) dx$ are both convergent, we say that the infinite integral $\int_{-\infty}^{\infty} f(x) dx$ is convergent and is equal to their sum.

Since
$$\int_a^x f(x) dx = \int_a^a f(x) dx + \int_a^x f(x) dx, \quad a < a < x,$$

it follows that, if one of the two integrals $\int_a^\infty f(x) dx$ or $\int_a^\infty f(x) dx$ converges, the other does.

$$\text{Also} \quad \int_a^\infty f(x) dx = \int_a^a f(x) dx + \int_a^\infty f(x) dx.$$

$$\text{Similarly,} \quad \int_x^a f(x) dx = \int_x^a f(x) dx + \int_a^a f(x) dx, \quad x < a < a,$$

and, if one of the two integrals $\int_{-\infty}^a f(x) dx$ or $\int_{-\infty}^a f(x) dx$ converges, the other does.

$$\text{Also} \quad \int_{-\infty}^a f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^a f(x) dx.$$

$$\text{Thus} \quad \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx,$$

and the value of $\int_{-\infty}^\infty f(x) dx$ is independent of the point a used in the definition.

$$\text{Ex.} \quad \int_{-\infty}^\infty \frac{dx}{1+x^2} = \pi, \quad \int_{-\infty}^\infty e^{-x^2} dx = 2 \int_0^\infty e^{-x^2} dx.$$

52. A necessary and sufficient condition for the convergence of $\int_a^\infty f(x) dx$.

$$\text{Let} \quad F(x) = \int_a^x f(x) dx.$$

The conditions under which $F(x)$ shall have a limit as $x \rightarrow \infty$ have been discussed in §§ 27 and 29. In the case of the infinite integral we are thus able to say that :

I. *The integral $\int_a^\infty f(x) dx$ is convergent and has the value I , when, any positive number ϵ having been chosen, as small as we please, there is a positive number X such that*

$$\left| I - \int_a^x f(x) dx \right| < \epsilon, \text{ provided that } x \geq X.$$

And further :

II. *A necessary and sufficient condition for the convergence of the integral $\int_a^\infty f(x) dx$ is that, when any positive number ϵ*

has been chosen, as small as we please, there shall be a positive number X such that

$$\left| \int_{x'}^{x''} f(x) dx \right| < \epsilon$$

for all values of x', x'' for which $x'' > x' \geq X$.

We have seen in § 51 that if $\int_a^\infty f(x) dx$ converges, then

$$\int_a^\infty f(x) dx = \int_a^a f(x) dx + \int_a^\infty f(x) dx, \quad a < a.$$

It follows from I. that, if $\int_a^\infty f(x) dx$ converges, to the arbitrary positive number ϵ there corresponds a positive number X such that

$$\left| \int_x^\infty f(x) dx \right| < \epsilon, \quad \text{when } x \geq X.$$

Also, if this condition is satisfied, the integral converges.

These results, and the others which follow, can be extended immediately to the infinite integral

$$\int_{-\infty}^a f(x) dx.$$

53. $\int_a^\infty f(x) dx$. Integrand Positive. If the integrand $f(x)$ is positive when $x > a$, it is clear that $\int_a^x f(x) dx$ is a monotonic increasing function of x . Thus $\int_a^\infty f(x) dx$ must either converge or diverge to ∞ .

I. *It will converge if there is a positive number A such that $\int_a^x f(x) dx < A$ when $x > a$, and in this case $\int_a^\infty f(x) dx \leq A$.*

It will diverge to ∞ if there is no such number.

These statements follow from the properties of monotonic functions (§ 34).

Further, there is an important "comparison test" for the convergence of integrals when the integrand is positive.

II. *Let $f(x), g(x)$ be two functions which are positive, bounded and integrable in the arbitrary interval (a, b) . Also let $g(x) \leq f(x)$ when $x \geq a$. Then, if $\int_a^\infty f(x) dx$ is convergent, it follows that $\int_a^\infty g(x) dx$ is convergent, and $\int_a^\infty g(x) dx \leq \int_a^\infty f(x) dx$.*

For from § 47 we know that

$$\int_a^x g(x) dx \leq \int_a^x f(x) dx, \text{ when } x > a.$$

Therefore
$$\int_a^x g(x) dx < \int_a^\infty f(x) dx.$$

Then, from I.,
$$\int_a^\infty g(x) dx \leq \int_a^\infty f(x) dx.$$

III. If $g(x) \geq f(x)$, and $\int_a^\infty f(x) dx$ diverges, so also does

$$\int_a^\infty g(x) dx.*$$

This follows at once, since $\int_a^x g(x) dx \geq \int_a^x f(x) dx$.

One of the most useful integrals for comparison is $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$.

We have
$$\int_a^x \frac{dx}{x^n} = \frac{1}{1-n} \{x^{1-n} - a^{1-n}\}, \text{ when } n \neq 1,$$

and
$$\int_a^x \frac{dx}{x} = \log x - \log a, \text{ when } n = 1.$$

Thus, when $n > 1$,

$$\text{Lt}_{x \rightarrow \infty} \int_a^x \frac{dx}{x^n} = \frac{a^{1-n}}{n-1},$$

$$\text{i.e. } \int_a^\infty \frac{dx}{x^n} = \frac{a^{1-n}}{n-1}.$$

And, when $n \leq 1$,

$$\text{Lt}_{x \rightarrow \infty} \int_a^x \frac{dx}{x^n} = \infty,$$

$$\text{i.e. } \int_a^\infty \frac{dx}{x^n} \text{ diverges.}$$

* Since the relative behaviour of the positive integrands $f(x)$ and $g(x)$ matters only as $x \rightarrow \infty$, these conditions may be expressed in terms of limits:

When $g(x)/f(x)$ has a limit as $x \rightarrow \infty$, $\int_a^\infty g(x) dx$ converges, if $\int_a^\infty f(x) dx$ converges.

When $g(x)/f(x)$ has a limit, not zero, or diverges, as $x \rightarrow \infty$, $\int_a^\infty g(x) dx$ diverges, if $\int_a^\infty f(x) dx$ diverges.

- Ex. 1.** $\int_a^\infty \frac{dx}{x\sqrt{1+x^2}}$ converges, since $\frac{1}{x\sqrt{1+x^2}} < \frac{1}{x^2}$, when $x \geq a > 0$.
2. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges, since $\frac{1}{\sqrt{x^2-1}} > \frac{1}{x}$, when $x \geq 2$.
3. $\int_a^\infty \frac{\sin^2 x}{x^2} dx$ converges, since $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$, when $x \geq a > 0$.

54. Absolute Convergence. The integral $\int_a^\infty f(x) dx$ is said to be **absolutely convergent** when $f(x)$ is bounded and integrable in the arbitrary interval (a, b) , and $\int_a^\infty |f(x)| dx$ is convergent.

Since $\left| \int_{x'}^{x''} f(x) dx \right| \leq \int_{x'}^{x''} |f(x)| dx$, for $x'' > x' \geq a$
(cf. § 47, Cor. I.),

it follows from § 52, II. that if $\int_a^\infty |f(x)| dx$ converges, so also does $\int_a^\infty f(x) dx$.

But the converse is not true. An infinite integral of this type may converge, and yet not converge **absolutely**.

For example, consider the integral

$$\int_0^\infty \frac{\sin x}{x} dx.$$

The Second Theorem of Mean Value (§ 50) shows that this integral converges.

For we have

$$\int_{x'}^{x''} \frac{\sin x}{x} dx = \frac{1}{x'} \int_{x'}^{\xi} \sin x dx + \frac{1}{x''} \int_{\xi}^{x''} \sin x dx,$$

where $0 < x' \leq \xi \leq x''$.

But

$\left| \int_{x'}^{\xi} \sin x dx \right|$ and $\left| \int_{\xi}^{x''} \sin x dx \right|$ are each less than or equal to 2.

Therefore $\left| \int_{x'}^{x''} \frac{\sin x}{x} dx \right| \leq 2 \left\{ \frac{1}{x'} + \frac{1}{x''} \right\}$
 $< \frac{4}{x'}$.

Thus $\left| \int_{x'}^{x''} \frac{\sin x}{x} dx \right| < \epsilon$, when $x'' > x' \geq X$,

provided that $X > \frac{4}{\epsilon}$.

Therefore $\int_0^{\infty} \frac{\sin x}{x} dx$ converges, and we shall find in § 88 that its value is $\frac{1}{2}\pi$.

But the integral $\int_0^{\infty} \frac{|\sin x|}{x} dx$ diverges.

To prove this, it is only necessary to consider the integral

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx,$$

where n is any positive integer.

We have
$$\int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_1^n \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx.$$

But
$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx = \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y} dy,$$

on putting $x = (r-1)\pi + y$.

Therefore
$$\begin{aligned} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx &> \frac{1}{r\pi} \int_0^{\pi} \sin y dy \\ &> \frac{2}{r\pi}. \end{aligned}$$

Thus
$$\int_0^{n\pi} \frac{|\sin x|}{x} dx > \frac{2}{\pi} \sum_1^n \frac{1}{r}.$$

But the series on the right hand diverges to ∞ as $n \rightarrow \infty$.

Therefore
$$\text{Lt}_{n \rightarrow \infty} \int_0^{n\pi} \frac{|\sin x|}{x} dx = \infty.$$

But when $x > n\pi$,

$$\int_0^x \frac{|\sin x|}{x} dx > \int_0^{n\pi} \frac{|\sin x|}{x} dx.$$

Therefore
$$\text{Lt}_{x \rightarrow \infty} \int_0^x \frac{|\sin x|}{x} dx = \infty.$$

When infinite integrals of this type converge, but do not converge absolutely, the convergence must be due to changes of sign in the integrand as $x \rightarrow \infty$.

55. The μ -Test for the Convergence of $\int_a^{\infty} f(x) dx$.

I. Let $f(x)$ be bounded and integrable in the arbitrary interval (a, b) where $a > 0$. If there is a number μ greater than 1 such

that $x^\mu f(x)$ is bounded when $x \geq a$, then $\int_a^\infty f(x) dx$ converges absolutely.

Here $|x^\mu f(x)| < A$, where A is some definite positive number and $x \geq a$.

Thus $|f(x)| < \frac{A}{x^\mu}$.

But we know that $\int_a^\infty \frac{dx}{x^\mu}$ converges.

It follows that $\int_a^\infty |f(x)| dx$ converges.

Therefore $\int_a^\infty f(x) dx$ converges, and the convergence is absolute.

II. Let $f(x)$ be bounded and integrable in the arbitrary interval (a, b) , where $a > 0$. If there is a number μ less than or equal to 1 such that $x^\mu f(x)$ has a positive lower bound when $x \geq a$, then $\int_a^\infty f(x) dx$ diverges to ∞ .

Here we have, as before,

$$x^\mu f(x) \geq A > 0, \text{ when } x \geq a.$$

It follows that $\frac{A}{x^\mu} \leq f(x)$.

But $\int_a^\infty \frac{dx}{x^\mu}$ diverges to ∞ when $\mu \leq 1$.

It follows that $\int_a^\infty f(x) dx$ diverges to ∞ .

III. Let $f(x)$ be bounded and integrable in the arbitrary interval (a, b) , where $a > 0$. If there is a number μ less than or equal to 1 such that $x^\mu f(x)$ has a negative upper bound when $x \geq a$, then $\int_a^\infty f(x) dx$ diverges to $-\infty$.

This follows from II., for in this case

$$-x^\mu f(x)$$

must have a positive lower bound when $x \geq a$.

But, if $\text{Lt}_{x \rightarrow \infty} (x^\mu f(x))$ exists, it follows that $x^\mu f(x)$ is bounded in $x \geq a$; also, by properly choosing the positive number X , $x^\mu f(x)$ will either have a positive lower bound, when this limit is

positive, or a negative upper bound, when this limit is negative provided that $x \geq X$.

Thus, from I.-III., the following theorem can be immediately deduced:

Let $f(x)$ be bounded and integrable in the arbitrary interval (a, b) , where $a > 0$.

If there is a number μ greater than 1 such that $\text{Lt}_{x \rightarrow \infty} (x^\mu f(x))$ exists, then $\int_a^\infty f(x) dx$ converges.

If there is a number μ less than or equal to 1 such that $\text{Lt}_{x \rightarrow \infty} (x^\mu f(x))$ exists and is not zero, then $\int_a^\infty f(x) dx$ diverges; and the same is true if $x^\mu f(x)$ diverges to $+\infty$, or to $-\infty$, as $x \rightarrow \infty$.

We shall make very frequent use of this test, and refer to it as the " μ -test." It is clear that we are simply comparing the integral $\int_a^\infty f(x) dx$ with the integral $\int_a^\infty \frac{dx}{x^\mu}$, and deducing the convergence or divergence of the former from that of the latter.

Ex. 1. $\int_0^\infty \frac{x^2}{(a^2 + x^2)^2} dx$ converges, since $\text{Lt}_{x \rightarrow \infty} \left(x^2 \times \frac{x^2}{(a^2 + x^2)^2} \right) = 1$.

2. $\int_0^\infty \frac{x^3}{(a^2 + x^2)^2} dx$ diverges, since $\text{Lt}_{x \rightarrow \infty} \left(x \times \frac{x^3}{(a^2 + x^2)^2} \right) = 1$.

3. $\int_a^\infty \frac{x^{\frac{3}{2}}}{b^2 x^2 + c^2} dx$ diverges, since $\text{Lt}_{x \rightarrow \infty} \left(x^{\frac{1}{2}} \times \frac{x^{\frac{3}{2}}}{b^2 x^2 + c^2} \right) = \frac{1}{b^2}$.

It should be noticed that the theorems of this section do not apply to the integral $\int_0^\infty \frac{\sin x}{x} dx$.

56. Further Tests for the Convergence of $\int_a^\infty f(x) dx$.

I. If $\phi(x)$ is bounded when $x \geq a$, and integrable in the arbitrary interval (a, b) , and $\int_a^\infty \psi(x) dx$ converges absolutely, then $\int_a^\infty \phi(x) \psi(x) dx$ is absolutely convergent.

For we have $|\phi(x)| < A$, where A is some definite positive number and $x \geq a$.

Also
$$\int_{x'}^{x''} |\phi(x)| |\psi(x)| dx < A \int_{x'}^{x''} |\psi(x)| dx,$$

when $x'' > x' > a$.

Since we are given that $\int_a^\infty |\psi(x)| dx$ converges, the result follows.

Ex. 1. $\int_a^\infty \frac{\sin x}{x^{1+n}} dx$, $\int_a^\infty \frac{\cos x}{x^{1+n}} dx$ converge absolutely, when n and a are positive.

2. $\int_a^\infty e^{-ax} \cos bx dx$ converges absolutely, when a is positive.

3. $\int_0^\infty \frac{\cos mx}{a^2 + x^2} dx$ converges absolutely.

II. Let $\phi(x)$ be monotonic and bounded when $x \geq a$. Let $\psi(x)$ be bounded and integrable in the arbitrary interval (a, b) , and not change sign more than a finite number of times in the interval. Also let $\int_a^\infty \psi(x) dx$ converge.

Then $\int_a^\infty \phi(x) \psi(x) dx$ converges.

This follows from the Second Theorem of Mean Value, since

$$\int_{x'}^{x''} \phi(x) \psi(x) dx = \phi(x') \int_{x'}^{\xi} \psi(x) dx + \phi(x'') \int_{\xi}^{x''} \psi(x) dx,$$

where $a < x' \leq \xi \leq x''$.

But $|\phi(x')|$ and $|\phi(x'')|$ are each less than some definite positive number A .

Also we can choose X so that

$$\left| \int_{x'}^{\xi} \psi(x) dx \right| \quad \text{and} \quad \left| \int_{\xi}^{x''} \psi(x) dx \right|$$

are each less than $\epsilon/2A$, when $x'' > x' \geq X$, and ϵ is any given positive number, as small as we please.

It follows that

$$\left| \int_{x'}^{x''} \phi(x) \psi(x) dx \right| < \epsilon, \quad \text{when } x'' > x' \geq X,$$

and the given integral converges.

Ex. 1. $\int_0^\infty e^{-x} \frac{\sin x}{x} dx$ converges.

2. $\int_a^\infty (1 - e^{-x}) \frac{\cos x}{x} dx$ converges when $a > 0$.

III. Let $\phi(x)$ be monotonic and bounded when $x \geq a$, and $\text{Lt}_{x \rightarrow \infty} \phi(x) = 0$.

Let $\psi(x)$ be bounded and integrable in the arbitrary interval (a, b) , and not change sign more than a finite number of times in the interval. Also let $\int_a^x \psi(x) dx$ be bounded when $x > a$.

Then $\int_a^\infty \phi(x) \psi(x) dx$ is convergent.

As above, in II., we know that

$$\int_{x'}^{x''} \phi(x) \psi(x) dx = \phi(x') \int_{x'}^{\xi} \psi(x) dx + \phi(x'') \int_{\xi}^{x''} \psi(x) dx,$$

where $a < x' \leq \xi \leq x''$

But $\left| \int_a^x \psi(x) dx \right| < A$, when $x > a$, where A is some definite positive number.

$$\begin{aligned} \text{And} \quad \left| \int_{x'}^{\xi} \psi(x) dx \right| &\leq \left| \int_a^{\xi} \psi(x) dx \right| + \left| \int_a^{x'} \psi(x) dx \right| \\ &< 2A. \end{aligned}$$

$$\text{Similarly} \quad \left| \int_{\xi}^{x''} \psi(x) dx \right| < 2A.$$

$$\text{Also} \quad \text{Lt}_{x \rightarrow \infty} \phi(x) = 0.$$

Therefore, if ϵ is any positive number, as small as we please, there will be a positive number X such that

$$|\phi(x)| < \frac{\epsilon}{4A}, \text{ when } x \geq X.$$

It follows that

$$\left| \int_{x'}^{x''} \phi(x) \psi(x) dx \right| < \epsilon, \text{ when } x'' > x' \geq X,$$

and $\int_a^\infty \phi(x) \psi(x) dx$ converges.

Ex. 1. $\int_a^\infty \frac{\sin x}{x^n} dx$, $\int_a^\infty \frac{\cos x}{x^n} dx$ converge when n and a are positive.

2. $\int_1^\infty \frac{x}{1+x^2} \sin x dx$ converges.

The Mean Value Theorems for the Infinite Integral.**57. The First Theorem of Mean Value.**

Let $\phi(x)$ be bounded when $x \geq a$, and integrable in the arbitrary interval (a, b) .

Let $\psi(x)$ keep the same sign in $x \geq a$, and $\int_a^\infty \psi(x) dx$ converge.

Then
$$\int_a^\infty \phi(x) \psi(x) dx = \mu \int_a^\infty \psi(x) dx,$$

where $m \leq \mu \leq M$, the upper and lower bounds of $\phi(x)$ in $x \geq a$ being M and m .

We have $m \leq \phi(x) \leq M$, when $x \geq a$,
and, if $\psi(x) \geq 0$,

$$m \psi(x) \leq \phi(x) \psi(x) \leq M \psi(x).$$

Therefore $m \int_a^x \psi(x) dx \leq \int_a^x \phi(x) \psi(x) dx \leq M \int_a^x \psi(x) dx$, when $x \geq a$.

But, by § 56, I., $\int_a^\infty \phi(x) \psi(x) dx$ converges, and we are given that $\int_a^\infty \psi(x) dx$ converges.

Thus we have from these inequalities

$$m \int_a^\infty \psi(x) dx \leq \int_a^\infty \phi(x) \psi(x) dx \leq M \int_a^\infty \psi(x) dx.$$

In other words,
$$\int_a^\infty \phi(x) \psi(x) dx = \mu \int_a^\infty \psi(x) dx,$$

where $m \leq \mu \leq M$.

58. The Second Theorem of Mean Value.

LEMMA. Let $\int_a^\infty f(x) dx$ be a convergent integral, and $F(x) = \int_x^\infty f(x) dx$ ($x \geq a$).

Then $F(x)$ is continuous when $x \geq a$, and bounded in the interval (a, ∞) . Also it takes at least once in that interval every value between its upper and lower bounds, these being included.

The continuity of $F(x)$ follows from the equation

$$F(x+h) - F(x) = - \int_x^{x+h} f(x) dx.$$

Further, $\lim_{x \rightarrow \infty} F(x)$ exists and is zero.

It follows from § 32 that $F(x)$ is bounded in the interval (a, ∞) , as defined in that section, and, if M, m are its upper and lower bounds, it takes at least once in (a, ∞) the values M and m and every value between M, m .

Let $\phi(x)$ be bounded and monotonic when $x \geq a$.

Let $\psi(x)$ be bounded and integrable in the arbitrary interval (a, b) , and not change sign more than a finite number of times in the interval. Also let $\int_a^\infty \psi(x) dx$ converge.

Then
$$\int_a^\infty \phi(x) \psi(x) dx = \phi(a+0) \int_a^\xi \psi(x) dx + \phi(\infty) \int_\xi^\infty \psi(x) dx,$$

where $a \leq \xi \leq \infty$.*

Suppose $\phi(x)$ to be monotonic increasing.

We apply the Second Theorem of Mean Value to the arbitrary interval (a, b) .

Then we have

$$\int_a^b \phi(x) \psi(x) dx = \phi(a+0) \int_a^\xi \psi(x) dx + \phi(b-0) \int_\xi^b \psi(x) dx,$$

where $a \leq \xi \leq b$.

Add to both sides
$$B = \phi(\infty) \int_b^\infty \psi(x) dx,$$

observing that $\phi(\infty)$ exists, since $\phi(x)$ is monotonic increasing in $x \geq a$ and does not exceed some definite number (§ 34).

Also $\lim_{b \rightarrow \infty} B = 0$ and $\int_a^\infty \phi(x) \psi(x) dx$ converges [§ 56, II.].

Then
$$\begin{aligned} B + \int_a^b \phi(x) \psi(x) dx &= \phi(a+0) \int_a^\xi \psi(x) dx + \phi(b-0) \int_\xi^b \psi(x) dx + \phi(\infty) \int_b^\infty \psi(x) dx \\ &= \phi(a+0) \left[\int_a^\infty \psi(x) dx - \int_\xi^\infty \psi(x) dx \right] + \phi(b-0) \left[\int_\xi^\infty \psi(x) dx - \int_b^\infty \psi(x) dx \right] \\ &\quad + \phi(\infty) \int_b^\infty \psi(x) dx \\ &= \phi(a+0) \int_a^\infty \psi(x) dx + U + V, \dots\dots\dots(1) \end{aligned}$$

where
$$U = \{ \phi(b-0) - \phi(a+0) \} \int_\xi^\infty \psi(x) dx,$$

and
$$V = \{ \phi(\infty) - \phi(b-0) \} \int_b^\infty \psi(x) dx.$$

Now we know from the above Lemma that $\int_x^\infty \psi(x) dx$ is bounded in (a, ∞) .

Let M, m be its upper and lower bounds.

Then
$$m \leq \int_\xi^\infty \psi(x) dx \leq M,$$

and
$$m \leq \int_b^\infty \psi(x) dx \leq M.$$

Therefore
$$\begin{aligned} \{ \phi(b-0) - \phi(a+0) \} m &\leq U \leq \{ \phi(b-0) - \phi(a+0) \} M, \\ \{ \phi(\infty) - \phi(b-0) \} m &\leq V \leq \{ \phi(\infty) - \phi(b-0) \} M. \end{aligned}$$

* Cf. Pierpont, *loc. cit.*, Vol. I., § 654.

Adding these, we see that

$$\{\phi(\infty) - \phi(a+0)\}m \leq U + V \leq \{\phi(\infty) - \phi(a+0)\}M.$$

Therefore $U + V = \mu\{\phi(\infty) - \phi(a+0)\}$, where $m \leq \mu \leq M$.

Insert this value for $U + V$ in (1), and proceed to the limit when $b \rightarrow \infty$.

$$\text{Then } \int_a^\infty \phi(x)\psi(x)dx = \phi(a+0) \int_a^\infty \psi(x)dx + \mu'\{\phi(\infty) - \phi(a+0)\},$$

where $\mu' = \text{Lt}_{b \rightarrow \infty} \mu$.

This limit must exist, since the other terms in (1) have limits when $b \rightarrow \infty$.

Also, since

$$m \leq \mu \leq M,$$

it follows that

$$m \leq \mu' \leq M.$$

But $\int_x^\infty \psi(x)dx$ takes the value μ' at least once in the interval (a, ∞) .

Thus we may put $\mu' = \int_{\xi'}^\infty \psi(x)dx$, where $a \leq \xi' \leq \infty$.

Therefore we have finally

$$\int_a^\infty \phi(x)\psi(x)dx = \phi(a+0) \int_a^{\xi'} \psi(x)dx + \phi(\infty) \int_{\xi'}^\infty \psi(x)dx,$$

where $a \leq \xi' \leq \infty$.

It is clear that we might have used the other forms I. and III., § 50, of the Second Theorem of Mean Value and obtained corresponding results.

INFINITE INTEGRALS. INTEGRAND INFINITE.

59. $\int_a^b f(x) dx$. In the preceding sections we have dealt with the infinite integrals $\int_a^\infty f(x) dx$, $\int_{-\infty}^a f(x) dx$ and $\int_{-\infty}^\infty f(x) dx$, when the integrand $f(x)$ is bounded in any arbitrary interval, however large.

A further extension of the definition of the integral is required so as to include the case in which $f(x)$ has a finite number of infinite discontinuities (cf. § 33) in the interval of integration.

First we take the case when a is the only point of infinite discontinuity in (a, b) . The integrand $f(x)$ is supposed bounded and integrable in the arbitrary interval $(a + \xi, b)$, where $a < a + \xi < b$.

On this understanding, if the integral $\int_{a+\xi}^b f(x) dx$ has a limit as $\xi \rightarrow 0$, we define the infinite integral $\int_a^b f(x) dx$ as $\text{Lt}_{\xi \rightarrow 0} \int_{a+\xi}^b f(x) dx$.

Similarly, when the point b is the only point of infinite discontinuity in (a, b) , and $f(x)$ is bounded and integrable in the arbitrary interval $(a, b - \xi)$, where $a < b - \xi < b$, we define the infinite integral $\int_a^b f(x) dx$ as $\text{Lt}_{\xi \rightarrow 0} \int_a^{b-\xi} f(x) dx$, when this limit exists.

Again, when a and b are both points of infinite discontinuity, we define the infinite integral $\int_a^b f(x) dx$ as the sum of the infinite integrals $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$, when these integrals exist, as defined above, c being a point between a and b .

This definition is independent of the position of c between a and b , since we have

$$\int_a^c f(x) dx = \int_a^{c'} f(x) dx + \int_{c'}^c f(x) dx,$$

where $a < c' < c$ (cf. § 51, III.).

Finally, let there be a finite number of points of infinite discontinuity in the interval (a, b) . Let these points be x_1, x_2, \dots, x_n , where $a \leq x_1 < x_2, \dots < x_n \leq b$. We define the infinite integral $\int_a^b f(x) dx$ by the equation

$$\int_a^b f(x) dx = \int_a^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_n}^b f(x) dx,$$

when the integrals on the right-hand exist, according to the definitions just given.

It should be noticed that with this definition there are only to be a finite number of points of infinite discontinuity, and $f(x)$ is to be bounded in any partial interval of (a, b) , which has not one of these points as an internal point or an end-point.

This definition was extended by Du Bois-Reymond, Dini and Harnack to certain cases in which the integrand has an infinite number of points of infinite discontinuity, but the case given in the text is amply sufficient for our purpose. The modern treatment of the integral, due chiefly to Lebesgue,* has rendered further generalisation of Riemann's discussion chiefly of historical interest.

It is convenient to speak of the infinite integrals of this and the succeeding section as *convergent*, as we did when one or other

* Cf. footnote, p. 77.

of the limits of integration was infinite, and the terms *divergent* and *oscillatory* are employed as before.

Some writers use the term *proper integral* for the ordinary integral $\int_a^b f(x) dx$, when $f(x)$ is bounded and integrable in the interval (a, b) , and *improper integral* for the case when it has points of infinite discontinuity in (a, b) , reserving the term *infinite integral* for

$$\int_a^\infty f(x) dx, \int_{-\infty}^a f(x) dx \text{ or } \int_{-\infty}^\infty f(x) dx.$$

French mathematicians refer to both as *intégrales généralisées*; Germans refer to both as *uneigentliche Integrale*, to distinguish them from *eigentliche Integrale* or ordinary integrals.

$$60. \quad \int_a^\infty \mathbf{f}(\mathbf{x}) d\mathbf{x}. \quad \int_{-\infty}^a \mathbf{f}(\mathbf{x}) d\mathbf{x}. \quad \int_{-\infty}^\infty \mathbf{f}(\mathbf{x}) d\mathbf{x}.$$

Let $f(x)$ have infinite discontinuities at a finite number of points in any interval, however large.

For example, let there be infinite discontinuities only at x_1, x_2, \dots, x_n in $x \geq a$, $f(x)$ being bounded in any interval (c, b) , where $c > x_n$.

$$\text{Let} \quad a \leq x_1 < x_2, \dots < x_n < b.$$

Then we have, as above (§ 59),

$$\int_a^b f(x) dx = \int_a^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_n}^c f(x) dx + \int_c^b f(x) dx,$$

where $x_n < c < b$, provided that the integrals on the right-hand side exist.

It will be noticed that the last integral $\int_c^b f(x) dx$ is an ordinary integral, $f(x)$ being bounded and integrable in (c, b) .

If the integral $\int_c^\infty f(x) dx$ also converges, we define the infinite integral $\int_a^\infty f(x) dx$ by the equation:

$$\int_a^\infty f(x) dx = \int_a^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_n}^c f(x) dx + \int_c^\infty f(x) dx.$$

It is clear that this definition is independent of the position of c , since we have

$$\int_{x_n}^c f(x) dx + \int_c^\infty f(x) dx = \int_{x_n}^{c'} f(x) dx + \int_{c'}^\infty f(x) dx,$$

where $x_n < c < c'$.

Also we may write the above in the form

$$\int_a^\infty f(x) dx = \int_a^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_n}^\infty f(x) dx.$$

The verbal alterations required in the definition of $\int_{-\infty}^a f(x) dx$ are obvious, and we define $\int_{-\infty}^\infty f(x) dx$, as before, as the sum of the integrals $\int_{-\infty}^a f(x) dx$ and $\int_a^\infty f(x) dx$.

It is easy to show that this definition is independent of the position of the point a .

61. Tests for Convergence of $\int_a^b f(x) dx$. It is clear that we need only discuss the case when there is a point of infinite discontinuity at an end of the interval of integration.

If $x = a$ is the only point of infinite discontinuity, we have

$$\int_a^b f(x) dx = \text{Lt}_{\xi \rightarrow 0} \int_{a+\xi}^b f(x) dx,$$

when this limit exists.

It follows at once, from the definition, that :

I. *The integral $\int_a^b f(x) dx$ is convergent and has the value I when, any positive number ϵ having been chosen, as small as we please, there is a positive number η such that*

$$\left| I - \int_{a+\xi}^b f(x) dx \right| < \epsilon, \text{ provided that } 0 < \xi \leq \eta.$$

And further :

II. *A necessary and sufficient condition for the convergence of the integral $\int_a^b f(x) dx$ is that, if any positive number ϵ has been chosen, as small as we please, there shall be a positive number η such that*

$$\left| \int_{a+\xi''}^{a+\xi'} f(x) dx \right| < \epsilon, \text{ when } 0 < \xi'' < \xi' \leq \eta.$$

Also, if this infinite integral $\int_a^b f(x) dx$ converges, we have

$$\int_a^b f(x) dx = \int_a^x f(x) dx + \int_x^b f(x) dx, \quad a < x < b.$$

It follows from I. that, if $\int_a^b f(x) dx$ converges, to the arbitrary positive number ϵ , there corresponds a positive number η such that

$$\left| \int_a^x f(x) dx \right| < \epsilon, \text{ when } 0 < (x - a) \leq \eta.$$

Absolute Convergence. *The infinite integral $\int_a^b f(x) dx$ is said to be **absolutely convergent**, if $f(x)$ is bounded and integrable in the arbitrary interval $(a + \xi, b)$, where $0 < \xi < b - a$, and $\int_a^b |f(x)| dx$ converges.*

It follows from II. that absolute convergence carries with it ordinary convergence. But the converse is not true. *An infinite integral of this kind may converge, but not converge **absolutely**,** as the following example shows.

An example of such an integral is suggested at once by § 54.

It is clear that $\int_0^1 \frac{\sin 1/x}{x} dx$

converges, but not absolutely, for this integral is reduced to

$$\int_1^\infty \frac{\sin x}{x} dx$$

by substituting $1/x$ for x .

Again, it is clear that $\int_a^b \frac{dx}{(x-a)^n}$ converges, if $0 < n < 1$.

For we have

$$\int_{a+\xi}^b \frac{dx}{(x-a)^n} = \frac{1}{1-n} \{(b-a)^{1-n} - \xi^{1-n}\}.$$

Therefore $\text{Lt}_{\xi \rightarrow 0} \int_{a+\xi}^b \frac{dx}{(x-a)^n} = \frac{(b-a)^{1-n}}{1-n}$, when $0 < n < 1$.

Also the integral diverges when $n \geq 1$.

From this we obtain results which correspond to those of § 55.

III. *Let $f(x)$ be bounded and integrable in the arbitrary interval $(a + \xi, b)$, where $0 < \xi < b - a$. If there is a number μ between 0 and 1 such that $(x - a)^\mu f(x)$ is bounded when $a < x \leq b$ then $\int_a^b f(x) dx$ converges absolutely.*

* Cf. § 43, V.; § 47, Cor. I.; and § 54.

Again,

IV. Let $f(x)$ be bounded and integrable in the arbitrary interval $(a + \xi, b)$, where $0 < \xi < b - a$. If there is a number μ greater than or equal to 1 such that $(x - a)^\mu f(x)$ has a positive lower bound when $a < x \leq b$, or a negative upper bound, then $\int_a^b f(x) dx$ diverges to $+\infty$ in the first case, and to $-\infty$ in the second case.

And finally,

V. Let $f(x)$ be bounded and integrable in the arbitrary interval $(a + \xi, b)$, where $0 < \xi < b - a$.

If there is a number μ between 0 and 1 such that $\text{Lt}_{x \rightarrow a+0} (x - a)^\mu f(x)$ exists, then $\int_a^b f(x) dx$ converges absolutely.

If there is a number μ greater than or equal to 1 such that $\text{Lt}_{x \rightarrow a+0} (x - a)^\mu f(x)$ exists and is not zero, then $\int_a^b f(x) dx$ diverges; and the same is true if $(x - a)^\mu f(x)$ tends to $+\infty$, or to $-\infty$, as $x \rightarrow a + 0$.

We shall speak of this test as the μ -test for the infinite integral $\int_a^b f(x) dx$, when $x = a$ is a point of infinite discontinuity. It is clear that in applying this test we are simply asking ourselves the order of the infinity that occurs in the integrand.

The results can be readily adapted to the case when the upper limit b is a point of infinite discontinuity.

Also, it is easy to show that

VI. If $\phi(x)$ is bounded and integrable in (a, b) , and $\int_a^b \psi(x) dx$ converges absolutely, then $\int_a^b \phi(x) \psi(x) dx$ is absolutely convergent. (Cf. § 56, I.)

The tests given in III.-VI. will cover most of the cases which we shall meet. But it would not be difficult to develop in detail the results which correspond to the other tests obtained for the convergence of the infinite integral $\int_a^\infty f(x) dx$.

No special discussion is required for the integral $\int_a^b f(x) dx$, when a certain number of points of infinite discontinuity occur

in (a, b) , or for $\int_a^\infty f(x) dx$, $\int_{-\infty}^a f(x) dx$, and $\int_{-\infty}^\infty f(x) dx$, as defined in § 60. These integrals all reduce to the sum of integrals of the types for which we have already obtained the required criteria.

We add some examples illustrating the points to which we have referred.

Ex. 1. Prove that $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ converges and that $\int_0^1 \frac{dx}{x(1+x)}$ diverges.

(i) Let
$$f(x) = \frac{1}{(1+x)\sqrt{x}}.$$

Then
$$\operatorname{Lt}_{x \rightarrow 0} \sqrt{x} f(x) = 1.$$

The μ -test thus establishes the convergence of $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$.

(ii) Let
$$f(x) = \frac{1}{x(1+x)}.$$

Then
$$\operatorname{Lt}_{x \rightarrow 0} x f(x) = 1.$$

Therefore the integral diverges by the same test.

Ex. 2. Prove that $\int_0^{\frac{1}{2}\pi} \frac{\sin x}{x^{1+n}} dx$ converges, when $0 < n < 1$.

The integral is an ordinary finite integral if $n \leq 0$.

Also
$$\operatorname{Lt}_{x \rightarrow 0} x^n \left(\frac{\sin x}{x^{1+n}} \right) = 1.$$

Therefore the integral converges when $0 < n < 1$. It diverges when $n \geq 1$.

Ex. 3. Prove that $\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}}$ converges.

The integrand has infinities at $x=0$ and $x=1$.

We have thus to examine the convergence of the two infinite integrals

$$\int_0^a \frac{dx}{\sqrt{\{x(1-x)\}}}, \quad \int_a^1 \frac{dx}{\sqrt{\{x(1-x)\}}},$$

where a is some number between 0 and 1.

The μ -test is sufficient in each case.

$$\int_0^a \frac{dx}{\sqrt{\{x(1-x)\}}} \text{ converges, since } \operatorname{Lt}_{x \rightarrow 0} \{x^{\frac{1}{2}} f(x)\} = 1,$$

$$\int_a^1 \frac{dx}{\sqrt{\{x(1-x)\}}} \text{ converges, since } \operatorname{Lt}_{x \rightarrow 1-0} \{(1-x)^{\frac{1}{2}} f(x)\} = 1,$$

where we have written $f(x) = \frac{1}{\sqrt{\{x(1-x)\}}}$.

Ex. 4. Show that $\int_0^{\frac{1}{2}\pi} \log \sin x \, dx$ converges and is equal to $-\frac{1}{2}\pi \log 2$.

The only infinity is at $x=0$, and the convergence of the integral follows from the μ -test.

Further,

$$\begin{aligned} \int_0^\pi \log \sin x \, dx &= 2 \int_0^{\frac{1}{2}\pi} \log \sin 2x \, dx \\ &= \pi \log 2 + 2 \int_0^{\frac{1}{2}\pi} \log \sin x \, dx + 2 \int_0^{\frac{1}{2}\pi} \log \cos x \, dx \\ &= \pi \log 2 + 4 \int_0^{\frac{1}{2}\pi} \log \sin x \, dx. \end{aligned}$$

But
$$\int_0^\pi \log \sin x \, dx = 2 \int_0^{\frac{1}{2}\pi} \log \sin x \, dx.$$

Therefore
$$\int_0^{\frac{1}{2}\pi} \log \sin x \, dx = -\frac{1}{2}\pi \log 2.$$

From this result it is easy to show that the convergent integrals

$$\int_0^\pi \log(1 - \cos x) \, dx \quad \text{and} \quad \int_0^\pi \log(1 + \cos x) \, dx$$

are equal to $-\pi \log 2$.

Ex. 5. Discuss the convergence or divergence of the Gamma Function integral $\int_0^\infty e^{-x} x^{n-1} \, dx$.

(i) Let $n \geq 1$.

Then the integrand is bounded in $0 < x \leq a$, where a is arbitrary, and we need only consider the convergence of $\int_a^\infty e^{-x} x^{n-1} \, dx$.

The μ -test of § 55 establishes that this integral converges, since the order of e^x is greater than any given power of x .

Or we might proceed as follows :

Since $e^x = 1 + x + \frac{x^2}{2!} + \dots$,

when $x > 0$,
$$e^x > \frac{x^r}{r!} \quad (r = \text{any positive integer}),$$

and
$$e^{-x} x^{n-1} < \frac{r!}{x^{r-n+1}}.$$

But whatever n may be, we can choose r so that $r - n + 1 > 1$.

It follows that, whatever n may be,

$$\int_a^\infty e^{-x} x^{n-1} \, dx \text{ converges.}$$

(ii) Let $0 < n < 1$.

In this case $e^{-x} x^{n-1}$ has an infinity at $x=0$.

The μ -test shows that $\int_0^1 e^{-x} x^{n-1} \, dx$ converges, and we have just shown that $\int_1^\infty e^{-x} x^{n-1} \, dx$ converges.

Therefore $\int_0^{\infty} e^{-x} x^{n-1} dx$ converges.

(iii) Let $n \leq 0$.

In this case $e^{-x} x^{n-1}$ has an infinity at $x=0$, and the μ -test shows that $\int_0^1 e^{-x} x^{n-1} dx$ diverges to $+\infty$.

Ex. 6. Discuss the integral $\int_0^1 x^{n-1} \log x dx$.*

Since $\text{Lt}_{x \rightarrow 0} (x^r \log x) = 0$, when $r > 0$, the integral is an ordinary integral, when $n > 1$.

Also we know that

$$\int_x^1 \log x dx = \left[x(\log x - 1) \right]_x^1 = x(1 - \log x) - 1.$$

It follows that $\int_0^1 \log x dx = \text{Lt}_{x \rightarrow 0} \{x(1 - \log x) - 1\} = -1$.

Again, $\text{Lt}_{x \rightarrow 0} (x^\mu \times x^{n-1} \log x) = \text{Lt}_{x \rightarrow 0} (x^{\mu+n-1} \log x) = 0$, if $\mu > 1 - n$.

And when $0 < n < 1$, we can choose a positive number μ less than 1 which satisfies this condition.

Therefore $\int_0^1 x^{n-1} \log x dx$ converges, when $0 < n \leq 1$.

Finally, we have

$$\text{Lt}_{x \rightarrow 0} (x \times x^{n-1} |\log x|) = \text{Lt}_{x \rightarrow 0} x^n |\log x| = \infty, \text{ when } n \leq 0.$$

Therefore $\int_0^1 x^{n-1} \log x dx$ diverges, when $n \leq 0$.

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* This integral can be reduced to the Gamma Function integral, and its convergence or divergence follows from Ex. 5. Also see below, Ex. 5, p. 120.

EXAMPLES ON CHAPTER IV.

1. Show that the following integrals converge :

$$\int_0^{\infty} \frac{\sin x}{1 + \cos x + e^x} dx, \quad \int_1^{\infty} \frac{(x-1)\sqrt{x}}{1+x+x^3+\sin x} dx, \quad \int_0^{\infty} e^{-x^2} dx,$$

$$\int_0^{\infty} e^{-a^2x^2} \cosh bx dx, \quad \int_0^1 \frac{\log x}{1+x} dx, \quad \int_0^1 \frac{\log x}{1-x^2} dx.$$

2. Discuss the convergence or divergence of the following integrals :

$$\int_a^b \frac{dx}{(x-a)\sqrt{(b-x)}}, \quad \int_0^c \frac{x^{a-1}}{x+1} dx, \quad \int_0^c \frac{x^{a-1}}{x-1} dx, \quad \text{where } 0 < c < 1,$$

$$\int_0^{\infty} \frac{x^{a-1}}{x+1} dx, \quad \int_0^{\infty} \frac{x^{a-1}}{x-1} dx, \quad \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta.$$

3. Show that the following integrals are absolutely convergent :

$$\int_0^b \sin \frac{1}{x} \frac{dx}{\sqrt{x}}, \quad \int_0^{\infty} e^{-a^2x^2} \cos bx dx, \quad \int_0^{\infty} e^{-a^2x^2} x^m \sin nx dx \quad (m > 0),$$

and

$$\int_a^{\infty} \frac{P(x)}{Q(x)} dx,$$

where $P(x)$ is a polynomial of the m^{th} degree, and $Q(x)$ a polynomial of the n^{th} degree, $n \geq m+2$, and a is a number greater than the largest root of $Q(x)=0$.

4. Let $f(x)$ be defined in the interval $0 < x \leq 1$ as follows :

$$f(x) = 2, \quad \frac{1}{2} < x \leq 1, \quad f(x) = -3, \quad \frac{1}{3} < x \leq \frac{1}{2},$$

$$f(x) = 4, \quad \frac{1}{4} < x \leq \frac{1}{3}, \quad f(x) = -5, \quad \frac{1}{5} < x \leq \frac{1}{4},$$

and so on, the values being alternately positive and negative.

Show that the infinite integral $\int_0^1 f(x) dx$ converges, but not absolutely.

5. Using the substitution $x = e^{-u}$, show that

$$\int_0^1 x^{m-1} (\log x)^n dx$$

converges, provided that $m > 0$ and $n > -1$.

And by means of a similar substitution, show that

$$\int_1^{\infty} x^{m-1} (\log x)^n dx$$

converges, provided that $m < 0$ and $n > -1$.

6. Show that $\int_a^{\infty} \frac{dx}{x(\log x)^{1+\mu}}$ converges when $\mu > 0$ and that it diverges when $\mu \leq 0$, the lower limit a of the integral being some number greater than unity.

Deduce that if there is a number $\mu > 0$, such that $\text{Lt}_{x \rightarrow \infty} \{x(\log x)^{1+\mu} f(x)\}$ exists, then $\int_a^{\infty} f(x) dx$ converges, and give a corresponding test for the divergence of

this integral, $f(x)$ being bounded and integrable in any arbitrary interval (a, b) , where $b > a$.

Show that $\int_2^{\infty} \frac{\cos x}{(x + \sin^2 x)(\log x)^2} dx$ converges,

and $\int_2^{\infty} \frac{dx}{(x + \sin^2 x) \log x}$ diverges.

7. On integrating $\int_1^x \cos x \log x dx$ by parts, we obtain

$$\int_1^x \cos x \log x dx = \sin x \log x - \int_1^x \frac{\sin x}{x} dx.$$

Deduce that $\int_1^{\infty} \cos x \log x dx$ oscillates infinitely.

Also show that $\int_0^1 \cos x \log x dx$ converges, and is equal to $-\int_0^1 \frac{\sin x}{x} dx$.

8. On integrating $\int_{x'}^{x''} \cos x^2 dx$ by parts, we obtain

$$\int_{x'}^{x''} \cos x^2 dx = \frac{1}{2x''} \sin x''^2 - \frac{1}{2x'} \sin x'^2 + \frac{1}{2} \int_{x'}^{x''} \frac{\sin x^2}{x^2} dx,$$

where $x'' > x' > 0$.

Deduce the convergence of $\int_0^{\infty} \cos x^2 dx$.

9. Let $f(x)$ and $g(x)$ be bounded and integrable in (a, b) , except at a certain number of points of infinite discontinuity, these points being different for the two functions.

Prove that $\int_a^b f(x)g(x) dx$ converges, if $\int_a^b |f(x)| dx$ and $\int_a^b |g(x)| dx$ converge.

10. Let $f(x)$ be monotonic when $x \geq a$, and $\text{Lt}_{x \rightarrow \infty} f(x) = 0$.

Then the series $f(a) + f(a+1) + f(a+2) + \dots$

is convergent or divergent according as $\int_a^{\infty} f(x) dx$ converges or diverges.

Prove that for all values of the positive integer n ,

$$2\sqrt{(n+1)} - 2 < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1.$$

Also show that

$$\frac{1}{2\sqrt{1}} + \frac{1}{3\sqrt{2}} + \frac{1}{4\sqrt{3}} \dots$$

converges to a value between $\frac{1}{2}(\pi + 1)$ and $\frac{1}{2}\pi$.

CHAPTER V

THE THEORY OF INFINITE SERIES, WHOSE TERMS ARE FUNCTIONS OF A SINGLE VARIABLE

62. We shall now consider some of the properties of series whose terms are functions of x .

We denote such a series by

$$u_1(x) + u_2(x) + u_3(x) + \dots,$$

and the terms of the series are supposed to be given for values of x in some interval, *e.g.* (a, b) .*

When we speak of the sum of the infinite series

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

it is to be understood: †

- (i) that we settle for what value of x we wish the sum of the series;
- (ii) that we then insert this value of x in the different terms of the series;
- (iii) that we then find the sum— $S_n(x)$ —of the first n terms; and
- (iv) that we then find the limit of this sum as $n \rightarrow \infty$, keeping x all the time at the value settled upon.

On this understanding, *the series*

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

is said to be convergent for the value x , and to have $f(x)$ for its sum, if, this value of x having been first inserted in the different

* As mentioned in §24, when we say that x lies in the interval (a, b) we mean that $a \leq x \leq b$. In some of the results of this chapter the ends of the interval are excluded from the range of x . When this is so, the fact that we are dealing with the *open* interval $(a < x < b)$ will be stated.

† Cf. Baker, *Nature*, 59, p. 319, 1899.

terms of the series, and any positive number ϵ having been chosen, as small as we please, there is a positive integer ν such that

$$|f(x) - S_n(x)| < \epsilon, \text{ when } n \geq \nu.$$

Further,

A necessary and sufficient condition for convergence is that, if any positive number ϵ has been chosen, as small as we please, there shall be a positive integer ν such that

$$|S_{n+p}(x) - S_n(x)| < \epsilon, \text{ when } n \geq \nu,$$

for every positive integer p .

A similar convention exists when we are dealing with other limiting processes. In the definition of the differential coefficient of $f(x)$ it is understood that we first agree for what value of x we wish to know $f'(x)$; that we then calculate $f(x)$ and $f(x+h)$ for this value of x ; then obtain the value of $\frac{f(x+h) - f(x)}{h}$; and finally take the limit of this fraction as $h \rightarrow 0$.

Again, in the case of the definite integral $\int_a^b f(x, \alpha) dx$, it is understood that we insert in $f(x, \alpha)$ the particular value of α for which we wish the integral before we proceed to the summation and limit involved in the integration.

We shall write, as before,

$$f(x) - S_n(x) = R_n(x),$$

where $f(x)$ is the sum of the series, and we shall call $R_n(x)$ the remainder after n terms.

As we have seen in § 19, $R_n(x)$ is the sum of the series

$$u_{n+1}(x) + u_{n+2}(x) + u_{n+3}(x) + \dots$$

Also we shall write

$${}_pR_n(x) = S_{n+p}(x) - S_n(x),$$

and call this a *partial remainder*.

With this notation, the two conditions for convergence are

$$(i) \quad |R_n(x)| < \epsilon, \text{ when } n \geq \nu;$$

$$(ii) \quad |{}_pR_n(x)| < \epsilon, \text{ when } n \geq \nu,$$

for every positive integer p .

A series may converge for every value of x in the open interval $a < x < b$ and not for the end-points a or b .

E.g. the series $1 + x + x^2 + \dots$

converges and has $\frac{1}{1-x}$ for its sum, when $-1 < x < 1$.

When $x=1$, it diverges to $+\infty$; when $x=-1$, it oscillates finitely.

63. The Sum of a Series whose Terms are Continuous Functions of x may be discontinuous. Until Abel* pointed out that the periodic function of x given by the series

$$2(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots),$$

which represents x in the interval $-\pi < x < \pi$, is discontinuous at the points $x = (2r+1)\pi$, r being any integer, it was supposed that a function defined by a convergent series of functions, continuous in a given interval, must itself be continuous in that interval. Indeed Cauchy † distinctly stated that this was the case, and later writers on Fourier's Series have sometimes tried to escape the difficulty by asserting that the sums of these trigonometrical series, at the critical values of x , passed continuously from the values just before those at the points of discontinuity to those just after. ‡

This mistaken view of the sum of such series was due to two different errors. The first consisted in the assumption that, as n increases, the curves $y = S_n(x)$ must approach more and more nearly to the curve $y = f(x)$, when the sum of the series is $f(x)$, an ordinary function capable of graphical representation. These curves $y = S_n(x)$ we shall call the approximation curves for the series, but we shall see that cases may arise where the approximation curves, even for large values of n , differ very considerably from the curve $y = f(x)$.

It is true that, in a certain sense, the curves

$$(i) \ y = S_n(x) \quad \text{and} \quad (ii) \ y = f(x)$$

approach towards coincidence; but the sense is that, if we choose any particular value of x in the interval, and the arbitrary small positive number ϵ , there will be a positive integer ν such that, for this value of x , the absolute value of the difference of the ordinates of the curves (i) and (ii) will be less than ϵ when $n \geq \nu$.

Still this is not the same thing as saying that the curves coincide geometrically. They do not, in fact, lie near to each other in the neighbourhood of a point of discontinuity of $f(x)$; and they may not do so, even where $f(x)$ is continuous.

* Abel, *J. Math.*, Berlin, 1, p. 316, 1826.

† Cauchy, *Cours d'Analyse*, 1^{re} Partie, p. 131, 1821. Also *Œuvres de Cauchy*, (Sér. 2), T. III., p. 120.

‡ Cf. Sachse, *loc. cit.*; Donkin, *Acoustics*, p. 53, 1870.

The following examples and diagrams illustrate these points :

Ex. 1. Consider the series

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \dots, \quad x \geq 0.$$

Here

$$u_n(x) = \frac{1}{(n-1)x+1} - \frac{1}{nx+1},$$

and

$$S_n(x) = 1 - \frac{1}{nx+1}.$$

Thus, when $x > 0$, $\text{Lt}_{n \rightarrow \infty} S_n(x) = 1$;

when $x = 0$, $\text{Lt}_{n \rightarrow \infty} S_n(x) = 0$, since $S_n(0) = 0$.

The curve $y = f(x)$, when $x \geq 0$, consists of the part of the line $y = 1$ for which $x > 0$, and the origin. The sum of the series is discontinuous at $x = 0$.

Now examine the approximation curves

$$y = S_n(x) = 1 - \frac{1}{nx+1}.$$

This equation may be written

$$(y-1)\left(x + \frac{1}{n}\right) = -\frac{1}{n}.$$

As n increases, this rectangular hyperbola (cf. Fig. 10) approaches more and more closely to the lines $y = 1$, $x = 0$. If we reasoned from the shape of the

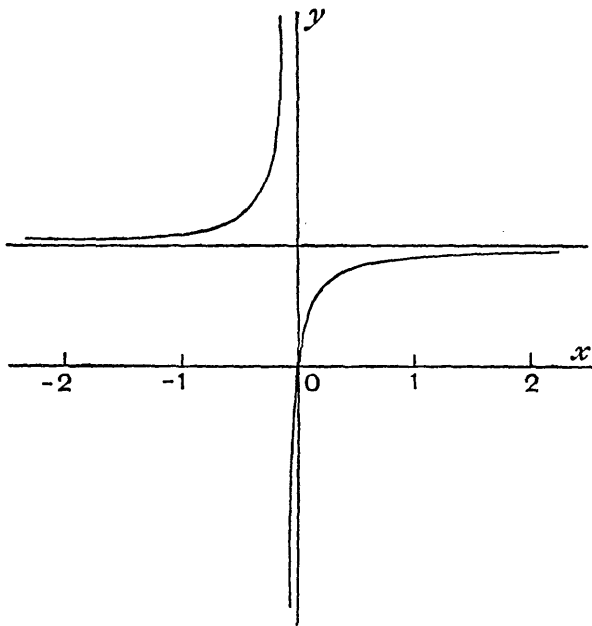


FIG. 10.

approximate curves, we should expect to find that part of the axis of y for which $0 < y < 1$ appearing as a portion of the curve $y = f(x)$ when $x \geq 0$.

As $S_n(x)$ is certainly continuous, when the terms of the series are continuous, the approximation curves will always differ very materially from the curve $y = f(x)$, when the sum of the series is discontinuous.

Ex. 2. Consider the series

$$u_1(x) + u_2(x) + u_3(x) + \dots, \quad x \begin{matrix} \geq \\ \leq \end{matrix} 0,$$

where

$$u_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}.$$

In this case

$$S_n(x) = \frac{nx}{1+n^2x^2},$$

and

$$\text{Lt}_{n \rightarrow \infty} S_n(x) = 0 \text{ for all values of } x.$$

Thus the sum of this series is continuous for all values of x , but we shall see that the approximation curves differ very materially from the curve $y=f(x)$ in the neighbourhood of the origin.

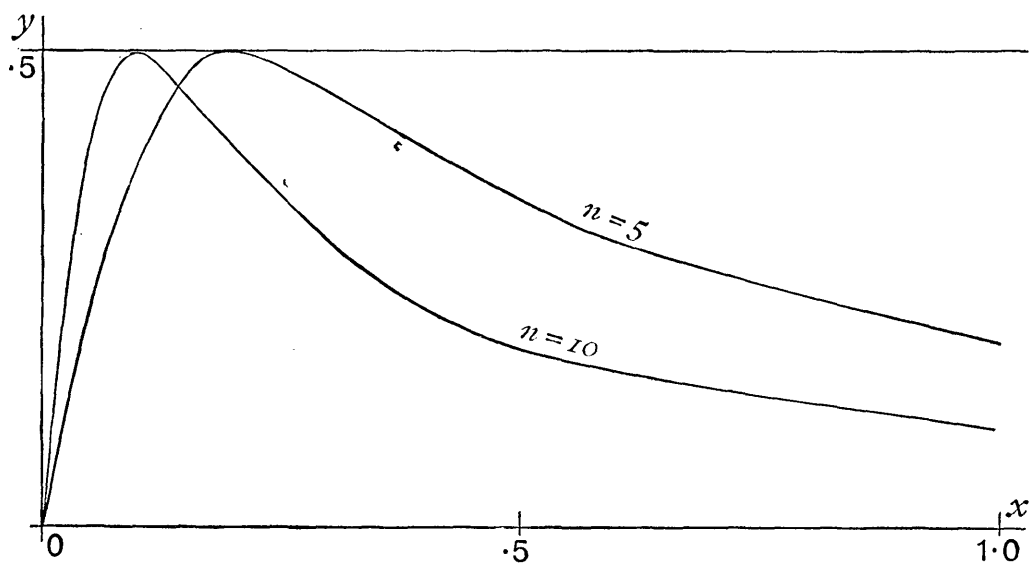


FIG. 11.

The curve

$$y = S_n(x) = \frac{nx}{1+n^2x^2}$$

has a maximum at $(1/n, \frac{1}{2})$ and a minimum at $(-1/n, -\frac{1}{2})$ (cf. Fig. 11). The points on the axis of x just below the maximum and minimum move in towards the origin as n increases. And if we reasoned from the shape of the curves $y=S_n(x)$, we should expect to find the part of the axis of y from $-\frac{1}{2}$ to $\frac{1}{2}$ appearing as a portion of the curve $y=f(x)$.

Ex. 3. Consider the series

$$u_1(x) + u_2(x) + u_3(x) + \dots, \quad x \begin{matrix} \geq \\ \leq \end{matrix} 0,$$

where

$$u_n(x) = \frac{n^2x}{1+n^3x^2} - \frac{(n-1)^2x}{1+(n-1)^3x^2}.$$

Here

$$S_n(x) = \frac{n^2x}{1+n^3x^2},$$

and

$$\text{Lt}_{n \rightarrow \infty} S_n(x) = 0 \text{ for all values of } x.$$

The sum of the series is again continuous, but the approximation curves (cf. Fig. 12), which have a maximum at $(1/\sqrt{n^3}, \frac{1}{2}\sqrt{n})$ and a minimum at

$(-1/\sqrt{n^3}, -\frac{1}{2}\sqrt{n})$, differ very greatly from the curve $y=f(x)$ in the neighbourhood of the origin. Indeed they would suggest that the whole of the axis of y should appear as part of $y=f(x)$.

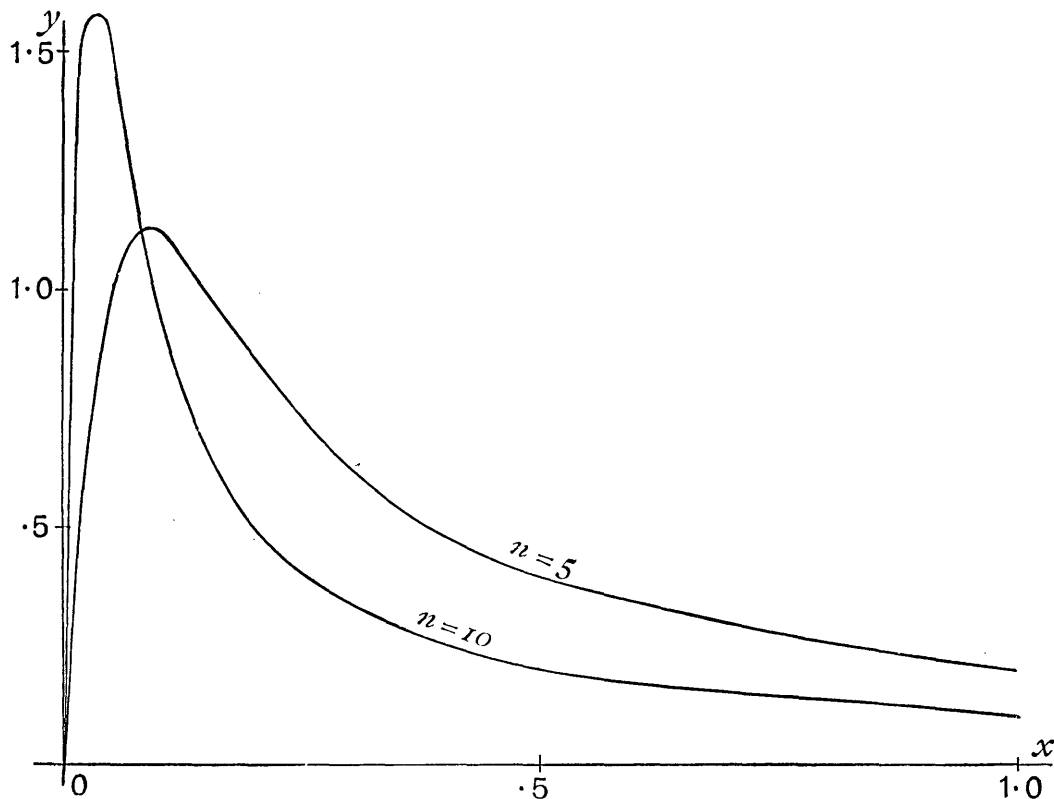


FIG. 12

64. Repeated Limits. These remarks dispose of the assumption referred to at the beginning of the previous section that the approximation curves $y=S_n(x)$, when n is large, must approach closely to the curve $y=f(x)$, where $f(x)$ is the sum of the series.

The second error alluded to above arose from neglect of the convention implied in the definition of the sum of an infinite series whose terms are functions of x . The proper method of finding the sum has been set out in § 62, but the mathematicians to whom reference is now made proceeded in quite a different manner. In finding the sum for a value of x , say x_0 , at which a discontinuity occurs, they replaced x by a function of n , which converges to x_0 as n increases. Then they took the limit when $n \rightarrow \infty$ of $S_n(x)$ in its new form. In this method x and n approach their limits concurrently, and the value of $\text{Lt}_{\{x \rightarrow x_0, n \rightarrow \infty\}} S_n(x)$ may

quite well differ from the actual sum for $x=x_0$. Indeed, by choosing the function of n suitably, this double limit may be made to take any value between $f(x_0+0)$ and $f(x_0-0)$.

For instance, in the series of § 63, Ex. 1,

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \dots, \quad x \geq 0,$$

we have seen that $x=0$ is a point of discontinuity.

If we put $x=p/n$ in the expression for $S_n(x)$, and then let $n \rightarrow \infty$, p remaining fixed, we can make $\text{Lt}_{\{x \rightarrow 0, n \rightarrow \infty\}} S_n(x)$ take any value between 0 and 1, according to our choice of p . For we have $S_n(p/n) = \frac{p}{p+1}$, which is independent of n , and

$$\text{Lt}_{\{x \rightarrow 0, n \rightarrow \infty\}} S_n(x) = \frac{p}{p+1},$$

which passes from 0 to 1 as p increases from 0 to ∞ .

It will be seen that the matter at issue was partly a question of words and a definition. The confusion can also be traced, in some cases, to ignorance of the care which must be exercised in any operation involving repeated limits, for we are really dealing here with two limiting processes.

If the series is convergent and its sum is $f(x)$, then

$$f(x) = \text{Lt}_{n \rightarrow \infty} S_n(x),$$

and the limit of $f(x)$ as x tends to x_0 , assuming that there is such a limit, is given by

$$\text{Lt}_{x \rightarrow x_0} f(x) = \text{Lt}_{x \rightarrow x_0} [\text{Lt}_{n \rightarrow \infty} S_n(x)]. \dots\dots\dots(1)$$

If we may use the curve as an illustration, this is the ordinate of the point towards which we move as we proceed along the curve $y=f(x)$, the abscissa getting nearer and nearer to x_0 , but not quite reaching x_0 . According as x approaches x_0 from the right or left, the limit given in (1) will be $f(x_0+0)$ or $f(x_0-0)$.

Now $f(x_0)$, the sum of the series for $x=x_0$, is, by definition,

$$\text{Lt}_{n \rightarrow \infty} [S_n(x_0)],$$

and since we are now dealing with a definite number of continuous functions, $S_n(x)$ is a continuous function of x in the interval with which we are concerned.

Thus
$$S_n(x_0) = \text{Lt}_{x \rightarrow x_0} S_n(x).$$

Therefore the sum of the series for $x = x_0$ may be written

$$\text{Lt}_{n \rightarrow \infty} [\text{Lt}_{x \rightarrow x_0} S_n(x)]. \dots\dots\dots(2)$$

The two expressions in (1) and (2) need not be the same. They are so only when $f(x)$ is continuous at x_0 .

65. Uniform Convergence.* When the question of changing the order of two limiting processes arises, the principle of uniform convergence, which we shall now explain for the case of infinite series whose terms are functions of x , is fundamental. What is involved in this principle will be seen most clearly by returning to the series

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \dots, \quad x \geq 0.$$

In this series
$$S_n(x) = 1 - \frac{1}{nx+1},$$

and
$$\text{Lt}_{n \rightarrow \infty} S_n(x) = 1, \text{ when } x > 0.$$

Also
$$R_n(x) = \frac{1}{nx+1}, \text{ when } x > 0, \text{ and } R_n(0) = 0.$$

If the arbitrary positive number ϵ is chosen, less than unity, and some positive x is taken, it is clear that $1/(nx+1) < \epsilon$ for a positive n , only if

$$n > \frac{\frac{1}{x} - 1}{\epsilon}.$$

E.g. let $\epsilon = \frac{1}{10^3+1}.$

If $x = 0.1, 0.01, 0.001, \dots, 10^{-p}$, respectively, $1/(nx+1) < \epsilon$ only when
$$n > 10^4, 10^5, 10^6, \dots, 10^{(p+3)}.$$

And when $\epsilon = \frac{1}{10^q+1}$, and $x = 10^{-p}$, n must be greater than 10^{p+q} if $1/(nx+1) < \epsilon.$

As we approach the origin we have to take more and more terms of the series to make the sum of n terms differ from the sum of the series by less than a given number. When $x = 10^{-8}$, the first million terms do not contribute 1% of the sum.

The inequality
$$n > \frac{\frac{1}{x} - 1}{\epsilon}$$

shows that when ϵ is any given positive number less than unity,

* The simplest treatment of uniform convergence will be found in a paper by Osgood, *Bull. Amer. Math. Soc.*, 3, 1896.

and x approaches nearer and nearer to zero, the smallest positive integer which will make $R_n(x)$, $R_{n+1}(x)$, ... all less than ϵ increases without limit.

There is no positive integer ν which will make $R_\nu(x)$, $R_{\nu+1}(x)$, ... all less than this ϵ in $x \geq 0$, *the same ν serving for all values of x in this range.*

On the other hand there is a positive integer ν which will satisfy this condition, if the range of x is given by $x \geq a$, where a is some definite positive number.

Such a value of ν would be the integer next above $\left(\frac{1}{\epsilon} - 1\right) / a$.

Our series is said to *converge uniformly* in $x \geq a$, but it does not *converge uniformly* in $x \geq 0$.

We turn now to the series

$$u_1(x) + u_2(x) + u_3(x) + \dots, \quad \text{~~~~~}$$

and define uniform convergence* in an interval as follows:

Let the series $u_1(x) + u_2(x) + u_3(x) + \dots$ converge for all values of x in the interval $a \leq x \leq b$ and its sum be $f(x)$. It is said to **converge uniformly in that interval**, if, any positive number ϵ having been chosen, as small as we please, there is a positive integer ν such that, **for all values of x in the interval**,

$$|f(x) - S_n(x)| < \epsilon, \text{ when } n \geq \nu. \dagger$$

It is true that, if the series converges, $|R_n(x)| < \epsilon$ for each x in (a, b) when $n \geq \nu$.

The additional point in the definition of uniform convergence is that, any positive number ϵ having been chosen, as small as we please, *the same value of ν is to serve for all the values of x in the interval.*

For this integer ν we must have

$$|R_\nu(x)|, |R_{\nu+1}(x)|, \dots$$

all less than ϵ , no matter where x lies in (a, b) .

* The property of uniform convergence was discovered independently by Stokes (cf. *Cambridge, Trans. Phil. Soc.*, **8**, p. 533, 1847) and Seidel (cf. *München, Abh. Ak. Wiss.*, **5**, p. 381, 1848). See also Hardy, *Cambridge, Proc. Phil. Soc.* **19**, p. 148, 1920.

† We can also have uniform convergence in the *open* interval $a < x < b$, or the half-open intervals $a < x \leq b$, $a \leq x < b$; but, when the terms are continuous in the closed interval, uniform convergence in the *open* interval carries with it uniform convergence in the *closed* interval (cf. § 68).

The series does not converge uniformly in (a, b) if we know that for some positive number (say ϵ_0) there is no positive integer ν which will make

$$|R_\nu(x)|, |R_{\nu+1}(x)|, \dots$$

all less than ϵ_0 for every x in (a, b) .

It will be seen that the series

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \dots$$

converges uniformly in any interval $a \leq x \leq b$, where a, b are any given positive numbers.

It may be said to converge infinitely slowly as x tends to zero, in the sense that, as we get nearer and still nearer to the origin, we cannot fix a limit to the number of terms which we must take to make $|R_n(x)| < \epsilon$. It is this property of *infinitely slow convergence* at a point (e.g. x_0) which prevents a series converging uniformly in an interval $(x_0 - \delta, x_0 + \delta)$ including that point.

Further, the above series converges uniformly in the *infinite interval* $x \geq a$, where a is any given positive number.

It is sometimes necessary to distinguish between uniform convergence in an *infinite interval* and uniform convergence in a *fixed interval, which may be as large as we please*.

The exponential series is convergent for all values of x , but it does not converge uniformly in the infinite interval $x \geq 0$.

For in this series $R_n(x)$ is greater than $x^n/n!$, when x is positive.

Thus, if the series were uniformly convergent in $x \geq 0$, $x^n/n!$ would need to be less than ϵ when $n \geq \nu$, the same ν serving for all values of x in the interval.

But it is clear that we need only take x greater than $(\nu! \epsilon)^{\frac{1}{\nu}}$ to make $R_n(x)$ greater than ϵ for n equal to ν .

However, the exponential series is uniformly convergent in the interval $(0, b)$, where b is fixed, but may be fixed as large as we please.

For take c greater than b . We know that the series converges for $x = c$.

Therefore $R_n(c) < \epsilon$, when $n \geq \nu$.

But $R_n(x) < R_n(c)$, when $0 \leq x \leq b < c$.

Therefore $R_n(x) < \epsilon$, when $n \geq \nu$, the same ν serving for all values of x in $(0, b)$.

From the uniform convergence of the exponential series in the interval $(0, b)$, it follows that the series also converges uniformly in the interval $(-b, b)$, where in both cases b is fixed, but may be fixed as large as we please.

Ex. 1. Prove that the series

$$1 + x + x^2 + \dots$$

converges uniformly to $1/(1-x)$ in $0 \leq x \leq x_0 < 1$.

Ex. 2. Prove that the series

$$(1-x) + x(1-x) + x^2(1-x) + \dots$$

converges uniformly to 1 in $0 \leq x \leq x_0 < 1$.

Ex. 3. Prove that the series

$$(1-x)^2 + x(1-x)^2 + x^2(1-x)^2 + \dots$$

converges uniformly to $(1-x)$ in $0 \leq x \leq 1$.

Ex. 4. Prove that the series

$$\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \dots$$

converges uniformly in the infinite interval $x \geq 0$.

Ex. 5. Prove that the series

$$\frac{x}{1 \cdot 2} + \frac{x}{2 \cdot 3} + \frac{x}{3 \cdot 4} + \dots$$

converges uniformly in the interval $(0, b)$, where b is fixed, but may be fixed as large as we please, and that it does not converge uniformly in the infinite interval $x \geq 0$.

66. A necessary and sufficient condition for Uniform Convergence. When the sum $f(x)$ is known, the above definition often gives a convenient means of deciding whether the convergence is uniform or not.

When the sum is not known, the following test, corresponding to the general principle of convergence (§ 15), is more suitable.

Let
$$u_1(x) + u_2(x) + u_3(x) + \dots$$

be an infinite series, whose terms are given in the interval (a, b) . A necessary and sufficient condition for the uniform convergence of the series in this interval is that, if any positive number ϵ has been chosen, as small as we please, there shall be a positive integer ν such that, for all values of x in the interval, $|{}_p R_n(x)| < \epsilon$, when $n \geq \nu$, for every positive integer p .

(i) *The condition is necessary.*

Let the positive number ϵ be chosen, as small as we please. Then take $\epsilon/2$.

Since the series is uniformly convergent, there is a positive integer ν , such that

$$|f(x) - S_n(x)| < \frac{\epsilon}{2}, \text{ when } n \geq \nu,$$

the same ν serving for all values of x in (a, b) , $f(x)$ being the sum of the series.

Let n'', n' be any two positive integers such that $n'' > n' \geq \nu$.

Then $|S_{n''}(x) - S_{n'}(x)| \leq |S_{n''}(x) - f(x)| + |f(x) - S_{n'}(x)|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$< \epsilon.$$

Thus $|S_{n+p}(x) - S_n(x)| < \epsilon$, when $n \geq \nu$, for every positive integer p , the same ν serving for all values of x in (a, b) .

(ii) *The condition is sufficient.*

We know that the series converges, when this condition is satisfied.

Let its sum be $f(x)$.

Again let the arbitrary positive number ϵ be chosen. Then there is a positive integer ν such that

$$|S_{n+p}(x) - S_n(x)| < \frac{\epsilon}{2}, \text{ when } n \geq \nu, \text{ for every positive integer } p,$$

the same ν serving for all values of x in (a, b) .

$$\text{Thus } S_\nu(x) - \frac{\epsilon}{2} < S_{\nu+p}(x) < S_\nu(x) + \frac{\epsilon}{2}.$$

$$\text{Also } \lim_{p \rightarrow \infty} S_{\nu+p}(x) = f(x).$$

$$\text{Therefore } S_\nu(x) - \frac{\epsilon}{2} \leq f(x) \leq S_\nu(x) + \frac{\epsilon}{2}.$$

$$\text{But } |S_n(x) - f(x)| \leq |S_n(x) - S_\nu(x)| + |S_\nu(x) - f(x)|.$$

It follows that, when n is greater than or equal to the value ν specified above,

$$|S_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$< \epsilon,$$

and this holds for all values of x in (a, b) .

Thus the series converges uniformly in this interval.

67. Weierstrass's M-Test for Uniform Convergence. The following simple test for uniform convergence is due to Weierstrass:

The series $u_1(x) + u_2(x) + u_3(x) + \dots$ will converge uniformly in (a, b) , if there is a convergent series of positive constants

$$M_1 + M_2 + M_3 + \dots,$$

such that, no matter what value x may have in (a, b) ,

$$|u_n(x)| \leq M_n \text{ for every positive integer } n.$$

Since the series $M_1 + M_2 + M_3 + \dots$ is convergent, with the usual notation,

$$M_{n+1} + M_{n+2} + \dots + M_{n+p} < \epsilon,$$

when $n \geq \nu$, for every positive integer p .

$$\text{But } |{}_p R_n(x)| \leq |u_{n+1}(x)| + |u_{n+2}(x)| + \dots + |u_{n+p}(x)|.$$

$$\text{Thus } |{}_p R_n(x)| \leq M_{n+1} + M_{n+2} + \dots + M_{n+p} < \epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p,$$

the inequality holding for all values of x in (a, b) .

Thus the given series is uniformly convergent in (a, b) .

For example, we know that the series

$$1 + 2a + 3a^2 + \dots$$

is convergent, when a is any given positive number less than unity.

It follows that the series

$$1 + 2x + 3x^2 + \dots$$

is uniformly convergent in the interval $(-a, a)$.

Ex. 1. Show that the series

$$x \cos \theta + x^2 \cos 2\theta + x^3 \cos 3\theta + \dots$$

is uniformly convergent for any interval (x_0, x_1) , where $-1 < x_0 < x_1 < 1$ and θ is any given number.

Ex. 2. Show that the series

$$x \cos \theta + x^2 \cos 2\theta + x^3 \cos 3\theta + \dots$$

$$\text{and } x \cos \theta + \frac{x^2}{2} \cos 2\theta + \frac{x^3}{3} \cos 3\theta + \dots$$

are uniformly convergent for all values of θ , when $|x|$ is any given positive number less than unity.

This theorem leads at once to the following result:

A series is absolutely and uniformly convergent in an interval, when the absolute values of its terms in the interval do not exceed the corresponding terms of a convergent series whose terms are all positive and independent of x .

It should be noted that Uniform Convergence does not imply Absolute Convergence.

E.g. the series $\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \dots$

is not absolutely convergent; but it is uniformly convergent in the infinite interval $x \geq 0$.*

Also a series may be absolutely convergent without being uniformly convergent.

E.g. $(1-x) + x(1-x) + x^2(1-x) + \dots$

is absolutely convergent in the interval $-c \leq x \leq 1$, where

$$-1 < -c < 0,$$

but it is not uniformly convergent in this interval.

68. Uniform Convergence of Series whose Terms are Continuous Functions of x . In the previous sections dealing with uniform convergence the terms of the series have not been assumed continuous in the given interval. We shall now prove some properties of these series when this condition is added:

I. *Uniform convergence implies continuity in the sum.*

If the terms of the series

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

are continuous in (a, b) , and the series converges uniformly to $f(x)$ in this interval, then $f(x)$ is a continuous function of x in (a, b) .

Since the series converges uniformly, we know that, however small the positive number ϵ may be, there is a positive integer ν , such that

$$|f(x) - S_n(x)| < \frac{\epsilon}{3}, \text{ when } n \geq \nu,$$

the same ν serving for all values of x in (a, b) .

* $|R_n(x)| < \frac{1}{n+1}$, for all values of x .

Choosing such a value of n , we have

$$f(x) = S_n(x) + R_n(x),$$

where $|R_n(x)| < \epsilon/3$, for all values of x in (a, b) .

Since $S_n(x)$ is the sum of n continuous functions, it is also continuous in (a, b) .

Thus we know from § 31 that there is a positive number η such that

$$|S_n(x') - S_n(x)| < \frac{\epsilon}{3},$$

when x, x' are any two values of x in the interval (a, b) for which $|x' - x| \leq \eta$.

But

$$f(x') = S_n(x') + R_n(x'),$$

where

$$|R_n(x')| < \epsilon/3.$$

$$\text{Also } f(x') - f(x) = (S_n(x') - S_n(x)) + R_n(x') - R_n(x).$$

$$\text{Thus } |f(x') - f(x)| \leq |S_n(x') - S_n(x)| + |R_n(x')| + |R_n(x)|.$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$$< \epsilon, \text{ when } |x' - x| \leq \eta.$$

Therefore $f(x)$ is continuous in (a, b) .

II. *If a series, whose terms are continuous functions, has a discontinuous sum, it cannot be uniformly convergent in an interval which contains a point of discontinuity.*

For if the series were uniformly convergent, we have just seen that its sum must be continuous in the interval of uniform convergence.

III. *Uniform convergence is thus a sufficient condition for the continuity of the sum of a series of continuous functions. It is not a necessary condition; since different non-uniformly convergent series are known, which represent continuous functions in the interval of non-uniform convergence.*

For example, the series discussed in Ex. 2 and Ex. 3 of § 63 are uniformly convergent in $x \geq a > 0$, for in both cases

$$|R_n(x)| < \frac{1}{nx} \leq \frac{1}{na}, \text{ when } x \geq a > 0.$$

Thus $|R_n(x)| < \epsilon$, when $n > 1/a\epsilon$, which is independent of x .

But the interval of uniform convergence does not extend up to and include $x=0$, even though the sum is continuous for all values of x .

This is clear in Ex. 2, where $R_n(x) = \frac{nx}{1+n^2x^2}$, for if it is asserted that $|R_n(x)| < \epsilon$, when $n \geq \nu$, the same ν serving for all values of x in $x \geq 0$, the statement is shown to be untrue by pointing out that for $x=1/\nu$, $R_\nu(x) = 1/2$, and thus $|R_n(x)| \not< \epsilon$, when $n \geq \nu$, right through the interval, if $\epsilon < 1/2$.

Similarly in Ex. 3, where $R_n(x) = \frac{n^2x}{1+n^3x^2}$, if it is asserted that $|R_n(x)| < \epsilon$, when $n \geq \nu$, the same ν serving for all values of x in $x \geq 0$, we need only point out that for $x=1/\sqrt{\nu^3}$, $R_\nu(x) = \frac{1}{2}\sqrt{\nu^3}$. Thus $|R_n(x)| \not< \epsilon$, when $n \geq \nu$, right through the interval, if $\epsilon < 1/2$.

There is, in both cases, a positive integer ν for which $R_n(1/m) < \epsilon < 1/2$, when $n \geq \nu$, but this integer is greater than $1/m$.

Thus it is clear that the convergence becomes infinitely slow as $x \rightarrow 0$.

IV. *If the terms of the series are continuous in the closed interval (a, b) , and the series converges uniformly in $a < x < b$, then it must converge for $x=a$ and $x=b$, and the uniformity of the convergence will hold for the closed interval (a, b) .*

Since the series is uniformly convergent in the open interval $a < x < b$, we have, with the usual notation,

$$|S_m(x) - S_n(x)| < \frac{\epsilon}{3}, \text{ when } m > n \geq \nu, \dots\dots\dots(1)$$

the same ν serving for every x in this open interval.

Let m, n be any two positive integers satisfying this relation.

Since the terms of the series are continuous in the closed interval (a, b) , there are positive numbers η_1 and η_2 , say, such that

$$|S_m(x) - S_m(a)| < \frac{\epsilon}{3}, \text{ when } 0 \leq (x-a) \leq \eta_1,$$

and

$$|S_n(x) - S_n(a)| < \frac{\epsilon}{3}, \text{ when } 0 \leq (x-a) \leq \eta_2.$$

Choose a positive number η not greater than η_1 or η_2 , and let $0 \leq (x-a) \leq \eta$.

Then $|S_m(a) - S_n(a)|$
 $\leq |S_m(a) - S_m(x)| + |S_m(x) - S_n(x)| + |S_n(x) - S_n(a)|$
 $< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$
 $< \epsilon, \text{ when } m > n \geq \nu. \dots\dots\dots(2)$

A similar argument shows that

$$|S_m(b) - S_n(b)| < \epsilon, \text{ when } m > n \geq \nu. \dots\dots\dots(3)$$

From (2) and (3) we see that the series converges for $x=a$ and $x=b$, and, combining (1), (2) and (3), we see that the condition for uniform convergence in the closed interval (a, b) is satisfied.

If the terms of the series

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

are continuous in (a, b) , and the series converges uniformly in every interval (α, β) , where $a < \alpha < \beta < b$, the series need not converge for $x=a$ or $x=b$.

E.g. the series $1 + 2x + 3x^2 + \dots$

converges uniformly in $(-a, a)$, where $a < 1$, but it does not converge for $x = -1$ or $x = 1$.

However we shall see that in the case of the Power Series, if it converges for $x=a$ or $x=b$, the uniform convergence in (a, β) extends up to a or b , as the case may be. (Cf. § 72.)

But this property is not true in general.

The series of continuous functions

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

may converge uniformly for every interval (α, β) within (a, b) , and converge for $x=a$ or $x=b$, while the range of uniform convergence does not extend up to and include the point a or b .

E.g. the series $x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{4}x^4 + \dots \dots\dots(1)$

formed from the logarithmic series

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \dots\dots(2)$$

by taking two consecutive positive terms and then one negative term, is convergent when $-1 < x \leq 1$, and its sum, when $x=1$, is $\frac{3}{2} \log 2$.*

Further, the series (2) is absolutely convergent when $|x| < 1$, and therefore the sum is not altered by taking the terms in any other order. (Cf. § 22.)

It follows that when $|x| < 1$ the sum of (1) is $\log(1+x)$, and when $x=1$ its sum is $\frac{3}{2} \log 2$.

Hence (1) is discontinuous at $x=1$ and therefore the interval of uniform convergence does not extend up to and include that point.

* Cf. Hobson, *Plane Trigonometry* (3rd Ed.), p. 251.

69. Uniform Convergence and the Approximation Curves. Let a series of continuous functions be uniformly convergent in (a, b) .

Then we have, as before,

$$|S_m(x) - S_n(x)| < \epsilon, \text{ when } m > n \geq \nu,$$

the same ν serving for all values of x in the interval,

In particular, $|S_m(x) - S_\nu(x)| < \epsilon, \text{ when } m > \nu,$

and we shall suppose ν the smallest positive integer which will satisfy this condition for the given ϵ and every x in the interval.

Plot the curve $y = S_\nu(x)$ and the two parallel curves $y = S_\nu(x) \pm \epsilon$, forming a strip σ of breadth 2ϵ , whose central line is $y = S_\nu(x)$. (Fig. 13.)

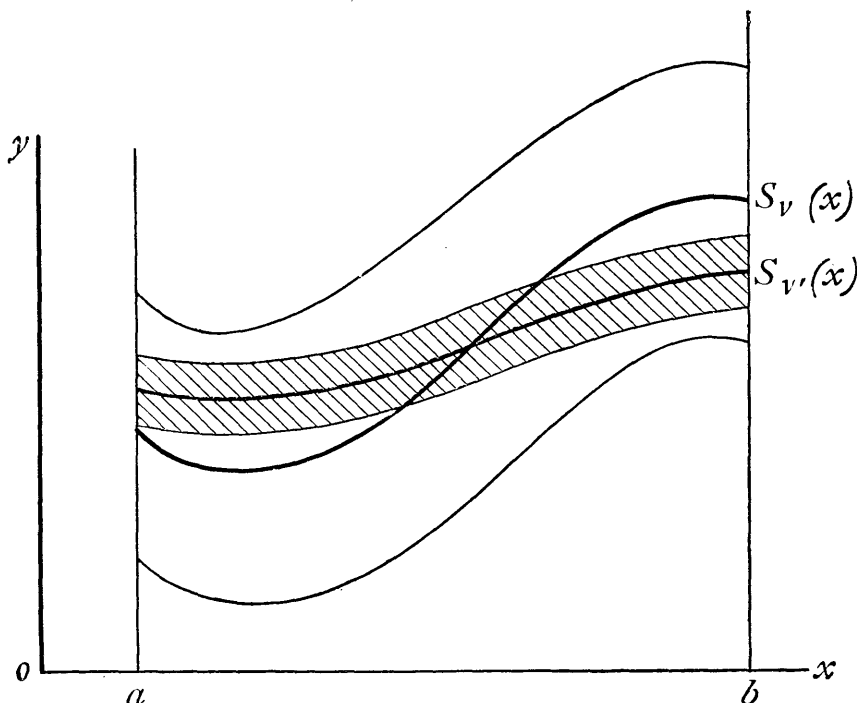


FIG. 13.

All the approximation curves $y = S_m(x), m > \nu$, lie in this strip, and the curve $y = f(x)$, where $f(x)$ is the sum of the series, also lies within the strip, or at most reaches its boundaries. (Cf. § 66 (ii).)

Next choose ϵ' less than ϵ , and let the corresponding smallest positive integer satisfying the condition for uniform convergence be ν' . Then ν' is greater than or equal to ν . The new curve $y = S_{\nu'}(x)$ thus lies in the first strip, and the new strip σ' of breadth $2\epsilon'$, formed as before, if it goes outside the first strip in any part, can have this portion blotted out, for we are concerned only with the region in which the approximation curves may lie as m increases from the value ν .

In this way, if we take the set of positive numbers

$$\epsilon > \epsilon' > \epsilon'' \dots, \text{ where } \lim_{\kappa \rightarrow \infty} \epsilon^{(\kappa)} = 0,$$

and the corresponding positive integers

$$\nu \leq \nu' \leq \nu'' \dots,$$

we obtain the set of strips $\sigma, \sigma', \sigma'', \dots$

Any strip lies within, or at most reaches the boundary of the preceding one, and their breadth tends to zero as their number increases.

Further, the curve $y=f(x)$ lies within, or at most reaches, the boundary of the strips.

This construction, therefore, not only establishes the continuity of the sum of the series of continuous functions, in an interval of uniform convergence, but it shows that the approximation curves, as the number of the terms increase, may be used as a guide to the shape of the curve for the sum right through the interval.*

70. A sufficient Condition for Term by Term Integration of a Series whose Terms are Continuous Functions of x . When the series of continuous functions

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

is uniformly convergent in the interval (a, b) , we have seen that its sum, $f(x)$, is continuous in (a, b) . It follows that $f(x)$ is integrable between x_0 and x_1 , when $a \leq x_0 < x_1 \leq b$.

But it does not follow, without further examination, that the series of integrals

$$\int_{x_0}^{x_1} u_1(x) dx + \int_{x_0}^{x_1} u_2(x) dx + \int_{x_0}^{x_1} u_3(x) dx + \dots$$

is convergent, and, even if it be convergent, it does not follow, without proof, that its sum is $\int_{x_0}^{x_1} f(x) dx$.

The geometrical treatment of the approximation curves in § 69 suggests that this result will be true, when the given series is uniformly convergent, arguing from the areas of the respective curves.

We shall now state the theorem more precisely and give its demonstration :

Let the functions $u_1(x), u_2(x), u_3(x), \dots$ be continuous in (a, b) , and let the series

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

be uniformly convergent in (a, b) and have $f(x)$ for its sum.

Then

$$\int_{x_0}^{x_1} f(x) dx = \int_{x_0}^{x_1} u_1(x) dx + \int_{x_0}^{x_1} u_2(x) dx + \int_{x_0}^{x_1} u_3(x) dx + \dots,$$

where $a \leq x_0 < x_1 \leq b$.

* Of course the argument of this section applies only to such functions as can be graphically represented.

Let the arbitrary positive number ϵ be chosen.

Since the series is uniformly convergent, we may put

$$f(x) = S_n(x) + R_n(x),$$

where $|R_n(x)| < \frac{\epsilon}{b-a}$, when $n \geq \nu$,

the same ν serving for all values of x in (a, b) .

Also $f(x)$ and $S_n(x)$ are continuous in (a, b) and therefore integrable.

Thus we have

$$\int_{x_0}^{x_1} f(x) dx = \int_{x_0}^{x_1} S_n(x) dx + \int_{x_0}^{x_1} R_n(x) dx,$$

where $a \leq x_0 < x_1 \leq b$.

$$\begin{aligned} \text{Therefore } \left| \int_{x_0}^{x_1} f(x) dx - \int_{x_0}^{x_1} S_n(x) dx \right| &= \left| \int_{x_0}^{x_1} R_n(x) dx \right| \\ &< \epsilon \frac{x_1 - x_0}{b - a} \\ &< \epsilon, \text{ when } n \geq \nu. \end{aligned}$$

$$\text{But } \int_{x_0}^{x_1} S_n(x) dx = \sum_1^n \int_{x_0}^{x_1} u_r(x) dx.$$

$$\text{Therefore } \left| \int_{x_0}^{x_1} f(x) dx - \sum_1^n \int_{x_0}^{x_1} u_r(x) dx \right| < \epsilon, \text{ when } n \geq \nu.$$

Thus the series of integrals is convergent and its sum is $\int_{x_0}^{x_1} f(x) dx$.

COROLLARY I. Let $u_1(x), u_2(x), u_3(x), \dots$ be continuous in (a, b) and the series

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

converge uniformly to $f(x)$ in (a, b) .

Then the series of integrals

$$\int_{x_0}^x u_1(x) dx + \int_{x_0}^x u_2(x) dx + \int_{x_0}^x u_3(x) dx + \dots$$

converges uniformly to $\int_{x_0}^x f(x) dx$ in (a, b) , when $a \leq x_0 < x \leq b$.

This follows at once from the argument above.

COROLLARY II. Let $u_1(x), u_2(x), u_3(x), \dots$ be continuous in (a, b) and the series

$$u_1(x) + u_2(x) + u_3(x) + \dots$$

converge uniformly to $f(x)$ in (a, b) .

Also let $g(x)$ be bounded and integrable in (a, b) .

Then
$$\int_{x_0}^x f(x)g(x) dx = \sum_1^\infty \int_{x_0}^x u_n(x)g(x) dx,$$

where $a \leq x_0 < x \leq b$, and the convergence of the series of integrals is uniform in (a, b) .

Let the arbitrary positive number ϵ be chosen, and let M be the upper bound of $|g(x)|$ in (a, b) .

Since the series $\sum_1^\infty u_n(x)$ converges uniformly to $f(x)$ in (a, b) ,

we may put
$$f(x) = S_n(x) + R_n(x),$$

where
$$|R_n(x)| < \frac{\epsilon}{M(b-a)}, \text{ when } n \geq \nu,$$

the same ν serving for all values of x in (a, b) .

Therefore we have

$$\int_{x_0}^x f(x)g(x) dx = \int_{x_0}^x S_n(x)g(x) dx + \int_{x_0}^x R_n(x)g(x) dx,$$

where
$$a \leq x_0 < x \leq b.$$

Thus
$$\left| \int_{x_0}^x f(x)g(x) dx - \int_{x_0}^x S_n(x)g(x) dx \right| = \left| \int_{x_0}^x R_n(x)g(x) dx \right|.$$

And
$$\left| \int_{x_0}^x f(x)g(x) dx - \sum_1^n \int_{x_0}^x u_r(x)g(x) dx \right|$$

$$< \frac{\epsilon}{M(b-a)} \times M(x-x_0)$$

$$< \epsilon, \text{ when } n \geq \nu,$$

which proves our theorem.

It is clear that these integrations can be repeated as many times as we wish.

Ex. To prove that

$$\int_0^\pi \log(1 - 2y \cos x + y^2) dx = 0, \text{ when } |y| < 1,^*$$

$$= \pi \log y^2, \text{ when } |y| > 1.$$

We know that

$$\frac{y - \cos x}{1 - 2y \cos x + y^2} = -\cos x - y \cos 2x - y^2 \cos 3x + \dots, \dots\dots\dots(1)$$

when $|y| < 1$.

* It follows from Ex. 4, p 118, that we may replace the symbols $<$, $>$ by \leq and \geq respectively.

Also the series (1) converges uniformly for any interval of y within $(-1, 1)$ (§ 67).

Therefore
$$\int_0^y \frac{y - \cos x}{1 - 2y \cos x + y^2} dy = - \sum_1^{\infty} \cos nx \int_0^y y^{n-1} dy, \text{ when } |y| < 1.$$

Therefore
$$\frac{1}{2} \log(1 - 2y \cos x + y^2) = - \sum_1^{\infty} \frac{\cos nx}{n} y^n, \text{ when } |y| < 1. \dots\dots\dots(2)$$

But the series (2) converges uniformly for all values of x , when $|y|$ is some positive number less than unity (§ 67).

Thus
$$\frac{1}{2} \int_0^{\pi} \log(1 - 2y \cos x + y^2) dx = - \sum_1^{\infty} \frac{y^n}{n} \int_0^{\pi} \cos nx dx, \text{ when } |y| < 1.$$

Therefore
$$\int_0^{\pi} \log(1 - 2y \cos x + y^2) dx = 0, \text{ when } |y| < 1.$$

But
$$\int_0^{\pi} \log(1 - 2y \cos x + y^2) dx = \int_0^{\pi} \left[\log y^2 + \log \left(1 - \frac{2}{y} \cos x + \frac{1}{y^2} \right) \right] dx.$$

Therefore
$$\int_0^{\pi} \log(1 - 2y \cos x + y^2) dx = \pi \log y^2, \text{ when } |y| > 1.$$

71. A sufficient Condition for Term by Term Differentiation.

If the series
$$u_1(x) + u_2(x) + u_3(x) + \dots$$

converges in (a, b) and each of its terms has a differential coefficient, continuous in (a, b) , and if the series of differential coefficients
$$u_1'(x) + u_2'(x) + u_3'(x) + \dots$$

converges uniformly in (a, b) , then $f(x)$, the sum of the original series, has a differential coefficient at every point of (a, b) , and

$$f'(x) = u_1'(x) + u_2'(x) + u_3'(x) + \dots$$

Let
$$\phi(x) = u_1'(x) + u_2'(x) + u_3'(x) + \dots$$

Since this series of continuous functions converges uniformly in (a, b) , we can integrate it term by term.

Thus we have

$$\int_{x_0}^{x_1} \phi(x) dx = \int_{x_0}^{x_1} u_1'(x) dx + \int_{x_0}^{x_1} u_2'(x) dx + \int_{x_0}^{x_1} u_3'(x) dx + \dots,$$

where $a \leq x_0 < x_1 \leq b$.

Therefore
$$\int_{x_0}^{x_1} \phi(x) dx = [u_1(x_1) - u_1(x_0)] + [u_2(x_1) - u_2(x_0)] + [u_3(x_1) - u_3(x_0)] + \dots$$

But
$$f(x_1) = u_1(x_1) + u_2(x_1) + u_3(x_1) + \dots$$

and
$$f(x_0) = u_1(x_0) + u_2(x_0) + u_3(x_0) + \dots$$

Therefore
$$\int_{x_0}^{x_1} \phi(x) dx = f(x_1) - f(x_0).$$

Now put $x_0 = x$ and $x_1 = x + \Delta x$.

Then, by the First Theorem of Mean Value,

$$\phi(\xi)\Delta x = f(x + \Delta x) - f(x),$$

where $x \leq \xi \leq x + \Delta x$.

Therefore
$$\phi(\xi) = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

But
$$\text{Lt}_{\Delta x \rightarrow 0} \phi(\xi) = \phi(x),$$

since $\phi(x)$ is continuous in (a, b) .

Therefore $f(x)$ has a differential coefficient $f'(x)$ in (a, b) ,

and
$$f'(x) = \phi(x) = u_1'(x) + u_2'(x) + u_3'(x) + \dots$$

It must be remembered that the conditions for continuity, for term by term differentiation and integration, which we have obtained are only *sufficient* conditions. They are not *necessary* conditions. We have imposed more restrictions on the functions than are required. But no other conditions of equal simplicity have yet been found, and for that reason these theorems are of importance.

It should also be noted that in these sections we have again been dealing with repeated limits (cf. § 64), and we have found that in certain cases the order in which the limits are taken may be reversed without altering the result.

In term by term integration, we have been led to the equality, in certain cases, of

$$\int_{x_0}^{x_1} \text{Lt}_{n \rightarrow \infty} S_n(x) dx \quad \text{and} \quad \text{Lt}_{n \rightarrow \infty} \int_{x_0}^{x_1} S_n(x) dx.$$

Similarly in term by term differentiation we have found that, in certain cases,

$$\text{Lt}_{h \rightarrow 0} \left[\text{Lt}_{n \rightarrow \infty} \left(\frac{S_n(x+h) - S_n(x)}{h} \right) \right] \quad \text{and} \quad \text{Lt}_{n \rightarrow \infty} \left[\text{Lt}_{h \rightarrow 0} \left(\frac{S_n(x+h) - S_n(x)}{h} \right) \right]$$

are equal.

72. The Power Series. The properties of the Power Series

$$a_0 + a_1x + a_2x^2 + \dots$$

are so important, and it offers so simple an illustration of the results we have just obtained, that a separate discussion of this series will now be given.

I. If the series $a_0 + a_1x + a_2x^2 + \dots$ is convergent for $x=x_0$, it is absolutely convergent for every value of x such that $|x| < |x_0|$.

Since the series is convergent for $x=x_0$, there is a positive number M such that $|a_n x_0^n| < M$, when $n \geq 0$.

But $|a_n x^n| = |a_n x_0^n| \times \left| \frac{x}{x_0} \right|^n$.

Therefore if $\left| \frac{x}{x_0} \right| = c < 1$, the terms of the series

$$|a_0| + |a_1x| + |a_2x^2| + \dots$$

are less than the corresponding terms of the convergent series

$$M\{1 + c + c^2 + \dots\},$$

and our theorem follows.

II. If the series does not converge for $x=x_0$, it does not converge for any value of x such that $|x| > |x_0|$.

This follows from I., since if the series converges for a value of x , such that $|x| > |x_0|$, it must converge for $x=x_0$.

III. It follows from I. and II. that only the following three cases can occur :

(i) The series converges for $x=0$ and no other value of x .

$$\begin{aligned} \text{E.g. } & 1 + 1!x + 2!x^2 + \dots, \\ & 1 + x + 2^2x^2 + \dots \end{aligned}$$

(ii) The series converges for all values of x .

$$\text{E.g. } 1 + x + \frac{x^2}{2!} + \dots$$

(iii) There is some positive number ρ such that, when $|x| < \rho$, the series converges, and, when $|x| > \rho$, the series does not converge.

$$\text{E.g. } x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$$

The interval $-\rho < x < \rho$ is called the *interval of convergence* of the series. Also it is convenient to say that, in the first case, the interval is zero, and, in the second, infinite. It will be seen that the interval of convergence of the following three series is $(-1, 1)$:

$$\begin{aligned} & 1 + x + x^2 + \dots, \\ & 1 + \frac{x}{1} + \frac{x^2}{2} + \dots, \\ & 1 + \frac{x}{1^2} + \frac{x^2}{2^2} + \dots \end{aligned}$$

But it should be noticed that the first of these does not converge at the ends of the interval; the second converges at one of the ends; and the third converges at both.

In the Power Series there cannot be first an interval of convergence, then an interval where the convergence fails, and then a return to convergence.

Also the interval of convergence is symmetrical with regard to the origin. We shall denote its ends by L' , L . The series need not converge at L' or L , but it may do so; and it must converge within $L'L$.

IV. *The series is absolutely convergent in the open interval $-\rho < x < \rho$.*

V. *The series is absolutely and uniformly convergent in the closed interval $-\rho + \delta \leq x \leq \rho - \delta$, where δ is any assigned positive number less than ρ .*

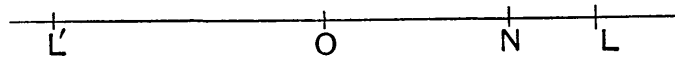


FIG. 14.

To prove IV., we have only to remark that if N is a point x_0 , where $-\rho < x_0 < \rho$, between N and the nearer boundary of the interval of convergence, there are values of x for which the series converges, and thus by I. it converges absolutely for $x = x_0$.

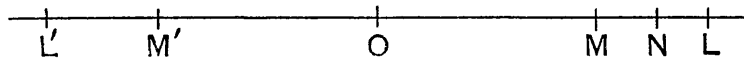


FIG. 15.

To prove V., let M' , M correspond to $x = -\rho + \delta$ and $x = \rho - \delta$ respectively. We now choose a point N (say x_0) between M and the nearer boundary L . The series converges absolutely for $x = x_0$, by IV.

Thus, with the usual notation,

$$|a_n x_0^n| + |a_{n+1} x_0^{n+1}| + \dots < \epsilon, \text{ when } n \geq \nu.$$

But $|a_n x^n| + |a_{n+1} x^{n+1}| + \dots$

is less than the above for every point in $M'M$, including the ends M' , M .

It follows that our series is absolutely and uniformly convergent in the closed interval (M', M) .* And the sum of the series is continuous in this closed interval.

It remains to examine the behaviour of the series at the ends of the interval of convergence, and we shall now prove Abel's Theorem : †

VI. *If the series converges for either of the ends of the interval of convergence, the interval of uniform convergence extends up to and includes that point, and the continuity of $f(x)$, the sum of the series, extends up to and includes that point.*

Let the series converge for $x = \rho$.

Then, with the usual notation,

$${}_p R_n(\rho) = a_n \rho^n + a_{n+1} \rho^{n+1} + \dots + a_{n+p-1} \rho^{n+p-1},$$

and $|{}_p R_n(\rho)| < \epsilon$, when $n \geq \nu$, for every positive integer p .

* When the interval of convergence extends to infinity, the series will be absolutely convergent for every value of x , but it need not be uniformly convergent in the *infinite interval*. However, it will be uniformly convergent in any interval $(-b, b)$, where b is fixed, but may be fixed as large as we please.

E.g. the exponential series converges uniformly in any fixed interval, which may be arbitrarily great, but not in an infinite interval [cf. § 65].

† *J. Math., Berlin*, **1**, p. 311, 1826.

$$\begin{aligned} \text{But } {}_pR_n(x) &= a_n x^n + a_{n+1} x^{n+1} + \dots + a_{n+p-1} x^{n+p-1} \\ &= a_n \rho^n \left(\frac{x}{\rho}\right)^n + a_{n+1} \rho^{n+1} \left(\frac{x}{\rho}\right)^{n+1} + \dots + a_{n+p-1} \rho^{n+p-1} \left(\frac{x}{\rho}\right)^{n+p-1}, \end{aligned}$$

and the factors $\left(\frac{x}{\rho}\right)^n, \left(\frac{x}{\rho}\right)^{n+1}, \dots, \left(\frac{x}{\rho}\right)^{n+p-1}$ are all positive and decreasing, when $0 < x < \rho$.

$$\begin{aligned} \text{Thus } {}_pR_n(x) &= {}_1R_n(\rho) c^n \\ &\quad + [{}_2R_n(\rho) - {}_1R_n(\rho)] c^{n+1} \\ &\quad + \dots \\ &\quad + [{}_pR_n(\rho) - {}_{(p-1)}R_n(\rho)] c^{n+p-1}, \end{aligned}$$

where $c = \frac{x}{\rho}$.

This expression may be re-written in the form

$$\begin{aligned} {}_pR_n(x) &= {}_1R_n(\rho) [c^n - c^{n+1}] \\ &\quad + {}_2R_n(\rho) [c^{n+1} - c^{n+2}] \\ &\quad + \dots \\ &\quad + {}_{(p-1)}R_n(\rho) [c^{n+p-2} - c^{n+p-1}] \\ &\quad + {}_pR_n(\rho) c^{n+p-1}. \end{aligned}$$

But $c^n - c^{n+1}, c^{n+1} - c^{n+2}, \dots, c^{n+p-2} - c^{n+p-1}$ are all positive, and

$$|{}_1R_n(\rho)|, |{}_2R_n(\rho)|, \dots, |{}_pR_n(\rho)|$$

are all less than ϵ , when $n \geq \nu$.

Therefore, when $n \geq \nu$,

$$\begin{aligned} |{}_pR_n(x)| &< \epsilon [(c^n - c^{n+1}) + (c^{n+1} - c^{n+2}) + \dots + (c^{n+p-2} - c^{n+p-1}) + c^{n+p-1}] \\ &< \epsilon c^n \\ &< \epsilon, \text{ provided that } 0 < x < \rho. \end{aligned}$$

But we started with $|{}_pR_n(\rho)| < \epsilon$, when $n \geq \nu$.

Thus we have shown that the series is uniformly convergent in the interval

$$0 < x \leq \rho.$$

Combining this result with V., we see that the series is now uniformly convergent in the interval $-\rho + \delta \leq x \leq \rho$, where δ is any assigned positive number less than ρ . And it follows that $f(x)$, the sum of the series, is continuous in this closed interval.

In particular, when the series converges at $x = \rho$,

$$\begin{aligned} \text{Lt}_{x \rightarrow \rho-0} f(x) &= f(\rho) \\ &= a_0 + a_1 \rho + a_2 \rho^2 + \dots \end{aligned}$$

In the case of the logarithmic series,

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots,$$

the interval of convergence is $-1 < x < 1$.

Further, when $x = 1$, the series converges.

It follows from Abel's Theorem that

$$\begin{aligned} \text{Lt}_{x \rightarrow 1} \log(1+x) &= 1 - \frac{1}{2} + \frac{1}{3} - \dots, \\ \text{i.e. } \log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \dots \end{aligned}$$

Similarly, in the Binomial Series,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots,$$

when $-1 < x < 1$.

And it is known* that $1 + m + \frac{m(m-1)}{2!} + \dots$

is conditionally convergent when $-1 < m < 0$, and absolutely convergent when $m > 0$.

Hence $\text{Lt}_{x \rightarrow 1} (1+x)^m = 1 + m + \frac{m(m-1)}{2!} + \dots,$

$$\text{i.e. } 2^m = 1 + m + \frac{m(m-1)}{2!} + \dots,$$

in both these cases.

On the other hand, if we put $x=1$ in the series for $(1+x)^{-1}$, we get a series which does not converge. The uniformity of the convergence of the series

$$1 - x + x^2 - \dots$$

is for the interval $-l \leq x \leq l$, where l is any given positive number less than 1.

VII. Term by term differentiation and integration of the Power Series.

We have seen in V. that the Power Series is uniformly convergent in any closed interval M' , $M'[-\rho + \delta \leq x \leq \rho - \delta]$ contained within its interval of convergence L , $L[-\rho < x < \rho]$.

It follows from § 70 that *the series may be integrated term by term in the interval M' , M' ; and the process may be repeated any number of times.*

We shall now show that a corresponding result holds for differentiation.

From the theorem proved in § 71, it is clear that we need only show that the interval of convergence $(-\rho < x < \rho)$ of the series

$$a_0 + a_1x + a_2x^2 + \dots$$

is also the interval of convergence of the series

$$a_1 + 2a_2x + 3a_3x^2 + \dots$$

To prove this, it will be sufficient to show that

$$|a_1| + 2|a_2x| + 3|a_3x^2| + \dots$$

is convergent when $|x| < \rho$ and is divergent when $|x| > \rho$.

When $|x| < \rho$, let ρ' lie between $|x|$ and ρ .

Then the series $\frac{1}{\rho'} + \frac{2}{\rho'} \left| \frac{x}{\rho'} \right| + \frac{3}{\rho'} \left| \frac{x}{\rho'} \right|^2 + \dots$

is convergent, because the ratio of the n^{th} term to the preceding has for its limit $\left| \frac{x}{\rho'} \right|$, which is less than 1.

If we multiply the different terms of this series by the factors

$$|a_1|\rho', \quad |a_2|\rho'^2, \quad |a_3|\rho'^3, \quad \dots,$$

* Cf. Chrystal, *Algebra*, Vol. II. p. 131.

which are all less than some fixed number, since (by IV.) the series

$$|a_0| + |a_1|\rho' + |a_2|\rho'^2 + \dots$$

is convergent, it is clear that the series which we thus obtain, namely

$$|a_1| + 2|a_2x| + 3|a_3x^2| + \dots,$$

is convergent when $|x| < \rho$.

We have yet to show that this last series diverges when $|x| > \rho$.

If this series were convergent for $x = |x_1|$, where $|x_1| > \rho$, the same would hold for the series

$$|a_1x_1| + 2|a_2x_1^2| + 3|a_3x_1^3| + \dots,$$

and also for the series $|a_1x_1| + |a_2x_1^2| + |a_3x_1^3| + \dots$,

since the terms of the latter are not greater than those of the former.

But this is impossible, since we are given that the interval of convergence of the original series is $-\rho < x < \rho$.

Thus the series $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

and the series $a_1 + 2a_2x + 3a_3x^2 + \dots$,

obtained by differentiating it term by term, have the same interval of convergence,* and it follows from § 71 that

$$\begin{aligned} \text{If} \quad f(x) &= a_0 + a_1x + a_2x^2 + \dots, \\ f'(x) &= a_1 + 2a_2x + \dots, \end{aligned}$$

when x is any point in the open interval $-\rho < x < \rho$.

Also this process can be repeated, as in the case of integration, as often as we please.

73. Extensions of Abel's Theorem on the Power Series.

I. We have seen in § 72 that if the series

$$a_0 + a_1 + a_2 + \dots$$

converges, the Power Series

$$a_0 + a_1x + a_2x^2 + \dots$$

is uniformly convergent, when $0 \leq x \leq 1$; and that, if

$$f(x) = a_0 + a_1x + a_2x^2 + \dots,$$

$$\text{Lt}_{x \rightarrow 1-0} f(x) = a_0 + a_1 + a_2 + \dots.$$

The above theorem of Abel's is a special case of the following :

Let the series $\sum_0^\infty a_n$ converge, and a_0, a_1, a_2, \dots be a sequence of positive numbers such that $0 \leq a_0 < a_1 < a_2 \dots$. Then the series $\sum_0^\infty a_n e^{-ant}$ is uniformly convergent, when $t \geq 0$, and if $f(t) = \sum_0^\infty a_n e^{-ant}$, we have $\text{Lt}_{t \rightarrow +0} f(t) = \sum_0^\infty a_n$.

Consider the partial remainder ${}_p R_n(t)$ for the series $\sum_0^\infty a_n e^{-ant}$.

* If we know that $\text{Lt}_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ exists, this result follows immediately from the ratio-test for convergence, since in each series

$$\text{Lt}_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1, \text{ if } |x| < \text{Lt}_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Then
$${}_pR_n(t) = a_n e^{-a_n t} + a_{n+1} e^{-a_{n+1} t} + \dots + a_{n+p-1} e^{-a_{n+p-1} t}$$

$$= {}_1r'_n e^{-a_n t} + ({}_2r'_n - {}_1r'_n) e^{-a_{n+1} t} + \dots + ({}_p r'_n - (p-1)r'_n) e^{-a_{n+p-1} t},$$

where ${}_1r'_n = a_n$, ${}_2r'_n = a_n + a_{n+1}$, ..., these being the partial remainders for the series $\sum_0^\infty a_n$.

Rewriting the expression for ${}_pR_n(t)$, we obtain, as in § 72, VI.,

$${}_pR_n(t) = {}_1r'_n (e^{-a_n t} - e^{-a_{n+1} t}) + {}_2r'_n (e^{-a_{n+1} t} - e^{-a_{n+2} t}) + \dots + {}_p r'_n e^{-a_{n+p-1} t}.$$

But the series $\sum_0^\infty a_n$ is convergent.

Therefore, with the usual notation,

$$|{}_p r'_n| < \epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p.$$

Also the numbers $e^{-a_n t}$, $e^{-a_{n+1} t}$, $e^{-a_{n+2} t}$, ... are all positive and decreasing, when $t > 0$.

It follows that, when $t > 0$,

$$|{}_pR_n(t)| < \epsilon [(e^{-a_n t} - e^{-a_{n+1} t}) + (e^{-a_{n+1} t} - e^{-a_{n+2} t}) + \dots + e^{-a_{n+p-1} t}]$$

$$< \epsilon e^{-a_n t}$$

$$< \epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p.$$

And this also holds when $t = 0$.

Therefore the series $\sum_0^\infty a_n e^{-a_n t}$ is uniformly convergent when $t \geq 0$.

Let its sum be $f(t)$.

Then
$$\text{Lt}_{t \rightarrow +0} f(t) = f(0) = \sum_0^\infty a_n .*$$

II. In Abel's Theorem and the extension proved above, the series $\sum_0^\infty a_n$ are supposed convergent. We proceed to prove Bromwich's Theorem dealing with series which need not converge.† In this discussion we shall adopt the following notation :

$$s_n = a_0 + a_1 + a_2 + \dots + a_n,$$

$$\sigma_n = s_0 + s_1 + s_2 + \dots + s_n,$$

and we write S_n for the Arithmetic Mean of the first n terms of the sequence s_0, s_1, s_2, \dots

Thus
$$S_n = \frac{s_0 + s_1 + s_2 + \dots + s_{n-1}}{n} = \frac{\sigma_{n-1}}{n}.$$

It can be shown (cf. § 102) that, if the series $\sum_0^\infty a_n$ converges and its sum is S , then, with the above notation, $\text{Lt}_{n \rightarrow \infty} S_n = S$. But the converse does not hold.

* If a_0, a_1, \dots are functions of x and the series $\sum_0^\infty a_n$ converges uniformly to $F(x)$ in a given interval. Then it follows from the above argument that $\text{Lt}_{t \rightarrow +0} \sum_0^\infty a_n e^{-a_n t}$ converges uniformly to $F(x)$ in this interval.

† *Math. Ann., Leipzig*, **65**, p. 350, 1908. See also a paper by C. N. Moore in *Bull. Amer. Math. Soc.*, **25**, p. 258, 1919.

The sequence of Arithmetic Means may converge, while the sum $\sum_0^\infty a_n$ fails to converge.*

Bromwich's Theorem. *Let the sequence of Arithmetic Means S_n for the series $\sum_0^\infty a_n$ converge to S . Also let u_n be a function of t with the following properties, when $t > 0$:*

- (α) $\sum_p^q n |\Delta^2 u_n| < K \dagger$ (p, q any positive integers; K , a positive number independent of p, q and t)
- (β) $\text{Lt}_{n \rightarrow \infty} n u_n = 0$,
- (γ) $\text{Lt}_{t \rightarrow +0} u_n = 1$.

Then the series $\sum_0^\infty a_n u_n$ converges when $t > 0$, and $\text{Lt}_{t \rightarrow +0} \sum_0^\infty a_n u_n = S$.

We have

$$\begin{aligned} \sigma_0 &= s_0 &= a_0, \\ \sigma_1 - 2\sigma_0 &= s_1 - s_0 = a_1, \\ \sigma_2 - 2\sigma_1 + \sigma_0 &= s_2 - s_1 = a_2, \\ \sigma_3 - 2\sigma_2 + \sigma_1 &= s_3 - s_2 = a_3, \text{ etc.} \end{aligned}$$

Thus $\sum_0^n a_n u_n = \sigma_0 u_0 + (\sigma_1 - 2\sigma_0)u_1 + (\sigma_2 - 2\sigma_1 + \sigma_0)u_2 + \dots + (\sigma_n - 2\sigma_{n-1} + \sigma_{n-2})u_n$.

Therefore

$$\sum_0^n a_n u_n = \sigma_0 \Delta^2 u_0 + \sigma_1 \Delta^2 u_1 + \dots + \sigma_n \Delta^2 u_n + 2\sigma_n u_{n+1} - \sigma_n u_{n+2} - \sigma_{n-1} u_{n+1}. \quad (1)$$

But the sequence of Arithmetic Means

$$s_1, s_2, s_3, \dots$$

converges, and $\text{Lt}_{n \rightarrow \infty} S_n = S$.

It follows that there is a number C , not less than $|S|$, such that $|\sigma_n| < (n+1)C$ for every integer n .

Also from (β) it is clear that

$$\text{Lt}_{n \rightarrow \infty} (\sigma_n u_{n+1}) = \text{Lt}_{n \rightarrow \infty} (\sigma_n u_{n+2}) = \text{Lt}_{n \rightarrow \infty} (\sigma_{n-1} u_{n+1}) = 0. \dots\dots\dots (2)$$

Further, the series $\sum_0^\infty (n+1) |\Delta^2 u_n|$ converges, since, from (α), the series $\sum_0^\infty n |\Delta^2 u_n|$ converges.

*If $a_n = (-1)^n$, $n \geq 0$, it is obvious that $\text{Lt}_{n \rightarrow \infty} S_n = \frac{1}{2}$, but the series $\sum_0^\infty a_n$ is not convergent. But see Hardy's Theorem, § 102, II. When $\text{Lt}_{n \rightarrow \infty} S_n = S$, the series $\sum_0^\infty a_n$ is often said to be "summable ($C1$)" and its sum ($C1$) is said to be S . For a discussion of this method of treating series, due to Cesàro, reference may be made to Whittaker and Watson, *Course of Modern Analysis*, p. 155, 1920. Also see below, §§ 101-103 and § 108.

† $\Delta^2 u_n$ is written for $(u_n - 2u_{n+1} + u_{n+2})$. Since all the terms in the series $\sum_0^\infty n |\Delta^2 u_n|$ are positive, this condition (α) implies the convergence of this series.

Also $|\sigma_n \Delta^2 u_n| < C(n+1) |\Delta^2 u_n|$.

Therefore the series $\sum_0^\infty \sigma_n \Delta^2 u_n$ converges absolutely.

It follows from (1) and (2) that

$$\sum_0^\infty a_n u_n = \sum_0^\infty \sigma_n \Delta^2 u_n. \dots\dots\dots(3)$$

Taking the special case $a_0 = 1, a_1 = a_2 = \dots = 0,$

we have $\sigma_n = n + 1$ and $u_0 = \sum_0^\infty (n + 1) \Delta^2 u_n. \dots\dots\dots(4)$

Thus, from (3) and (4),

$$\sum_0^\infty a_n u_n - S u_0 = \sum_0^\infty (\sigma_n - (n + 1) S) \Delta^2 u_n. \dots\dots\dots(5)$$

Now
$$\text{Lt}_{n \rightarrow \infty} \frac{\sigma_n}{n + 1} = S.$$

Therefore, to the arbitrary positive number ϵ , there corresponds a positive integer ν such that

$$\left| \frac{\sigma_n}{n + 1} - S \right| < \frac{\epsilon}{4K}, \text{ when } n \geq \nu.$$

Thus $|\sigma_n - (n + 1) S| < \frac{\epsilon}{4K} (n + 1), \text{ when } n \geq \nu.$

Also $|\sigma_n| < (n + 1) C$, for every positive integer, and $|S| \leq C$.

It follows, from these inequalities and (5), that

$$\begin{aligned} \left| \sum_0^\infty a_n u_n - S u_0 \right| &\leq \left| \sum_0^{\nu-1} (\sigma_n - (n + 1) S) \Delta^2 u_n \right| + \left| \sum_\nu^\infty (\sigma_n - (n + 1) S) \Delta^2 u_n \right| \\ &< 2C \sum_0^{\nu-1} (n + 1) |\Delta^2 u_n| + \frac{\epsilon}{4K} \sum_\nu^\infty (n + 1) |\Delta^2 u_n|. \dots\dots\dots(6) \end{aligned}$$

But
$$\begin{aligned} \sum_\nu^\infty (n + 1) |\Delta^2 u_n| &< 2 \sum_\nu^\infty n |\Delta^2 u_n| \\ &< 2K, \text{ by (a)}. \dots\dots\dots(7) \end{aligned}$$

And
$$\text{Lt}_{t \rightarrow +0} \Delta^2 u_n = 0, \text{ since } \text{Lt}_{t \rightarrow +0} u_n = 1, \text{ by } (\gamma).$$

It follows that, ν being fixed, there is a positive number η such that

$$|\Delta^2 u_n| < \frac{\epsilon}{2\nu(\nu + 1)C}, \text{ when } 0 < t \leq \eta \text{ and } n \leq \nu - 1. \dots\dots\dots(8)$$

Thus from (6), (7) and (8), we see that

$$\left| \sum_0^\infty a_n u_n - S u_0 \right| < \frac{1}{2} \epsilon + \frac{1}{2} \epsilon < \epsilon, \text{ when } 0 < t \leq \eta.$$

Therefore
$$\text{Lt}_{t \rightarrow +0} \left(\sum_0^\infty a_n u_n - S u_0 \right) = 0.$$

And, finally,
$$\text{Lt}_{t \rightarrow +0} \sum_0^\infty a_n u_n = S,$$

since, from $(\gamma), \text{Lt}_{t \rightarrow +0} u_0 = 1.$

III. Let the sequence of Arithmetic Means for the series $\sum_0^\infty a_n$ converge to S , and let u_n be e^{-nt} (or $e^{-n^2 t}$). Then the series $\sum_0^\infty a_n u_n$ converges, when $t > 0$, and

$$\text{Lt}_{t \rightarrow +0} \sum_0^\infty a_n u_n = S.$$

This follows at once from Bromwich's Theorem above, if e^{-nt} (or e^{-n^2t}) satisfy the conditions (α), (β) and (γ) of that theorem.

It is obvious that (β) and (γ) are satisfied, so it only remains to establish that (α) is satisfied.

(i) Let $u_n = e^{-nt}, \quad t > 0.$
 Then
$$\begin{aligned} \Delta^2 u_n &= u_n - 2u_{n+1} + u_{n+2} \\ &= e^{-nt} (1 - e^{-t})^2. \end{aligned}$$

Therefore $\Delta^2 u_n$ is positive.

Also
$$\begin{aligned} \sum_1^n n |\Delta^2 u_n| &= \sum_1^n n \Delta^2 u_n \\ &= n u_{n+2} - (n+1) u_{n+1} + u_1. \end{aligned}$$

Therefore $\sum_1^\infty n |\Delta^2 u_n| = e^{-t}$, and the condition (α) is satisfied.

(ii) Let $u_n = e^{-n^2t}, \quad t > 0.$

In this case
$$\Delta^2 u_n = e^{-(n+\theta)^2t} (4(n+\theta)^2t^2 - 2t),$$

where $0 < \theta < 2$.*

Therefore the sign of $\Delta^2 u_n$ depends upon that of $(4(n+\theta)^2t^2 - 2t)$.

It follows that it is positive or negative according as

$$2(n+\theta)t - \sqrt{2t} \geq 0.$$

Also
$$(n+\theta) > \frac{1}{\sqrt{2t}}, \quad \text{when } n > \frac{1}{\sqrt{2t}},$$

and
$$(n+\theta) < \frac{1}{\sqrt{2t}}, \quad \text{when } n+2 < \frac{1}{\sqrt{2t}}.$$

Therefore $\Delta^2 u_n$ cannot change sign more than three times for any positive value of t .

But it follows at once from the equation

$$n \Delta^2 u_n = n e^{-(n+\theta)^2t} (4(n+\theta)^2t^2 - 2t)$$

that a positive number, independent of t , can be assigned such that $n |\Delta^2 u_n|$ is less than this number for all values of n .

Hence K can be chosen so that the condition (α) is satisfied, provided that the sum of any sequence of terms, all of the same sign, that we can choose from $\sum n \Delta^2 u_n$, is less in absolute value than some fixed positive number for all values of t .

Let $\sum_r^s n \Delta^2 u_n$ be the sum of such a set of consecutive terms.

Then we have

$$\sum_r^s n \Delta^2 u_n = r e^{-r^2t} - (r-1) e^{-(r+1)^2t} - (s+1) e^{-(s+1)^2t} + s e^{-(s+2)^2t},$$

* This follows from the fact that

$$\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = f''(x+\theta h),$$

where $0 < \theta < 2$, provided that $f(x)$, $f'(x)$, $f''(x)$ are continuous from x to $x+2h$. (Cf. Goursat, *loc. cit.*, T. I., § 22.)

which differs from

$$r(e^{-r^2t} - e^{-(r+1)^2t}) - (s+1)(e^{-(s+1)^2t} - e^{-(s+2)^2t})$$

by at most unity.

But, when n is a positive integer and $t > 0$,

$$\begin{aligned} 0 < n(e^{-n^2t} - e^{-(n+1)^2t}) &= 2n(n+\theta)te^{-(n+\theta)^2t} \dots \quad (0 < \theta < 1) \\ &< 2(n+\theta)^2te^{-(n+\theta)^2t} \\ &< 2e^{-1}. \end{aligned}$$

Therefore, for the set of terms considered,

$$\sum_r^s |n\Delta^2 u_n| < 4e^{-1} + 1.$$

Then the argument above shows that the condition (a) is satisfied.

IV. Let the sequence of Arithmetic Means for the series $\sum_0^\infty a_n$ converge to S .

Then the series $\sum_0^\infty a_n x^n$ will converge when $0 < x < 1$, and

$$\text{Lt}_{x \rightarrow 1-0} \sum_0^\infty a_n x^n = S.$$

This follows from the first part of III. on putting $x = e^{-t}$.

V. Let the terms of the series $\sum_0^\infty a_n$ be functions of x , and the sequence of Arithmetic Means for this series converge uniformly to the bounded function $S(x)$ in $a \leq x \leq b$. Then $\text{Lt}_{t \rightarrow +0} \sum_0^\infty a_n u_n$ converges uniformly to $S(x)$ in this interval, provided that u_n is a function of t satisfying the conditions (a), (β) and (γ) of Bromwich's Theorem, when $t > 0$.

This follows at once by making slight changes in the argument of II.

The theorems proved in this section will be found useful in the solution of problems in Applied Mathematics, when the differential equation, which corresponds to the problem, is solved by series. The solution has to satisfy certain initial and boundary conditions. What we really need is that, as we approach the boundary, or as the time tends to zero, our solution shall have the given value as its limit. What happens upon the boundaries, or at the instant $t=0$, is not discussed. (See below § 123.)

74. Integration of Series. Infinite Integrals. Finite Interval. In the discussion of § 70 we dealt only with ordinary finite integrals. We shall now examine the question of term by term integration, both when the integrand has points of infinite discontinuity in the interval of integration, supposed finite, and when the integrand is bounded in any finite interval, but the interval of integration itself extends to infinity. In this section we shall deal with the first of these forms, and it will be sufficient to confine the discussion to the case when the infinity occurs at one end of the interval (a, b), say $x=b$.

I. Let $u_1(x), u_2(x), \dots$ be continuous in (a, b) and the series $\sum_1^\infty u_n(x)$ converge uniformly to $f(x)$ in (a, b).

Also let $g(x)$ have an infinite discontinuity at $x=b$ and $\int_a^b g(x) dx$ be absolutely convergent.*

Then
$$\int_a^b f(x) g(x) dx = \sum_1^\infty \int_a^b u_n(x) g(x) dx.$$

From the uniform convergence of $\sum_1^\infty u_n(x)$ in (a, b) , we know that its sum $f(x)$ is continuous in (a, b) , and thus bounded and integrable.

Also $\int_a^b f(x) g(x) dx$ is absolutely convergent, since $\int_a^b g(x) dx$ is so (§61, VI.).

Let
$$\int_a^b |g(x)| dx = A.$$

Then, having chosen the positive number ϵ , as small as we please, we may put

$$f(x) = S_n(x) + R_n(x),$$

where $|R_n(x)| < \frac{\epsilon}{A}$, when $n \geq \nu$, the same ν serving for all values of x in (a, b) .

But $\int_a^b f(x) g(x) dx$ and $\int_a^b S_n(x) g(x) dx$ both exist.

It follows that $\int_a^b R_n(x) g(x) dx$ also exists, and that

$$\int_a^b f(x) g(x) dx = \int_a^b S_n(x) g(x) dx + \int_a^b R_n(x) g(x) dx.$$

Thus we see that

$$\begin{aligned} \left| \int_a^b f(x) g(x) dx - \sum_1^n \int_a^b u_r(x) g(x) dx \right| &= \left| \int_a^b R_n(x) g(x) dx \right| \\ &< \frac{\epsilon}{A} \int_a^b |g(x)| dx \\ &< \epsilon, \text{ when } n \geq \nu, \end{aligned}$$

which proves that the series $\sum_1^\infty \int_a^b u_n(x) g(x) dx$ is convergent and that its sum is $\int_a^b f(x) g(x) dx$.

Ex. This case is illustrated by

$$\begin{aligned} \int_0^1 \log x \log(1+x) dx &= \sum_1^\infty (-1)^{n-1} \int_0^1 \frac{x^n}{n} \log x dx \\ &= \sum_1^\infty \frac{(-1)^n}{n(n+1)^2}, \text{ since } \int_0^1 x^n \log x dx = -\frac{1}{(n+1)^2}, \\ &= \sum_1^\infty (-1)^n \left[\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right] \\ &= 2 - 2 \log 2 - \frac{1}{12} \pi^2, \text{ using the series for } \frac{1}{12} \pi^2. \dagger \end{aligned}$$

* It is clear that this proof also applies when $g(x)$ has a finite number of infinite discontinuities in (a, b) and $\int_a^b g(x) dx$ is absolutely convergent.

† Cf. Carslaw, *Plane Trigonometry* (2nd Ed.), p. 279.

Here the series for $\log(1+x)$ converges uniformly in $0 \leqq x \leqq 1$, and

$$\int_0^1 \log x \, dx$$

converges absolutely (as a matter of fact $\log x$ is always of the same sign in $0 < x \leqq 1$), while $|\log x| \rightarrow \infty$ as $x \rightarrow 0$.

On the other hand, we may still apply term by term integration in certain cases when the above conditions are not satisfied, as will be seen from the following theorems :

II. Let $u_1(x), u_2(x), \dots$ be continuous and positive and the series $\sum_1^\infty u_n(x)$ converge uniformly to $f(x)$ in the arbitrary interval (a, a) , where $a < a < b$.

Further, let $g(x)$ be positive, bounded and integrable in (a, a) .

Then
$$\int_a^b f(x)g(x) \, dx = \sum_1^\infty \int_a^b u_n(x)g(x) \, dx,$$

provided that either the integral $\int_a^b f(x)g(x) \, dx$ or the series $\sum_1^\infty \int_a^b u_n(x)g(x) \, dx$ converges.

Let us suppose that $\int_a^b f(x)g(x) \, dx$ converges.

In other words, we are given that the repeated limit

$$\text{Lt}_{\xi \rightarrow 0} \int_a^{b-\xi} \left[\text{Lt}_{n \rightarrow \infty} \sum_1^n u_r(x) \right] g(x) \, dx \text{ exists.}$$

Since the functions $u_1(x), u_2(x), \dots$ are all positive, as well as $g(x)$, in (a, a) , from the convergence of $\int_a^b f(x)g(x) \, dx$ there follows at once the convergence of

$$\int_a^b u_r(x)g(x) \, dx \quad (r=1, 2, \dots).$$

Again, let $f(x) = u_1(x) + u_2(x) + \dots + u_n(x) + R_n(x)$.

Then $\int_a^b R_n(x)g(x) \, dx$ also converges, and for every positive integer n

$$\int_a^b f(x)g(x) \, dx - \sum_1^n \int_a^b u_r(x)g(x) \, dx = \int_a^b R_n(x)g(x) \, dx. \dots\dots\dots(1)$$

But from the convergence of $\int_a^b f(x)g(x) \, dx$ it follows that, when the arbitrary positive number ϵ has been chosen, as small as we please, there will be a positive number ξ such that

$$0 < \int_{b-\xi}^b f(x)g(x) \, dx < \frac{\epsilon}{2}.$$

A fortiori,
$$0 < \int_{b-\xi}^b R_n(x)g(x) \, dx < \frac{\epsilon}{2}, \dots\dots\dots(2)$$

and this holds for all positive integers n .

Let the upper bound of $g(x)$ in $(a, b - \xi)$ be M .

The series $\sum_1^\infty u_n(x)$ converges uniformly in $(a, b - \xi)$.

Keeping the number ϵ we have chosen above, there will be a positive integer ν such that

$$0 < R_n(x) < \frac{\epsilon}{2M(b-a)}, \text{ when } n \geq \nu,$$

the same ν serving for all values of x in $(a, b - \xi)$.

Thus

$$\begin{aligned} 0 < \int_a^{b-\xi} R_n(x) g(x) dx &< \frac{\epsilon}{2M(b-a)} \int_a^{b-\xi} g(x) dx \\ &< \frac{\epsilon}{2(b-a)} \int_a^{b-\xi} dx \\ &< \frac{\epsilon}{2}, \text{ when } n \geq \nu. \dots\dots\dots(3) \end{aligned}$$

Combining these results (2) and (3), we have

$$0 < \int_a^b R_n(x) g(x) dx < \epsilon, \text{ when } n \geq \nu.$$

Then, from (1),

$$0 < \int_a^b f(x) g(x) dx - \sum_1^n \int_a^b u_r(x) g(x) dx < \epsilon, \text{ when } n \geq \nu.$$

Therefore $\int_a^b f(x) g(x) dx = \sum_1^\infty \int_a^b u_r(x) g(x) dx.$

The other alternative, stated in the enunciation, may be treated in the same way.

Bromwich has pointed out that in this case where the terms are all positive, as well as the multiplier $g(x)$, the argument is substantially the same as that employed in dealing with the convergence of a Double Series of positive terms, and the same remark applies to the corresponding theorem in § 75.*

Ex. 1. Show that

$$\begin{aligned} \int_0^1 \frac{\log x}{1-x} dx &= \sum_0^\infty \int_0^1 x^n \log x dx + \\ &= - \sum_0^\infty \frac{1}{(n+1)^2} \\ &= - \frac{\pi^2}{6}. \end{aligned}$$

Ex. 2. Show that $\int_0^1 \log \frac{1+x}{1-x} \frac{dx}{x} = 2 \sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{4}.$

The conditions imposed upon the terms $u_1(x), u_2(x), \dots$ and $g(x)$, that they are positive, may be removed, and the following more general theorem stated :

III. Let $u_1(x), u_2(x), \dots$ be continuous and the series $\sum_1^\infty |u_n(x)|$ converge uniformly in the arbitrary interval $(a, a),$ where $a < a < b.$ Also let $\sum_1^\infty u_n(x) = f(x).$

* Cf. Bromwich, *Infinite Series*, p. 449, and *Mess. Math.*, Cambridge, **36**, p. 1, 1906.

† The interval $(0, 1)$ has to be broken up into two parts, $(0, a)$ and $(a, 1).$ In the first we use Theorem I. and in the second Theorem II. Or we may apply § 72, VI.

Further, let $g(x)$ be bounded and integrable in (a, a) .

Then
$$\int_a^b f(x)g(x)dx = \sum_1^{\infty} \int_a^b u_n(x)g(x)dx,$$

provided that either the integral $\int_a^b \sum_1^{\infty} |u_n(x)| \cdot |g(x)| dx$ or the series $\sum_1^{\infty} \int_a^b |u_n(x)| |g(x)| dx$ converge.

This can be deduced from II., using the identity :

$$u_n g = \{u_n + |u_n|\} \{g + |g|\} - \{u_n + |u_n|\} |g| - |u_n| \{g + |g|\} + |u_n| |g|,$$

since that theorem can be applied to each term on the right-hand side.

Ex. 1. Show that
$$\int_0^1 \frac{\log x}{1+x} dx = \sum_0^{\infty} (-1)^n \int_0^1 x^n \log x dx$$
$$= \sum_0^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$$
$$= -\frac{\pi^2}{12}.$$

Ex. 2. Show that

$$\int_0^1 \frac{x^p}{1+x} \log x dx = \sum_1^{\infty} (-1)^{n-1} \int_0^1 x^{n+p-1} \log x dx = \sum_1^{\infty} \frac{(-1)^n}{(n+p)^2},$$

when $p+1 > 0$.*

75. Integration of Series. Infinite Integrals. Interval Infinite.

For the second form of infinite integral we have results corresponding to the theorems proved in § 74.

I. Let $u_1(x), u_2(x), \dots$ be continuous and bounded in $x \geq a$, and let the series $\sum_1^{\infty} u_n(x)$ converge uniformly to $f(x)$ in $x \geq a$. Further, let $g(x)$ be bounded and integrable in the arbitrary interval (a, a) , where $a < a$, and a may be chosen as large as we please; and let $\int_a^{\infty} g(x) dx$ converge absolutely.

Then
$$\int_a^{\infty} f(x)g(x)dx = \sum_1^{\infty} \int_a^{\infty} u_n(x)g(x)dx.$$

The proof of this theorem follows exactly the same lines as I. of § 74.

Ex.
$$\int_1^{\infty} \frac{e^{-x}}{x} dx = \sum_1^{\infty} \int_1^{\infty} \frac{e^{-x} dx}{(x+n-1)(x+n)}.$$

This follows from the fact that the series

$$\frac{1}{x(x+1)} + \frac{1}{(x+1)(x+2)} + \dots$$

converges uniformly to $1/x$ in $x \geq 1$.

* If $p > 0$, Theorem III. can be used at once.

If $0 > p > -1$, the interval has to be broken up into two parts $(0, a)$ and $(a, 1)$. In the first we use Theorem I. and in the second Theorem III. Or we may apply § 72, VI.

But it is often necessary to justify term by term integration when either $\int_a^\infty |g(x)| dx$ is divergent or $\sum_1^\infty u_n(x)$ can only be shown to converge uniformly in the arbitrary interval (a, a) , where a can be taken as large as we please.

Many important cases are included in the following theorems, which correspond to II. and III. of § 74 :

II. Let $u_1(x), u_2(x), \dots$ be continuous and positive and the series $\sum_1^\infty u_n(x)$ converge uniformly to $f(x)$ in the arbitrary interval (a, a) , where a may be taken as large as we please.

Also let $g(x)$ be positive, bounded and integrable in (a, a) .

Then
$$\int_a^\infty f(x)g(x) dx = \sum_1^\infty \int_a^\infty u_n(x)g(x) dx,$$

provided that either the integral $\int_a^\infty f(x)g(x) dx$ or the series $\sum_1^\infty \int_a^\infty u_n(x)g(x) dx$ converge.

Let us suppose that $\int_a^\infty f(x)g(x) dx$ converges.

In other words we are given that the repeated limit

$$\text{Lt}_{a \rightarrow \infty} \int_a^a [\text{Lt}_{n \rightarrow \infty} \sum_1^n u_r(x)] g(x) dx \text{ exists.}$$

Since the terms of the series $\sum_1^\infty u_r(x)$ are all positive, as well as $g(x)$, in $x \geqq a$, from the convergence of $\int_a^\infty f(x)g(x) dx$ there follows at once that of

$$\int_a^\infty u_r(x)g(x) dx \quad (r=1, 2, \dots).$$

Again let
$$f(x) = u_1(x) + u_2(x) + \dots + u_n(x) + R_n(x).$$

Then $\int_a^\infty R_n(x)g(x) dx$ also converges, and for every positive integer n

$$\int_a^\infty f(x)g(x) dx - \sum_1^n \int_a^\infty u_r(x)g(x) dx = \int_a^\infty R_n(x)g(x) dx. \dots\dots\dots(1)$$

But from the convergence of $\int_a^\infty f(x)g(x) dx$, it follows that, when the arbitrary positive number ϵ has been chosen, as small as we please, there will be a positive number a such that

$$0 < \int_a^\infty f(x)g(x) dx < \frac{\epsilon}{2}.$$

A fortiori,
$$0 < \int_a^\infty R_n(x)g(x) dx < \frac{\epsilon}{2},$$

and this holds for all positive integers n .

With this choice of a , let the upper bound of $g(x)$ in (a, a) be M .

The series $\sum_1^\infty u_n(x)$ converges uniformly in (a, a) .

Keeping the number ϵ we have chosen above, there will be a positive integer ν such that

$$0 < R_n(x) < \frac{\epsilon}{2M(a-a)}, \text{ when } n \geq \nu,$$

the same ν serving for all values of x in (a, a) .

$$\text{Thus } 0 < \int_a^a R_n(x) g(x) dx < \frac{\epsilon}{2}, \text{ when } n \geq \nu.$$

$$\text{But } 0 < \int_a^\infty R_n(x) g(x) dx < \frac{\epsilon}{2},$$

and this holds for all values of n .

Combining these two results, we have

$$0 < \int_a^\infty R_n(x) g(x) dx < \epsilon, \text{ when } n \geq \nu,$$

and from (1),

$$0 < \int_a^\infty f(x) g(x) dx - \sum_1^n \int_a^\infty u_r(x) g(x) dx < \epsilon, \text{ when } n \geq \nu.$$

$$\text{Therefore } \int_a^\infty f(x) g(x) dx = \sum_1^\infty \int_a^\infty u_r(x) g(x) dx.$$

The other alternative can be treated in the same way.

$$\text{Ex. 1. } \int_0^\infty e^{-ax} \cosh bx dx = \sum_0^\infty \frac{b^{2n}}{2n!} \int_0^\infty e^{-ax} x^{2n} dx \text{ if } 0 < |b| < a.$$

$$\text{Ex. 2. } \int_0^\infty e^{-a^2x^2} \cosh bx dx = \sum_0^\infty \frac{b^{2n}}{2n!} \int_0^\infty e^{-a^2x^2} x^{2n} dx.$$

Further, the conditions imposed upon $u_1(x)$, $u_2(x)$, ... and $g(x)$, that they shall be positive, may be removed, leading to the theorem :

III. Let $u_1(x)$, $u_2(x)$, ... be continuous and the series $\sum_1^\infty |u_n(x)|$ converge formly in the arbitrary interval (a, a) , where a may be taken as large as please. Also let $\sum_1^\infty u_n(x) = f(x)$.

Further, let $g(x)$ be bounded and integrable in (a, a) .

$$\text{Then } \int_a^\infty f(x) g(x) dx = \sum_1^\infty \int_a^\infty u_n(x) g(x) dx,$$

provided that either the integral $\int_a^\infty \sum_1^\infty |u_n(x)| |g(x)| dx$ or the series

$$\sum_1^\infty \int_a^\infty |u_n(x)| |g(x)| dx \text{ converges.}$$

This is deduced, as before, from the identity

$$u_n g = \{u_n + |u_n|\} \{g + |g|\} - \{u_n + |u_n|\} |g| - |u_n| \{g + |g|\} + |u_n| |g|,$$

since the Theorem II. can be applied to each term in the right-hand side.

$$\text{Ex. 1. Show that } \int_0^\infty e^{-ax} \cos bx dx = \sum_0^\infty \frac{(-1)^n b^{2n}}{2n!} \int_0^\infty e^{-ax} x^{2n} dx, \text{ if } 0 < |b| < a$$

Ex. 2. Show that $\int_0^{\infty} e^{-a^2x^2} \cos bx \, dx = \sum_0^{\infty} \frac{(-1)^n b^{2n}}{2n!} \int_0^{\infty} e^{-a^2x^2} x^{2n} \, dx$.

76. Certain cases to which the theorems of § 75 do not apply are covered by the following test :

Let $u_1(x), u_2(x), \dots$ be continuous in $x \geq a$, and let the series $\sum_1^{\infty} u_n(x)$ converge uniformly to $f(x)$ in the arbitrary interval (a, a) , where a may be taken as large as we please.

Further, let the integrals

$$\int_a^{\infty} u_1(x) \, dx, \int_a^{\infty} u_2(x) \, dx, \text{ etc.},$$

converge, and the series of integrals

$$\int_a^x u_1(x) \, dx + \int_a^x u_2(x) \, dx + \dots$$

converge uniformly in $x \geq a$.

Then the series of integrals

$$\int_a^{\infty} u_1(x) \, dx + \int_a^{\infty} u_2(x) \, dx + \dots$$

converges, and the integral $\int_a^{\infty} f(x) \, dx$ converges.

Also $\int_a^{\infty} f(x) \, dx = \int_a^{\infty} u_1(x) \, dx + \int_a^{\infty} u_2(x) \, dx + \dots$

Let $S_n(x) = \int_a^x u_1(x) \, dx + \int_a^x u_2(x) \, dx + \dots + \int_a^x u_n(x) \, dx$.

Then we know that $\text{Lt}_{n \rightarrow \infty} S_n(x)$ exists in $x \geq a$, and we denote it by $F(x)$.

Also we know that $\text{Lt}_{x \rightarrow \infty} S_n(x)$ exists, and we denote it by $G(n)$.

We shall now show that $\text{Lt}_{x \rightarrow \infty} F(x)$ and $\text{Lt}_{n \rightarrow \infty} G(n)$ both exist, and that the two limits are equal.

From this result our theorem as to term by term integration will follow at once.

I. To prove $\text{Lt}_{x \rightarrow \infty} F(x)$ exists.

Since $\text{Lt}_{n \rightarrow \infty} S_n(x)$ converges uniformly to $F(x)$ in $x \geq a$, with the usual notation, we have

$$|F(x) - S_n(x)| < \frac{\epsilon}{3}, \text{ when } n \geq \nu,$$

the same ν serving for every x greater than or equal to a .

Choose some value of n in this range.

Then we have $\text{Lt}_{x \rightarrow \infty} S_n(x) = G(n)$.

Therefore we can choose X so that

$$|S_n(x'') - S_n(x')| < \frac{\epsilon}{3}, \text{ when } x'' > x' \geq X > a.$$

$$\begin{aligned}
\text{But} \quad & |F(x'') - F(x')| \\
& \leq |F(x'') - S_n(x'')| + |S_n(x'') - S_n(x')| + |S_n(x') - F(x')| \\
& < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
& < \epsilon, \text{ when } x'' > x' \geq X > a.
\end{aligned}$$

Thus $F(x)$ has a limit as $x \rightarrow \infty$.

II. To prove $\text{Lt}_{n \rightarrow \infty} G(n)$ exists.

Since $\text{Lt}_{n \rightarrow \infty} S_n(x)$ converges uniformly to $F(x)$ in $x \geq a$,

$$|S_{n''}(x) - S_{n'}(x)| < \frac{\epsilon}{3}, \text{ when } n'' > n' \geq \nu,$$

the same ν serving for every x greater than or equal to a .

But $\text{Lt}_{x \rightarrow \infty} S_n(x) = G(n)$.

Therefore we can choose X_1, X_2 such that

$$|G(n') - S_{n'}(x)| < \frac{\epsilon}{3}, \text{ when } x \geq X_1 > a,$$

$$|G(n'') - S_{n''}(x)| < \frac{\epsilon}{3}, \text{ when } x \geq X_2 > a.$$

Then, taking a value of x not less than X_1 or X_2 ,

$$\begin{aligned}
& |G(n'') - G(n')| \\
& \leq |G(n'') - S_{n''}(x)| + |S_{n''}(x) - S_{n'}(x)| + |S_{n'}(x) - G(n')| \\
& < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
& < \epsilon, \text{ when } n'' > n' \geq \nu.
\end{aligned}$$

Therefore $\text{Lt}_{n \rightarrow \infty} G(n)$ exists.

III. To prove $\text{Lt}_{x \rightarrow \infty} F(x) = \text{Lt}_{n \rightarrow \infty} G(n)$.

Since $\text{Lt}_{n \rightarrow \infty} G(n)$ exists, we can choose ν_1 so that

$$\left| \sum_{n+1}^{\infty} \int_a^{\infty} u_r(x) dx \right| < \frac{\epsilon}{3}, \text{ when } n \geq \nu_1.$$

Again $\sum_1^{\infty} \int_a^x u_r(x) dx$ converges uniformly to $F(x)$ in $x \geq a$.

Therefore we can choose ν_2 so that

$$\left| \sum_{n+1}^{\infty} \int_a^x u_r(x) dx \right| < \frac{\epsilon}{3}, \text{ when } n \geq \nu_2,$$

the same ν_2 serving for every x greater than or equal to a .

Choose ν not less than ν_1 or ν_2 .

From the convergence of the integral $\int_a^{\infty} \sum_1^{\nu} u_r(x) dx$, we can choose X so that

$$\left| \int_x^{\infty} \sum_1^{\nu} u_r(x) dx \right| < \frac{\epsilon}{3}, \text{ when } x \geq X > a.$$

$$\begin{aligned}
\text{But} \quad & \left| \sum_1^{\infty} \int_a^{\infty} u_r(x) dx - F(x) \right| \\
& = \left| \int_x^{\infty} \sum_1^{\nu} u_r(x) dx + \sum_{\nu+1}^{\infty} \int_a^{\infty} u_r(x) dx - \sum_{\nu+1}^{\infty} \int_a^x u_r(x) dx \right| \\
& \leq \left| \int_x^{\infty} \sum_1^{\nu} u_r(x) dx \right| + \left| \sum_{\nu+1}^{\infty} \int_a^{\infty} u_r(x) dx \right| + \left| \sum_{\nu+1}^{\infty} \int_a^x u_r(x) dx \right| \\
& < \frac{\epsilon}{3} \qquad \qquad \qquad + \frac{\epsilon}{3} \qquad \qquad \qquad + \frac{\epsilon}{3}, \\
& < \epsilon, \text{ when } x \geq X > a.
\end{aligned}$$

$$\text{Therefore} \quad \text{Lt}_{x \rightarrow \infty} F(x) = \sum_1^{\infty} \int_a^{\infty} u_r(x) dx = \text{Lt}_{n \rightarrow \infty} G(n).$$

IV. But we are given that the series

$$u_1(x) + u_2(x) + \dots$$

converges uniformly to $f(x)$ in any arbitrary interval (a, α) .

Therefore we have

$$\int_a^x f(x) dx = \int_a^x u_1(x) dx + \int_a^x u_2(x) dx + \dots \text{ in } x \leq \alpha.$$

Thus, with the above notation,

$$F(x) = \int_a^x f(x) dx.$$

It follows from I. that $\int_a^{\infty} f(x) dx$ converges, and from III. that

$$\int_a^{\infty} f(x) dx = \int_a^{\infty} u_1(x) dx + \int_a^{\infty} u_2(x) dx + \dots$$

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EXAMPLES ON CHAPTER V.

UNIFORM CONVERGENCE.

1. Examine the convergency of the series

$$\sum_1^{\infty} \frac{x}{(nx+1)[(n+1)x+1][(n+2)x+1]}, \quad x \geq 0,$$

and by its means illustrate the effect of non-uniform convergence upon the continuity of a function of x represented by an infinite series.

2. Prove that the series

$$x^2 + \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \dots$$

is convergent for all values of x , but is not uniformly convergent in an interval including the origin.

3. Determine whether the series

$$\sum_1^{\infty} \frac{1}{n^3 + n^4 x^2}$$

is uniformly convergent for all values of x .

4. Find for what values of x the series $\sum_1^{\infty} u_n$ converges where

$$\begin{aligned} u_n &= \frac{x^n - x^{-n-1}}{(x^n + x^{-n})(x^{n+1} + x^{-n-1})} \\ &= \frac{1}{(x-1)(x^n + x^{-n})} - \frac{1}{(x-1)(x^{n+1} + x^{-(n+1)})}. \end{aligned}$$

Find also whether the series is

- (i) uniformly convergent through an interval including $+1$;
- (ii) continuous when x passes through the value $+1$.

5. Discuss the uniformity or non-uniformity of the convergence of the series whose general term is

$$u_n = \frac{1 - (1+x)^n}{1 + (1+x)^n} - \frac{1 - (1+x)^{n-1}}{1 + (1+x)^{n-1}}.$$

6. Let

$$a_0 + a_1 + \dots$$

be an absolutely convergent series of constant terms, and let

$$f_0(x), f_1(x), \dots$$

be a set of functions each continuous in the interval $a \leq x \leq \beta$, and each comprised between certain fixed limits,

$$A \leq f_i(x) \leq B, \quad i=0, 1, \dots,$$

where A, B are constants. Show that the series

$$a_0 f_0(x) + a_1 f_1(x) + \dots$$

represents a continuous function of x in the interval $a \leq x \leq \beta$.

7. Show that the function defined by the series

$$\sum_1^{\infty} \frac{x}{n(1+nx^2)}$$

is finite and continuous for all values of x . Examine whether the series is uniformly convergent for all such values.

8. Show that if $u_0(x) + u_1(x) + \dots$

is a series of functions each continuous and having no roots in the interval $a \leq x \leq b$, and if

$$\left| \frac{u_{n+1}(x)}{u_n(x)} \right| \leq \gamma < 1, \text{ when } n \geq \nu,$$

where γ, ν do not depend on x , then the given series is uniformly convergent in this interval.

Apply this test to the series

$$1 + xa + x(x-1)\frac{a^2}{2!} + x(x-1)(x-2)\frac{a^3}{3!} + \dots,$$

where $0 < a < 1$.

THE POWER SERIES.

9. From the equation $\sin^{-1}x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$,

show that the series for $\sin^{-1}x$ is

$$x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \dots \text{ when } |x| < 1.$$

Prove that the expansion is also valid when $|x|=1$.*

10. From the equation $\tan^{-1}x = \int_0^x \frac{dx}{1+x^2}$,

obtain Gregory's Series,

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$$

Within what range of x does this hold?

11. Show that we may substitute the series for $\sin^{-1}x$ and $\tan^{-1}x$ in the integrals

$$\int_0^1 \frac{\sin^{-1}x}{x} dx \quad \text{and} \quad \int_0^1 \frac{\tan^{-1}x}{x} dx,$$

and integrate term by term, when $|x| < 1$.

Also show that

$$\int_0^1 \frac{\sin^{-1}x}{x} dx = \sum_0^{\infty} \frac{1 \cdot 3 \dots 2n-1}{2 \cdot 4 \dots 2n} \frac{1}{(2n+1)^2}, \quad \int_0^1 \frac{\tan^{-1}x}{x} dx = \sum_0^{\infty} (-1)^n \frac{1}{(2n+1)^2},$$

and from the integration by parts of $\frac{\sin^{-1}x}{x}$ and $\frac{\tan^{-1}x}{x}$, prove that these series also represent

$$\int_0^1 \frac{|\log x|}{\sqrt{1-x^2}} dx \quad \text{and} \quad \int_0^1 \frac{|\log x|}{1+x^2} dx.$$

* When $x=1$, $\text{Lt}_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$, but Raabe's test (cf. Bromwich, *loc. cit.*, p. 35, or Goursat, *loc. cit.*, p. 404) shows that the series converges for this value of x .

12. If $|x| < 1$, prove that

$$\int_0^x \tan^{-1} x \, dx = \frac{x^2}{1 \cdot 2} - \frac{x^4}{3 \cdot 4} + \frac{x^6}{5 \cdot 6} - \dots$$

Show that the result also holds for $x=1$, and deduce that

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \dots = 0.43882\dots$$

13. If $|x| < 1$, prove that

$$\int_0^x \log(1+x) \, dx = \frac{x^2}{1 \cdot 2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} - \dots$$

Does the result hold for $x = \pm 1$?

14. Prove that

$$\int_0^x \log(1-x) \frac{dx}{x} = -\sum_1^{\infty} \frac{x^n}{n^2}, \quad 0 < x < 1.$$

$$\int_0^x \log(1+x) \frac{dx}{x} = \sum_1^{\infty} (-1)^{n-1} \frac{x^n}{n^2}, \quad 0 < x < 1.$$

$$\int_0^x \log\left(\frac{1+x}{1-x}\right) \frac{dx}{x} = 2 \sum_1^{\infty} \frac{x^{2n-1}}{(2n-1)^2}, \quad 0 < x < 1.$$

Express the integrals

$$\int_0^1 \log(1-x) \frac{dx}{x}, \quad \int_0^1 \log(1+x) \frac{dx}{x} \quad \text{and} \quad \int_0^1 \log\left(\frac{1+x}{1-x}\right) \frac{dx}{x}$$

as infinite series.

15. Show that

$$1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} + \frac{1}{13} - \frac{1}{17} - \frac{1}{19} + \dots$$

$$= \int_0^1 \frac{1-x^4}{1+x^6} dx$$

$$= \frac{1}{\sqrt{3}} \log(2+\sqrt{3}).$$

INTEGRATION AND DIFFERENTIATION OF SERIES.

16. Prove that the series $\sum_1^{\infty} (ae^{-nax} - \beta e^{-n\beta x})$ is uniformly convergent in (c, γ) , where a, β, γ and c are positive and $c < \gamma$.

Verify that

$$\int_c^{\gamma} \sum_1^{\infty} (ae^{-nax} - \beta e^{-n\beta x}) \, dx = \sum_1^{\infty} \int_c^{\gamma} (ae^{-nax} - \beta e^{-n\beta x}) \, dx.$$

Is

$$\int_0^{\gamma} \sum_1^{\infty} (ae^{-nax} - \beta e^{-n\beta x}) \, dx = \sum_1^{\infty} \int_0^{\gamma} (ae^{-nax} - \beta e^{-n\beta x}) \, dx?$$

17. If it be given that for values of x between 0 and π ,

$$\pi \cosh ax = 2 \sinh a\pi \left\{ \frac{1}{2a} - \frac{a \cos x}{1+a^2} + \frac{a \cos 2x}{2^2+a^2} - \dots \right\},$$

prove rigorously that

$$\pi \sinh ax = 2 \sinh a\pi \left\{ \frac{\sin x}{1+a^2} - \frac{2 \sin 2x}{2^2+a^2} + \frac{3 \sin 3x}{3^2+a^2} - \dots \right\}.$$

18. Show that if $f(x) = \sum_1^{\infty} \frac{1}{n^3 + n^4 x^2}$, then it has a differential coefficient equal to $-2x \sum_1^{\infty} \frac{1}{n^2(1 + nx^2)^2}$ for all values of x .

19. When a stands for a positive number, then the series

$$\sum_0^{\infty} \frac{1}{r!} \frac{a^{-r}}{a^{-2r} + x^2}, \quad \sum_1^{\infty} \frac{(-1)^r}{r!} \frac{a^{-r}}{a^{-2r} + x^2},$$

are uniformly convergent for all values of x ; and if their sums are $f(x)$ and $F(x)$ respectively,

$$f'(x) = \sum_0^{\infty} \frac{1}{r!} \frac{d}{dx} \left(\frac{a^{-r}}{a^{-2r} + x^2} \right),$$

$$F'(x) = \sum_0^{\infty} \frac{1}{r!} \frac{d}{dx} \left(\frac{(-1)^r a^{-r}}{a^{-2r} + x^2} \right).$$

20. Find all the values of x for which the series

$$e^x \sin x + e^{2x} \sin 2x + \dots$$

converges. Does it converge uniformly for these values? For what values of x can the series be differentiated term by term?

21.* Show that the series

$$2xe^{-x^2} = \sum_1^{\infty} 2x \left(\frac{1}{n^2} e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2} e^{-\frac{x^2}{(n+1)^2}} \right)$$

can be integrated term by term between any two finite limits. Can the function defined by the series be integrated between the limits 0 and ∞ ? If so, is the value of this integral given by integrating the series term by term between these limits?

22. If each of the terms of the series

$$u_1(x) + u_2(x) + \dots$$

is a continuous function of x in $x \geq a > 0$, and if the series

$$x^\kappa u_1(x) + x^\kappa u_2(x) + \dots \quad (\kappa > 1)$$

satisfies the M -test (§ 67), then the original series may be integrated term by term from a to ∞ .

23. Show that the series

$$\frac{1}{(1+x)^2} + \frac{1}{(2+x)^2} + \dots \quad (x \geq 0)$$

can be integrated term by term between any two positive finite limits. Can this series be integrated term by term between the limits 0 and ∞ ? Show that the function defined by the series cannot be integrated between these limits.

24. Show that the function defined by the series

$$\frac{1}{(1+x)^3} + \frac{1}{(2+x)^3} + \dots$$

can be integrated from 0 to ∞ , and that its value is given by the term by term integration of the series.

* Ex. 21-27 depend upon the theorems of §§ 74-76.

25. Prove that

$$\int_0^1 \frac{x^{a-1}}{1+x} dx = \frac{1}{a} - \frac{1}{a+1} + \frac{1}{a+2} - \dots \quad (a > 0).$$

Explain the nature of the difficulties involved in your proof, and justify the process you have used.

26. By expansion in powers of a , prove that, if $|a| < 1$,

$$\int_0^\infty e^{-x}(1 - e^{-ax}) \frac{dx}{x} = \log(1+a),$$

$$\int_0^{\frac{1}{2}\pi} \tan^{-1}(a \sin x) \frac{dx}{\sin x} = \frac{1}{2}\pi \sinh^{-1} a,$$

examining carefully the legitimacy of term by term integration in each case.

27. Assuming that

$$J_0(bx) = \sum_0^\infty \frac{(-1)^n \left(\frac{bx}{2}\right)^{2n}}{(n!)^2},$$

show that

$$\int_0^\infty e^{-ax} J_0(bx) dx = \frac{1}{\sqrt{a^2 + b^2}},$$

when $a > 0$.

CHAPTER VI.

DEFINITE INTEGRALS CONTAINING AN ARBITRARY PARAMETER.

77. Continuity of the Integral. Finite Interval. In the ordinary definite integral $\int_a^{a'} \phi(x, y) dx$ let a, a' be constants. Then the integral will be a function of y .*

The properties of such integrals will be found to correspond very closely to those of infinite series whose terms are functions of a single variable. Indeed this chapter will follow almost the same lines as the preceding one, in which such infinite series were treated.

I. *If $\phi(x, y)$ is a continuous function of (x, y) in the region*

$$a \leq x \leq a', \quad b \leq y \leq b',$$

then $\int_a^{a'} \phi(x, y) dx$ is a continuous function of y in the interval (b, b') .

Since $\phi(x, y)$ is a continuous function of $(x, y)^\dagger$, as defined in § 37, it is also a continuous function of x and a continuous function of y .

Thus $\phi(x, y)$ is integrable with regard to x .

Let
$$f(y) = \int_a^{a'} \phi(x, y) dx.$$

* As already remarked in § 62, it is understood that before proceeding to the limit involved in the integration, the value of y , for which the integral is required, is to be inserted in the integrand.

† When a function of two variables x, y is continuous with respect to the two variables, as defined in § 37, we speak of it as a continuous function of (x, y) .

It will be noticed that we do not use the full consequences of this continuity in the following argument.

We know that, since $\phi(x, y)$ is a continuous function of (x, y) in the given region, to any positive number ϵ , chosen as small as we please, there corresponds a positive number η such that

$$|\phi(x, y + \Delta y) - \phi(x, y)| < \epsilon, \text{ when } |\Delta y| \leq \eta,$$

the same η serving for all values of x in (a, a') .*

Let Δy satisfy this condition, and write

$$f(y + \Delta y) = \int_a^{a'} \phi(x, y + \Delta y) dx.$$

$$\text{Then } f(y + \Delta y) - f(y) = \int_a^{a'} [\phi(x, y + \Delta y) - \phi(x, y)] dx.$$

Therefore

$$\begin{aligned} |f(y + \Delta y) - f(y)| &\leq \int_a^{a'} |\phi(x, y + \Delta y) - \phi(x, y)| dx \\ &< (a' - a)\epsilon, \text{ when } |\Delta y| \leq \eta. \end{aligned}$$

Thus $f(y)$ is continuous in the interval (b, b') .

II. If $\phi(x, y)$ is a continuous function of (x, y) in $a \leq x \leq a'$, $b \leq y \leq b'$, and $\psi(x)$ is bounded and integrable in (a, a') , then $\int_a^{a'} \phi(x, y) \psi(x) dx$ is a continuous function of y in (b, b') .

$$\text{Let } f(y) = \int_a^{a'} \phi(x, y) \psi(x) dx.$$

The integral exists, since the product of two integrable functions is integrable.

Also, with the same notation as in I.,

$$f(y + \Delta y) - f(y) = \int_a^{a'} [\phi(x, y + \Delta y) - \phi(x, y)] \psi(x) dx.$$

Let M be the upper bound of $|\psi(x)|$ in (a, a') .

$$\text{Then } |f(y + \Delta y) - f(y)| < M(a' - a)\epsilon, \text{ when } |\Delta y| \leq \eta.$$

Thus $f(y)$ is continuous in (b, b') .

78. Differentiation of the Integral.

I. Let $f(y) = \int_a^{a'} \phi(x, y) dx$, where $\phi(x, y)$ is a continuous function of (x, y) in $a \leq x \leq a'$, $b \leq y \leq b'$, and $\frac{\partial \phi}{\partial y}$ exists and satisfies the same condition.

* This follows from the theorem on the uniform continuity of a continuous function (cf. § 37, p. 75).

Then $f'(y)$ exists and is equal to $\int_a^{a'} \frac{\partial \phi}{\partial y} dx$.

Since $\frac{\partial \phi}{\partial y}$ is a continuous function of (x, y) in the given region, to any positive number ϵ , chosen as small as we please, there corresponds a positive number η , such that, with the usual notation,

$$\left| \frac{\partial \phi(x, y + \Delta y)}{\partial y} - \frac{\partial \phi(x, y)}{\partial y} \right| < \epsilon, \text{ when } |\Delta y| \leq \eta,$$

the same η serving for all values of x in (a, a') .

Let Δy satisfy this condition.

Then

$$\begin{aligned} \frac{f(y + \Delta y) - f(y)}{\Delta y} &= \int_a^{a'} \frac{\phi(x, y + \Delta y) - \phi(x, y)}{\Delta y} dx \\ &= \int_a^{a'} \frac{\partial \phi(x, y + \theta \Delta y)}{\partial y} dx, \text{ where } 0 < \theta < 1, \\ &= \int_a^{a'} \frac{\partial \phi(x, y)}{\partial y} dx + \int_a^{a'} \left[\frac{\partial \phi(x, y + \theta \Delta y)}{\partial y} - \frac{\partial \phi(x, y)}{\partial y} \right] dx. \end{aligned}$$

Thus we have

$$\begin{aligned} \left| \frac{f(y + \Delta y) - f(y)}{\Delta y} - \int_a^{a'} \frac{\partial \phi}{\partial y} dx \right| &= \left| \int_a^{a'} \left[\frac{\partial \phi(x, y + \theta \Delta y)}{\partial y} - \frac{\partial \phi(x, y)}{\partial y} \right] dx \right| \\ &< (a' - a)\epsilon, \text{ when } |\Delta y| \leq \eta. \end{aligned}$$

And this establishes that $\text{Lt}_{\Delta y \rightarrow 0} \left\{ \frac{f(y + \Delta y) - f(y)}{\Delta y} \right\}$ exists and is equal to $\int_a^{a'} \frac{\partial \phi}{\partial y} dx$ at any point in (b, b') .

II. Let $f(y) = \int_a^{a'} \phi(x, y) \psi(x) dx$, where $\phi(x, y)$ and $\frac{\partial \phi}{\partial y}$ are as in I., and $\psi(x)$ is bounded and integrable in (a, a') .

Then $f'(y)$ exists and is equal to $\int_a^{a'} \frac{\partial \phi}{\partial y} \psi(x) dx$.

Let the upper bound of $|\psi(x)|$ in (a, a') be M .

Then we find, as above, that

$$\left| \frac{f(y + \Delta y) - f(y)}{\Delta y} - \int_a^{a'} \frac{\partial \phi}{\partial y} \psi(x) dx \right| < M(a' - a)\epsilon, \text{ when } |\Delta y| \leq \eta.$$

And the result follows.

The theorems of this section show that if we have to differentiate the integral $\int_a^{a'} F(x, y) dx$, where $F(x, y)$ is of the form $\phi(x, y)$ or $\phi(x, y) \psi(x)$, and these functions satisfy the conditions named above, we may put the symbol of differentiation under the integral sign. In other words, we may reverse the order of the two limiting operations involved without affecting the result.

It will be noticed that so far we are dealing with ordinary integrals. The interval of integration is finite, and the function has no points of infinite discontinuity in the interval.

79. Integration of the Integral.

Let $f(y) = \int_a^{a'} \phi(x, y) \psi(x) dx$, where $\phi(x, y)$ is a continuous function of (x, y) in $a \leq x \leq a'$, $b \leq y \leq b'$, and $\psi(x)$ is bounded and integrable in (a, a') .

$$\text{Then} \quad \int_{y_0}^y f(y) dy = \int_a^{a'} dx \int_{y_0}^y \phi(x, y) \psi(x) dy,$$

where y_0, y are any two points in (b, b') .

$$\text{Let} \quad \Phi(x, y) = \int_b^y \phi(x, y) dy.$$

$$\text{Then we know that} \quad \frac{\partial \Phi}{\partial y} = \phi(x, y) \quad [\S 49],$$

and it is easy to show that $\Phi(x, y)$ is a continuous function of (x, y) in the region $a \leq x \leq a'$, $b \leq y \leq b'$.

$$\text{Now let} \quad g(y) = \int_a^{a'} \Phi(x, y) \psi(x) dx.$$

From § 78, we know that

$$\begin{aligned} g'(y) &= \int_a^{a'} \frac{\partial \Phi}{\partial y} \psi(x) dx \\ &= \int_a^{a'} \phi(x, y) \psi(x) dx. \end{aligned}$$

Also $g'(y)$ is continuous in the interval (b, b') by § 77.

$$\text{Therefore} \quad \int_{y_0}^y g'(y) dy = \int_{y_0}^y dy \int_a^{a'} \phi(x, y) \psi(x) dx,$$

where y_0, y are any two points in (b, b') .

$$\begin{aligned}
\text{Thus } \int_{y_0}^y dy \int_a^{a'} \phi(x, y) \psi(x) dx \\
&= g(y) - g(y_0) \\
&= \int_a^{a'} [\Phi(x, y) - \Phi(x, y_0)] \psi(x) dx \\
&= \int_a^{a'} \left[\int_b^y \phi(x, y) dy - \int_b^{y_0} \phi(x, y) dy \right] \psi(x) dx \\
&= \int_a^{a'} dx \int_{y_0}^y \phi(x, y) \psi(x) dy.
\end{aligned}$$

Thus we have shown that we may invert the order of integration with respect to x and y in the repeated integral

$$\int_a^{a'} dx \int_{y_0}^y F(x, y) dy,$$

when the integrand satisfies the above conditions; and in particular, since we may put $\psi(x) = 1$, when $F(x, y)$ is a continuous function of (x, y) in the region with which the integral deals.

80. In the preceding sections of this chapter the intervals (a, a') and (b, b') have been supposed finite, and the integrand bounded in $a \leq x \leq a'$, $b \leq y \leq b'$. The argument employed does not apply to infinite integrals.

For example, the infinite integral

$$f(y) = \int_0^{\infty} ye^{-xy} dx$$

converges when $y \geq 0$, but it is discontinuous at $y = 0$, since $f(y) = 1$ when $y > 0$, and $f(0) = 0$.

Similarly
$$\int_0^{\infty} \sin \pi y e^{-x \sin^2 \pi y} dx$$

converges for all values of y , but it is discontinuous for every positive and negative integral value of y , as well as for $y = 0$.

Under what conditions then, it may be asked, will the infinite integrals

$$\int_a^{\infty} F(x, y) dx \quad \text{and} \quad \int_a^{a'} F(x, y) dx,$$

if convergent when $b \leq y \leq b'$, define continuous functions of y in (b, b') ? And when can we differentiate and integrate under the sign of integration?

In the case of infinite series, we have met with the same questions and partly answered them [cf. §§ 68, 70, 71]. We proceed to discuss them for both types of infinite integral. The discussion requires the definition of the form of convergence of infinite integrals which corresponds to uniform convergence in infinite series.

81. Uniform Convergence of Infinite Integrals. We deal first with the convergent infinite integral

$$\int_a^\infty F(x, y) dx,$$

where $F(x, y)$ is bounded in the region $a \leq x \leq a'$, $b \leq y \leq b'$, the number a' being arbitrary.

I. *The integral $\int_a^\infty F(x, y) dx$ is said to converge uniformly to its value $f(y)$ in the interval (b, b') , if, any positive number ϵ having been chosen, as small as we please, there is a positive number X such that*

$$\left| f(y) - \int_a^x F(x, y) dx \right| < \epsilon, \text{ when } x \geq X,$$

the same X serving for every y in (b, b') .

And, just as in the case of infinite series, we have a useful test for uniform convergence, corresponding to the general principle of convergence (§ 15):

II. *A necessary and sufficient condition for the uniform convergence of the integral $\int_a^\infty F(x, y) dx$ in the interval (b, b') is that, if any positive number ϵ has been chosen, as small as we please, there shall be a positive number X such that*

$$\left| \int_{x'}^{x''} F(x, y) dx \right| < \epsilon, \text{ when } x'' > x' \geq X,$$

the same X serving for every y in (b, b') .

The proof that II. forms a necessary and sufficient condition for the uniform convergence of the integral, as defined in I., follows exactly the same lines as the proof in § 66 for the corresponding theorem in infinite series.

Further, it will be seen that if $\int_a^\infty F(x, y) dx$ converges uniformly in (b, b') , to the arbitrary positive number ϵ there corresponds a positive number X such that

$$\left| \int_x^\infty F(x, y) dx \right| < \epsilon, \text{ when } x \geq X,$$

the same X serving for every y in (b, b') .

The definition and theorem given above correspond exactly to those for the series $u_1(x) + u_2(x) + \dots$, uniformly convergent in $a \leq x \leq b$; namely,

$$|R_n(x)| < \epsilon, \text{ when } n \geq \nu,$$

and $|{}_p R_n(x)| < \epsilon$, when $n \geq \nu$, for every positive integer p , the same ν serving for every value of x in the interval (a, b) .

82. Uniform Convergence of Infinite Integrals (continued).

We now consider the convergent infinite integral

$$\int_a^{a'} F(x, y) dx,$$

where the interval (a, a') is finite, but the integrand is not bounded in the region $a \leq x \leq a', b \leq y \leq b'$. This case is more complex than the preceding, since the points of infinite discontinuity can be distributed in more or less complicated fashion over the given region. We shall confine ourselves in our definition, and in the theorems which follow, to the simplest case, which is also the most important, where the integrand $F(x, y)$ has points of infinite discontinuity only on certain lines $x = a_1, a_2, \dots, a_n$ ($a \leq a_1 < a_2 < \dots < a_n \leq a'$), and is bounded in the given region, except in the neighbourhood of these lines.

This condition can be realised in two different ways. The infinities may be at isolated points, or they may be distributed right along the lines.

$$E.g. \text{ (i) } \int_0^a \frac{dx}{\sqrt{(x+y)}}, \text{ when } 0 \leq y \leq b.$$

$$\text{(ii) } \int_0^1 x^{y-1} e^{-x} dx, \text{ when } 0 \leq y < 1.$$

In the first of these integrals there is a single infinity in the given region, at the origin; in the second, there are infinities right along the line $x = 0$ from the origin up to but not including $y = 1$.

In the definitions and theorems which follow there is no need for any distinction between the two cases.

Consider, first of all, the convergent integral

$$\int_a^{a'} F(x, y) dx,$$

where $F(x, y)$ has points of infinite discontinuity on $x = a'$, and is bounded in $a \leq x \leq a' - \xi$, $b \leq y \leq b'$, where $a < a' - \xi < a'$.

For this integral we have the following definition of uniform convergence:

I. *The integral $\int_a^{a'} F(x, y) dx$ is said to converge uniformly to its value $f(y)$ in the interval (b, b') , if, any positive number ϵ having been chosen, as small as we please, there is a positive number η such that*

$$\left| f(y) - \int_a^{a' - \xi} F(x, y) dx \right| < \epsilon, \text{ when } 0 < \xi \leq \eta,$$

the same η serving for every y in (b, b') .

And, from this definition, the following test for uniform convergence can be established as before:

II. *A necessary and sufficient condition for the uniform convergence of the integral $\int_a^{a'} F(x, y) dx$ in the interval (b, b') is that, if any positive number ϵ has been chosen, as small as we please, there shall be a positive number η such that*

$$\left| \int_{a' - \xi'}^{a' - \xi''} F(x, y) dx \right| < \epsilon, \text{ when } 0 < \xi'' < \xi' \leq \eta,$$

the same η serving for every y in (b, b') .

Also we see that if this infinite integral is uniformly convergent in (b, b') , to the arbitrary positive number ϵ there will correspond a positive number η such that

$$\left| \int_{a' - \xi}^{a'} F(x, y) dx \right| < \epsilon, \text{ when } 0 < \xi \leq \eta,$$

the same η serving for every y in (b, b) .

The definition, and the above condition, require obvious modifications when the points of infinite discontinuity lie on $x = a$, instead of $x = a'$.

And when they lie on lines $x = a_1, a_2, \dots, a_n$ in the given region, by the definition of the integral it can be broken up into several others in which $F(x, y)$ has points of infinite discontinuity only at one of the limits.

In this case the integral is said to converge uniformly in (b, b') , when each of these integrals converges uniformly in that interval.

And, as before, if the integrand $F(x, y)$ has points of infinite discontinuity on $x = a_1, a_2, \dots, a_n$, and we are dealing with the integral $\int_a^\infty F(x, y) dx$, this integral must be broken up into several integrals of the preceding type, followed by an integral of the form discussed in §81.

The integral is now said to be uniformly convergent in (b, b') when the integrals into which it has been divided are each uniformly convergent in this interval.

83. Tests for Uniform Convergence. The simplest test for the uniform convergence of the integral $\int_a^\infty F(x, y) dx$, taking the first type of infinite integral, corresponds to Weierstrass's M -test for the uniform convergence of infinite series (§67).

I. Let $F(x, y)$ be bounded in $a \leq x \leq a'$, $b \leq y \leq b'$ and integrable in (a, a') , where a' is arbitrary, for every y in (b, b') .

Then the integral $\int_a^\infty F(x, y) dx$ will converge uniformly in (b, b') , if there is a function $\mu(x)$, independent of y , such that

$$(i) \quad \mu(x) \geq 0, \text{ when } x \geq a;$$

$$(ii) \quad |F(x, y)| \leq \mu(x), \text{ when } x \geq a \text{ and } b \leq y \leq b';$$

$$\text{and } (iii) \quad \int_a^\infty \mu(x) dx \text{ exists.}$$

For, by (i) and (ii), when $x'' > x' \geq a$ and $b \leq y \leq b'$,

$$\left| \int_{x'}^{x''} F(x, y) dx \right| \leq \int_{x'}^{x''} \mu(x) dx,$$

and, from (iii), there is a positive number X such that

$$\int_{x'}^{x''} \mu(x) dx < \epsilon, \text{ when } x'' > x' \geq X.$$

These conditions will be satisfied if $x^n F(x, y)$ is bounded when $x \geq a$, and $b \leq y \leq b'$ for some constant n greater than 1.

COROLLARY. Let $F(x, y) = \phi(x, y)\psi(x)$, where $\phi(x, y)$ is bounded in $x \geq a$ and $b \leq y \leq b'$, and integrable in the interval (a, a') , where a' is arbitrary, for every y in (b, b') . Also let $\int_a^\infty \psi(x) dx$ be absolutely convergent. Then $\int_a^\infty F(x, y) dx$ is uniformly convergent in (b, b') .

Ex. 1. $\int_1^\infty \frac{dx}{x^{1+y}}$, $\int_0^\infty e^{-xy} dx$ converge uniformly in $y \geq y_0 > 0$.

Ex. 2. $\int_0^\infty e^{-\frac{x}{y}} dx$ converges uniformly in $0 < y \leq Y$, where Y is an arbitrary positive number.

Ex. 3. $\int_1^\infty \frac{\cos xy}{x^{1+n}} dx$, $\int_1^\infty \frac{\sin xy}{x^{1+n}} dx$, $\int_0^\infty \frac{\cos xy}{1+x^2} dx$, $\int_0^\infty \frac{\sin xy}{1+x^2} dx$ converge uniformly for all values of y , where $n > 0$.

II. Let $\phi(x, y)$ be bounded in $x \geq a$, $b \leq y \leq b'$, and a monotonic function of x for every y in (b, b') . Also let $\psi(x)$ be bounded and not change sign more than a finite number of times in the arbitrary interval (a, a') ,* and let $\int_a^\infty \psi(x) dx$ exist.

Then $\int_a^\infty \phi(x, y)\psi(x) dx$ converges uniformly in (b, b') .

This follows immediately from the Second Theorem of Mean Value, which gives, subject to the conditions named above,

$$\int_{x'}^{x''} \phi(x, y)\psi(x) dx = \phi(x', y) \int_{x'}^{\xi} \psi(x) dx + \phi(x'', y) \int_{\xi}^{x''} \psi(x) dx,$$

where ξ satisfies $a < x' \leq \xi \leq x''$.

But $\phi(x, y)$ is bounded in $x \geq a$ and $b \leq y \leq b'$, and $\int_a^\infty \psi(x) dx$ converges.

Thus it follows from the relation

$$\begin{aligned} & \left| \int_{x'}^{x''} \phi(x, y)\psi(x) dx \right| \\ & \leq \left| \phi(x', y) \right| \left| \int_{x'}^{\xi} \psi(x) dx \right| + \left| \phi(x'', y) \right| \left| \int_{\xi}^{x''} \psi(x) dx \right| \end{aligned}$$

that $\int_a^\infty \phi(x, y)\psi(x) dx$ converges uniformly in (b, b') .

* This condition is borrowed from the enunciation in the Second Theorem of Mean Value, as proved in §50. If a more general proof had been given, a corresponding extension of II. would have been treated here.

It is evident that $\psi(x)$ in this theorem may be replaced by $\psi(x, y)$, if $\int_a^\infty \psi(x, y) dx$ converges uniformly in (b, b') .

Ex. $\int_0^\infty e^{-xy} \frac{\sin x}{x} dx, \int_a^\infty e^{-xy} \frac{\cos x}{x} dx (a > 0)$ converge uniformly in $y \geq 0$.

III. Let $\phi(x, y)$ be a monotonic function of x for each y in (b, b') , and tend uniformly to zero as x increases, y being kept constant. Also let $\psi(x)$ be bounded and integrable in the arbitrary interval (a, a') , and not change sign more than a finite number of times in such an interval. Further, let $\int_a^x \psi(x) dx$ be bounded in $x \geq a$, without converging as $x \rightarrow \infty$.

Then $\int_a^\infty \phi(x, y) \psi(x) dx$ is uniformly convergent in (b, b') .

This follows at once from the Second Theorem of Mean Value, as in II. Also it will be seen that $\psi(x)$ may be replaced by $\psi(x, y)$, if $\int_a^x \psi(x, y) dx$ is bounded in $x \geq a$ and $b \leq y \leq b'$.*

Ex. 1. $\int_0^\infty e^{-xy} \sin x dx, \int_0^\infty e^{-xy} \cos x dx$ converge uniformly in $y \geq y_0 > 0$.

Ex. 2. $\int_0^\infty \frac{\sin xy}{x} dx$ and $\int_0^\infty \frac{x \sin xy}{1+x^2} dx$ both converge uniformly in $y \geq y_0 > 0$ and $y \leq -y_0 < 0$.

It can be left to the reader to enunciate and prove similar theorems for the second type of infinite integral $\int_a^{a'} F(x, y) dx$. The most useful test for uniform convergence in this case is that corresponding to I. above.

Ex. 1. $\int_0^1 x^{y-1} dx, \int_0^1 x^{y-1} e^{-x} dx$ converge uniformly in $1 > y \geq y_0 > 0$.

Ex. 2. $\int_0^1 \frac{\sin x}{x^{1+y}} dx$ converges uniformly in $0 < y \leq y_0 < 1$.

84. Continuity of the Integral $\int_a^\infty \mathbf{F}(x, y) dx$. We shall now consider, to begin with, the infinite integral $\int_a^\infty F(x, y) dx$, where $F(x, y)$ is bounded in the region $a \leq x \leq a', b \leq y \leq b'$, a' being arbitrary. Later we shall return to the other form of infinite integral in which the region contains points of infinite discontinuity.

* In Examples 1 and 2 of § 88 illustrations of this theorem will be found.

Let $f(y) = \int_a^\infty F(x, y) dx$, where $F(x, y)$ is either a continuous function of (x, y) in $a \leq x \leq a'$, $b \leq y \leq b'$, a' being arbitrary, or is of the form $\phi(x, y)\psi(x)$, where $\phi(x, y)$ is continuous as above, and $\psi(x)$ is bounded and integrable in the arbitrary interval (a, a') .

Also let $\int_a^\infty F(x, y) dx$ converge uniformly in (b, b') .

Then $f(y)$ is a continuous function of y in (b, b') .

Let the positive number ϵ be chosen, as small as we please.

Then to $\epsilon/3$ there corresponds a positive number X such that

$$\left| \int_x^\infty F(x, y) dx \right| < \frac{\epsilon}{3}, \text{ when } x \geq X,$$

the same X serving for every y in (b, b') .

But we have proved in § 77 that, under the given conditions, $\int_a^x F(x, y) dx$ is continuous in y in (b, b') .

Therefore, for some positive number η ,

$$\left| \int_a^x F(x, y + \Delta y) dx - \int_a^x F(x, y) dx \right| < \frac{\epsilon}{3}, \text{ when } |\Delta y| \leq \eta.$$

$$\text{Also } f(y) = \int_a^x F(x, y) dx + \int_x^\infty F(x, y) dx.$$

$$\begin{aligned} \text{Thus } f(y + \Delta y) - f(y) &= \left[\int_a^x F(x, y + \Delta y) dx - \int_a^x F(x, y) dx \right] \\ &\quad + \int_x^\infty F(x, y + \Delta y) dx - \int_x^\infty F(x, y) dx. \end{aligned}$$

$$\text{Also we have } \left| \int_x^\infty F(x, y) dx \right| < \frac{\epsilon}{3},$$

$$\text{and } \left| \int_x^\infty F(x, y + \Delta y) dx \right| < \frac{\epsilon}{3}.$$

Therefore, finally,

$$\begin{aligned} |f(y + \Delta y) - f(y)| &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon, \text{ when } |\Delta y| \leq \eta. \end{aligned}$$

Thus $f(y)$ is continuous in (b, b') .

85. Integration of the Integral $\int_a^\infty \mathbf{F}(x, y) dx$.

Let $F(x, y)$ satisfy the same conditions as in § 84.

$$\text{Then} \quad \int_{y_0}^y dy \int_a^\infty F(x, y) dx = \int_a^\infty dx \int_{y_0}^y F(x, y) dy,$$

where y_0, y are any two points in (b, b') .

$$\text{Let} \quad f(y) = \int_a^\infty F(x, y) dx.$$

Then we have shown that under the given conditions $f(y)$ is continuous, and therefore integrable, in (b, b') .

Also from § 79, for any arbitrary interval (a, x) , where x can be taken as large as we please,

$$\int_a^x dx \int_{y_0}^y F(x, y) dy = \int_{y_0}^y dy \int_a^x F(x, y) dx,$$

y_0, y being any two points in (b, b') .

Therefore

$$\int_a^\infty dx \int_{y_0}^y F(x, y) dx = \text{Lt}_{x \rightarrow \infty} \int_{y_0}^y dy \int_a^x F(x, y) dx,$$

provided we can show that the limit on the right-hand exists.

But

$$\int_{y_0}^y dy \int_a^x F(x, y) dx = \int_{y_0}^y dy \int_a^\infty F(x, y) dx - \int_{y_0}^y dy \int_x^\infty F(x, y) dx.$$

Thus we have only to show that

$$\text{Lt}_{x \rightarrow \infty} \int_{y_0}^y dy \int_x^\infty F(x, y) dx = 0.$$

Of course we cannot reverse the order of these limiting processes and write this as

$$\int_{y_0}^y dy \text{Lt}_{x \rightarrow \infty} \int_x^\infty F(x, y) dx,$$

for we have not shown that this inversion would not alter the result.

But we are given that $\int_a^\infty F(x, y) dx$ is uniformly convergent in (b, b') .

Let the positive number ϵ be chosen, as small as we please. Then take $\epsilon/(b' - b)$. To this number there corresponds a positive number X such that

$$\left| \int_x^\infty F(x, y) dx \right| < \frac{\epsilon}{b' - b}, \text{ when } x \geq X,$$

the same X serving for every y in (b, b') .

It follows that

$$\left| \int_{y_0}^y dy \int_x^\infty F(x, y) dx \right| < \epsilon, \text{ when } x \geq X,$$

if y_0, y lie in (b, b') .

In other words,

$$\text{Lt}_{x \rightarrow \infty} \left[\int_{y_0}^y dy \int_x^\infty F(x, y) dx \right] = 0.$$

And from the preceding remarks this establishes our result.

86. Differentiation of the Integral $\int_a^\infty \mathbf{F}(x, y) dx$.

Let $F(x, y)$ be either a continuous function of (x, y) in the region $a \leq x \leq a', b \leq y \leq b'$, a' being arbitrary, or be of the form $\phi(x, y)\psi(x)$, where $\phi(x, y)$ is continuous as above, and $\psi(x)$ is bounded and integrable in the arbitrary interval (a, a') . Also let $F(x, y)$ have a partial differential coefficient $\frac{\partial F}{\partial y}$ which satisfies the same conditions.

Then, if the integral $\int_a^\infty F(x, y) dx$ converges to $f(y)$, and the integral $\int_a^\infty \frac{\partial F}{\partial y} dx$ converges uniformly in (b, b') , $f(y)$ has a differential coefficient at every point in (b, b') , and

$$f'(y) = \int_a^\infty \frac{\partial F}{\partial y} dx.$$

We know from § 84 that, on the assumption named above, $\int_a^\infty \frac{\partial F}{\partial y} dx$ is a continuous function of y in (b, b') .

$$\text{Let } g(y) = \int_a^\infty \frac{\partial F}{\partial y} dx.$$

Then, by § 85,

$$\int_{y_0}^{y_1} g(y) dy = \int_a^\infty dx \int_{y_0}^{y_1} \frac{\partial F}{\partial y} dy,$$

where y_0, y_1 are any two points in (b, b') .

$$\text{Let } y_0 = y \text{ and } y_1 = y + \Delta y.$$

$$\text{Then } \int_y^{y+\Delta y} g(y) dy = \int_a^\infty [F(x, y + \Delta y) - F(x, y)] dx.$$

$$\text{Therefore } g(\xi) \Delta y = f(y + \Delta y) - f(y),$$

$$\text{where } y \leq \xi \leq y + \Delta y \text{ and } f(y) = \int_a^\infty F(x, y) dx.$$

Thus
$$g(\xi) = \frac{f(y + \Delta y) - f(y)}{\Delta y}.$$

But $\lim_{\Delta y \rightarrow 0} g(\xi) = g(y)$, since $g(x)$ is continuous.

It follows that $f(y)$ is differentiable, and that

$$f'(y) = \int_a^{\infty} \frac{\partial F}{\partial y} dx,$$

where y is any point in (b, b') .

87. Properties of the Infinite Integral $\int_a^{a'} \mathbf{F}(\mathbf{x}, \mathbf{y}) d\mathbf{x}$. The results of §§ 84-86 can be readily extended to the second type of infinite integral. It will be sufficient to state the theorems without proof. The steps in the argument are in each case parallel to those in the preceding discussion. As before, the region with which we deal is

$$a \leq x \leq a', \quad b \leq y \leq b'.$$

I. Continuity of $\int_a^{a'} F(x, y) dx$.

Let $f(y) = \int_a^{a'} F(x, y) dx$, where $F(x, y)$ has points of infinite discontinuity on certain lines (e.g., $x = a_1, a_2, \dots, a_n$) between $x = a$ and $x = a'$, and is either a continuous function of (x, y) , or the product of a continuous function $\phi(x, y)$ and a bounded and integrable function $\psi(x)$, except in the neighbourhood of the said lines.

Then, if $\int_a^{a'} F(x, y) dx$ is uniformly convergent in (b, b') , $f(y)$ is a continuous function of y in (b, b') .

II. Integration of the Integral $\int_a^{a'} F(x, y) dx$.

Let $F(x, y)$ satisfy the same conditions as in I.

Then
$$\int_{y_0}^y dy \int_a^{a'} F(x, y) dx = \int_a^{a'} dx \int_{y_0}^y F(x, y) dy,$$

where y_0, y are any two points in (b, b') .

III. Differentiation of the Integral $\int_a^{a'} F(x, y) dx$.

Let $f(y) = \int_a^{a'} F(x, y) dx$, where $F(x, y)$ has points of infinite

discontinuity on certain lines (e.g., $x = a_1, a_2, \dots, a_n$) between $x = a$ and $x = a'$, and is either a continuous function of (x, y) , or the product of a continuous function $\phi(x, y)$ and a bounded and integrable function $\psi(x)$, except in the neighbourhood of the said lines.

Also let $F(x, y)$ have a partial differential coefficient $\frac{\partial F}{\partial y}$, which satisfies the same conditions.

Further, let $\int_a^{a'} F(x, y) dx$ converge, and $\int_a^{a'} \frac{\partial F}{\partial y} dx$ converge uniformly in (b, b') .

Then $f'(y)$ exists and is equal to $\int_a^{a'} \frac{\partial F}{\partial y} dx$ in (b, b') .

88. Applications of the preceding Theorems.

Ex. 1. To prove $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

(i) Let $F(a) = \int_0^\infty e^{-ax} \frac{\sin x}{x} dx \quad (a \geq 0)$.

This integral converges uniformly when $a \geq 0$. (Cf. § 83, III.)

For e^{-ax}/x is a monotonic function of x when $x > 0$.

Thus, by the Second Theorem of Mean Value,

$$\int_{x'}^{x''} \frac{e^{-ax}}{x} \sin x dx = \frac{e^{-ax'}}{x'} \int_{x'}^{\xi} \sin x dx + \frac{e^{-ax''}}{x''} \int_{\xi}^{x''} \sin x dx,$$

where $0 < x' \leq \xi \leq x''$.

Therefore

$$\begin{aligned} \left| \int_{x'}^{x''} \frac{e^{-ax}}{x} \sin x dx \right| &\leq \frac{e^{-ax'}}{x'} \left| \int_{x'}^{\xi} \sin x dx \right| + \frac{e^{-ax''}}{x''} \left| \int_{\xi}^{x''} \sin x dx \right| \\ &< 4 \frac{e^{-ax'}}{x'}, \quad \text{since } \left| \int_p^q \sin x dx \right| \leq 2 \text{ for all values of } p \text{ and } q, \\ &< \frac{4}{x'}, \quad \text{since } a \geq 0. \end{aligned}$$

It follows that $\left| \int_{x'}^{x''} e^{-ax} \frac{\sin x}{x} dx \right| < \epsilon$, when $x'' > x' \geq X$,

provided that $X > 4/\epsilon$, and this holds for every a greater than or equal to 0.

This establishes the uniform continuity of the integral, and from § 84 $F(a)$ must be continuous in $a \geq 0$.

Thus $F(0) = \text{Lt}_{a \rightarrow 0} \int_0^\infty e^{-ax} \frac{\sin x}{x} dx,$

$$\text{i.e. } \int_0^\infty \frac{\sin x}{x} dx = \text{Lt}_{a \rightarrow 0} \int_0^\infty e^{-ax} \frac{\sin x}{x} dx.$$

(ii) Again, the integral $\int_0^{\infty} e^{-ax} \sin x \, dx$

is uniformly convergent in $a \geq a_0 > 0$.

This follows as above, and is again an example of § 83, III.

Thus, by § 86, when $a > 0$,

$$\begin{aligned} F'(a) &= \int_0^{\infty} \frac{\partial}{\partial a} \left(e^{-ax} \frac{\sin x}{x} \right) dx \\ &= - \int_0^{\infty} e^{-ax} \sin x \, dx. \end{aligned}$$

But $\frac{d}{dx} e^{-ax} (\cos x + a \sin x) = -(\alpha^2 + 1) e^{-ax} \sin x$.

Therefore $\int_0^{\infty} e^{-ax} \sin x \, dx = -\frac{1}{\alpha^2 + 1}$.

Thus $F'(a) = -\frac{1}{\alpha^2 + 1}$.

And $F(a) = -\tan^{-1} a + \frac{\pi}{2}$,

since $\lim_{a \rightarrow \infty} F(a) = 0$.*

It follows from (i) that $\int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}$.

Ex. 2. To prove

$$\int_0^{\infty} \frac{\cos ax}{1+x^2} \, dx = \frac{\pi}{2} e^{-a} \quad \text{and} \quad \int_0^{\infty} \frac{\sin ax}{x(1+x^2)} \, dx = \frac{\pi}{2} (1 - e^{-a}) \quad (a \geq 0).$$

* If a formal proof of this is required, we might proceed as follows:

Let the arbitrary positive ϵ be chosen, as small as we please.

Since $\int_0^{\infty} e^{-ax} \frac{\sin x}{x} \, dx$ converges uniformly in $a \geq 0$, there is a positive number X such that

$$\left| \int_X^{\infty} e^{-ax} \frac{\sin x}{x} \, dx \right| < \frac{\epsilon}{3},$$

and this number X is independent of a .

Also we can choose x_0 , independent of a , so that

$$\left| \int_0^{x_0} e^{-ax} \frac{\sin x}{x} \, dx \right| < \frac{\epsilon}{3}.$$

But $\int_{x_0}^X e^{-ax} \frac{\sin x}{x} \, dx = e^{-ax_0} \int_{x_0}^{\xi} \frac{\sin x}{x} \, dx + e^{-aX} \int_{\xi}^X \frac{\sin x}{x} \, dx$,

where $x_0 \leq \xi \leq X$.

And we know that

$$\left| \int_p^q \frac{\sin x}{x} \, dx \right| < \pi \quad \text{for } q > p > 0 \quad (\text{cf. p. 202}).$$

Therefore $\left| \int_{x_0}^X e^{-ax} \frac{\sin x}{x} \, dx \right| < 2\pi e^{-ax_0}$.

Thus we can choose A so large that

$$\left| \int_{x_0}^X e^{-ax} \frac{\sin x}{x} \, dx \right| < \frac{\epsilon}{3}, \quad \text{when } a \geq A.$$

It follows that $\left| \int_0^{\infty} e^{-ax} \frac{\sin x}{x} \, dx \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$, when $a \geq A$.

(i) Let
$$f(\alpha) = \int_0^{\infty} \frac{\cos \alpha x}{1+x^2} dx.$$

The integral is uniformly convergent for every α , so that, by § 84, $f(\alpha)$ is continuous for all values of α , and we can integrate under the sign of integration (§ 85).

(ii) Let
$$\phi(\alpha) = \int_0^{\alpha} f(\alpha) d\alpha.$$

Then $\phi(\alpha)$ is a continuous function of α for all values of α (§ 49).

Also
$$\begin{aligned} \phi(\alpha) &= \int_0^{\infty} dx \int_0^{\alpha} \frac{\cos \alpha x}{1+x^2} d\alpha \\ &= \int_0^{\infty} \frac{\sin \alpha x}{x(1+x^2)} dx. \end{aligned}$$

(iii) Again, we know that $f'(\alpha)$ will be equal to $-\int_0^{\infty} \frac{x}{1+x^2} \sin \alpha x dx$, if α lies within an interval of uniform convergence of this integral.

But $\frac{x}{1+x^2}$ is monotonic when $x \geq 1$, and $\text{Lt}_{x \rightarrow \infty} \frac{x}{1+x^2} = 0$.

Thus we have, when $x'' > x' \geq 1$,

$$\int_{x'}^{x''} \frac{x}{1+x^2} \sin \alpha x dx = \frac{x'}{1+x'^2} \int_{x'}^{\xi} \sin \alpha x dx + \frac{x''}{1+x''^2} \int_{\xi}^{x''} \sin \alpha x dx,$$

where $x' \leq \xi \leq x''$.

It follows that

$$\int_{x'}^{x''} \frac{x}{1+x^2} \sin \alpha x dx = \frac{x'}{\alpha(1+x'^2)} \int_{\alpha x'}^{\alpha \xi} \sin x dx + \frac{x''}{\alpha(1+x''^2)} \int_{\alpha \xi}^{\alpha x''} \sin x dx.$$

Therefore
$$\left| \int_{x'}^{x''} \frac{x}{1+x^2} \sin \alpha x dx \right| \leq \frac{4x'}{\alpha(1+x'^2)}.$$

Thus $\int_0^{\infty} \frac{x}{1+x^2} \sin \alpha x dx$ is uniformly convergent when $\alpha \geq \alpha_0 > 0$, and

$$f'(\alpha) = - \int_0^{\infty} \frac{x}{1+x^2} \sin \alpha x dx.$$

(iv) Now
$$\phi(\alpha) = \int_0^{\alpha} f(\alpha) d\alpha.$$

Therefore
$$\phi'(\alpha) = f(\alpha)$$

and
$$\phi''(\alpha) = f'(\alpha) = - \int_0^{\infty} \frac{x}{1+x^2} \sin \alpha x dx.$$

Thus
$$\begin{aligned} \phi''(\alpha) &= - \int_0^{\infty} \left[1 - \frac{1}{1+x^2} \right] \frac{\sin \alpha x}{x} dx \\ &= - \int_0^{\infty} \frac{\sin \alpha x}{x} dx + \int_0^{\infty} \frac{\sin \alpha x}{x(1+x^2)} dx \\ &= - \frac{\pi}{2} + \phi(\alpha). \end{aligned}$$

This result has been established on the understanding that $\alpha > 0$.

(v) From (iv) we have

$$\phi(\alpha) = Ae^{\alpha} + Be^{-\alpha} + \frac{\pi}{2}, \text{ when } \alpha > 0.$$

But $\phi(\alpha)$ is continuous in $\alpha \geq 0$, and $\phi(0) = 0$.

Therefore $\text{Lt}_{\alpha \rightarrow 0} \phi(\alpha) = 0$,
and $A + B + \frac{\pi}{2} = 0$.

Also $\phi'(a)$ is continuous in $a \geq 0$, and $\phi'(0) = \frac{\pi}{2}$.

Therefore $A - B - \frac{\pi}{2} = 0$.

It follows that $A = 0$ and $B = -\frac{\pi}{2}$.

Thus $\phi(a) = \frac{\pi}{2}(1 - e^{-a})$, when $a > 0$.

And $f(a) = \phi'(a) = \frac{\pi}{2}e^{-a}$, when $a > 0$.

Both these results obviously hold for $a = 0$ as well.

Ex. 3. *The Gamma Function* $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1} dx$, $n > 0$, and its derivatives.

(i) *To prove $\Gamma(n)$ is uniformly convergent when $N \geq n \geq n_0 > 0$, however large N may be and however near zero n_0 may be.*

When $n \geq 1$, the integral $\int_0^{\infty} e^{-x}x^{n-1} dx$ has to be examined for convergence only at the upper limit. When $0 < n < 1$, the integrand becomes infinite at $x = 0$. In this case we break up the integral into

$$\int_0^1 e^{-x}x^{n-1} dx + \int_1^{\infty} e^{-x}x^{n-1} dx.$$

Take first the integral $\int_0^1 e^{-x}x^{n-1} dx$.

When $0 < x < 1$, $x^{n-1} \leq x^{n_0-1}$, if $n \geq n_0 > 0$.

Therefore $e^{-x}x^{n-1} \leq e^{-x}x^{n_0-1}$, if $n \geq n_0 > 0$.

It follows from the theorem which corresponds to § 83, I. that $\int_0^1 e^{-x}x^{n-1} dx$ converges uniformly when $n \geq n_0 > 0$.

Again consider $\int_1^{\infty} e^{-x}x^{n-1} dx$, $n > 0$.

When $x > 1$, $x^{n-1} \leq x^{N-1}$, if $0 < n \leq N$.

Therefore $e^{-x}x^{n-1} \leq e^{-x}x^{N-1}$, if $0 < n \leq N$.

It follows as above from § 83, I. that $\int_1^{\infty} e^{-x}x^{n-1} dx$ converges uniformly for $0 < n \leq N$.

Combining these two results, we see that $\int_0^{\infty} e^{-x}x^{n-1} dx$ converges uniformly when $N \geq n \geq n_0 > 0$, however large N may be and however near zero n_0 may be.

(ii) *To prove $\Gamma'(n) = \int_0^{\infty} e^{-x}x^{n-1} \log x dx$, $n > 0$.*

We know that $\text{Lt}_{x \rightarrow 0} (x^r \log x) = 0$, when $r > 0$,

so that the integrand has an infinity at $x = 0$ for positive values of n only when $0 < n \leq 1$.

But when $0 < x < 1$, $x^{n-1} \leq x^{n_0-1}$, if $n \geq n_0 > 0$.

Therefore $e^{-x}x^{n-1}|\log x| \leq x^{n_0-1}|\log x|$, if $n \geq n_0 > 0$.

And we have seen [Ex. 6, p. 119] that $\int_0^1 x^{n_0-1} \log x \, dx$ converges when $0 < n_0 \leq 1$.

It follows as above that $\int_0^1 e^{-x}x^{n-1} \log x \, dx$ is uniformly convergent when $n \geq n_0 > 0$.

Also for $\int_1^\infty e^{-x}x^{n-1} \log x \, dx$, we proceed as follows:

When $x > 1$, $x^{n-1} \leq x^{N-1}$, if $0 < n \leq N$.

Therefore $e^{-x}x^{n-1} \log x \leq e^{-x}x^{N-1} \log x$
 $< e^{-x}x^N$, since $\frac{\log x}{x} < 1$ when $x > 1$.

But $\int_1^\infty e^{-x}x^N \, dx$ is convergent.

Therefore $\int_1^\infty e^{-x}x^{n-1} \log x \, dx$ is uniformly convergent when $0 < n \leq N$.

Combining these two results, we see that $\int_0^\infty e^{-x}x^{n-1} \log x \, dx$ is uniformly convergent when $N \geq n \geq n_0 > 0$, however large N may be and however near zero n_0 may be.

We are thus able to state, relying on §§ 86, 87, that

$$\Gamma'(n) = \int_0^\infty e^{-x}x^{n-1} \log x \, dx \text{ for } n > 0.$$

It can be shown in the same way that the successive derivatives of $\Gamma(n)$ can be obtained by differentiating under the integral sign.

Ex. 4. (i) To prove $\int_0^\pi \log(1 - 2y \cos x + y^2) \, dx$ is uniformly convergent for any interval of y (e.g. $b \leq y \leq b'$); and (ii) to deduce that $\int_0^{\frac{\pi}{2}} \log \sin x \, dx = -\frac{1}{2}\pi \log 2$.

(i) Since $1 - 2y \cos x + y^2 = (y - \cos x)^2 + \sin^2 x$, this expression is positive for all values of x, y , unless when $x = m\pi$ and $y = (-1)^m$, $m = 0, \pm 1, \pm 2$, etc., and for these values it is zero.

It follows that the integrand becomes infinite at $x = 0$ and $x = \pi$; in the one case when $y = 1$, and in the other when $y = -1$.

We consider first the infinity at $x = 0$.

As the integrand is bounded in any strip $0 \leq x \leq X$, where $X < \pi$, for any interval of y which does not include $y = 1$, we have only to examine the integral

$$\int_0^x \log(1 - 2y \cos x + y^2) \, dx$$

in the neighbourhood of $y = 1$.

Put $y = 1 + h$, where $|h| \leq \alpha$ and α is some positive number less than unity, to be fixed more definitely later.

$$\begin{aligned} \text{Since } & \int_0^x \log(1 - 2y \cos x + y^2) \, dx \\ &= \int_0^x \log\left(1 - \cos x + \frac{h^2}{2(1+h)}\right) \, dx + x \log 2(1+h), \end{aligned}$$

it is clear that we need only discuss the convergence of the integral

$$\int_0^x \log \left(1 - \cos x + \frac{h^2}{2(1+h)} \right) dx.$$

Take a value of a ($0 < a < 1$) such that $\frac{a^2}{2(1-a)} < 1$.

Then $1 > \frac{a^2}{2(1-a)} \geq \frac{h^2}{2(1+h)} \geq 0$, since $|h| \leq a$.

Now let $\beta = \cos^{-1} \frac{a^2}{2(1-a)}$.

It will be seen that, when $|h| \leq a$,

$$0 < 1 - \cos x \leq 1 - \cos x + \frac{h^2}{2(1+h)} \leq 1,$$

provided that $0 < x \leq \beta$.

Therefore, under the same conditions,

$$\left| \log \left(1 - \cos x + \frac{h^2}{2(1+h)} \right) \right| \leq \left| \log(1 - \cos x) \right|.$$

But the μ -test shows that the integral

$$\int_0^\beta \log(1 - \cos x) dx$$

converges.

It follows that $\int_0^\beta \log \left(1 - \cos x + \frac{h^2}{2(1+h)} \right) dx$

converges uniformly for $|h| \leq a$. [Cf. § 83, I.]

And therefore $\int_0^\beta \log(1 - 2y \cos x + y^2) dx$

converges uniformly for any interval (b, b') of y .

The infinity at $x = \pi$ can be treated in the same way, and the uniform convergence of the integral

$$\int_0^\pi \log(1 - 2y \cos x + y^2) dx$$

is thus established for any interval (b, b') of y .

(ii) Let $f(y) = \int_0^\pi \log(1 - 2y \cos x + y^2) dx$.

We know from § 70 that

$$f(y) = 0, \quad \text{when } |y| < 1,$$

and

$$f(y) = \pi \log y^2, \quad \text{when } |y| > 1.$$

But we have just seen that the integral converges uniformly for any finite interval of y .

It follows from § 87, I., that

$$f(1) = \lim_{y \rightarrow 1} f(y) = 0$$

and

$$f(-1) = \lim_{y \rightarrow -1} f(y) = 0.$$

$$\begin{aligned}
 \text{But} \quad f(1) &= \int_0^\pi \log 2(1 - \cos x) dx \\
 &= 2\pi \log 2 + 2 \int_0^\pi \log \sin \frac{x}{2} dx \\
 &= 2\pi \log 2 + 4 \int_0^{\frac{\pi}{2}} \log \sin x dx.
 \end{aligned}$$

$$\text{Thus} \quad \int_0^{\frac{\pi}{2}} \log \sin x dx = -\frac{1}{2}\pi \log 2.*$$

From $f(-1)=0$, we find in the same way that

$$\int_0^{\frac{\pi}{2}} \log \cos x dx = -\frac{1}{2}\pi \log 2,$$

a result which, of course, could have been deduced from the preceding.

89. The Repeated Integral $\int_a^\infty dx \int_b^\infty f(x, y) dy$. It is not easy to determine general conditions under which the equation

$$\int_a^\infty dx \int_b^\infty f(x, y) dy = \int_b^\infty dy \int_a^\infty f(x, y) dx$$

is satisfied.

The problem is closely analogous to that of term by term integration of an infinite series between infinite limits. We shall discuss only a case somewhat similar to that in infinite series given in § 76.

Let $f(x, y)$ be a continuous function of (x, y) in $x \geq a$, $y \geq b$, and let the integrals

$$(i) \int_a^\infty f(x, y) dx, \quad (ii) \int_b^\infty f(x, y) dy,$$

respectively, converge uniformly in the arbitrary intervals

$$b \leq y \leq b', \quad a \leq x \leq a'.$$

Also let the integral (iii) $\int_a^\infty dx \int_b^y f(x, y) dy$ converge uniformly in $y \geq b$.

Then the integrals

$$\int_a^\infty dx \int_b^\infty f(x, y) dy \quad \text{and} \quad \int_b^\infty dy \int_a^\infty f(x, y) dx$$

exist and are equal.

Since we are given that $\int_a^\infty f(x, y) dx$ converges uniformly in the arbitrary interval $b \leq y \leq b'$, we know from § 85 that

$$\int_b^y dy \int_a^\infty f(x, y) dx = \int_a^\infty dx \int_b^y f(x, y) dy, \quad \text{when } y > b.$$

It follows that

$$\int_b^\infty dy \int_a^\infty f(x, y) dx = \text{Lt}_{y \rightarrow \infty} \int_a^\infty dx \int_b^y f(x, y) dy,$$

provided that the limit on the right-hand side exists.

* This integral was obtained otherwise in Ex. 4, p. 118.

To prove the existence of this limit, it is sufficient to show that to the arbitrary positive number ϵ there corresponds a positive number Y such that

$$\left| \int_a^\infty dx \int_{y'}^{y''} f(x, y) dy \right| < \epsilon, \text{ when } y'' > y' \geq Y.$$

But from the uniform convergence of

$$\int_a^\infty dx \int_b^y f(x, y) dy$$

in $y \geq b$, we can choose the positive number X such that

$$\left| \int_x^\infty dx \int_b^y f(x, y) dy \right| < \frac{\epsilon}{3}, \text{ when } x \geq X, \dots\dots\dots(1)$$

the same X serving for every y greater than or equal to b .

Also we are given that $\int_b^\infty f(x, y) dy$

is uniformly convergent in the arbitrary interval (a, a') .

Therefore we can choose the positive number Y so that

$$\left| \int_{y'}^{y''} f(x, y) dy \right| < \frac{\epsilon}{3(X-a)}, \text{ when } y'' > y' \geq Y,$$

the same Y serving for every x in (a, X) .

Thus we have $\left| \int_a^X dx \int_{y'}^{y''} f(x, y) dy \right| < \frac{\epsilon}{3}, \text{ when } y'' > y' \geq Y. \dots\dots\dots(2)$

But it is clear that

$$\int_a^\infty dx \int_{y'}^{y''} f(x, y) dy = \int_a^X dx \int_{y'}^{y''} f(x, y) dy + \int_X^\infty dx \int_b^{y''} f(x, y) dy - \int_X^\infty dx \int_b^{y'} f(x, y) dy.$$

Therefore from (1) and (2) we have

$$\left| \int_a^\infty dx \int_{y'}^{y''} f(x, y) dy \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon, \text{ when } y'' > y' \geq Y.$$

We have thus shown that

$$\int_b^\infty dy \int_a^\infty f(x, y) dx = \text{Lt}_{y \rightarrow \infty} \int_a^\infty dx \int_b^y f(x, y) dy. \dots\dots\dots(3)$$

It remains to prove that

$$\text{Lt}_{y \rightarrow \infty} \int_a^\infty dx \int_b^y f(x, y) dy = \int_a^\infty dx \int_b^\infty f(x, y) dy.$$

Let the limit on the left-hand side be l .

Then ϵ being any positive number, as small as we please, there is a positive number Y_1 such that

$$\left| l - \int_a^\infty dx \int_b^y f(x, y) dy \right| < \frac{\epsilon}{3}, \text{ when } y \geq Y_1. \dots\dots\dots(4)$$

Also, from the uniform convergence of

$$\int_a^\infty dx \int_b^y f(x, y) dy, \text{ when } y \geq b,$$

we know that there is a positive number X such that

$$\left| \int_a^\infty dx \int_b^y f(x, y) dy - \int_a^{X'} dx \int_b^y f(x, y) dy \right| < \frac{\epsilon}{3}, \text{ when } X' \geq X, \dots\dots(5)$$

the same X serving for every y greater than or equal to b .

Choose a number X' such that $X' \geq X > a$.

Then, from the uniform convergence of $\int_b^\infty f(x, y) dy$ in any arbitrary interval, we know that there is a positive number Y_2 such that

$$\left| \int_b^y f(x, y) dy - \int_b^\infty f(x, y) dy \right| < \frac{\epsilon}{3(X' - a)}, \text{ when } y \geq Y_2,$$

the same Y_2 serving for every x in (a, X') .

$$\text{Thus } \left| \int_a^{X'} dx \int_b^y f(x, y) dy - \int_a^{X'} dx \int_b^\infty f(x, y) dy \right| < \frac{\epsilon}{3}, \text{ when } y \geq Y_2. \dots\dots(6)$$

Now take a number Y greater than Y_1 and Y_2 .

Equations (4), (5) and (6) hold for this number Y .

But

$$\begin{aligned} & \left| l - \int_a^{X'} dx \int_b^\infty f(x, y) dy \right| \\ & \leq \left| l - \int_a^Y dx \int_b^Y f(x, y) dy \right| + \left| \int_a^Y dx \int_b^Y f(x, y) dy - \int_a^{X'} dx \int_b^Y f(x, y) dy \right| \\ & \quad + \left| \int_a^{X'} dx \int_b^Y f(x, y) dy - \int_a^{X'} dx \int_b^\infty f(x, y) dy \right| \\ & < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}, \text{ from (4), (5) and (6),} \\ & < \epsilon. \end{aligned}$$

This result holds for every number X' greater than or equal to X .

Thus we have shown that

$$l = \text{Lt}_{x \rightarrow \infty} \int_a^x dx \int_b^\infty f(x, y) dy = \int_a^\infty dx \int_b^\infty f(x, y) dy.$$

Also, from (3), we have

$$\int_a^\infty dx \int_b^\infty f(x, y) dy = \int_b^\infty dy \int_a^\infty f(x, y) dx$$

under the conditions stated in the theorem.

It must be noticed that the conditions we have taken are *sufficient*, but not *necessary*. For a more complete discussion of the conditions under which the integrals

$$\int_a^\infty dx \int_b^\infty f(x, y) dy, \quad \int_b^\infty dy \int_a^\infty f(x, y) dx,$$

when they both exist, are equal, reference should be made to the works of de la Vallée Poussin,* to whom the above treatment is due. A valuable dis-

* His investigations are contained in three memoirs, the first in *Bruxelles, Ann. Soc. scient.*, **17**; the second in *J. math., Paris*, (Sér. 4) **8**, 1892; and the third in *J. math., Paris*, (Sér. 5) **5**.

See also Bromwich, *London, Proc. Math. Soc.* (Ser. 2), **1**, 1903.

cussion of the whole subject is also given in Pierpont's *Theory of Functions of a Real Variable*. The question is dealt with in Hobson's *Theory of Functions of a Real Variable*, but from a more difficult standpoint.

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EXAMPLES ON CHAPTER VI.

1. Prove that $\int_0^{\infty} e^{-ax^2} dx$ is uniformly convergent in $a \geq a_0 > 0$, and that $\frac{1}{1+a^2}$ is uniformly convergent in $a \geq 0$, when $b > 0$.
2. Prove that $\int_0^{\infty} e^{-xy} \frac{\sin xy}{y} dx$ is uniformly convergent in $y \geq y_0 > 0$, and that $\int_0^{\infty} e^{-ax} \frac{\sin xy}{y} dx$ is uniformly convergent in $y \geq 0$, when $a > 0$.
3. Prove that $\int_0^{\infty} e^{-ax} x^{n-1} \cos x dx$ is uniformly convergent in the interval $a \geq a_0 > 0$, when $n \geq 1$, and in the interval $a \geq 0$, when $0 < n < 1$.
4. Prove that $\int_0^{\infty} e^{-ax} x^{n-1} \sin x dx$ is uniformly convergent in the interval $a \geq a_0 > 0$, when $n \geq 0$, and in the interval $a \geq 0$, when $-1 < n < 1$.
5. Using the fact that $\int_0^{\infty} \frac{\sin xy}{x} dx$ is uniformly convergent in $y \geq y_0 < 0$ and $y \leq -y_0 < 0$, show that $\int_0^{\infty} \frac{\sin xy \cos ax}{x} dx$ is uniformly convergent in any interval of y which does not include $y = \pm a$.
6. Show that (i) $\int_0^{\infty} (1 - e^{-xy}) \frac{\sin x}{x} dx$,
 (ii) $\int_0^{\infty} e^{-xy} \frac{(\cos ax - \cos bx)}{x} dx$, $b \neq a$,
 are uniformly convergent for $y \geq 0$.

7. Discuss the uniform convergence of the integrals :

$$\begin{aligned} \text{(i)} \int_0^1 \frac{\tan^{-1} xy}{x\sqrt{(1-x^2)}} dx, \quad \text{(ii)} \int_0^1 \frac{dx}{(1+v^2y^2)\sqrt{(1-x^2)}}, \quad \text{(iii)} \int_0^1 x^y dx. \\ \text{(iv)} \int_0^1 \frac{x^y - 1}{\log x} dx, \quad \text{(v)} \int_0^1 x^{y-1}(\log x)^n dx, \quad n > 0. \end{aligned}$$

8. Show that differentiation under the integral sign is allowable in the following integrals, and hence obtain the results that are given opposite each :

$$\begin{aligned} \text{(i)} \int_0^\infty e^{-ax^2} dx &= \frac{1}{2} \sqrt{\frac{\pi}{a}}; & \int_0^\infty x^{2n} e^{-ax^2} dx &= \frac{\sqrt{\pi}}{2} \cdot \frac{1 \cdot 3 \dots (2n-1)}{2^n a^{n+\frac{1}{2}}}. \\ \text{(ii)} \int_0^\infty \frac{dx}{x^2+a} &= \frac{\pi}{2\sqrt{a}}, \quad a > 0; & \int_0^\infty \frac{dx}{(x^2+a)^{n+1}} &= \frac{\pi}{2} \cdot \frac{1 \cdot 3 \dots (2n-1)}{2^n n! a^{n+\frac{1}{2}}}. \\ \text{(iii)} \int_0^1 x^n dx &= \frac{1}{n+1}, \quad n > -1; & \int_0^1 x^n (-\log x)^m dx &= \frac{m!}{(n+1)^{m+1}}. \\ \text{(iv)} \int_0^\infty \frac{x^n}{1+x^2} dx &= \frac{\pi}{2 \cos \frac{n\pi}{2}}, \quad 0 < n < 1; & \int_0^\infty \frac{x^n \log x}{1+x^2} dx &= \frac{\pi^2}{4} \sec \frac{n\pi}{2} \tan \frac{n\pi}{2}. \end{aligned}$$

9. Establish the right to integrate under the integral sign in the following integrals :

$$\begin{aligned} \text{(i)} \int_0^\infty e^{-ax} dx; \quad \text{interval } a \geq a_0 > 0. \\ \text{(ii)} \int_0^\infty e^{-ax} \cos bx dx; \quad \text{interval } a \geq a_0 > 0, \text{ or any interval of } b. \\ \text{(iii)} \int_0^\infty e^{-ax} \sin bx dx; \quad \text{interval } a \geq a_0 > 0, \text{ or any interval of } b. \\ \text{(iv)} \int_0^1 x^a dx; \quad \text{interval } a \geq a_0 > -1. \end{aligned}$$

10. Assuming that $\int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2+b^2}$, $a > 0$, show that

$$\int_0^\infty \frac{e^{-fx} - e^{-gx}}{x} \sin bx dx = \tan^{-1} \frac{g}{b} - \tan^{-1} \frac{f}{b}, \quad g > f > 0.$$

Deduce that (i) $\int_0^\infty \frac{1 - e^{-gx}}{x} \sin bx dx = \tan^{-1} \frac{g}{b}$.

$$\text{(ii)} \int_0^\infty \frac{\sin bx}{x} dx = \frac{1}{2}\pi, \quad b > 0.$$

11. Show that the integrals

$$\int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2+b^2}, \quad \int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2+b^2}, \quad a > 0,$$

can be differentiated under the integral sign, either with regard to a or b , and hence obtain the values of

$$\begin{aligned} \int_0^\infty x e^{-ax} \cos bx dx, \quad \int_0^\infty x e^{-ax} \sin bx dx, \\ \int_0^\infty x^2 e^{-ax} \cos bx dx, \quad \int_0^\infty x^2 e^{-ax} \sin bx dx. \end{aligned}$$

12. Let
$$f(y) = \int_0^{\infty} \frac{1 - \cos xy}{x} e^{-x} dx.$$

Show that $f'(y) = \int_0^{\infty} \sin xy e^{-x} dx$ for all values of y , and deduce that

$$f(y) = \frac{1}{2} \log(1 + y^2).$$

13. Let
$$f(y) = \int_0^{\infty} e^{-x^2} \cos 2xy dx.$$

Show that $f'(y) = -2 \int_0^{\infty} x e^{-x^2} \sin 2xy dx$ for all values of y .

On integrating by parts, it will be found that

$$f'(y) + 2yf(y) = 0.$$

From this result, show that $f(y) = \frac{1}{2} \sqrt{\pi} e^{-y^2}$, assuming that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Also show that
$$\int_0^{\infty} dx \int_0^y e^{-x^2} \cos 2xy dy = \int_0^y f(y) dy,$$

and deduce that
$$\int_0^{\infty} e^{-x^2} \frac{\sin 2xy}{x} dx = \sqrt{\pi} \int_0^y e^{-y^2} dy.$$

14. Let
$$U = \int_0^{\infty} e^{-ax} x^{n-1} \cos bx dx, \quad V = \int_0^{\infty} e^{-ax} x^{n-1} \sin bx dx,$$
 where $a > 0, n > 0$.

Make the following substitutions :

$$\begin{aligned} a &= r \cos \theta, & b &= r \sin \theta, & \text{where } -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi, \\ r x &= y, & U r^n &= u, & V r^n &= v. \end{aligned}$$

Then show that
$$\frac{du}{d\theta} = -nv, \quad \frac{dv}{d\theta} = nu.$$

From these it follows that
$$\frac{d^2u}{d\theta^2} + n^2u = 0.$$

Deduce that
$$u = \Gamma(n) \cos n\theta, \quad v = \Gamma(n) \sin n\theta.$$

Thus
$$U = \Gamma(n) \frac{\cos n\theta}{r^n}, \quad V = \Gamma(n) \frac{\sin n\theta}{r^n}.$$

Also show that, if $0 < n < 1$,
$$\text{Lt}_{a \rightarrow 0} U = \int_0^{\infty} x^{n-1} \cos bx dx,$$

$$\text{Lt}_{a \rightarrow 0} V = \int_0^{\infty} x^{n-1} \sin bx dx.$$

And deduce that

$$(i) \int_0^{\infty} \frac{\cos bx}{x^{1-n}} dx = \frac{\Gamma(n) \cos \frac{n\pi}{2}}{b^n}, \quad (ii) \int_0^{\infty} \frac{\sin bx}{x^{1-n}} dx = \frac{\Gamma(n) \sin \frac{n\pi}{2}}{b^n},$$

$$(iii) \int_0^{\infty} \frac{\cos x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}} = \int_0^{\infty} \frac{\sin x}{\sqrt{x}} dx.$$

[Compare Gibson, *Treatise on the Calculus*, p. 471.]

15. Prove that

$$\int_0^{\infty} dx \int_b^{\infty} e^{-xy} \sin x dy = \int_b^{\infty} dy \int_0^{\infty} e^{-xy} \sin x dx,$$

where b is any positive number.

CHAPTER VII

FOURIER'S SERIES

90. Trigonometrical Series and Fourier's Series. We have already discussed some of the properties of infinite series whose terms are functions of x , confining our attention chiefly to those whose terms are continuous functions.

The trigonometrical series,

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots, \dots\dots(1)$$

where a_0, a_1, b_1 , etc., are constants, is a special type of such series.

Let $f(x)$ be given in the interval $(-\pi, \pi)$. If bounded, let it be integrable in this interval; if unbounded, let the infinite integral $\int_{-\pi}^{\pi} f(x) dx$ be absolutely convergent. Then

$$\int_{-\pi}^{\pi} f(x') \cos nx' dx' \quad \text{and} \quad \int_{-\pi}^{\pi} f(x') \sin nx' dx'$$

exist for all values of n . (§ 61, VI.)

The trigonometrical series (1) is called a *Fourier's Series*, when the coefficients a_0, a_1, b_1 , etc., are given by

$$\left. \begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx', \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx', \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx', \end{aligned} \right\} \dots(2)$$

and these coefficients are called *Fourier's Constants* for the function $f(x)$.

The important thing about the *Fourier's Series* is that, when $f(x)$ satisfies very general conditions in the interval $(-\pi, \pi)$, the sum of this series is equal to $f(x)$, or in special cases to $\frac{1}{2}[f(x+0) + f(x-0)]$, when x lies in this interval.

If we assume that the arbitrary function $f(x)$, given in the interval $(-\pi, \pi)$, can be expanded in a trigonometrical series of the form (1), and that the series may be integrated term by term after multiplying both sides by $\cos nx$ or $\sin nx$, we obtain these values for the coefficients.

For, multiply both sides of the equation

$$f(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots, \\ -\pi \leq x \leq \pi, \dots\dots(3)$$

by $\cos nx$, and integrate from $-\pi$ to π .

Then
$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \pi a_n,$$

since
$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0,$$

when m, n are different integers, and

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi.$$

Thus we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' \, dx', \text{ when } n \geq 1.$$

And similarly,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' \, dx',$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \, dx'.$$

Inserting these values in the series (3), the result may be written

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \, dx' + \frac{1}{\pi} \sum_1^{\infty} \int_{-\pi}^{\pi} f(x') \cos n(x' - x) \, dx', \\ -\pi \leq x \leq \pi. \dots\dots(4)$$

This is the *Fourier's Series* for $f(x)$.

If the arbitrary function, given in $(-\pi, \pi)$, is an even function—in other words, if $f(x) = f(-x)$ when $0 < x < \pi$ —the Fourier's Series becomes the Cosine Series:

$$f(x) = \frac{1}{\pi} \int_0^{\pi} f(x') \, dx' + \frac{2}{\pi} \sum_1^{\infty} \cos nx \int_0^{\pi} f(x') \cos nx' \, dx', \\ 0 \leq x \leq \pi. \dots\dots(5)$$

Again, if it is an odd function—*i.e.* if $f(x) = -f(-x)$ when $0 < x < \pi$ —the Fourier's Series becomes the Sine Series:

$$f(x) = \frac{2}{\pi} \sum_1^{\infty} \sin nx \int_0^{\pi} f(x') \sin nx' \, dx', \quad 0 \leq x \leq \pi. \dots\dots(6)$$

The expansions in (5) and (6) could have been obtained in the same way as the expansion in (3) by assuming a series in cosines only, or a series in sines only, and multiplying by $\cos nx$ or $\sin nx$, as the case may be, integrating now from 0 to π .

Further, if we take the interval $(-l, l)$ instead of $(-\pi, \pi)$, we find the following expansions, corresponding to (4), (5) and (6):

$$f(x) = \frac{1}{2l} \int_{-l}^l f(x') dx' + \frac{1}{l} \sum_1^{\infty} \int_{-l}^l f(x') \cos \frac{n\pi}{l} (x' - x) dx',$$

$$-l \leq x \leq l. \dots\dots(7)$$

$$f(x) = \frac{1}{l} \int_0^l f(x') dx' + \frac{2}{l} \sum_1^{\infty} \cos \frac{n\pi}{l} x \int_0^l f(x') \cos \frac{n\pi}{l} x' dx',$$

$$0 \leq x \leq l. \dots\dots(8)$$

$$f(x) = \frac{2}{l} \sum_1^{\infty} \sin \frac{n\pi}{l} x \int_0^l f(x') \sin \frac{n\pi}{l} x' dx', \quad 0 \leq x \leq l. \dots\dots(9)$$

However, this method does not give a rigorous proof of these very important expansions for the following reasons:

- (i) We have assumed the possibility of the expansion of the function in the series.
- (ii) We have integrated the series term by term.

This would have been allowable if the convergence of the series were uniform, since multiplying right through by $\cos nx$ or $\sin nx$ does not affect the uniformity; but this property has not been proved, and indeed is not generally applicable to the whole interval in these expansions.

- (iii) The discussion does not give us any information as to the behaviour of the series at points of discontinuity, if such arise, nor does it give any suggestion as to the conditions to which $f(x)$ must be subject if the expansion is to hold.

Another method of obtaining the coefficients, due to Lagrange,* may be illustrated by the case of the Sine Series.

Consider the curve

$$y = a_1 \sin x + a_2 \sin 2x + \dots + a_{n-1} \sin (n-1)x.$$

We can obtain the values of the coefficients

$$a_1, \quad a_2, \quad \dots \quad a_{n-1},$$

* Lagrange, *Œuvres*, T. I., p. 533; Byerly, *Fourier's Series*, etc., p. 30.

so that this curve shall pass through the points of the curve

$$y = f(x),$$

at which the abscissae are

$$\frac{\pi}{n}, \frac{2\pi}{n}, \dots, (n-1) \frac{\pi}{n}.$$

In this way we find a_1, a_2, \dots, a_{n-1} as functions of n . Proceeding to the limit as $n \rightarrow \infty$, we have the values of the coefficients in the infinite series

$$f(x) = a_1 \sin x + a_2 \sin 2x + \dots$$

But this passage from a finite number of equations to an infinite number requires more complete examination before the results can be accepted.

The most satisfactory method of discussing the possibility of expressing an arbitrary function $f(x)$, given in the interval $(-\pi, \pi)$ by the corresponding Fourier's Series, is to take the series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

where the constants have the values given in (2), and sum the terms up to $(a_n \cos nx + b_n \sin nx)$; then to find the limit of this sum, if it has a limit, as $n \rightarrow \infty$.

In this way we shall show that when $f(x)$ satisfies very general conditions, the Fourier's Series for $f(x)$ converges to $f(x)$ at every point in $(-\pi, \pi)$, where $f(x)$ is continuous; that it converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ at every point of ordinary discontinuity; also that it converges to $\frac{1}{2}[f(-\pi+0) + f(\pi-0)]$ at $x = \pm\pi$, when these limits exist.

Since the series is periodic in x with period 2π , when the sum is known in $(-\pi, \pi)$, it is also known for every value of x .

If it is more convenient to take the interval in which $f(x)$ is defined as $(0, 2\pi)$, the values of the coefficients in the corresponding expansion would be

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x') dx',$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x') \cos nx' dx', \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x') \sin nx' dx', \quad n \geq 1.$$

It need hardly be added that the function $f(x)$ can have different analytical expressions in different parts of the given interval. And in particular we can obtain any number of such

expansions which will hold in the interval $(0, \pi)$, since we can give $f(x)$ any value we please, subject to the general conditions we shall establish, in the interval $(-\pi, 0)$

The following discussion of the possibility of the expansion of an arbitrary function in the corresponding Fourier's Series depends upon a modified form of the integrals by means of which Dirichlet* gave the first rigorous proof that, for a large class of functions, the Fourier's Series converges to $f(x)$. With the help of the Second Theorem of Mean Value the sum of the series can be deduced at once from these integrals, which we shall call Dirichlet's Integrals.

91. Dirichlet's Integrals (First Form).

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{x} dx = \frac{\pi}{2} f(+0), \quad \text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{x} dx = 0,$$

where $0 < a < b$.

When we apply the Second Theorem of Mean Value to the integral

$$\int_{b'}^{c'} \frac{\sin x}{x} dx, \quad 0 < b' < c',$$

we see that $\int_{b'}^{c'} \frac{\sin x}{x} dx = \frac{1}{b'} \int_{b'}^{\xi} \sin x dx + \frac{1}{c'} \int_{\xi}^{c'} \sin x dx$,

where $b' \leq \xi \leq c'$.

$$\begin{aligned} \text{Thus} \quad \left| \int_{b'}^{c'} \frac{\sin x}{x} dx \right| &\leq 2 \left(\frac{1}{b'} + \frac{1}{c'} \right) \\ &< \frac{4}{b'}. \end{aligned}$$

It follows that the integral

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

is convergent. Its value has been found in § 88 to be $\frac{1}{2}\pi$.

The integrals $\int_0^a \frac{\sin \mu x}{x} dx$ and $\int_a^b \frac{\sin \mu x}{x} dx$ can be transformed, by putting $\mu x = x'$, into

$$\int_0^{\mu a} \frac{\sin x}{x} dx, \quad \int_{\mu a}^{\mu b} \frac{\sin x}{x} dx,$$

respectively.

* *J. Math.*, Berlin, 4, p. 157, 1829, and Dove's *Repertorium der Physik*, Bd. I., p. 152, 1837.

It follows that

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a \frac{\sin \mu x}{x} dx = \int_0^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \pi, \quad 0 < a,$$

and
$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b \frac{\sin \mu x}{x} dx = \text{Lt}_{\mu \rightarrow \infty} \int_{\mu a}^{\mu b} \frac{\sin x}{x} dx = 0, \quad 0 < a < b.$$

These results are special cases of the theorem that, *when $f(x)$ satisfies certain conditions, given below,*

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{x} dx = \frac{\pi}{2} f(+0), \quad 0 < a,$$

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{x} dx = 0, \quad 0 < a < b.$$

In the discussion of this theorem we shall, first of all, assume that $f(x)$ satisfies the conditions we have imposed upon $\phi(x)$ in the statement of the Second Theorem of Mean Value (§ 50); viz., it is to be bounded and monotonic (and therefore integrable) in the interval with which we are concerned.

It is clear that $\frac{\sin \mu x}{x}$ satisfies the conditions imposed upon $\psi(x)$ in that theorem. It is bounded and integrable, and does not change sign more than a finite number of times in the interval.

We shall remove some of the restrictions placed upon $f(x)$ later.

I. Consider the integral

$$\int_a^b f(x) \frac{\sin \mu x}{x} dx, \quad 0 < a < b.$$

From the Second Theorem of Mean Value

$$\int_a^b f(x) \frac{\sin \mu x}{x} dx = f(a+0) \int_a^{\xi} \frac{\sin \mu x}{x} dx + f(b-0) \int_{\xi}^b \frac{\sin \mu x}{x} dx,$$

where ξ is some definite value of x in $a \leq x \leq b$.

Since $f(x)$ is monotonic in $a \leq x \leq b$, the limits $f(a+0)$ and $f(b-0)$ exist.

And we have seen that the limits of the integrals on the right-hand are zero as $\mu \rightarrow \infty$.

It follows that, under the conditions named above,

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{x} dx = 0, \quad \text{when } 0 < a < b.$$

II. Consider the integral

$$\int_0^a f(x) \frac{\sin \mu x}{x} dx, \quad 0 < a.$$

Put $f(x) = \phi(x) + f(+0)$. The limit $f(+0)$ exists, since $f(x)$ is monotonic in $0 \leq x \leq a$.

Then $\phi(x)$ is monotonic and $\phi(+0) = 0$.

Also

$$\int_0^a f(x) \frac{\sin \mu x}{x} dx = f(+0) \int_0^a \frac{\sin \mu x}{x} dx + \int_0^a \phi(x) \frac{\sin \mu x}{x} dx.$$

As $\mu \rightarrow \infty$ the first integral on the right-hand has the limit $\frac{1}{2}\pi$. We shall now show that the second integral has the limit zero.

To prove this, it is sufficient to show that, to the arbitrary positive number ϵ , there corresponds a positive number ν such that

$$\left| \int_0^a \phi(x) \frac{\sin \mu x}{x} dx \right| < \epsilon \quad \text{when } \mu \geq \nu.$$

Let us break up the interval $(0, a)$ into two parts, $(0, \alpha)$ and (α, a) , where α is chosen so that

$$|\phi(\alpha - 0)| < \epsilon/2\pi.$$

We can do this, since we are given that $\phi(+0) = 0$, and thus there is a positive number α such that

$$|\phi(x)| < \epsilon/2\pi, \quad \text{when } 0 < x \leq \alpha.$$

Then, by the Second Theorem of Mean Value,

$$\int_0^a \phi(x) \frac{\sin \mu x}{x} dx = \phi(\alpha - 0) \int_{\xi}^a \frac{\sin \mu x}{x} dx,$$

since $\phi(+0) = 0$, ξ being some definite value of x in $0 \leq x \leq a$.

But, in the curve $y = \frac{\sin x}{x}$, $x \geq 0$,

the successive waves have the same breadth and diminishing amplitude, and the area between 0 and π is greater than that between π and 2π in absolute value: that between π and 2π is greater than that between 2π and 3π , and so on; since $|\sin x|$ goes through the same set of values in each case, and $1/x$ diminishes as x increases.

Thus
$$\int_0^x \frac{\sin x}{x} dx \leq \int_0^{\pi} \frac{\sin x}{x} dx < \pi,$$

whatever positive value x may have.

Also
$$\int_p^q \frac{\sin x}{x} dx = \int_0^q \frac{\sin x}{x} dx - \int_0^p \frac{\sin x}{x} dx,$$

and each of the integrals on the right-hand is positive and less than π for $0 < p < q$.

Therefore
$$\left| \int_p^q \frac{\sin x}{x} dx \right| < \pi, \text{ when } 0 \leq p < q.$$

It follows that
$$\left| \int_0^a \phi(x) \frac{\sin \mu x}{x} dx \right| < \frac{\epsilon}{2\pi} \times \pi$$

$$< \frac{\epsilon}{2},$$

and this is independent of μ .

But we have seen in I. that

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^a \phi(x) \frac{\sin \mu x}{x} dx = 0, \quad 0 < a < a.$$

Therefore, to the arbitrary positive number $\epsilon/2$, there corresponds a positive number ν such that

$$\left| \int_a^a \phi(x) \frac{\sin \mu x}{x} dx \right| < \frac{\epsilon}{2}, \text{ when } \mu \geq \nu.$$

Also

$$\left| \int_0^a \phi(x) \frac{\sin \mu x}{x} dx \right| \leq \left| \int_0^a \phi(x) \frac{\sin \mu x}{x} dx \right| + \left| \int_a^a \phi(x) \frac{\sin \mu x}{x} dx \right|.$$

Therefore

$$\left| \int_0^a \phi(x) \frac{\sin \mu x}{x} dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$< \epsilon, \text{ when } \mu \geq \nu.$$

Thus
$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a \phi(x) \frac{\sin \mu x}{x} dx = 0.$$

And, finally, under the conditions named above,

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{x} dx = \frac{\pi}{2} f(+0).$$

92. In the preceding section we have assumed that $f(x)$ is bounded and monotonic in the intervals $(0, a)$ and (a, b) . We shall now show that these restrictions may be somewhat relaxed.

I. *Dirichlet's Integrals still hold when $f(x)$ is bounded, and the interval of integration can be broken up into a finite number of open partial intervals, in each of which $f(x)$ is monotonic.**

This follows at once from the fact that under these conditions we may write

$$f(x) = F(x) - G(x),$$

* This condition is sometimes, but less exactly, expressed by the phrase: $f(x)$ shall have only a finite number of maxima and minima in the interval.

where $F(x)$ and $G(x)$ are positive, bounded, and monotonic increasing in the interval with which we are concerned [cf. § 36].

This result can be obtained, as follows, without the use of the theorem of § 36 :

Let the interval $(0, a)$ be broken up into the n open intervals,

$$(0, a_1), (a_1, a_2), \dots, (a_{n-1}, a),$$

in each of which $f(x)$ is bounded and monotonic.

Then, writing $a_0 = 0$ and $a_n = a$, we have

$$\int_0^a f(x) \frac{\sin \mu x}{x} dx = \sum_{r=1}^n \int_{a_{r-1}}^{a_r} f(x) \frac{\sin \mu x}{x} dx.$$

The first integral in this sum has the limit $\frac{\pi}{2} f(+0)$, and the others have the limit zero when $\mu \rightarrow \infty$.

It follows that, under the given conditions,

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{x} dx = \frac{\pi}{2} f(+0), \quad 0 < a.$$

The proof that, under the same conditions,

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{x} dx = 0, \quad 0 < a < b,$$

is practically contained in the above.

It will be seen that we have used the condition that the number of partial intervals is finite, as we have relied upon the theorem that the limit of a sum is equal to the sum of the limits.

II. *The integrals still hold for certain cases where a finite number of points of infinite discontinuity of $f(x)$ (as defined in § 33) occur in the interval of integration.*

We shall suppose that, when arbitrarily small neighbourhoods of these points of infinite discontinuity are excluded, the remainder of the interval of integration can be broken up into a finite number of open partial intervals, in each of which $f(x)$ is bounded and monotonic.

Further, we shall assume that the infinite integral $\int f(x) dx$ is absolutely convergent in the interval of integration, and that $x=0$ is not a point of infinite discontinuity.

We may take first the case when an infinite discontinuity occurs at the upper limit b of the integral $\int_a^b f(x) \frac{\sin \mu x}{x} dx$, and only there.

Since we are given that $\int_a^b f(x) dx$ is absolutely convergent, we know that $\int_a^b f(x) \frac{\sin \mu x}{x} dx$ also converges, for $\left| \frac{\sin \mu x}{x} \right| \leq \frac{1}{a}$ in (a, b) . And this convergence is uniform.

To the arbitrary positive number ϵ there corresponds a positive number η , which we take less than $(b - a)$, such that

$$\left| \int_{b-\xi}^b f(x) \frac{\sin \mu x}{x} dx \right| < \frac{\epsilon}{2}, \text{ when } 0 < \xi \leq \eta, \dots\dots\dots(1)$$

and the same η serves for all values of μ .

But

$$\int_a^b f(x) \frac{\sin \mu x}{x} dx = \int_a^{b-\eta} f(x) \frac{\sin \mu x}{x} dx + \int_{b-\eta}^b f(x) \frac{\sin \mu x}{x} dx. \quad (2)$$

And, by I. above,

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^{b-\eta} f(x) \frac{\sin \mu x}{x} dx = 0.$$

It follows that there is a positive number ν such that

$$\left| \int_a^{b-\eta} f(x) \frac{\sin \mu x}{x} dx \right| < \frac{\epsilon}{2}, \text{ when } \mu \geq \nu. \dots\dots\dots(3)$$

From (1), (2) and (3), we have at once

$$\begin{aligned} \left| \int_a^b f(x) \frac{\sin \mu x}{x} dx \right| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon, \text{ when } \mu \geq \nu. \end{aligned}$$

Thus we have shown that, with the conditions described above,

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{x} dx = 0, \quad 0 < a < b.$$

A similar argument applies to the case when an infinite discontinuity occurs at the lower limit a of the integral, and only there.

When there is an infinite discontinuity at a and at b , and only there, the result follows from these two, since

$$\int_a^b f(x) \frac{\sin \mu x}{x} dx = \int_a^c f(x) \frac{\sin \mu x}{x} dx + \int_c^b f(x) \frac{\sin \mu x}{x} dx, \quad a < c < b.$$

When an infinite discontinuity occurs between a and b we proceed in the same way; and as we have assumed that the number of points of infinite discontinuity is finite, we can break up the given interval into a definite number of partial intervals, to which we can apply the results just obtained.

Thus, under the conditions stated above in II.,

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{x} dx = 0, \text{ when } 0 < a < b.$$

Further, we have assumed that $x=0$ is not a point of infinite discontinuity of $f(x)$. Thus the interval $(0, a)$ can be broken up into two intervals, $(0, \alpha)$ and (α, a) , where $f(x)$ is bounded in $(0, \alpha)$, and satisfies the conditions given in I. of this section in $(0, \alpha)$.

It follows that

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{x} dx = \frac{\pi}{2} f(+0),$$

and we have just shown that

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^a f(x) \frac{\sin \mu x}{x} dx = 0.$$

Therefore, under the conditions stated above in II.,

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{x} dx = \frac{\pi}{2} f(+0).$$

93. Dirichlet's Conditions. The results which we have obtained in §§ 91, 92 can be conveniently expressed in terms of what we shall call Dirichlet's Conditions.

A function $f(x)$ will be said to satisfy Dirichlet's Conditions in an interval (a, b) , in which it is defined, when it is subject to one of the two following conditions:

- (i) $f(x)$ is bounded in (a, b) , and the interval can be broken up into a finite number of open partial intervals, in each of which $f(x)$ is monotonic.
- (ii) $f(x)$ has a finite number of points of infinite discontinuity in the interval, but, when arbitrarily small neighbourhoods of these points are excluded, $f(x)$ is bounded in the remainder of the interval, and this can be broken up into a finite number of open partial intervals, in each of which $f(x)$ is monotonic. Further, the infinite integral $\int_a^b f(x) dx$ is to be absolutely convergent.

We may now say that:

When $f(x)$ satisfies Dirichlet's Conditions in the intervals $(0, a)$ and (a, b) respectively, where $0 < a < b$, and $f(+0)$ exists, then

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{x} dx = \frac{\pi}{2} f(+0),$$

and

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{x} dx = 0.$$

It follows from the properties of monotonic functions (cf. § 34) that except at the points, if any, where $f(x)$ becomes infinite, or oscillates infinitely, a function which satisfies Dirichlet's Conditions, as defined above, can only have *ordinary* discontinuities.* But we have not assumed that the function $f(x)$ shall have only a finite number of ordinary discontinuities. A bounded function which is monotonic in an open interval can have an infinite number of ordinary discontinuities in that interval [cf. § 34].

Perhaps it should be added that the conditions which Dirichlet himself imposed upon the function $f(x)$ in a given interval (a, b) were not so general as those to which we have given the name *Dirichlet's Conditions*. He contemplated at first only bounded functions, continuous, except at a finite number of ordinary discontinuities, and with only a finite number of maxima and minima. Later he extended his results to the case in which there are a finite number of points of infinite discontinuity in the interval, provided that the infinite integral $\int_a^b f(x) dx$ is absolutely convergent.

If the somewhat difficult idea of a *function of bounded variation*, due to Jordan, is introduced, the statement of Dirichlet's Conditions can be simplified. But, at least in this place, it seems unadvisable to complicate the discussion by further reference to this class of function.

94. Dirichlet's Integrals (Second Form).

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{\sin x} dx = \frac{\pi}{2} f(+0), \quad \text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{\sin x} dx = 0,$$

where $0 < a < b < \pi$.

In the discussion of Fourier's Series the integrals which we shall meet are slightly different from Dirichlet's Integrals, the properties of which we have just established.

* These conditions can be further extended so as to include a finite number of points of oscillatory discontinuity in the neighbourhood of which the function is bounded [e.g. $\sin 1/(x-c)$ at $x=c$], or of continuity, with an infinite number of maxima and minima in their neighbourhood [e.g. $(x-c) \sin 1/(x-c)$ at $x=c$].

This generalisation would also apply to the sections in which Dirichlet's Conditions are employed.

The second type of integral—and this is the one which Dirichlet himself used in his classical treatment of Fourier's Series—is

$$\int_0^a f(x) \frac{\sin \mu x}{\sin x} dx, \quad \int_a^b f(x) \frac{\sin \mu x}{\sin x} dx,$$

where $0 < a < b < \pi$.

We shall now prove that :

When $f(x)$ satisfies Dirichlet's Conditions (as defined in § 93) in the intervals $(0, a)$ and (a, b) respectively, where $0 < a < b < \pi$, and $f(+0)$ exists, then

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{\sin x} dx = \frac{\pi}{2} f(+0),$$

and
$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{\sin x} dx = 0.$$

Let us suppose that $f(x)$ satisfies the first of the two conditions given in § 93 as Dirichlet's Conditions :

$f(x)$ is bounded, and the intervals $(0, a)$ and (a, b) can be broken up into a finite number of open partial intervals, in each of which $f(x)$ is monotonic.

Then, by § 36, we can write

$$f(x) = F(x) - G(x),$$

where $F(x)$, $G(x)$ are positive, bounded and monotonic increasing in the interval with which we are concerned.

$$\text{Then } f(x) \frac{\sin \mu x}{\sin x} = \left[F(x) \frac{x}{\sin x} - G(x) \frac{x}{\sin x} \right] \frac{\sin \mu x}{x}.$$

But $x/\sin x$ is bounded, positive and monotonic increasing in $(0, a)$ or (a, b) , when $0 < a < b < \pi$.*

Thus $F(x) \frac{x}{\sin x}$ and $G(x) \frac{x}{\sin x}$ will both be bounded, positive and monotonic increasing in the interval $(0, a)$ or (a, b) , as the case may be, provided that $0 < a < b < \pi$.

It follows from § 91 that

$$\begin{aligned} \text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{\sin x} dx &= \frac{\pi}{2} (F(+0) - G(+0)) \\ &= \frac{\pi}{2} f(+0), \end{aligned}$$

and
$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{\sin x} dx = 0, \text{ when } 0 < a < b < \pi.$$

* We assign to $x/\sin x$ the value 1 at $x=0$.

Next, let $f(x)$ satisfy the second of the conditions given in § 93, and let $f(+0)$ exist.

We can prove, just as in § 92, II., that

$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{\sin x} dx = 0 \quad \text{when } 0 < a < b < \pi.$$

For we are given that $\int_a^b f(x) dx$ is absolutely convergent, and we know that $x/\sin x$ is bounded and integrable in (a, b) .

It follows that
$$\int_a^b f(x) \frac{x}{\sin x} dx$$

is absolutely convergent; and the preceding proof [§ 92, II.] applies to the neighbourhood of the point, or points, of infinite discontinuity, when we write $f(x) \frac{x}{\sin x}$ in place of $f(x)$.

Also, for the case
$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{\sin x} dx,$$

we need only, as before, break up the interval $(0, a)$ into $(0, \alpha)$ and (α, a) , where $f(x)$ is bounded in $(0, \alpha)$, and from the results we have already obtained in this section the limit is found as stated.

If it is desired to obtain the second form of Dirichlet's Integrals without the use of the theorem of § 36, the reader may proceed as follows :

(i) Let $f(x)$ be positive, bounded and monotonic increasing in $(0, a)$ and (a, b) .

Then $\frac{x}{\sin x}$ is so also, and $\phi(x) = f(x) \frac{x}{\sin x}$ is so also, $0 < a < b < \pi$.

But, by § 91,

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a \phi(x) \frac{\sin \mu x}{x} dx = \frac{\pi}{2} \phi(+0) = \frac{\pi}{2} f(+0).$$

Therefore

$$\text{Lt}_{\mu \rightarrow \infty} \int_0^a f(x) \frac{\sin \mu x}{\sin x} dx = \frac{\pi}{2} f(+0).$$

Also

$$\int_a^b f(x) \frac{\sin \mu x}{\sin x} dx = \int_0^b f(x) \frac{\sin \mu x}{\sin x} dx - \int_0^a f(x) \frac{\sin \mu x}{\sin x} dx.$$

Therefore
$$\text{Lt}_{\mu \rightarrow \infty} \int_a^b f(x) \frac{\sin \mu x}{\sin x} dx = 0.$$

(ii) Let $f(x)$ be positive, bounded and monotonic decreasing.

Then for some value of c the function $c - f(x)$ is positive, bounded and monotonic increasing.

Also

$$\int_0^a [c - f(x)] \frac{\sin \mu x}{\sin x} dx = c \int_0^a \frac{\sin \mu x}{\sin x} dx - \int_0^a f(x) \frac{\sin \mu x}{\sin x} dx.$$

Using (i), the result follows.

- (iii) If $f(x)$ is bounded and monotonic increasing, but not positive all the time, by adding a constant we can make it positive, and proceed as in (ii); and a similar remark applies to the case of the monotonic decreasing function.
- (iv) When $f(x)$ is bounded and the interval can be broken up into a finite number of open partial intervals in which it is monotonic, the result follows from (i)-(iii).
- (v) And if $f(x)$ has a finite number of points of infinite discontinuity, as stated in the second of Dirichlet's Conditions, so far as these points are concerned the proof is similar to that given above.

95. Proof of the Convergence of Fourier's Series. In the opening sections of this chapter we have given the usual elementary, but quite incomplete, argument, by means of which the coefficients in the expansion

$$f(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

$-\pi \leq x \leq \pi$

are obtained.

We now return to this question, which we approach in quite a different way.

We take the Fourier's Series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

where the coefficients are given by

$$\left. \begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx', \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx', \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx'. \end{aligned} \right\}$$

We find the sum of the terms of this series up to $\cos nx$ and $\sin nx$, and we then examine whether this sum has a limit as $n \rightarrow \infty$.

We shall prove that, *when $f(x)$ is given in the interval $(-\pi, \pi)$, and satisfies Dirichlet's Conditions in that interval, this sum has a limit as $n \rightarrow \infty$. It is equal to $f(x)$ at any point in $-\pi < x < \pi$, where $f(x)$ is continuous; and to*

$$\frac{1}{2} [f(x+0) + f(x-0)],$$

when there is an ordinary discontinuity at the point; and to

$\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$ at $x = \pm\pi$, when the limits $f(\pi-0)$ and $f(-\pi+0)$ exist.

Let

$$S_n(x) = a_0 + (a_1 \cos x + b_1 \sin x) + \dots + (a_n \cos nx + b_n \sin nx),$$

where a_0, a_1, b_1 , etc., have the values given above.

Then we find, without difficulty, that

$$\begin{aligned} S_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') [1 + 2 \cos(x' - x) + \dots + 2 \cos n(x' - x)] dx' \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \frac{\sin \frac{1}{2}(2n+1)(x' - x)}{\sin \frac{1}{2}(x' - x)} dx' \\ &= \frac{1}{2\pi} \int_{-\pi}^x f(x') \frac{\sin \frac{1}{2}(2n+1)(x' - x)}{\sin \frac{1}{2}(x' - x)} dx' \\ &\quad + \frac{1}{2\pi} \int_x^{\pi} f(x') \frac{\sin \frac{1}{2}(2n+1)(x' - x)}{\sin \frac{1}{2}(x - x')} dx'. \end{aligned}$$

Thus

$$\begin{aligned} S_n(x) &= \frac{1}{\pi} \int_0^{\frac{1}{2}\pi + \frac{1}{2}x} f(x - 2a) \frac{\sin (2n+1)a}{\sin a} da \\ &\quad + \frac{1}{\pi} \int_0^{\frac{1}{2}\pi - \frac{1}{2}x} f(x + 2a) \frac{\sin (2n+1)a}{\sin a} da, \dots\dots\dots(1) \end{aligned}$$

on changing the variable by the substitutions $x' - x = \mp 2a$.

If $-\pi < x < \pi$ and $f(x)$ satisfies Dirichlet's Conditions in the interval $(-\pi, \pi)$, $f(x \mp 2a)$ considered as functions of a in the integrals of (1) satisfy Dirichlet's Conditions in the intervals $(0, \frac{1}{2}\pi + \frac{1}{2}x)$ and $(0, \frac{1}{2}\pi - \frac{1}{2}x)$ respectively, and these functions of a have limits as $a \rightarrow 0$, provided that at the point x with which we are concerned $f(x+0)$ and $f(x-0)$ exist.

It follows from § 94 that, when x lies between $-\pi$ and π and $f(x-0)$ and $f(x+0)$ both exist,

$$\begin{aligned} \text{Lt}_{n \rightarrow \infty} S_n(x) &= \frac{1}{\pi} \left[\frac{\pi}{2} f(x-0) + \frac{\pi}{2} f(x+0) \right] \\ &= \frac{1}{2} [f(x-0) + f(x+0)], \end{aligned}$$

giving the value $f(x)$ at a point where $f(x)$ is continuous.

We have yet to examine the cases $x = \pm\pi$.

In finding the sum of the series for $x = \pi$, we must insert this value for x in $S_n(x)$ before proceeding to the limit.

$$\text{Thus} \quad S_n(\pi) = \frac{1}{\pi} \int_0^{\pi} f(\pi - 2a) \frac{\sin (2n+1)a}{\sin a} da,$$

since the second integral in (1) is zero.

It follows that

$$\begin{aligned} S_n(\pi) &= \frac{1}{\pi} \int_0^{\pi-\xi} f(\pi-2\alpha) \frac{\sin(2n+1)\alpha}{\sin\alpha} d\alpha \\ &\quad + \frac{1}{\pi} \int_{\pi-\xi}^{\pi} f(\pi-2\alpha) \frac{\sin(2n+1)\alpha}{\sin\alpha} d\alpha \\ &= \frac{1}{\pi} \int_0^{\pi-\xi} f(\pi-2\alpha) \frac{\sin(2n+1)\alpha}{\sin\alpha} d\alpha \\ &\quad + \frac{1}{\pi} \int_0^{\xi} f(-\pi+2\alpha) \frac{\sin(2n+1)\alpha}{\sin\alpha} d\alpha, \end{aligned}$$

where ξ is any number between 0 and π .

We can apply the theorem of § 94 to these integrals, if $f(x)$ satisfies Dirichlet's Conditions in $(-\pi, \pi)$, and the limits $f(\pi-0)$, $f(-\pi+0)$ exist.

Thus we have

$$\text{Lt}_{n \rightarrow \infty} S_n(\pi) = \frac{1}{2} [f(-\pi+0) + f(\pi-0)].$$

A similar discussion gives the same value for the sum at $x = -\pi$, which is otherwise obvious since the series has a period 2π .

Thus we have shown that *when the arbitrary function $f(x)$ satisfies Dirichlet's Conditions in the interval $(-\pi, \pi)$, and*

$$\left. \begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx', \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx', \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx', \end{aligned} \right\}$$

the Fourier's Series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

converges to

$$\frac{1}{2} [f(x+0) + f(x-0)]$$

at every point in $-\pi < x < \pi$ where $f(x+0)$ and $f(x-0)$ exist; and at $x = \pm\pi$ it converges to

$$\frac{1}{2} [f(-\pi+0) + f(\pi-0)],$$

*when $f(-\pi+0)$ and $f(\pi-0)$ exist.**

* If the reader refers to § 101, he will see that, if $f(x)$ is defined outside the interval $(-\pi, \pi)$ by the equation $f(x+2\pi) = f(x)$, we can replace (1) by

$$S_n(x) = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} f(x-2\alpha) \frac{\sin(2n+1)\alpha}{\sin\alpha} d\alpha + \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} f(x+2\alpha) \frac{\sin(2n+1)\alpha}{\sin\alpha} d\alpha.$$

In this form we can apply the result of § 94 at once to every point in the closed interval $(-\pi, \pi)$, except points of infinite discontinuity.

There is, of course, no reason why the arbitrary function should be defined by the same analytical expression in all the interval [cf. Ex. 2 below].

Also it should be noticed that if we first sum the series, and then let x approach a point of ordinary discontinuity x_0 , we would obtain $f(x_0+0)$ or $f(x_0-0)$, according to the side from which we approach the point. On the other hand, if we insert the value x_0 in the terms of the series and then sum the series, we obtain $\frac{1}{2}[f(x_0+0)+f(x_0-0)]$.

We have already pointed out more than once that when we speak of the sum of the series for any value of x , it is understood that we first insert this value of x in the terms of the series, then find the sum of n terms, and finally obtain the limit of this sum.

Ex. 1. Find a series of sines and cosines of multiples of x which will represent $\frac{\pi}{2 \sinh \pi} e^x$ in the interval $-\pi < x < \pi$.

What is the sum of the series for $x = \pm \pi$?

Here $f(x) = \frac{\pi}{2 \sinh \pi} e^x$ and $a_n = \frac{1}{2 \sinh \pi} \int_{-\pi}^{\pi} e^x \cos nx \, dx, \quad n \geq 1.$



FIG. 16.

Integrating by parts,

$$(1+n^2) \int_{-\pi}^{\pi} e^x \cos nx \, dx = (e^{\pi} - e^{-\pi}) \cos n\pi.$$

Therefore

$$a_n = \frac{(-1)^n}{1+n^2}.$$

Also we find

$$a_0 = \frac{1}{2}.$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{2 \sinh \pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx \\ &= \frac{n}{1+n^2} (-1)^{n-1}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\pi}{2 \sinh \pi} e^x &= \frac{1}{2} + \left(-\frac{1}{1+1^2} \cos x + \frac{1}{1+1^2} \sin x \right) \\ &\quad + \left(\frac{1}{1+2^2} \cos 2x - \frac{2}{1+2^2} \sin 2x \right) + \dots, \end{aligned}$$

when $-\pi < x < \pi$.

When $x = \pm\pi$, the sum of the series is $\frac{1}{2}\pi \coth \pi$, since

$$f(-\pi+0) + f(\pi-0) = \pi \coth \pi.$$

In Fig. 16, the curves

$$\left. \begin{aligned} y &= \frac{\pi}{2 \sinh \pi} e^x, \\ y &= \frac{1}{2} + \left(-\frac{1}{1+1^2} \cos x + \frac{1}{1+1^2} \sin x \right) + (\dots) + \left(-\frac{1}{1+3^2} \cos 3x + \frac{3}{1+3^2} \sin 3x \right) \end{aligned} \right\}$$

are drawn for the interval $(-\pi, \pi)$.

It will be noticed that the expansion we have obtained converges very slowly, and that more terms would have to be taken to bring the approximation curves $[y = S_n(x)]$ near the curve of the given function in $-\pi < x < \pi$.

At $x = \pm\pi$, the sum of the series is discontinuous. The behaviour of the approximation curves at a point of discontinuity of the sum is examined in Chapter IX.

Ex. 2. Find a series of sines and cosines of multiples of x which will represent $f(x)$ in the interval $-\pi < x < \pi$, when

$$\left. \begin{aligned} f(x) &= 0, & -\pi < x \leq 0, \\ f(x) &= \frac{1}{4}\pi x, & 0 < x < \pi. \end{aligned} \right\}$$

Here

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^\pi \frac{1}{4}\pi x \, dx = \frac{\pi^2}{16}, \\ a_n &= \frac{1}{\pi} \int_0^\pi \frac{1}{4}\pi x \cos nx \, dx = \frac{1}{4n^2}(\cos n\pi - 1), \\ b_n &= \frac{1}{\pi} \int_0^\pi \frac{1}{4}\pi x \sin nx \, dx = -\frac{\pi}{4n} \cos n\pi. \end{aligned}$$

Therefore $f(x) = \frac{\pi^2}{16} + \frac{1}{2} \left[-\cos x + \frac{\pi}{2} \sin x \right] - \frac{\pi}{8} \sin 2x + \dots$,

when $-\pi < x < \pi$.

When $x = \pm\pi$ the sum of the series is $\frac{1}{8}\pi^2$, and we obtain the well-known result,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Ex. 3. Find a series of sines and cosines of multiples of x which will represent $x + x^2$ in the interval $-\pi < x < \pi$.

Here

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^\pi (x + x^2) \, dx = \frac{1}{\pi} \int_0^\pi x^2 \, dx = \frac{\pi^2}{3}, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^\pi (x + x^2) \cos nx \, dx = \frac{2}{\pi} \int_0^\pi x^2 \cos nx \, dx, \end{aligned}$$

and, after integration by parts, we find that

$$a_n = \frac{4}{n^2} \cos n\pi.$$

Also,

$$b_n = \frac{1}{\pi} \int_{-\pi}^\pi (x + x^2) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx,$$

which reduces to $b_n = (-1)^{n-1} \frac{2}{n}$.

Therefore

$$x + x^2 = \frac{\pi^2}{3} + 4 \left(-\cos x + \frac{1}{2} \sin x \right) - 4 \left(\frac{1}{2^2} \cos 2x + \frac{1}{4} \sin 2x \right) + \dots,$$

when $-\pi < x < \pi$.

When $x = \pm\pi$ the sum of the series is π^2 , and we obtain the well-known result that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

96. The Cosine Series. Let $f(x)$ be given in the interval $(0, \pi)$, and satisfy Dirichlet's Conditions in that interval. Define $f(x)$ in $-\pi \leq x < 0$ by the equation $f(-x) = f(x)$. The function thus defined for $(-\pi, \pi)$ satisfies Dirichlet's Conditions in this interval, and we can apply to it the results of § 95.

But it is clear that in this case

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx' \quad \text{leads to} \quad a_0 = \frac{1}{\pi} \int_0^{\pi} f(x') dx',$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx' \quad \text{leads to} \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x') \cos nx' dx',$$

$$\text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx' \quad \text{leads to} \quad b_n = 0.$$

Thus the sine terms disappear from the Fourier's Series for this function.

Also, from the way in which $f(x)$ was defined in $-\pi \leq x < 0$ we have

$$\frac{1}{2} [f(+0) + f(-0)] = f(+0),$$

$$\text{and} \quad \frac{1}{2} [f(-\pi + 0) + f(\pi - 0)] = f(\pi - 0),$$

provided the limits $f(+0)$ and $f(\pi - 0)$ exist.

In this case the sum of the series for $x = 0$ is $f(+0)$, and for $x = \pi$ it is $f(\pi - 0)$.

It follows that, *when $f(x)$ is an arbitrary function satisfying Dirichlet's Conditions in the interval $(0, \pi)$, the sum of the Cosine Series*

$$\frac{1}{\pi} \int_0^{\pi} f(x') dx' + \frac{2}{\pi} \sum_1^{\infty} \cos nx \int_0^{\pi} f(x') \cos nx' dx'$$

is equal to $\frac{1}{2} [f(x+0) + f(x-0)]$

at every point between 0 and π where $f(x+0)$ and $f(x-0)$ exist; and, when $f(+0)$ and $f(\pi-0)$ exist, the sum is $f(+0)$ at $x=0$ and $f(\pi-0)$ at $x=\pi$.

Thus, when $f(x)$ is continuous and satisfies Dirichlet's Conditions in the interval $(0, \pi)$, the Cosine Series represents it in this *closed* interval.

Ex. 1. Find a series of cosines of multiples of x which will represent x in the interval $(0, \pi)$.

Here
$$a_0 = \frac{1}{\pi} \int_0^\pi x \, dx = \frac{1}{2}\pi,$$

and
$$a_n = \frac{2}{\pi} \int_0^\pi x \cos nx \, dx = \frac{2}{n^2\pi} (\cos n\pi - 1).$$

Therefore
$$x = \frac{\pi}{2} - \frac{4}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \dots \right], \quad 0 \leq x \leq \pi.$$

Since the sum of the series is zero at $x=0$, we have again

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

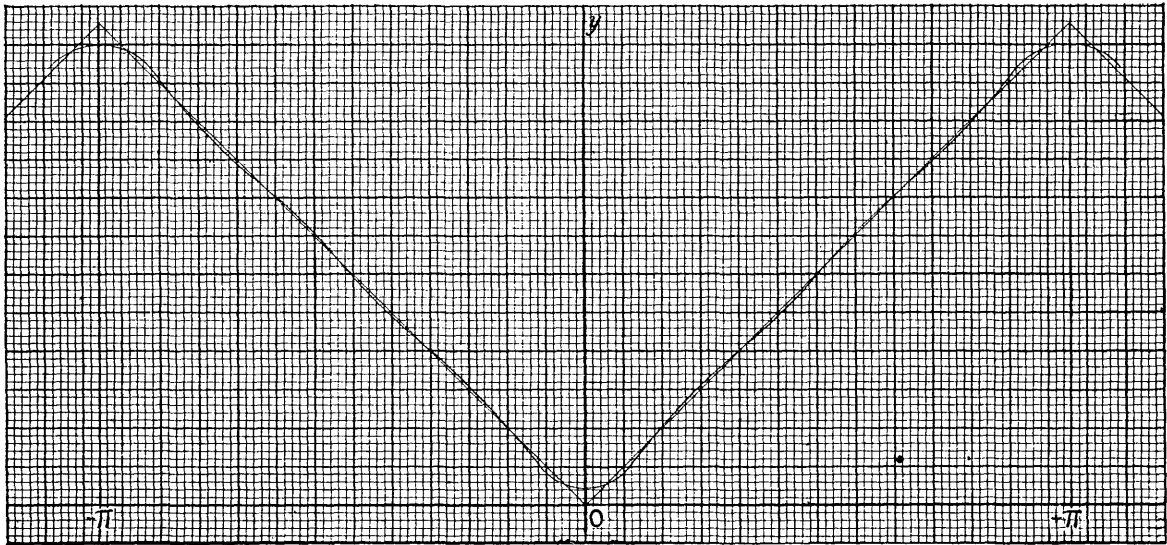


FIG. 17.

In Fig. 17, the lines
$$\left. \begin{aligned} y &= x, & 0 \leq x \leq \pi, \\ y &= -x, & -\pi \leq x \leq 0, \end{aligned} \right\}$$

and the approximation curve

$$y = \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x \right), \quad -\pi \leq x \leq \pi,$$

are drawn.

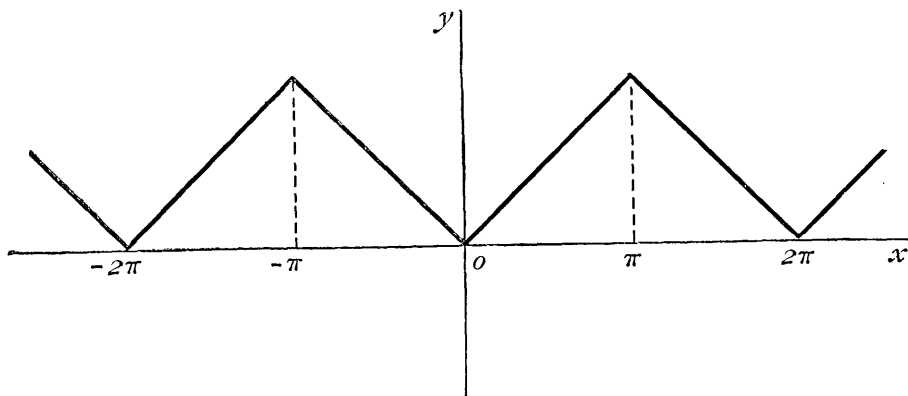


FIG. 18.

It will be seen how closely this approximation curve approaches the lines $y = \pm x$ in the whole interval.

Since the Fourier's Series has a period 2π , this series for unrestricted values of x represents the ordinates of the lines shown in Fig. 18, the part from the interval $(-\pi, \pi)$ being repeated indefinitely in both directions.

The sum is continuous for all values of x .

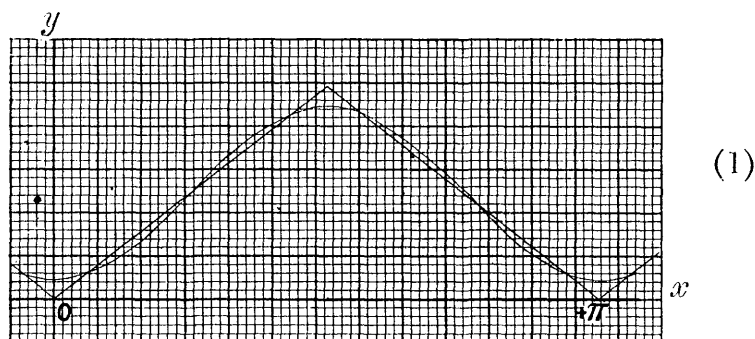
Ex. 2. Find a series of cosines of multiples of x which will represent $f(x)$ in the interval $(0, \pi)$, where

$$\left. \begin{aligned} f(x) &= \frac{1}{4}\pi x, & 0 \leq x \leq \frac{1}{2}\pi, \\ f(x) &= \frac{1}{4}\pi(\pi - x), & \frac{1}{2}\pi < x \leq \pi. \end{aligned} \right\}$$

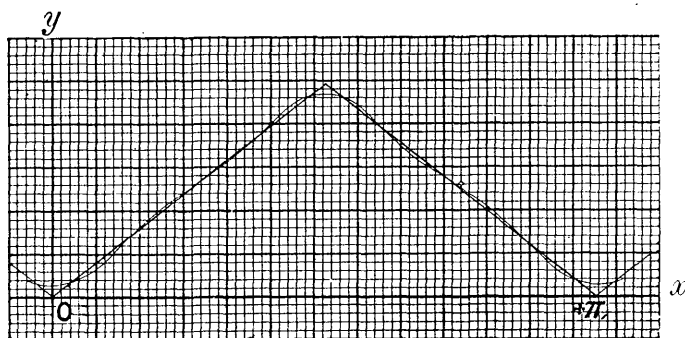
Here
$$a_0 = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} \frac{1}{4}\pi x \, dx + \frac{1}{\pi} \int_{\frac{1}{2}\pi}^{\pi} \frac{1}{4}\pi(\pi - x) \, dx = \frac{1}{16}\pi^2,$$

and
$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{1}{4}\pi x \cos nx \, dx + \frac{2}{\pi} \int_{\frac{1}{2}\pi}^{\pi} \frac{1}{4}\pi(\pi - x) \cos nx \, dx \\ &= \frac{1}{2} \int_0^{\frac{1}{2}\pi} x \cos nx \, dx + \frac{1}{2} \int_{\frac{1}{2}\pi}^{\pi} (\pi - x) \cos nx \, dx, \end{aligned}$$

which gives
$$a_n = -\frac{1}{2n^2} [1 + \cos n\pi - 2 \cos \frac{1}{2}n\pi] = \frac{2}{n^2} \cos \frac{n\pi}{2} \operatorname{si}_n^2 \frac{n\pi}{4}.$$



(1)



(2)

FIG. 19.

Thus a_n vanishes when n is odd or a multiple of 4.

Also
$$f(x) = \frac{\pi^2}{16} - 2 \left[\frac{1}{2^2} \cos 2x + \frac{1}{6^2} \cos 6x + \dots \right], \quad 0 \leq x \leq \pi.$$

In Fig. 19, the lines
$$\left. \begin{aligned} y &= \frac{1}{4}\pi x, & 0 \leq x \leq \frac{1}{2}\pi, \\ y &= \frac{1}{4}\pi(\pi - x), & \frac{1}{2}\pi \leq x \leq \pi, \end{aligned} \right\}$$

and the approximation curves

$$\left. \begin{aligned} y &= \frac{1}{16}\pi^2 - \frac{1}{2}\cos 2x, \\ y &= \frac{1}{16}\pi^2 - \frac{1}{2}\cos 2x - \frac{1}{18}\cos 6x, \end{aligned} \right\} 0 \leq x \leq \pi,$$

are drawn.

It will be noticed that the approximation curves, corresponding to the terms up to and including $\cos 6x$, approaches the given lines closely, except at the sharp corner, right through the interval $(0, \pi)$.

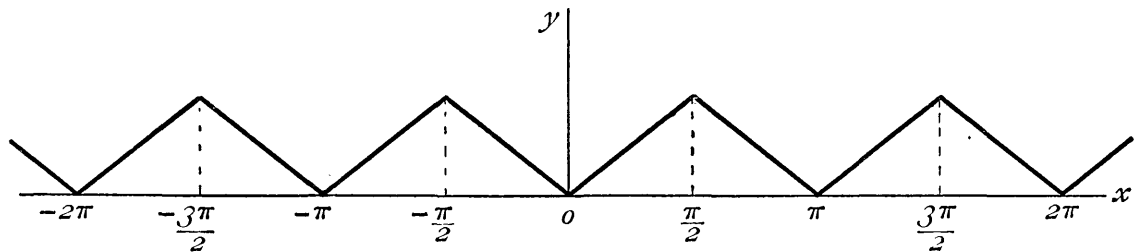


FIG. 20.

For unrestricted values of x the series represents the ordinates of the lines shown in Fig. 20, the part from $-\pi$ to π being repeated indefinitely in both directions.

The sum is continuous for all values of x .

Ex. 3. Find a series of cosines of multiples of x which will represent $f(x)$ in the interval $(0, \pi)$, where

$$\left. \begin{aligned} f(x) &= 0, & 0 \leq x < \frac{1}{2}\pi, \\ f(\frac{1}{2}\pi) &= \frac{1}{4}\pi, \\ f(x) &= \frac{1}{2}\pi, & \frac{1}{2}\pi < x \leq \pi. \end{aligned} \right\}$$

Here
$$a_0 = \frac{1}{\pi} \int_0^\pi f(x) dx = \frac{1}{\pi} \int_{\frac{1}{2}\pi}^\pi \frac{1}{2}\pi dx = \frac{1}{4}\pi.$$

Also
$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx = \int_{\frac{1}{2}\pi}^\pi \cos nx dx = -\frac{1}{n} \sin \frac{1}{2}n\pi.$$

Thus
$$f(x) = \frac{1}{4}\pi - [\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \dots], \quad 0 \leq x \leq \pi,$$

since, when $x = \frac{1}{2}\pi$, the sum of the Fourier's Series is $\frac{1}{2}[f(\frac{1}{2}\pi + 0) + f(\frac{1}{2}\pi - 0)]$.

From the values at $x = 0$ and $x = \pi$, we have the well-known result,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

In Fig. 21, the graph of the given function, and the approximation curves

$$\left. \begin{aligned} y &= \frac{1}{4}\pi - \cos x, \\ y &= \frac{1}{4}\pi - \cos x + \frac{1}{3}\cos 3x, \\ y &= \frac{1}{4}\pi - \cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x, \end{aligned} \right\} 0 \leq x \leq \pi,$$

are drawn.

The points $x = 0$ and $x = \pi$ are points of continuity in the sum of the series : the point $x = \frac{1}{2}\pi$ is a point of discontinuity.

The behaviour of the approximation curves at a point of discontinuity, when n is large, will be treated fully in Chapter IX. It will be sufficient to say now that it is proved in § 117 that just before $x = \frac{1}{2}\pi$ the approxima-

tion curve for a large value of n will have a minimum at a depth nearly 0.14 below $y=0$: that it will then ascend at a steep gradient, passing near the point $(\frac{1}{2}\pi, \frac{1}{2}\pi)$, and rising to a maximum just after $x=\frac{1}{2}\pi$ at a height nearly 0.14 above $\frac{1}{2}\pi$.

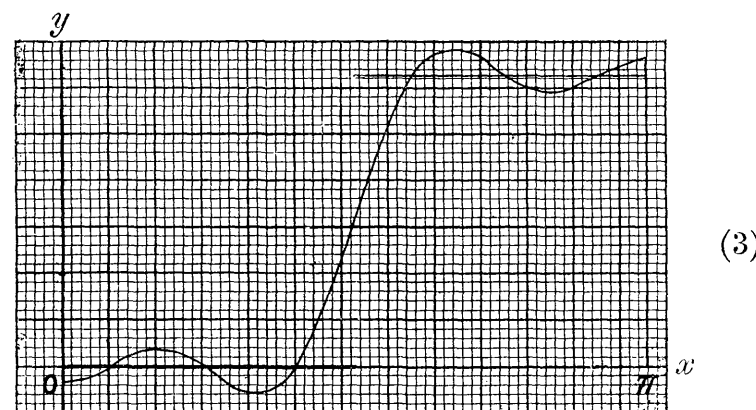
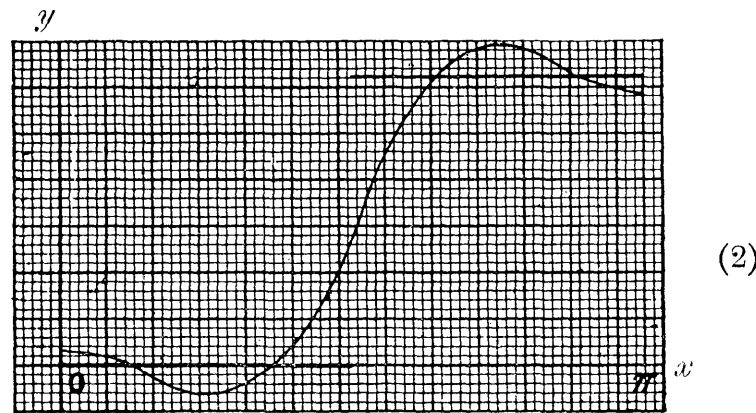
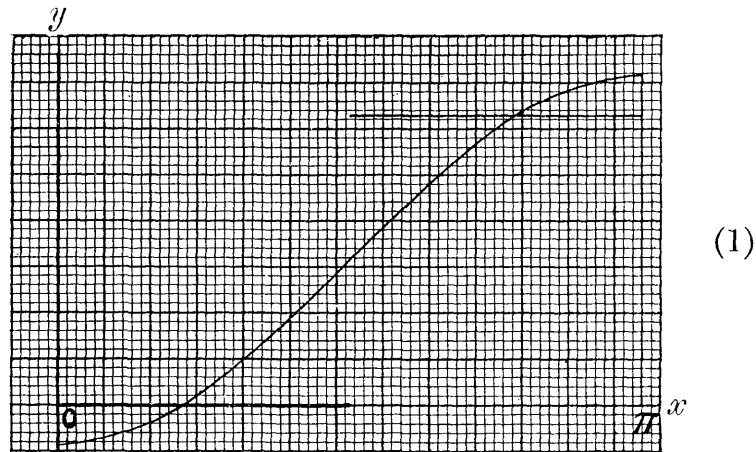


FIG. 21

Ex. 4. Find a series of cosines of multiples of x which will represent $f(x)$ in the interval $(0, \pi)$, where

$$\left. \begin{aligned} f(x) &= \frac{1}{3}\pi, & 0 \leq x < \frac{1}{3}\pi, \\ f(x) &= 0, & \frac{1}{3}\pi < x < \frac{2}{3}\pi, \\ f(x) &= -\frac{1}{3}\pi, & \frac{2}{3}\pi < x \leq \pi. \\ f(\frac{1}{3}\pi) &= \frac{1}{6}\pi, & f(\frac{2}{3}\pi) &= -\frac{1}{6}\pi. \end{aligned} \right\}$$

Also

Here $a_0 = \frac{1}{3} \int_0^{\frac{1}{3}\pi} dx - \frac{1}{3} \int_{\frac{2}{3}\pi}^{\pi} dx = 0,$

and $a_n = \frac{2}{3} \int_0^{\frac{1}{3}\pi} \cos nx \, dx - \frac{2}{3} \int_{\frac{2}{3}\pi}^{\pi} \cos nx \, dx$

$$= \frac{2}{3n} [\sin \frac{1}{3}n\pi + \sin \frac{2}{3}n\pi]$$

$$= \frac{4}{3n} \sin \frac{1}{2}n\pi \cos \frac{1}{6}n\pi.$$

Thus a_n vanishes when n is even or a multiple of 3.

And $f(x) = \frac{2\sqrt{3}}{3} [\cos x - \frac{1}{5} \cos 5x + \frac{1}{7} \cos 7x - \frac{1}{11} \cos 11x + \dots], \quad 0 \leq x \leq \pi.$

The points $x=0$ and $x=\pi$ are points of continuity in the sum of the series. The points $x=\frac{1}{3}\pi$ and $x=\frac{2}{3}\pi$ are points of discontinuity.

Fig. 22 contains the graph of the given function, and the approximation curves

$$\left. \begin{aligned} y &= \frac{2\sqrt{3}}{3} \cos x, \\ y &= \frac{2\sqrt{3}}{3} [\cos x - \frac{1}{5} \cos 5x], \\ y &= \frac{2\sqrt{3}}{3} [\cos x - \frac{1}{5} \cos 5x + \frac{1}{7} \cos 7x], \\ y &= \frac{2\sqrt{3}}{3} [\cos x - \frac{1}{5} \cos 5x + \frac{1}{7} \cos 7x - \frac{1}{11} \cos 11x], \end{aligned} \right\} 0 \leq x \leq \pi.$$

97. The Sine Series. Again let $f(x)$ be given in the interval $(0, \pi)$, and satisfy Dirichlet's Conditions in that interval. Define $f(x)$ in $-\pi \leq x < 0$ by the equation $f(-x) = -f(x)$. The function thus defined for $(-\pi, \pi)$ satisfies Dirichlet's Conditions in this interval, and we can apply to it the results of § 95.

But it is clear that in this case

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' \, dx' \text{ leads to } b_n = \frac{2}{\pi} \int_0^{\pi} f(x') \sin nx' \, dx',$$

and that $a_n = 0$ when $n \geq 0$.

Thus the cosine terms disappear from the Fourier's Series.

Since all the terms of the series

$$b_1 \sin x + b_2 \sin 2x + \dots$$

vanish when $x=0$ and $x=\pi$, the sum of the series is zero at these points.

It follows that, when $f(x)$ is an arbitrary function satisfying Dirichlet's Conditions in the interval $(0, \pi)$, the sum of the Sine Series,

$$\frac{2}{\pi} \sum_1^{\infty} \sin nx \int_0^{\pi} f(x') \sin nx' \, dx',$$

is equal to

$$\frac{1}{2} [f(x+0) + f(x-0)]$$

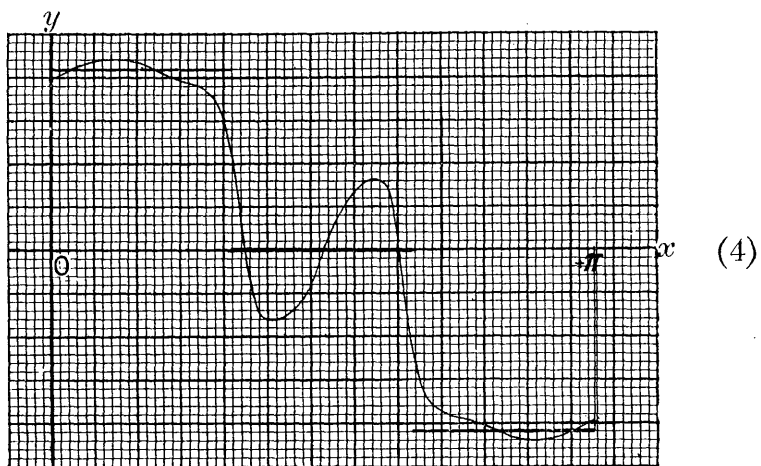
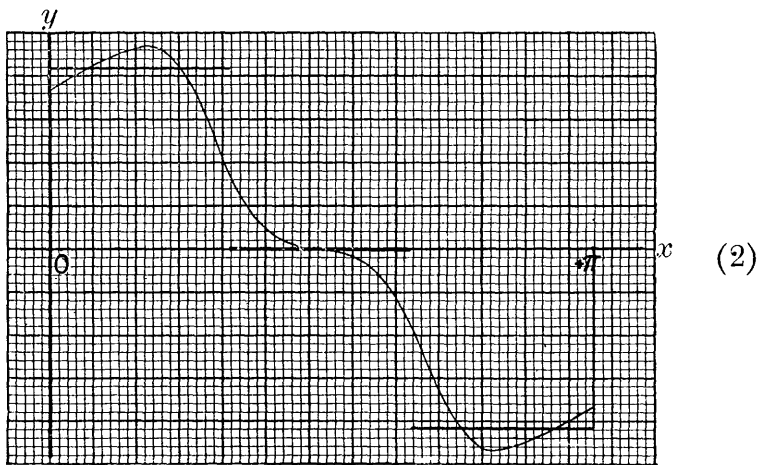
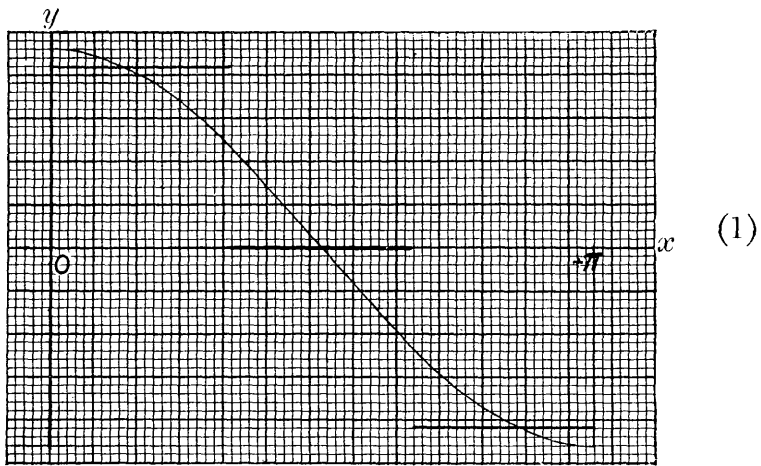


FIG. 22.

at every point between 0 and π where $f(x+0)$ and $f(x-0)$ exist; and, when $x=0$ and $x=\pi$, the sum is zero.

It will be noticed that, when $f(x)$ is continuous at the end-points $x=0$ and $x=\pi$, the Cosine Series gives the value of the function at these points. The Sine Series only gives the value of $f(x)$ at these points if $f(x)$ is zero there.

Ex. 1. Find a series of sines of multiples of x which will represent x in the interval $0 < x < \pi$.

Here
$$b_n = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx = (-1)^{n-1} \frac{2}{n}.$$

Therefore
$$x = 2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right], \quad 0 \leq x < \pi.$$

At $x=\pi$ the sum is discontinuous.

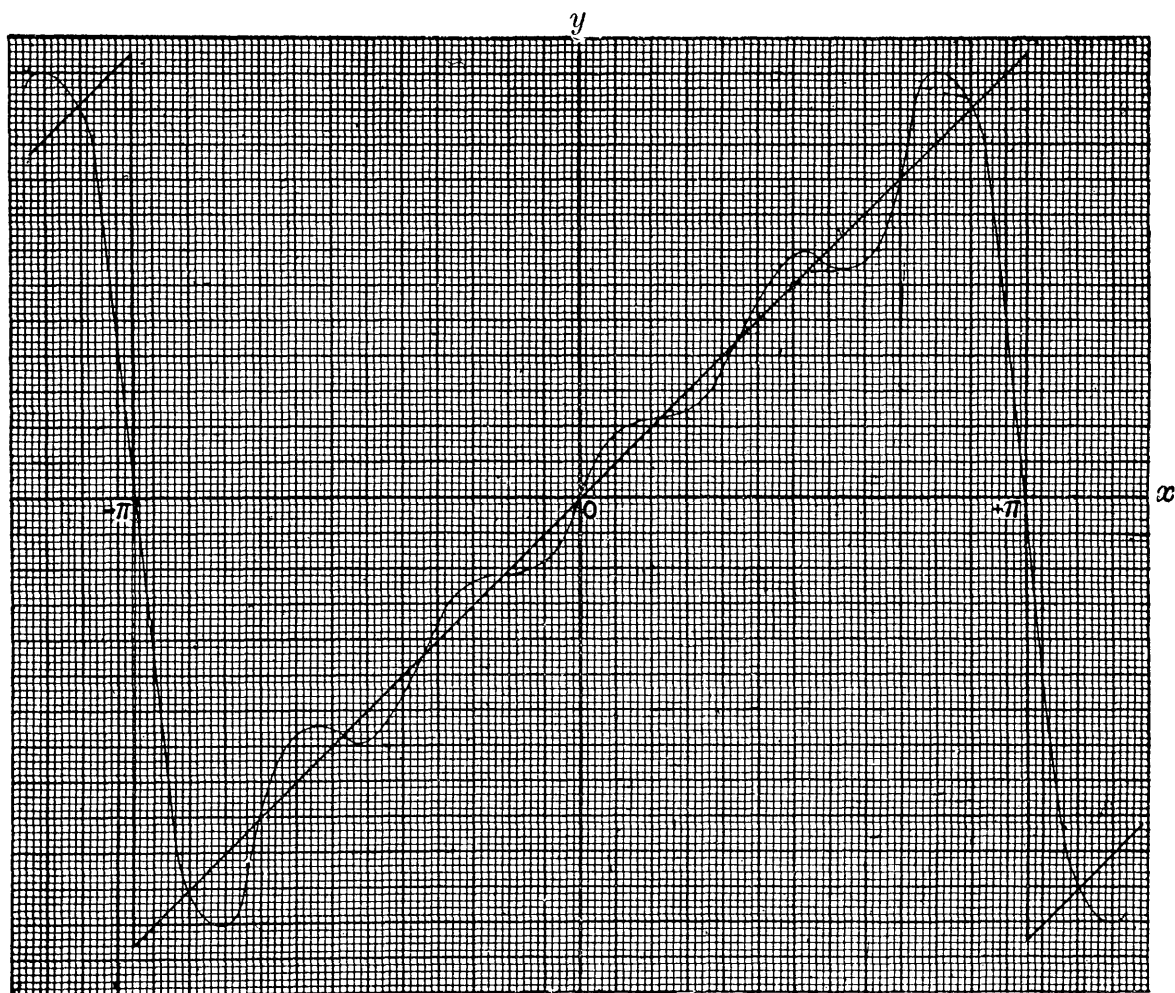


FIG. 23.

In Fig. 23, the line $y=x, \quad -\pi \leq x \leq \pi,$
and the approximation curve

$$y = 2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \frac{1}{5} \sin 5x \right], \quad -\pi \leq x \leq \pi,$$

are drawn.

The convergence of the series is so slow that this curve does not approach $y=x$ between $-\pi$ and π nearly as closely as the corresponding approximation

curve in the cosine series approached $y = \pm x$. If n is taken large enough the curve $y = S_n(x)$ will be a wavy curve oscillating about the line $y = x$ from $-\pi$ to $+\pi$, but we would be wrong if we were to say that it descends at a steep gradient from $x = -\pi$ to the end of $y = x$, and again descends from the other end of $y = x$ to $x = \pi$ at a steep gradient. As a matter of fact the summit of the first wave is some distance below $y = x$ at $x = -\pi$, and the summit of the last wave a corresponding distance above $y = x$ at $x = \pi$ when n is large.

To this question we return in Chapter IX.

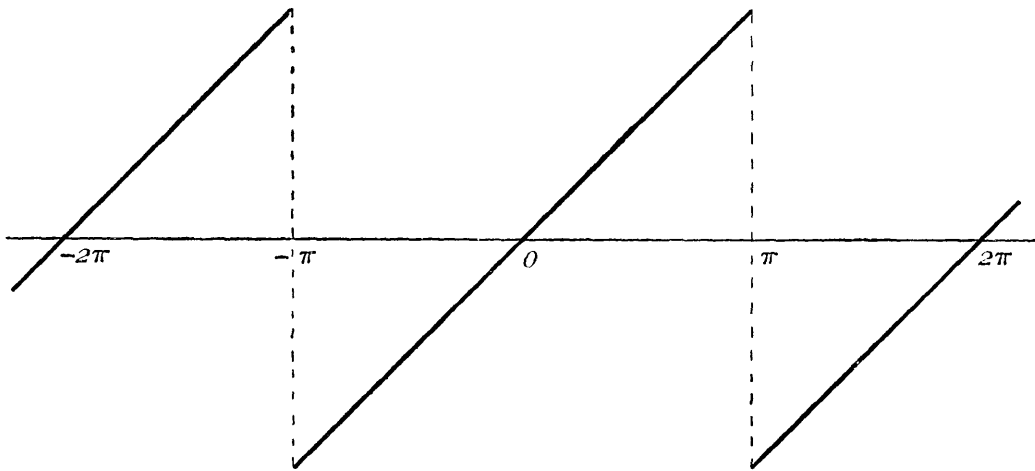


FIG. 24.

Since the Fourier's Series has a period 2π , this series for unrestricted values of x represents the ordinates of the lines shown in Fig. 24, the part from the open interval $(-\pi, \pi)$ being repeated indefinitely in both directions. The points $\pm\pi, \pm 3\pi, \dots$ are points of discontinuity. At these the sum is zero.

Ex. 2. Find a series of sines of multiples of x which will represent $f(x)$ in the interval $0 \leq x \leq \pi$, where

$$\left. \begin{aligned} f(x) &= \frac{1}{4}\pi x, & 0 \leq x \leq \frac{1}{2}\pi, \\ f(x) &= \frac{1}{4}\pi(\pi - x), & \frac{1}{2}\pi \leq x \leq \pi. \end{aligned} \right\}$$

Here
$$b_n = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{1}{4}\pi x \sin nx \, dx + \frac{2}{\pi} \int_{\frac{1}{2}\pi}^{\pi} \frac{1}{4}\pi(\pi - x) \sin nx \, dx$$

$$= \frac{1}{2} \int_0^{\frac{1}{2}\pi} x \sin nx \, dx + \frac{1}{2} \int_{\frac{1}{2}\pi}^{\pi} (\pi - x) \sin nx \, dx,$$

which gives
$$b_n = \frac{1}{n^2} \sin \frac{n\pi}{2}.$$

Thus
$$f(x) = \sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \dots$$

Fig. 25 contains the lines
$$\left. \begin{aligned} y &= \frac{1}{4}\pi x, & 0 \leq x \leq \frac{1}{2}\pi, \\ y &= \frac{1}{4}\pi(\pi - x), & \frac{1}{2}\pi \leq x \leq \pi, \end{aligned} \right\}$$

and the approximation curves

$$\left. \begin{aligned} y &= \sin x, \\ y &= \sin x - \frac{1}{3^2} \sin 3x, \\ y &= \sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x, \end{aligned} \right\} 0 \leq x \leq \pi.$$

It will be noticed that the last of these curves approaches the given lines closely, except at the sharp corner, right through the interval.

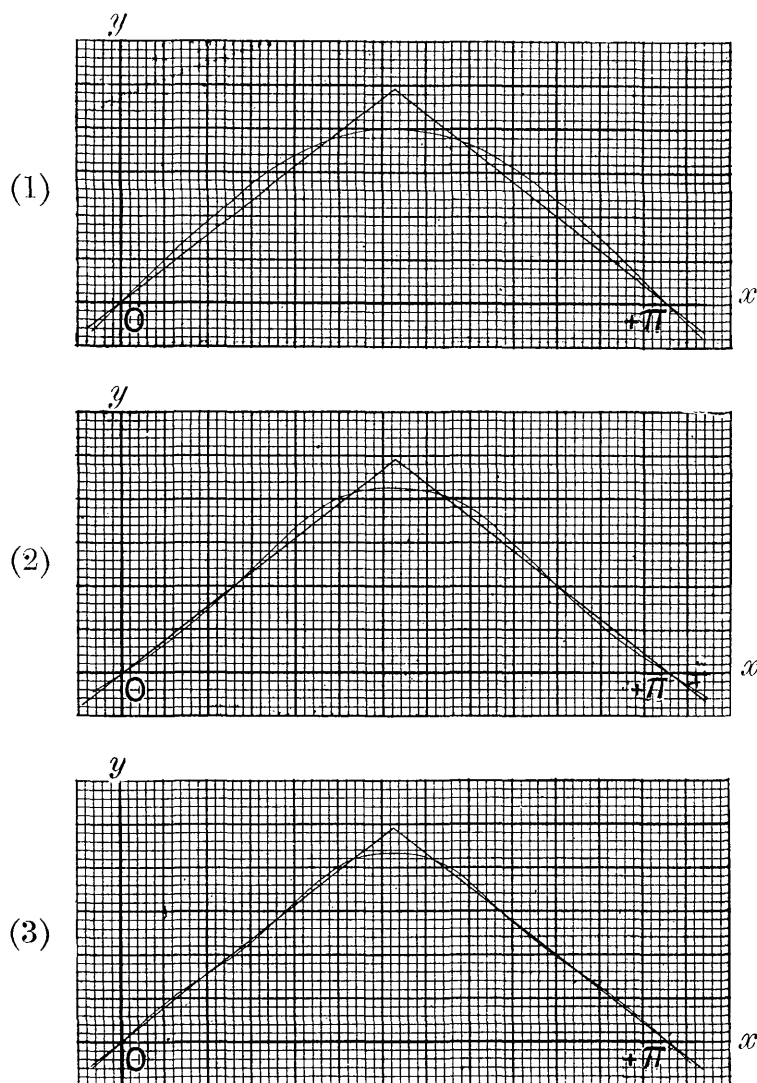


FIG. 25.

For unrestricted values of x the series represents the ordinates of the lines shown in Fig. 26, the part from $-\pi$ to $+\pi$ being repeated indefinitely in both directions.

The sum is continuous for all values of x .

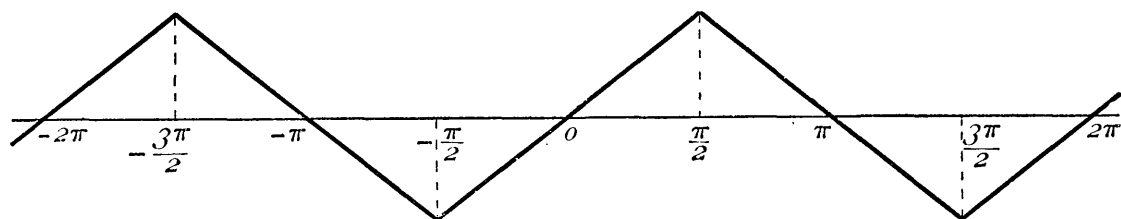


FIG. 26.

Ex. 3. Find a series of sines of multiples of x which will represent $f(x)$ in the interval $(0, \pi)$, where

$$\left. \begin{aligned} f(x) &= 0, & 0 \leq x < \frac{1}{2}\pi, \\ f(\frac{1}{2}\pi) &= \frac{1}{4}\pi, \\ f(x) &= \frac{1}{2}\pi, & \frac{1}{2}\pi < x < \pi, \\ f(\pi) &= 0. \end{aligned} \right\}$$

Here

$$\begin{aligned} b_n &= \int_{\frac{1}{2}\pi}^{\pi} \sin nx \, dx \\ &= \frac{1}{n} \left(\cos \frac{n\pi}{2} - \cos n\pi \right) \\ &= \frac{2}{n} \sin \frac{3n\pi}{4} \sin \frac{n\pi}{4}. \end{aligned}$$

Therefore b_n vanishes when n is a multiple of 4.

And $f(x) = \sin x - \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \frac{1}{3} \sin 6x + \dots$ $0 \leq x \leq \pi$.

Fig. 27 contains the graph of the given function, and the approximation curves

$$\left. \begin{aligned} y &= \sin x, \\ y &= \sin x - \sin 2x, \\ y &= \sin x - \sin 2x + \frac{1}{3} \sin 3x, \\ y &= \sin x - \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x, \end{aligned} \right\} 0 \leq x \leq \pi.$$

The points $x = \frac{1}{2}\pi$ and $x = \pi$ are points of discontinuity in the sum of the series. The behaviour of the approximation curves for large values of n at these points will be examined in Chapter IX.

Ex. 4. Find a series of sines of multiples of x which will represent $f(x)$ in the interval $(0, \pi)$, where

$$\left. \begin{aligned} f(x) &= \frac{1}{3}\pi, & 0 < x < \frac{1}{3}\pi, \\ f(x) &= 0, & \frac{1}{3}\pi < x < \frac{2}{3}\pi, \\ f(x) &= -\frac{1}{3}\pi, & \frac{2}{3}\pi < x < \pi. \end{aligned} \right\}$$

Also

$$f(0) = f(\pi) = 0; \quad f\left(\frac{1}{3}\pi\right) = \frac{1}{6}\pi; \quad f\left(\frac{2}{3}\pi\right) = -\frac{1}{6}\pi.$$

Here

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^{\frac{1}{3}\pi} \sin nx \, dx - \frac{2}{3} \int_{\frac{2}{3}\pi}^{\pi} \sin nx \, dx \\ &= \frac{2}{3n} [1 - \cos \frac{1}{3}n\pi - \cos \frac{2}{3}n\pi + \cos n\pi] \\ &= \frac{8}{3n} \cos^2 \frac{n\pi}{2} \sin^2 \frac{n\pi}{6}. \end{aligned}$$

Therefore a_n vanishes when n is odd or a multiple of 6.

And $f(x) = \sin 2x + \frac{1}{2} \sin 4x + \frac{1}{4} \sin 8x + \frac{1}{5} \sin 10x + \dots$, $0 \leq x < \pi$.

The points $x = 0$, $x = \frac{1}{3}\pi$, $x = \frac{2}{3}\pi$ and $x = \pi$ are points of discontinuity in the sum of the series.

Fig. 28 contains the graph of the given function, and the approximation curves

$$\left. \begin{aligned} y &= \sin 2x, \\ y &= \sin 2x + \frac{1}{2} \sin 4x, \\ y &= \sin 2x + \frac{1}{2} \sin 4x + \frac{1}{4} \sin 8x, \\ y &= \sin 2x + \frac{1}{2} \sin 4x + \frac{1}{4} \sin 8x + \frac{1}{5} \sin 10x, \end{aligned} \right\} 0 \leq x \leq \pi.$$

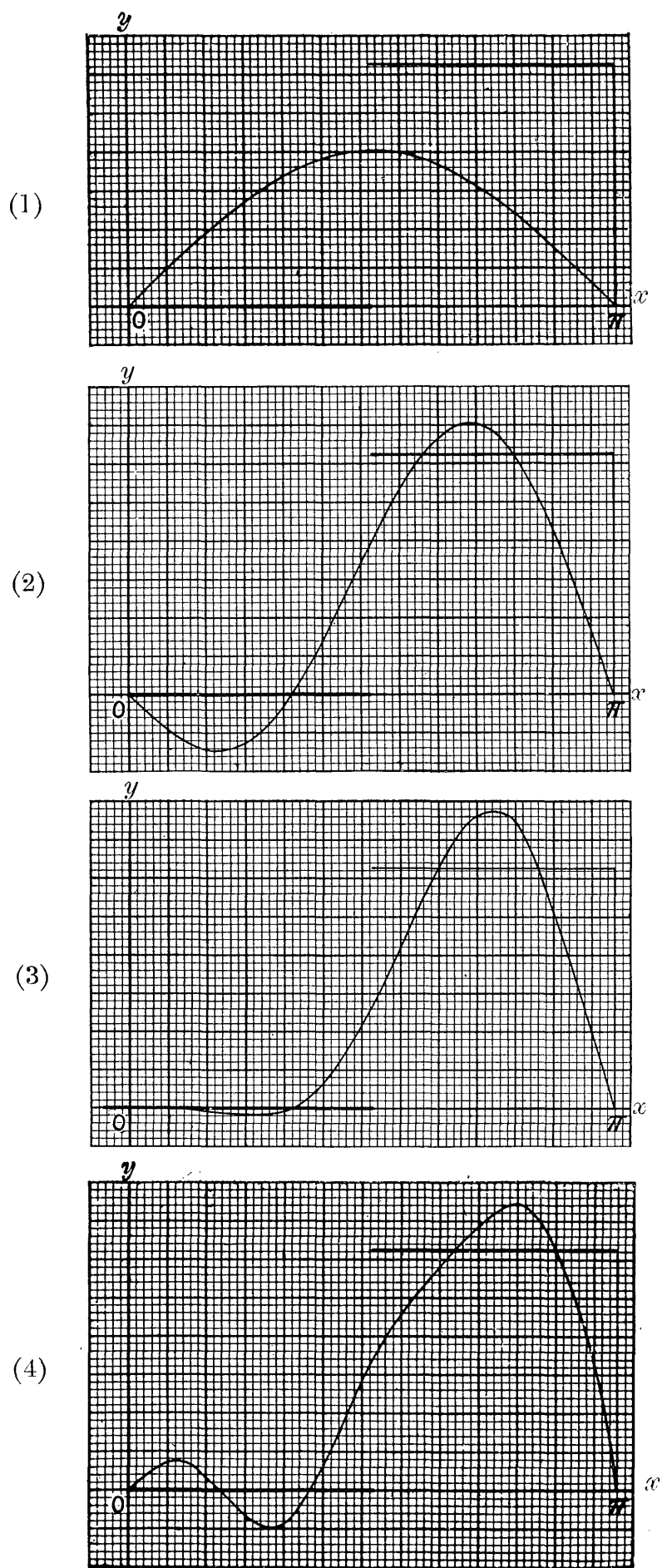


FIG. 27.

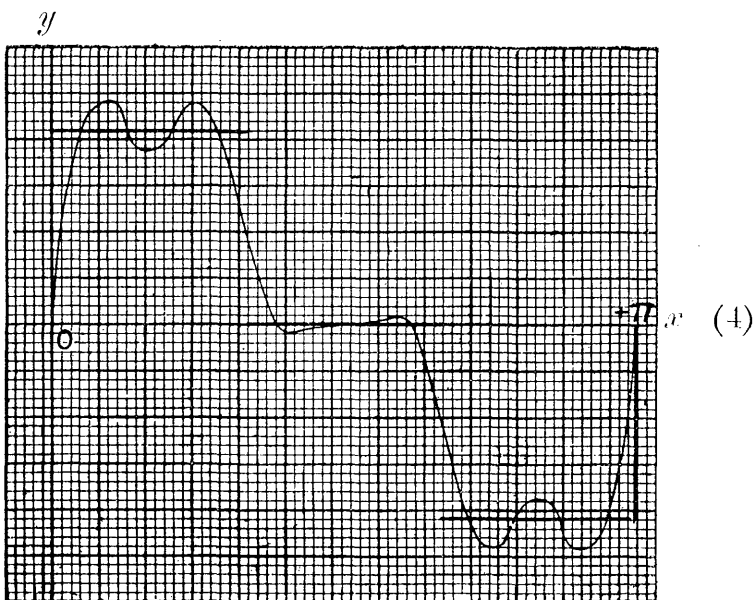
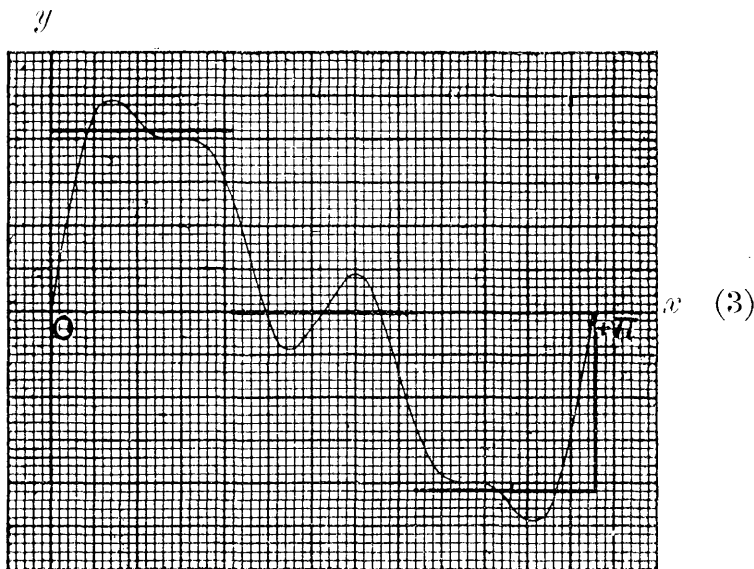
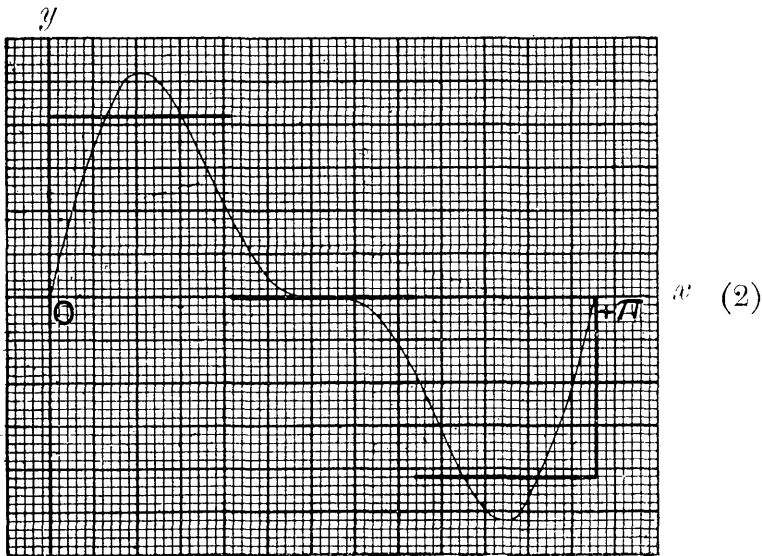
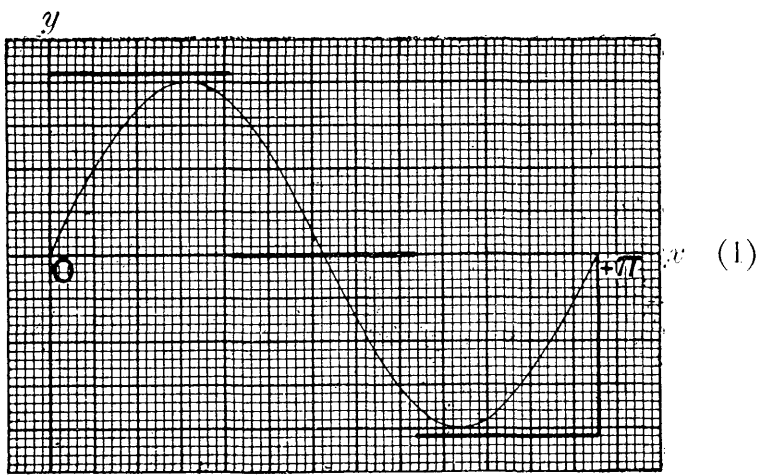


FIG. 28.

98. Other Forms of Fourier's Series. When the arbitrary function is given in the interval $(-l, l)$, we can change this interval to $(-\pi, \pi)$ by the substitution $u = \pi x/l$.

In this way we may deduce the following expansions from those already obtained:

$$\frac{1}{2l} \int_{-l}^l f(x') dx' + \frac{1}{l} \sum_1^{\infty} \int_{-l}^l f(x') \cos \frac{n\pi}{l} (x' - x) dx', \quad -l \leq x \leq l, \dots(1)$$

$$\frac{1}{l} \int_0^l f(x') dx' + \frac{2}{l} \sum_1^{\infty} \cos \frac{n\pi}{l} x \int_0^l f(x') \cos \frac{n\pi}{l} x' dx', \quad 0 \leq x \leq l, \dots(2)$$

$$\frac{2}{l} \sum_1^{\infty} \sin \frac{n\pi}{l} x \int_0^l f(x') \sin \frac{n\pi}{l} x' dx', \quad 0 \leq x \leq l. \dots\dots\dots(3)$$

When $f(x)$ satisfies Dirichlet's Conditions in $(-l, l)$, the sum of the series (1) is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point in $-l < x < l$ where $f(x+0)$ and $f(x-0)$ exist; and at $x = \pm l$ its sum is $\frac{1}{2}[f(-l+0)+f(l-0)]$, when the limits $f(-l+0)$ and $f(l-0)$ exist.

When $f(x)$ satisfies Dirichlet's Conditions in $(0, l)$, the sum of the series (2) is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point in $0 < x < l$ where $f(x+0)$ and $f(x-0)$ exist; and at $x=0$ its sum is $f(+0)$, at $x=l$ its sum is $f(l-0)$, when these limits exist.

When $f(x)$ satisfies Dirichlet's Conditions in $(0, l)$ the sum of the series (3) is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point in $0 < x < l$ where $f(x+0)$ and $f(x-0)$ exist; and at $x=0$ and $x=l$ its sum is zero.

It is sometimes more convenient to take the interval in which the arbitrary function is given as $(0, 2\pi)$. We may deduce the corresponding series for this interval from that already found for $(-\pi, \pi)$.

Consider the Fourier's Series

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F(x') dx' + \frac{1}{\pi} \sum_1^{\infty} \int_{-\pi}^{\pi} F(x') \cos n(x' - x) dx',$$

where $F(x)$ satisfies Dirichlet's Conditions in $(-\pi, \pi)$.

Let $u = \pi + x, \quad u' = \pi + x' \quad$ and $f(u) = F(u - \pi)$.

Then we obtain the series for $f(u)$,

$$\frac{1}{2\pi} \int_0^{2\pi} f(u') du' + \frac{1}{\pi} \sum_1^{\infty} \int_0^{2\pi} f(u') \cos n(u' - u) du',$$

for the interval $(0, 2\pi)$.

On changing u into x , we have the series for $f(x)$,

$$\frac{1}{2\pi} \int_0^{2\pi} f(x') dx' + \frac{1}{\pi} \sum_1^{\infty} \int_0^{2\pi} f(x') \cos n(x' - x) dx', \quad 0 \leq x \leq 2\pi. \dots(4)$$

The sum of the series (4) is $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point between 0 and 2π where $f(x+0)$ and $f(x-0)$ exist; and at $x=0$ and $x=2\pi$ its sum is

$$\frac{1}{2}[f(+0)+f(2\pi-0)],$$

when these limits exist.

In (4), it is assumed that $f(x)$ satisfies Dirichlet's Conditions in the interval $(0, 2\pi)$.

Again, it is sometimes convenient to take the interval in which the function is defined as (a, b) . We can deduce the corresponding series for this interval from the result just obtained.

Taking the series

$$\frac{1}{2\pi} \int_0^{2\pi} F(x') dx' + \frac{1}{\pi} \sum_1^{\infty} \int_0^{2\pi} F(x') \cos n(x' - x) dx', \quad 0 \leq x \leq 2\pi,$$

we write $u = a + \frac{(b-a)x}{2\pi}$ and $f(u) = F\left\{\frac{2\pi(u-a)}{b-a}\right\}$.

Then for the interval $a \leq u \leq b$ we have the series for $f(u)$,

$$\frac{1}{b-a} \int_a^b f(u') du' + \frac{2}{b-a} \sum_1^{\infty} \int_a^b f(u') \cos \frac{2n\pi}{b-a}(u' - u) du'.$$

On changing u into x , we obtain the series for $f(x)$ in $a \leq x \leq b$, namely,

$$\frac{1}{b-a} \int_a^b f(x') dx' + \frac{2}{b-a} \sum_1^{\infty} \int_a^b f(x') \cos \frac{2n\pi}{b-a}(x' - x) dx'. \dots(5)$$

The sum of the series (5) is $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point in $a < x < b$ where $f(x+0)$ and $f(x-0)$ exist; and at $x=a$ and $x=b$ its sum is $\frac{1}{2}[f(a+0)+f(b-0)]$, when these limits exist.

Of course $f(x)$ is again subject to Dirichlet's Conditions in the interval (a, b) .

The corresponding Cosine Series and Sine Series are, respectively,

$$\begin{aligned} & \frac{1}{(b-a)} \int_a^b f(x') dx' + \frac{2}{b-a} \sum_1^{\infty} \cos \frac{n\pi}{b-a}(x-a) \\ & \quad \times \int_a^b f(x') \cos \frac{n\pi}{b-a}(x' - a) dx', \quad a \leq x \leq b, \dots\dots(6) \end{aligned}$$

and $\frac{2}{(b-a)} \sum_1^{\infty} \sin \frac{n\pi}{(b-a)}(x-a) \int_a^b f(x') \sin \frac{n\pi}{b-a}(x' - a) dx'$,

$$a \leq x \leq b. \dots\dots(7)$$

Ex. 1. Show that the series

$$\frac{4}{\pi} \left(\sin \frac{\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} + \dots \right)$$

is equal to 1 when $0 < x < l$.

Ex. 2. Show that the series

$$\frac{ca}{l} + \frac{2c}{\pi} \left(\sin \frac{\pi a}{l} \cos \frac{\pi x}{l} + \frac{1}{2} \sin \frac{2\pi a}{l} \cos \frac{2\pi x}{l} + \dots \right)$$

is equal to c when $0 < x < a$ and to zero when $a < x < l$.

Ex. 3. Show that the series

$$\frac{v_1 + v_2 + v_3}{3} + \frac{1}{\pi} \sum_1^{\infty} \frac{1}{n} \sin \frac{n\pi}{3} \left\{ 2(v_3 - v_1) \sin \frac{2n\pi}{3} \sin \frac{n\pi x}{l} + (2v_2 - v_1 - v_3) \cos \frac{n\pi x}{l} \right\}$$

is equal to

$$\begin{aligned} v_1 & \text{ when } -l < x < -\frac{l}{3}, \\ v_2 & \text{ when } -\frac{l}{3} < x < \frac{l}{3}, \\ v_3 & \text{ when } \frac{l}{3} < x < l. \end{aligned}$$

Ex. 4. Show that the series

$$2 \left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right]$$

represents $(\pi - x)$ in the interval $0 < x < 2\pi$.

99. Poisson's Discussion of Fourier's Series. As has been mentioned in the introduction, within a few years of Fourier's discovery of the possibility of representing an arbitrary function by what is now called its Fourier's Series, Poisson discussed the subject from a quite different standpoint.

He began with the equation

$$\frac{1 - r^2}{1 - 2r \cos(x' - x) + r^2} = 1 + 2 \sum_1^{\infty} r^n \cos n(x' - x),$$

where $|r| < 1$, and he obtained, by integration,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(x' - x) + r^2} f(x') dx' \\ = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx' + \frac{1}{\pi} \sum_1^{\infty} r^n \int_{-\pi}^{\pi} f(x') \cos n(x' - x) dx'. \end{aligned}$$

Poisson proceeded to show that, as $r \rightarrow 1$, the integral on the left-hand side of this equation has the limit $f(x)$, supposing $f(x)$ continuous at that point, and he argued that $f(x)$ must then be the sum of the series on the right-hand side when $r = 1$. Apart from the incompleteness of his discussion of the questions connected

with the limit of the integral as $r \rightarrow 1$, the conclusion he sought to draw is invalid until it is shown that the series does converge when $r=1$, and this, in fact, is the real difficulty. In accordance with Abel's Theorem on the Power Series (§ 72), if the series converges when $r=1$, its sum is continuous up to and including $r=1$. In other words, if we write

$$F(r, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx' + \frac{1}{\pi} \sum_1^{\infty} r^n \int_{-\pi}^{\pi} f(x') \cos n(x' - x) dx',$$

we know that, if $F(1, x)$ converges, then

$$\text{Lt}_{r \rightarrow 1} F(r, x) = F(1, x).$$

But we have no right to assume, from the convergence of

$$\text{Lt}_{r \rightarrow 1} F(r, x),$$

that $F(1, x)$ does converge.

Poisson's method, however, has a definite value in the treatment of Fourier's Series, and we shall now give a presentation of it on the more exact lines which we have followed in the discussion of series and integrals in the previous pages of this book.

100. Poisson's Integral. The integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(x' - x) + r^2} f(x') dx', \quad |r| < 1,$$

is called Poisson's Integral.

We shall assume that $f(x)$ is either bounded and integrable in the interval $(-\pi, \pi)$, or that the infinite integral $\int_{-\pi}^{\pi} f(x) dx$ is absolutely convergent.

$$\text{Now we know that } \frac{1 - r^2}{1 - 2r \cos \theta + r^2} = 1 + 2 \sum_1^{\infty} r^n \cos n\theta$$

when $|r| < 1$, and that this series is uniformly convergent for any interval of θ , when r has any given value between -1 and $+1$.

It follows that
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(x' - x) + r^2} dx' = 1,$$
 and that

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(x' - x) + r^2} f(x') dx' \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx' + \frac{1}{\pi} \sum_1^{\infty} r^n \int_{-\pi}^{\pi} f(x') \cos n(x' - x) dx' \dots\dots\dots(1) \end{aligned}$$

under the limitations above imposed upon $f(x)$. (Cf. § 70, Cor. II., and § 74, I.)

Now let us choose a number x between $-\pi$ and π for which we wish the sum of this series, or, what is the same thing, the value of Poisson's Integral,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r \cos(x'-x)+r^2} f(x') dx'$$

Denote this sum, or the integral, by $F(r, x)$.

Let us assume that, for the value of x chosen,

$$\text{Lt}_{t \rightarrow 0} [f(x+t)+f(x-t)]$$

exists.

Also, let the function $\phi(x')$ be defined when $-\pi \leq x' \leq \pi$ by the equation

$$\phi(x') = f(x') - \frac{1}{2} \text{Lt}_{t \rightarrow 0} [f(x+t)+f(x-t)].$$

Then

$$\begin{aligned} F(r, x) - \frac{1}{2} \text{Lt}_{t \rightarrow 0} [f(x+t)+f(x-t)] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r \cos(x'-x)+r^2} \{f(x') - \frac{1}{2} \text{Lt}_{t \rightarrow 0} [f(x+t)+f(x-t)]\} dx' \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r \cos(x'-x)+r^2} \phi(x') dx' \dots\dots\dots(2) \end{aligned}$$

But we are given that $\text{Lt}_{t \rightarrow 0} [f(x+t)+f(x-t)]$ exists.

Let the arbitrary positive number ϵ be chosen, as small as we please. Then to $\epsilon/2$ there will correspond a positive number η such that

$$|f(x+t)+f(x-t) - \text{Lt}_{t \rightarrow 0} [f(x+t)+f(x-t)]| < \frac{\epsilon}{2}, \dots\dots\dots(3)$$

when $0 < t \leq \eta$.

The number η fixed upon will be such that $(x-\eta, x+\eta)$ does not go beyond $(-\pi, \pi)$.

Then

$$\begin{aligned} \frac{1}{2\pi} \int_{x-\eta}^{x+\eta} \frac{1-r^2}{1-2r \cos(x'-x)+r^2} \phi(x') dx' &= \frac{1}{2\pi} \int_0^{\eta} \frac{1-r^2}{1-2r \cos t+r^2} [\phi(x+t)+\phi(x-t)] dt \\ &= \frac{1}{2\pi} \int_0^{\eta} \frac{1-r^2}{1-2r \cos t+r^2} \{f(x+t)+f(x-t) - \text{Lt}_{t \rightarrow 0} [f(x+t)+f(x-t)]\} dt. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \frac{1}{2\pi} \int_{x-\eta}^{x+\eta} \frac{1-r^2}{1-2r \cos(x'-x)+r^2} \phi(x') dx' \right| &< \frac{\epsilon}{4\pi} \int_0^{\eta} \frac{1-r^2}{1-2r \cos t+r^2} dt \\ &< \frac{\epsilon}{4\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r \cos t+r^2} dt \\ &< \frac{\epsilon}{2} \dots\dots\dots(4) \end{aligned}$$

Also, when $0 < r < 1$,

$$\begin{aligned} & \left| \frac{1}{2\pi} \left(\int_{-\pi}^{x-\eta} + \int_{x+\eta}^{\pi} \right) \frac{1-r^2}{1-2r\cos(x'-x)+r^2} \phi(x') dx' \right| \\ & < \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\eta+r^2} \int_{-\pi}^{\pi} |\phi(x')| dx' \\ & < \frac{1-r^2}{1-2r\cos\eta+r^2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x')| dx' + \frac{1}{2} \left| \text{Lt}_{t \rightarrow 0} [f(x+t) + f(x-t)] \right| \right) \\ & < \frac{1-r^2}{1-2r\cos\eta+r^2} \times A, \text{ say.} \dots\dots\dots(5) \end{aligned}$$

But

$$\begin{aligned} \frac{1-r^2}{1-2r\cos\eta+r^2} & < \frac{2(1-r)}{(1-r)^2 + 4r\sin^2\frac{\eta}{2}}, \text{ if } 0 < r < 1, \\ & < \frac{1-r}{2r\sin^2\frac{\eta}{2}}. \end{aligned}$$

And

$$\frac{1-r}{2r\sin^2\frac{\eta}{2}} < \frac{\epsilon}{2A},$$

provided that

$$r > \frac{1}{1 + \frac{\epsilon}{A} \sin^2\frac{\eta}{2}}.$$

It follows that

$$\left| \frac{1}{2\pi} \left(\int_{-\pi}^{x-\eta} + \int_{x+\eta}^{\pi} \right) \frac{1-r^2}{1-2r\cos(x'-x)+r^2} \phi(x') dx' \right| < \frac{\epsilon}{2},$$

if

$$1 > r > \frac{1}{1 + \frac{\epsilon}{A} \sin^2\frac{\eta}{2}}. \dots\dots\dots(6)$$

Combining (4) and (6), it will be seen that when any positive number ϵ has been chosen, as small as we please, there is a positive number ρ such that

$$\left| F(r, x) - \frac{1}{2} \text{Lt}_{t \rightarrow 0} [f(x+t) + f(x-t)] \right| < \epsilon,$$

when $\rho \leq r < 1$, provided that for the value of x considered $\text{Lt}_{t \rightarrow 0} [f(x+t) + f(x-t)]$ exists.

We have thus established the following theorem :

Let $f(x)$, given in the interval $(-\pi, \pi)$, be bounded and integrable, or have an absolutely convergent infinite integral, in this range. Then for any value of x in $-\pi < x < \pi$ for which $\text{Lt}_{t \rightarrow 0} [f(x+t) + f(x-t)]$

exists, Poisson's Integral converges to that limit as $r \rightarrow 1$ from below.

In particular, at a point of ordinary discontinuity of $f(x)$, Poisson's Integral converges to

$$\frac{1}{2}[f(x+0) + f(x-0)],$$

and, at a point where $f(x)$ is continuous, it converges to $f(x)$.

It has already been pointed out that no conclusion can be drawn from this as to the convergence, or non-convergence, of the Fourier's Series at this point. But if we know that the Fourier's Series does converge, it follows from Abel's Theorem that it must converge to the limit to which Poisson's Integral converges as $r \rightarrow 1$.

We have thus the following theorem :

If $f(x)$ is any function, given in $(-\pi, \pi)$, which is either bounded and integrable, or has an absolutely convergent infinite integral $\int_{-\pi}^{\pi} f(x) dx$, then, at any point x in $-\pi < x < \pi$ at which the Fourier's Series is convergent, its sum must be equal to

$$\text{Lt}_{t \rightarrow 0} [f(x+t) + f(x-t)],$$

provided that this limit exists.

With certain obvious modifications these theorems can be made to apply to the points $-\pi$ and π as well as points between $-\pi$ and π .

It follows immediately from this theorem that :

If all the Fourier's Constants are zero for a function, continuous in the interval $(-\pi, \pi)$, then the function vanishes identically.

If the constants vanish but the function only satisfies the conditions ascribed to $f(x)$ in the earlier theorems of this section, we can only infer that the function must vanish at all points where it is continuous, and that at points where $\text{Lt}_{t \rightarrow 0} [f(x+t) + f(x-t)]$, exists, this limit must be zero.

Further, if (a, b) is an interval in which $f(x)$ is continuous, the same number ρ , corresponding to the arbitrary ϵ , may be chosen to serve all the values of x in the interval (a, b) ; for this is true, first of the number η in (3), then of A in (5), and thus finally of ρ .

It follows that Poisson's Integral converges as $r \rightarrow 1$ uniformly to the value $f(x)$ in any interval (a, b) in which $f(x)$ is continuous.*

This last theorem has an important application in connection with the approximate representation of functions by finite trigonometrical series.†

101. Fejér's Theorem.‡

Let $f(x)$ be given in the interval $(-\pi, \pi)$. If bounded, let it be integrable; if unbounded, let the infinite integral $\int_{-\pi}^{\pi} f(x) dx$ be absolutely convergent. Denote by s_n the sum of the $(n+1)$ terms

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') + \frac{1}{\pi} \sum_1^n \int_{-\pi}^{\pi} f(x') \cos n(x' - x) dx'.$$

Also let
$$S_n(x) = \frac{s_0 + s_1 + \dots + s_{n-1}}{n}.$$

* It is assumed in this that $f(a-0) = f(a) = f(a+0)$ and $f(b-0) = f(b) = f(b+0)$. Also $f(x)$ is subject to the conditions given at the beginning of this section. Cf. § 107.

† Cf. Picard, *Traité d'Analyse*, (2^e éd.), T. I., p. 275, 1905; Bôcher, *Ann. Math.*, Princeton, N.J. (Ser. 2), 7, p. 102, 1906; Hobson, *loc. cit.*, p. 722.

‡ Cf. *Math. Ann.*, Leipzig, 58, p. 51, 1904.

Then at every point x in the interval $-\pi < x < \pi$ at which $f(x+0)$ and $f(x-0)$ exist,

$$\lim_{n \rightarrow \infty} S_n(x) = \frac{1}{2} [f(x+0) + f(x-0)].$$

With the above notation,

$$s_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \frac{\cos n(x'-x) - \cos (n+1)(x'-x)}{1 - \cos(x'-x)} dx'.$$

Therefore

$$\begin{aligned} S_n(x) &= \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(x') \frac{1 - \cos n(x'-x)}{1 - \cos(x'-x)} dx' \\ &= \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(x') \frac{\sin^2 \frac{1}{2} n(x'-x)}{\sin^2 \frac{1}{2} (x'-x)} dx' \\ &= \frac{1}{2n\pi} \int_{-\pi+x}^{\pi+x} f(x') \frac{\sin^2 \frac{1}{2} n(x'-x)}{\sin^2 \frac{1}{2} (x'-x)} dx', \dots\dots\dots(1) \end{aligned}$$

if $f(x)$ is defined outside the interval $(-\pi, \pi)$ by the equation

$$f(x+2\pi) = f(x).$$

Dividing the range of integration into $(-\pi+x, x)$ and $(x, \pi+x)$, and substituting $x' = x - 2\alpha$ in the first, and $x' = x + 2\alpha$ in the second, we obtain

$$S_n(x) = \frac{1}{n\pi} \int_0^{\frac{1}{2}\pi} f(x-2\alpha) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha + \frac{1}{n\pi} \int_0^{\frac{1}{2}\pi} f(x+2\alpha) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha. \dots\dots(2)$$

Now suppose that x is a point in $(-\pi, \pi)$ at which $f(x+0)$ and $f(x-0)$ exist.

Let ϵ be any positive number, chosen as small as we please.

Then to ϵ there corresponds a positive number η chosen less than $\frac{1}{2}\pi$ such that

$$|f(x+2\alpha) - f(x+0)| < \epsilon \text{ when } 0 < \alpha \leq \eta.$$

Also

$$\begin{aligned} \frac{1}{n\pi} \int_0^{\frac{1}{2}\pi} f(x+2\alpha) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha &= \frac{1}{n\pi} \int_0^{\eta} \{f(x+2\alpha) - f(x+0)\} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha \\ &\quad + \frac{1}{n\pi} f(x+0) \int_0^{\frac{1}{2}\pi} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha \\ &\quad + \frac{1}{n\pi} \int_{\eta}^{\frac{1}{2}\pi} f(x+2\alpha) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha \\ &\quad - \frac{1}{n\pi} f(x+0) \int_{\eta}^{\frac{1}{2}\pi} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha \\ &= I_1 + I_2 + I_3 + I_4, \text{ say.}^* \dots\dots\dots(3) \end{aligned}$$

* This discussion also applies when the upper limit of the integral on the left is any positive number less than $\frac{1}{2}\pi$.

Putting $\sigma_{n-1} = \frac{1}{2} + \cos 2a + \cos 4a + \dots + \cos 2(n-1)a$,

we have $\sigma_{n-1} = \frac{1}{2} \frac{(\cos 2(n-1)a - \cos 2na)}{4(1 - \cos 2a)}$,

and $\sigma_0 + \sigma_1 + \dots + \sigma_{n-1} = \frac{1}{2} \frac{(1 - \cos 2na)}{(1 - \cos 2a)}$
 $= \frac{1}{2} \frac{\sin^2 na}{\sin^2 a}$.

Thus $\int_0^{\frac{1}{2}\pi} \frac{\sin^2 na}{\sin^2 a} da = 2 \int_0^{\frac{1}{2}\pi} (\sigma_0 + \sigma_1 + \dots + \sigma_{n-1}) da$
 $= \frac{1}{2} n\pi$,

since all the terms on the right-hand side disappear on integration except the first in each of the σ 's.

It follows that $I_2 = \frac{1}{2} f(x+0)$.

Also $|I_1| \leq \frac{1}{n\pi} \int_0^\eta |f(x+2a) - f(x+0)| \frac{\sin^2 na}{\sin^2 a} da$
 $< \frac{\epsilon}{n\pi} \int_0^\eta \frac{\sin^2 na}{\sin^2 a} da$
 $< \frac{\epsilon}{n\pi} \int_0^{\frac{1}{2}\pi} \frac{\sin^2 na}{\sin^2 a} da$
 $< \frac{1}{2}\epsilon$(4)

Further, $|I_3| \leq \frac{1}{n\pi} \int_\eta^{\frac{1}{2}\pi} |f(x+2a)| \frac{\sin^2 na}{\sin^2 a} da$
 $< \frac{1}{n\pi \sin^2 \eta} \int_\eta^{\frac{1}{2}\pi} |f(x+2a)| da$
 $< \frac{1}{2n\pi \sin^2 \eta} \int_{x+2\eta}^{x+\pi} |f(x')| dx'$(5)

But we are given that $\int_{-\pi}^\pi |f(x')| dx'$ converges, and we have defined $f(x)$ outside the interval $(-\pi, \pi)$ by the equation $f(x+2\pi) = f(x)$.

Let $\int_{-\pi}^\pi |f(x')| dx' = \pi J$, say.

Then we have $|I_3| < \frac{J}{2n \sin^2 \eta}$.

Also $|I_4| < \frac{1}{2n \sin^2 \eta} |f(x+0)|$(6)

Combining these results, it follows from (3) that

$$\left| \frac{1}{n\pi} \int_0^{\frac{1}{2}\pi} f(x+2a) \frac{\sin^2 na}{\sin^2 a} da - \frac{1}{2} f(x+0) \right| < \frac{1}{2}\epsilon + \frac{1}{2n \sin^2 \eta} \{J + |f(x+0)|\}.$$

Now let ν be a positive integer such that

$$\frac{1}{\nu \sin^2 \eta} \{J + |f(x+0)|\} < \epsilon. \dots\dots\dots(7)$$

Then

$$\left| \frac{1}{n\pi} \int_0^{\frac{1}{2}\pi} f(x+2a) \frac{\sin^2 na}{\sin^2 a} da - \frac{1}{2} f(x+0) \right| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon < \epsilon, \text{ when } n \geq \nu.$$

In other words,

$$\text{Lt}_{n \rightarrow \infty} \frac{1}{n\pi} \int_0^{\frac{1}{2}\pi} f(x+2a) \frac{\sin^2 na}{\sin^2 a} da = \frac{1}{2} f(x+0),$$

when $f(x+0)$ exists.

In precisely the same way we find that

$$\text{Lt}_{n \rightarrow \infty} \frac{1}{n\pi} \int_0^{\frac{1}{2}\pi} f(x-2a) \frac{\sin^2 na}{\sin^2 a} da = \frac{1}{2} f(x-0),$$

when $f(x-0)$ exists.

Then, returning to (2), we have

$$\text{Lt}_{n \rightarrow \infty} S_n(x) = \frac{1}{2} [f(x+0) + f(x-0)],$$

when $f(x \pm 0)$ exist.

This proof applies also to the points $x = \pm \pi$, when $f(\pi-0)$ and $f(-\pi+0)$ exist. Since we have defined $f(x)$ outside the interval $(-\pi, \pi)$ by the equation $f(x+2\pi) = f(x)$, it is clear that $f(-\pi+0) = f(\pi+0)$ and $f(-\pi-0) = f(\pi-0)$.

In this way we obtain

$$\text{Lt}_{n \rightarrow \infty} S_n(\pm \pi) = \frac{1}{2} [f(-\pi+0) + f(\pi-0)],$$

when $f(-\pi+0)$ and $f(\pi-0)$ exist.

COROLLARY. If $f(x)$ is continuous in $a \leq x \leq b$, including the end-points, when the arbitrary positive number ϵ is chosen, the same η will do for all values of x from a to b , including the end-points. Then, from (7), it follows that *the sequence of arithmetic means*

$$S_1, S_2, S_3, \dots$$

converges uniformly to the sum $f(x)$ in the interval (a, b) .

It is assumed in this statement that $f(x)$ is continuous at $x=a$ and $x=b$ as well as in the interval (a, b) ; i.e. $f(a-0)=f(a)=f(a+0)$, and $f(b-0)=f(b)=f(b+0)$.

102. Two Theorems on the Arithmetic Means. Before applying this very important theorem to the discussion of Fourier's Series, we shall prove two theorems regarding the sequence of arithmetic means for any series

$$u_1 + u_2 + u_3 + \dots$$

In this connection we adopt the notation

$$s_n = u_1 + u_2 + \dots + u_n,$$

$$S_n = \frac{s_1 + s_2 + \dots + s_n}{n}.$$

THEOREM I. *If the series $u_1 + u_2 + u_3 + \dots$ converges and its sum is s , then $\text{Lt}_{n \rightarrow \infty} S_n = \text{Lt}_{n \rightarrow \infty} s_n = s$.*

We are given that the series

$$u_1 + u_2 + u_3 + \dots$$

converges, and that its sum is s .

Thus, to the arbitrary positive number ϵ there corresponds a positive integer ν such that $|s - s_n| < \epsilon$, when $n \geq \nu$.

Thus, with the usual notation,

$$|{}_p R_n| = |s_{n+p} - s_n| < 2\epsilon, \text{ when } n \geq \nu, \text{ for every positive integer } p.$$

Also it follows, from the definition of S_n , that, when $n > \nu$,

$$\begin{aligned} S_n - \left[u_1 + u_2 \left(1 - \frac{1}{n} \right) + \dots + u_\nu \left(1 - \frac{\nu-1}{n} \right) \right] \\ = u_{\nu+1} \left(1 - \frac{\nu}{n} \right) + u_{\nu+2} \left(1 - \frac{\nu+1}{n} \right) + \dots + u_n \left(1 - \frac{n-1}{n} \right). \end{aligned}$$

But $\left(1 - \frac{\nu}{n} \right), \left(1 - \frac{\nu+1}{n} \right), \dots, \left(1 - \frac{n-1}{n} \right)$

are all positive and decreasing; and

$$|u_{\nu+1}|, |u_{\nu+1} + u_{\nu+2}|, \dots, |u_{\nu+1} + u_{\nu+2} + \dots + u_n|$$

are all less than 2ϵ when $n > \nu$.

It follows, as in § 50, that

$$\left| u_{\nu+1} \left(1 - \frac{\nu}{n} \right) + u_{\nu+2} \left(1 - \frac{\nu+1}{n} \right) + \dots + u_n \left(1 - \frac{n-1}{n} \right) \right| < 2\epsilon \left(1 - \frac{\nu}{n} \right) < \epsilon, \text{ when } n > \nu.$$

Thus $\left| S_n - \left\{ u_1 + u_2 \left(1 - \frac{1}{n} \right) + \dots + u_\nu \left(1 - \frac{\nu-1}{n} \right) \right\} \right| < 2\epsilon$, when $n > \nu$.

Therefore $|S_n - s_\nu| < 2\epsilon + \frac{|u_2 + 2u_3 + \dots + (\nu-1)u_\nu|}{n}$, when $n > \nu$.

But ν being fixed, we can choose the positive integer N so that

$$\frac{|u_2 + 2u_3 + \dots + (\nu-1)u_\nu|}{n} < \epsilon, \text{ when } n \geq N > \nu.$$

Therefore $|S_n - s_\nu| < 3\epsilon$, when $n \geq N > \nu$.
 But $|s - s_\nu| < \epsilon$.
 And $|S_n - s| \leq |S_n - s_\nu| + |s - s_\nu|$.
 Therefore $|S_n - s| < 4\epsilon$, when $n \geq N > \nu$.
 Thus $\lim_{n \rightarrow \infty} S_n = s$.

THEOREM II.* *Let the sequence of arithmetic means S_n of the series*

$$u_1 + u_2 + u_3 + \dots \dots \dots (1)$$

converge to S . Then, if a positive integer n_0 exists such that

$$|u_n| < K/n, \text{ when } n > n_0,$$

where K is some positive number independent of n , the series (1) converges and its sum is S .

With the same notation as above, let $t_n = s_n - S$.

We have to prove that $\lim_{n \rightarrow \infty} t_n = 0$.

If t_n has not the limit zero as $n \rightarrow \infty$, there must be a positive number h such that there are an infinite number of the terms t_n which satisfy either (i) $t_n > h$ or (ii) $t_n < -h$.

We shall show that neither of these hypotheses can be admitted.

Take the former to be true, and let

$$\sigma_n = t_1 + t_2 + \dots + t_n.$$

Then $\sigma_n/n = S_n - S$,

and $\lim_{n \rightarrow \infty} \sigma_n/n = 0$.

Thus, to the arbitrary positive number ϵ , there corresponds a positive integer n_1 such that $|\sigma_n/n| < \epsilon$, when $n \geq n_1$.

Choose this number ϵ so that $h/K + 2 < \frac{1}{2}h^2/K\epsilon$.

Now let n be any positive integer greater than both n_0 and n_1 such that $t_n > h$.

Then, when $r = 0, 1, 2, \dots$, $|u_{n+r}| < K/n$.

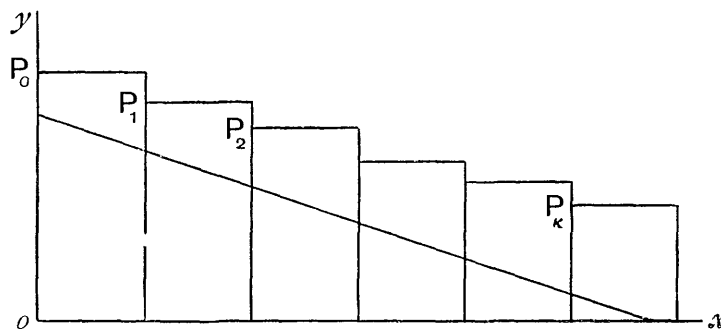


FIG. 29.

Now plot the points P_r , whose coordinates are (r, t_{n+r}) in a Cartesian diagram. Since $t_{n+r+1} - t_{n+r} = u_{n+r+1}$, the slope of the line $P_r P_{r+1}$ is less

* This theorem was published by Hardy in *London, Proc. Math. Soc. (Ser. 2)*, 8, pp. 302-304, 1910. This proof, due to Littlewood, is given in Whittaker and Watson, *Modern Analysis*, p. 157, 1920. Cf. also de la Vallée Poussin, *loc. cit.* T. II., §151.

in absolute value than $\tan^{-1}K/n$. Therefore the points P_0, P_1, P_2, \dots lie above the line $y = h - x \tan \theta$, where $\tan \theta = K/n$.

Let P_κ be the last of the points P_0, P_1, P_2, \dots , which lies to the left of, or upon, the line $x = h \cot \theta$, so that $\kappa \leq h \cot \theta = hn/K$.

Draw rectangles as shown in the diagram. The area of these rectangles exceeds the area of the triangle bounded by the axes and $y = h - x \tan \theta$.

Thus we have

$$\sigma_{n+\kappa} - \sigma_{n-1} = t_n + t_{n+1} + \dots + t_{n+\kappa} > \frac{1}{2}h^2 \cot \theta = \frac{1}{2}h^2 n/K.$$

But $|\sigma_{n+\kappa} - \sigma_{n-1}| \leq |\sigma_{n+\kappa}| + |\sigma_{n-1}| < [(n+\kappa) + (n-1)]\epsilon$, since $n > n_1$.

Therefore $(\kappa + 2n - 1)\epsilon > \frac{1}{2}h^2 n/K$.

But $\kappa \leq h \cot \theta = hn/K$.

Thus $hn/K \geq \kappa > \frac{1}{2}h^2 n/K\epsilon - (2n - 1)$.

Therefore $n[h/K - \frac{1}{2}h^2/K\epsilon + 2] > 1$.

But $h/K - \frac{1}{2}h^2/K\epsilon + 2 < 0$.

Therefore $n < \frac{1}{\frac{1}{2}h^2/K\epsilon - h/K - 2}$.

But we have assumed that there are an infinite number of values of n such that $t_n > h$.

The hypothesis (i) thus leads to a contradiction ; and a precisely similar argument shows that the hypothesis (ii) does the same.

Thus $\text{Lt}_{n \rightarrow \infty} t_n = 0$,

and it follows that $\text{Lt}_{n \rightarrow \infty} s_n = \text{Lt}_{n \rightarrow \infty} S_n = S$.

COROLLARY.

Let $u_1(x) + u_2(x) + u_3(x) + \dots \dots \dots (1)$ be a series whose terms are functions of x , and let the sequence of arithmetic means $S_n(x)$ for the series converge uniformly to $S(x)$ in an interval (a, b) . Then, if a positive integer n_0 exists such that

$$|u_n(x)| < K/n, \text{ when } n > n_0,$$

where K is independent of n and x , and the same n_0 serves for all values of x in the interval, the series (1) converges uniformly to $S(x)$ in (a, b) .

For, with the notation of the theorem just proved, if $t_n(x)$ does not tend uniformly to zero through (a, b) , there must be a positive number h , independent of x and n , such that an infinite sequence of values of n can be found for which $t_n(x_n) > h$, or $t_n(x_n) < -h$ for some point x_n in the interval ; the value of x_n depends on the particular value of n under consideration.

We then find, as in the original theorem, that both these hypotheses are inadmissible.

103. Fejér's Theorem and Fourier's Series.* We shall now use Fejér's Theorem to establish the convergence of Fourier's Series under the limitations imposed in our previous discussion ; that is we shall show that :

*Cf. Whittaker and Watson, *loc. cit.*, p. 167, 1915, and the note on p. 263 below.

When $f(x)$ satisfies Dirichlet's Conditions in the interval $(-\pi, \pi)$, and

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx', \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx',$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx' \quad (n \geq 1),$$

the sum of the series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

is $\frac{1}{2}[f(x+0) + f(x-0)]$ at every point in $-\pi < x < \pi$ where $f(x+0)$ and $f(x-0)$ exist; and at $x = \pm\pi$ the sum is $\frac{1}{2}[f(-\pi+0) + f(\pi-0)]$, when these limits exist.

I. First, let $f(x)$ be bounded in $(-\pi, \pi)$ and otherwise satisfy Dirichlet's Conditions in this interval.

Then the interval $(-\pi, \pi)$ can be broken up into a finite number (say p) of open partial intervals in which $f(x)$ is monotonic, and it follows at once from the Second Theorem of Mean Value that each of these intervals contributes to $|a_n|$ or $|b_n|$ a part less than $4M/n\pi$, where $|f(x)| < M$ in $(-\pi, \pi)$.

$$\text{Thus we have} \quad |a_n \cos nx + b_n \sin nx| < 8pM/n\pi,$$

where M is independent of n .

It follows from Fejér's Theorem, combined with Theorem II. of § 102, that the Fourier's Series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

converges, and its sum is $\frac{1}{2}[f(x+0) + f(x-0)]$ at every point in $-\pi < x < \pi$ at which $f(x \pm 0)$ exist, and at $x = \pm\pi$ its sum is $\frac{1}{2}[f(-\pi+0) + f(\pi-0)]$, provided that $f(-\pi+0)$ and $f(\pi-0)$ exist.

II. Next, let there be a finite number of points of infinite discontinuity in $(-\pi, \pi)$, but, when arbitrarily small neighbourhoods of these points are excluded, let $f(x)$ be bounded in the remainder of the interval, which can be broken up into a finite number of open partial intervals in each of which $f(x)$ is monotonic. And in addition let the infinite integral $\int_{-\pi}^{\pi} f(x') dx'$ be absolutely convergent.

In this case, let x be a point between $-\pi$ and π at which $f(x+0)$ and $f(x-0)$ exist. Then we may suppose it an internal point of an interval (a, b) , where $b-a < \pi$, and $f(x)$ is bounded in (a, b) and otherwise satisfies Dirichlet's Conditions therein.

$$\text{Let} \quad \left. \begin{aligned} \pi a_n' &= \int_a^b f(x') \cos nx' dx' \\ \pi b_n' &= \int_a^b f(x') \sin nx' dx' \end{aligned} \right\} n \geq 1,$$

and

$$\text{while} \quad 2\pi a_0' = \int_a^b f(x') dx'.$$

Then, forming the arithmetic means for the series

$$a_0' + (a_1' \cos x + b_1' \sin x) + (a_2' \cos 2x + b_2' \sin 2x) + \dots,$$

we have, with the notation of § 101,

$$\begin{aligned} S_n(x) &= \frac{s_0 + s_1 + \dots + s_{n-1}}{n} \\ &= \frac{1}{2n\pi} \int_a^b f(x') \frac{\sin^2 \frac{1}{2}n(x' - x)}{\sin^2 \frac{1}{2}(x' - x)} dx' \\ &= \frac{1}{n\pi} \left\{ \int_0^{\frac{1}{2}(x-a)} f(x-2\alpha) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha + \int_0^{\frac{1}{2}(b-x)} f(x+2\alpha) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha \right\}, \end{aligned}$$

where $\frac{1}{2}(x-a)$ and $\frac{1}{2}(b-x)$ are each positive and less than $\frac{1}{2}\pi$.

But it will be seen that the argument used in Fejér's Theorem with regard to the integrals

$$\frac{1}{n\pi} \int_0^{\frac{1}{2}\pi} f(x \pm 2\alpha) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha$$

applies equally well when the upper limits of the integrals are positive and less than $\frac{1}{2}\pi$.*

Therefore, in this case,

$$\text{Lt}_{n \rightarrow \infty} S_n(x) = \frac{1}{2} [f(x+0) + f(x-0)].$$

And, as the terms $(a_n' \cos nx + b_n' \sin nx)$

satisfy the condition of Theorem II. of § 102, it follows that the series

$$a_0' + (a_1' \cos x + b_1' \sin x) + (a_2' \cos 2x + b_2' \sin 2x) + \dots$$

converges and that its sum is $\text{Lt}_{n \rightarrow \infty} S_n(x)$.

But $(a_0 - a_0') + \sum_1^n \{(a_n - a_n') \cos nx + (b_n - b_n') \sin nx\}$

$$\begin{aligned} &= \frac{1}{\pi} \left\{ \int_{-\pi}^a + \int_b^{\pi} \right\} f(x') \left\{ \frac{1}{2} + \sum_1^n \cos n(x' - x) \right\} dx' \\ &= \frac{1}{2\pi} \left\{ \int_{-\pi+x}^a + \int_b^{\pi+x} \right\} f(x') \frac{\sin \frac{1}{2}(2n+1)(x' - x)}{\sin \frac{1}{2}(x' - x)} dx' + \\ &= \frac{1}{\pi} \int_{\frac{1}{2}(x-a)}^{\frac{1}{2}\pi} f(x-2\alpha) \frac{\sin (2n+1)\alpha}{\sin \alpha} d\alpha + \frac{1}{\pi} \int_{\frac{1}{2}(b-x)}^{\frac{1}{2}\pi} f(x+2\alpha) \frac{\sin (2n+1)\alpha}{\sin \alpha} d\alpha. \end{aligned}$$

By § 94 both of these integrals vanish in the limit as $n \rightarrow \infty$.

It follows that the series

$$(a_0 - a_0') + \sum_1^{\infty} \{(a_n - a_n') \cos nx + (b_n - b_n') \sin nx\}$$

converges, and that its sum is zero.

But we have already shown that

$$a_0' + \sum_1^{\infty} (a_n' \cos nx + b_n' \sin nx)$$

converges, and that its sum is

$$\frac{1}{2} [f(x+0) + f(x-0)].$$

* Cf. footnote, p. 235.

† Outside the interval $(-\pi, \pi)$ $f(x)$ is defined by the equation $f(x+2\pi) = f(x)$.

It follows, by adding the two series, that

$$a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges, and that its sum is

$$\frac{1}{2}[f(x+0) + f(x-0)]$$

at any point between $-\pi$ and π at which these limits exist.

When the limits $f(-\pi+0)$ and $f(\pi-0)$ exist, we can reduce the discussion of the sum of the series for $x = \pm\pi$ to the above argument, using the equation $f(x+2\pi) = f(x)$.

We can then treat $x = \pm\pi$ as inside an interval (a, b) , as above.

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EXAMPLES ON CHAPTER VII.

1. In the interval $0 < x < \frac{l}{2}$, $f(x) = \frac{1}{4}l - x$,

and in the interval $\frac{l}{2} < x < l$, $f(x) = x - \frac{3}{4}l$.

Prove that $f(x) = \frac{2l}{\pi^2} \left(\cos \frac{2\pi x}{l} + \frac{1}{9} \cos \frac{6\pi x}{l} + \frac{1}{25} \cos \frac{10\pi x}{l} + \dots \right)$.

2. The function $f(x)$ is defined as follows for the interval $(0, \pi)$;

$$f(x) = \frac{2}{3}x, \text{ when } 0 \leq x \leq \frac{1}{3}\pi,$$

$$f(x) = \frac{1}{3}\pi, \text{ when } \frac{1}{3}\pi < x < \frac{2}{3}\pi,$$

$$f(x) = \frac{2}{3}(\pi - x), \text{ when } \frac{2}{3}\pi \leq x \leq \pi.$$

Show that

$$f(x) = \frac{6}{\pi} \sum_1^{\infty} \frac{\sin \frac{1}{3}(2n-1)\pi \sin (2n-1)x}{(2n-1)^2}, \text{ when } 0 \leq x \leq \pi.$$

3. Expand $f(x)$ in a series of sines of multiples of $\pi x/a$, given that

$$f(x) = mx, \quad \text{when } 0 \leq x \leq \frac{1}{2}a,$$

$$f(x) = m(a-x), \quad \text{when } \frac{1}{2}a \leq x \leq a.$$

4. Prove that

$$\frac{1}{2}l - x = \frac{l}{\pi} \sum_1^{\infty} \frac{\sin \frac{2n\pi x}{l}}{n}, \quad \text{when } 0 < x < l,$$

and

$$\left(\frac{1}{2}l - x\right)^2 = \frac{l^2}{12} + \frac{l^2}{\pi^2} \sum_1^{\infty} \frac{\cos \frac{2n\pi x}{l}}{n^2}, \quad \text{when } 0 \leq x \leq l.$$

5. Obtain an expansion in a mixed series of sines and cosines of multiples of x which is zero between $-\pi$ and 0 , and is equal to e^x between 0 and π , and gives its values at the three limits.

6. Show that between the values $-\pi$ and $+\pi$ of x the following expansions hold:

$$\sin mx = \frac{2}{\pi} \sin m\pi \left(\frac{\sin x}{1^2 - m^2} - \frac{2 \sin 2x}{2^2 - m^2} + \frac{3 \sin 3x}{3^2 - m^2} + \dots \right),$$

$$\cos mx = \frac{2}{\pi} \sin m\pi \left(\frac{1}{2m} + \frac{m \cos x}{1^2 - m^2} - \frac{m \cos 2x}{2^2 - m^2} + \frac{m \cos 3x}{3^2 - m^2} - \dots \right),$$

$$\frac{\cosh mx}{\sinh m\pi} = \frac{2}{\pi} \left(\frac{1}{2m} - \frac{m \cos x}{1^2 + m^2} + \frac{m \cos 2x}{2^2 + m^2} - \frac{m \cos 3x}{3^2 + m^2} + \dots \right).$$

7. Express x^2 for values of x between $-\pi$ and π as the sum of a constant and a series of cosines of multiples of x .

Prove that the locus represented by

$$\sum_1^{\infty} \frac{(-1)^{n-1}}{n^2} \sin nx \sin ny = 0$$

is two systems of lines at right angles dividing the plane of x, y into squares of area π^2 .

8. Prove that

$$y^2 = \frac{2}{3} \frac{c^3}{d} + \sum_1^{\infty} \frac{4d}{n^3 \pi^3} \left(d \sin \frac{n\pi c}{d} - n\pi c \cos \frac{n\pi c}{d} \right) \cos \frac{n\pi}{d} x$$

represents a series of circles of radius c with their centres on the axis of x at distances $2d$ apart, and also the portions of the axis exterior to the circles, one circle having its centre at the origin.

9. A polygon is inscribed in a circle of radius a , and is such that the alternate sides beginning at $\theta=0$ subtend angles α and β at the centre of the circle. Prove that the first, third, ... pairs of sides of the polygon may be represented, except at angular points, by the polar equation

$$r = a_1 \sin \frac{\pi\theta}{\alpha + \beta} + a_2 \sin \frac{2\pi\theta}{\alpha + \beta} + \dots,$$

where

$$\frac{a_n(a+\beta)}{4a} = \sin \frac{n\pi a}{2(a+\beta)} \cos \frac{a}{2} \int_0^{\frac{\pi}{2}} \cos \frac{n\pi\phi}{a+\beta} \sec \phi d\phi$$

$$+ \sin \frac{n\pi(2a+\beta)}{2(a+\beta)} \cos \frac{\beta}{2} \int_0^{\frac{\pi}{2}} \cos \frac{n\pi\phi}{a+\beta} \sec \phi d\phi.$$

Find a similar equation to represent the other sides.

10. A regular hexagon has a diagonal lying along the axis of x . Investigate a trigonometrical series which shall represent the value of the ordinate of any point of the perimeter lying above the axis of x .

11. If $0 < x < 2\pi$, prove that

$$\frac{\pi \sinh \alpha(\pi - x)}{2 \sinh \alpha\pi} = \frac{\sin x}{a^2 + 1^2} + \frac{2 \sin 2x}{a^2 + 2^2} + \frac{3 \sin 3x}{a^2 + 3^2} + \dots$$

12. Prove that the equation in rectangular coordinates

$$y = \frac{2}{3}h + \frac{4h}{\pi^2} \left(\cos \frac{\pi x}{\kappa} - \frac{1}{2^2} \cos \frac{2\pi x}{\kappa} + \frac{1}{3^2} \cos \frac{3\pi x}{\kappa} - \dots \right)$$

represents a series of equal and similar parabolic arcs of height h and span 2κ standing in contact along the axis of x .

13. The arcs of equal parabolas cut off by the latera recta of length $4a$ are arranged alternately on opposite sides of a straight line formed by placing the latera recta end to end, so as to make an undulating curve. Prove that the equation of the curve can be written in the form

$$\frac{\pi^3 y}{64a} = \sin \frac{\pi x}{4a} + \frac{1}{3^3} \sin \frac{3\pi x}{4a} + \frac{1}{5^3} \sin \frac{5\pi x}{4a} + \dots$$

14. If circles be drawn on the sides of a square as diameters, prove that the polar equation of the quatrefoil formed by the external semicircles, referred to the centre as origin, is

$$\frac{\pi r}{4a\sqrt{2}} = \frac{1}{2} + \frac{1}{15} \cos 4\theta - \frac{1}{63} \cos 8\theta + \frac{1}{143} \cos 12\theta + \dots,$$

where a is the side of the square.

15. On the sides of a regular pentagon remote from the centre are described segments of circles which contain angles equal to that of the pentagon; prove that the equation to the cinquefoil thus obtained is

$$\pi r = 5a \tan \frac{\pi}{5} \left[1 - 2 \sum_1^{\infty} (-1)^n \frac{\cos 5n\theta}{25n^2 - 1} \right],$$

a being the radius of the circle circumscribing the pentagon.

16. In the interval $0 < x < \frac{l}{2}$, $f(x) = x^2$;
and in the interval $\frac{l}{2} < x < l$, $f(x) = 0$.

Express the function by means of a series of sines and also by means of a series of cosines of multiples of $\frac{\pi x}{l}$. Draw figures showing the functions represented by the two series respectively for all values of x not restricted to lie between 0 and l . What are the sums of the series for the value $x = \frac{l}{2}$?

17. A point moves in a straight line with a velocity which is initially u , and which receives constant increments each equal to u at equal intervals τ . Prove that the velocity at any time t after the beginning of the motion is

$$v = \frac{1}{2}u + \frac{u\tau}{\tau} + \frac{u}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi}{\tau} t,$$

and that the distance traversed is

$$\frac{u\tau}{2\tau}(t + \tau) + \frac{u\tau}{12} - \frac{u\tau}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi}{\tau} t. \quad [\text{See Ex. 4 above.}]$$

18. A curve is formed by the positive halves of the circles

$$(x - (4n + 1)a)^2 + y^2 = a^2$$

and the negative halves of the circles

$$(x - (4n - 1)a)^2 + y^2 = a^2,$$

n being an integer. Prove that the equation for the complete curve obtained by Fourier's method is

$$y = \frac{2}{a} \sum_{\kappa=1}^{\infty} (-1)^{\kappa-1} \sin\left(\kappa - \frac{1}{2}\right) \frac{\pi x}{a} \int_0^a \sin\left(\kappa - \frac{1}{2}\right) \frac{\pi x'}{a} \sqrt{a^2 - x'^2} dx'.$$

19. Having given the form of the curve $y = f(x)$, trace the curves

$$y = \frac{2}{\pi} \sum_{r=1}^{\infty} \sin rx \int_0^{\frac{\pi}{2}} f(t) \sin rt dt,$$

$$y = \frac{4}{\pi} \sum_{r=1}^{\infty} \sin(2r-1)x \int_0^{\frac{\pi}{2}} f(t) \sin(2r-1)t dt,$$

and show what these become when the upper limit is $\frac{\pi}{4}$ instead of $\frac{\pi}{2}$.

20. Prove that for all values of t between 0 and $\frac{b}{a}$ the value of the series

$$\sum_{r=1}^{\infty} \frac{1}{r} \sin \frac{r\pi b}{l} \sin \frac{r\pi x}{l} \sin \frac{r\pi at}{l}$$

is zero for all values of x between 0 and $b - at$ and between $at + b$ and l , and is $\frac{\pi}{4}$ for all values of x between $b - at$ and $at + b$, when $b < \frac{l}{2}$.

21. Find the sum of the series

$$u = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n},$$

$$v = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1},$$

and hence prove that the greatest integer in the positive number x is represented by $x + u - 8v^2$. [See Ex. 1, p. 230, and Ex. 4 above.]

22. If x, y, z are the rectangular coordinates of a point which moves so that from $y=0$ to $y=x$ the value of z is $\kappa(a^2-x^2)$, and from $y=x$ to $y=a$ the value of z is $\kappa(a^2-y^2)$, show that for all values of x and y between 0 and a , z may be expressed by a series in the form

$$\sum A_{p,q} \sin \frac{p\pi}{2} \left(\frac{x-a}{a} \right) \sin \frac{q\pi}{2} \left(\frac{y+a}{a} \right),$$

and find the values of $A_{p,q}$ for the different types of terms.

23. If $f(x) = \frac{\pi}{2} \sin x$, when $0 \leq x \leq \frac{\pi}{2}$,

and $f(x) = \frac{\pi}{2}$, when $\frac{\pi}{2} < x \leq \pi$,

prove that, when $0 \leq x < \pi$,

$$f(x) = S_1 + S_2 - S_3,$$

where $S_1 = \frac{2}{1 \cdot 3} \sin 2x - \frac{4}{3 \cdot 5} \sin 4x + \frac{6}{5 \cdot 7} \sin 6x - \dots$,

$$S_2 = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots,$$

$$S_3 = \sin 2x + \frac{1}{3} \sin 6x + \frac{1}{5} \sin 10x + \dots,$$

and find the values of S_1, S_2 , and S_3 separately for values of x lying within the assigned interval. [Cf. Ex. 1, p. 230.]

24. If $f(x) = \frac{4}{\pi^2} \left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots \right)$
 $+ \frac{2}{\pi} \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right),$

show that $f(x)$ is continuous between 0 and π , and that $f(\pi-0)=1$. Also show that $f'(x)$ has a sudden change of value $\frac{2}{\pi}$ at the point $\frac{\pi}{2}$. [See Ex. 1, 2, pp. 222-3.]

25. Let

$$f(x) = \sum_1^{\infty} \frac{\sin 3(2n-1)x}{2n-1} - 2 \sum_1^{\infty} \frac{\sin (2n-1)x}{2n-1} + \frac{6}{\pi} \sum_1^{\infty} \frac{\sin \frac{1}{3}(2n-1)\pi \sin (2n-1)x}{(2n-1)^2},$$

when $0 \leq x \leq \pi$.

Show that

$$f(+0) = f(\pi-0) = -\frac{1}{4}\pi,$$

$$f\left(\frac{1}{3}\pi+0\right) - f\left(\frac{1}{3}\pi-0\right) = -\frac{1}{2}\pi,$$

$$f\left(\frac{2}{3}\pi+0\right) - f\left(\frac{2}{3}\pi-0\right) = \frac{1}{2}\pi;$$

also that

$$f(0) = f\left(\frac{1}{3}\pi\right) = f\left(\frac{2}{3}\pi\right) = f(\pi) = 0.$$

Draw the graph of $f(x)$ in the interval $(0, \pi)$. See Ex. 1, p. 230, and Ex. 2 above.]

CHAPTER VIII

THE NATURE OF THE CONVERGENCE OF FOURIER'S SERIES

104. The Order of the Terms. Before entering upon the discussion of the nature of the convergence of the Fourier's Series for a function satisfying Dirichlet's Conditions, we shall show that in certain cases the order of the terms may be determined easily.

I. *If $f(x)$ is bounded and otherwise satisfies Dirichlet's Conditions in the interval $(-\pi, \pi)$, the coefficients in the Fourier's Series for $f(x)$ are less in absolute value than K/n , where K is some positive number independent of n .*

Since
$$\pi a_n = \int_{-\pi}^{\pi} f(x) \cos nx \, dx,$$

and the interval may be broken up into a finite number of open partial intervals (c_r, c_{r+1}) in which $f(x)$ is monotonic, it follows, from the Second Theorem of Mean Value, that

$$\begin{aligned} \pi a_n &= \sum \int_{c_r}^{c_{r+1}} f(x) \cos nx \, dx \\ &= \sum \left\{ f(c_r + 0) \int_{c_r}^{\xi_r} \cos nx \, dx + f(c_{r+1} - 0) \int_{\xi_r}^{c_{r+1}} \cos nx \, dx \right\}, \end{aligned}$$

where ξ_r is some definite number in (c_r, c_{r+1}) .

Thus
$$\begin{aligned} \pi |a_n| &< \frac{2}{n} \sum \{ |f(c_r + 0)| + |f(c_{r+1} - 0)| \} \\ &< \frac{4pM}{n}, \end{aligned}$$

where p is the number of partial intervals and M is the upper bound of $|f(x)|$ in the interval $(-\pi, \pi)$.

Therefore $|a_n| < K/n$,
where K is some positive number independent of n .

And similarly we obtain

$$|b_n| < K/n.$$

We may speak of the terms of this series as of the order $1/n$. When the terms are of the order $1/n$, the series will, in general, be only conditionally convergent, the convergence being due to the presence of both positive and negative terms.

II. *If $f(x)$ is bounded and continuous, and otherwise satisfies Dirichlet's Conditions in $-\pi < x < \pi$, while $f(\pi-0) = f(-\pi+0)$, and if $f'(x)$ is bounded and otherwise satisfies Dirichlet's Conditions in the same interval, the coefficients in the Fourier's Series for $f(x)$ are less in absolute value than K/n^2 , where K is some positive number independent of n .*

In this case we can make $f(x)$ continuous in the closed interval $(-\pi, \pi)$ by giving to it the values $f(-\pi+0)$ and $f(\pi-0)$ at $x = -\pi$ and π respectively.

$$\begin{aligned} \text{Then } \pi a_n &= \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{n} \left[f(x) \sin nx \right]_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx \\ &= -\frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx. \end{aligned}$$

But we have just seen that with the given conditions

$$\int_{-\pi}^{\pi} f'(x) \sin nx \, dx$$

is of the order $1/n$.

It follows that $|a_n| < K/n^2$,

where K is some positive number independent of n .

A similar argument, in which it will be seen that the condition $f(\pi-0) = f(-\pi+0)$ is used, shows that

$$|b_n| < K/n^2.$$

Since the terms of this Fourier's Series are of the order $1/n^2$, it follows that it is absolutely convergent, and also uniformly convergent in any interval.

The above result can be generalised as follows: *If the function $f(x)$ and its differential coefficients, up to the $(p-1)^{\text{th}}$, are bounded, continuous and otherwise satisfy Dirichlet's Conditions in the interval $-\pi < x < \pi$, and*

$$f^{(r)}(-\pi+0) = f^{(r)}(\pi-0), \quad [r = 0, 1, \dots, (p-1)],$$

and if the p^{th} differential coefficient is bounded and otherwise satisfies Dirichlet's Conditions in the same interval, the coefficients in the Fourier's Series for $f(x)$ will be less in absolute value than K/n^{p+1} , where K is some positive number independent of n .

105. Again suppose that the interval $(-\pi, \pi)$ can be broken up into a certain number of open partial intervals $(-\pi, c_1), (c_1, c_2), \dots (c_m, \pi)$, in each of which $f(x)$ and $f'(x)$ are bounded, continuous and otherwise satisfy Dirichlet's Conditions. Also let $f''(x)$ be bounded and otherwise satisfy Dirichlet's Conditions in the whole interval.

In this case $f(-\pi+0), \dots f(c_r \pm 0), \dots f(\pi-0)$ and $f'(-\pi+0), \dots, f'(c_r \pm 0), \dots f'(\pi-0)$ exist, when $r \leq m$.

$$\text{Also } \pi a_n = \int_{-\pi}^{c_1} f(x) \cos nx \, dx + \int_{c_1}^{c_2} f(x) \cos nx \, dx + \dots + \int_{c_m}^{\pi} f(x) \cos nx \, dx.$$

Integrating by parts, we find that

$$a_n = \frac{A_n}{n} - \frac{b_n'}{n}, \dots \dots \dots (1)$$

where $\pi A_n = \sum_1^m \sin nc_r \{f(c_r-0) - f(c_r+0)\}$

and $\pi b_n' = \int_{-\pi}^{\pi} f'(x) \sin nx \, dx.$

Similarly, from the equation

$$\pi b_n = \int_{-\pi}^{c_1} f(x) \sin nx \, dx + \int_{c_1}^{c_2} f(x) \sin nx \, dx + \dots + \int_{c_m}^{\pi} f(x) \sin nx \, dx,$$

we find that

$$b_n = \frac{B_n}{n} + \frac{a_n'}{n}, \dots \dots \dots (2)$$

where $\pi B_n = -\sum_1^m \cos nc_r \{f(c_r-0) - f(c_r+0)\} + \cos n\pi \{f(-\pi+0) - f(\pi-0)\}$

and $\pi a_n' = \int_{-\pi}^{\pi} f'(x) \cos nx \, dx.$

In the same way we get

$$a_n' = \frac{A_n'}{n} - \frac{b_n''}{n}, \quad b_n' = \frac{B_n'}{n} + \frac{a_n''}{n},$$

where

$$\pi A_n' = \sum_1^m \sin nc_r \{f'(c_r-0) - f'(c_r+0)\},$$

$$\pi B_n' = -\sum_1^m \cos nc_r \{f'(c_r-0) - f'(c_r+0)\} + \cos n\pi \{f'(-\pi+0) - f'(\pi-0)\},$$

and $\pi a_n'' = \int_{-\pi}^{\pi} f''(x) \cos nx \, dx,$

$$\pi b_n'' = \int_{-\pi}^{\pi} f''(x) \sin nx \, dx.$$

Thus

$$a_n = \frac{A_n}{n} - \frac{B_n'}{n^2} - \frac{a_n''}{n^2},$$

$$b_n = \frac{B_n}{n} + \frac{A_n'}{n^2} - \frac{b_n''}{n^2}.$$

This result includes that of § 104 as a special case. It is also clear that $\pi|A_n|$, $\pi|B_n|$, $\pi|A_n'|$ and $\pi|B_n'|$ are at most equal to the sum of the "jumps" in $f(x)$ and $f'(x)$, as the case may be, including the points $x = \pm\pi$ as points of discontinuity, when $f(\pi-0) \neq f(-\pi+0)$, and $f'(\pi-0) \neq f'(-\pi+0)$. And under the given conditions a_n'' and b_n'' are of the order $1/n$.

The expressions for a_n and b_n can be used in determining the points at which the sum of a given Fourier's Series may be discontinuous.

Thus, if we are given the series

$$\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots,$$

and assume that it is a Fourier's Series, and not simply a trigonometrical series (cf. § 90), we see that

$$a_n = 0, \quad b_n = (1 - \cos n\pi)/2n.$$

So that (1) and (2) above would be satisfied if

$$A_n = 0, \quad B_n = (1 - \cos n\pi)/2, \quad a_n' = 0 = b_n'.$$

Hence, if c_1, c_2, \dots are the points of discontinuity,

$$\pi A_n = 0 = [\sin nc_1 \{f(c_1-0) - f(c_1+0)\} + \sin nc_2 \{f(c_2-0) - f(c_2+0)\} + \dots].$$

Since this is true for all integral values of n , the numbers c_1, c_2, \dots would need to be multiples of π ; and, since c_1, c_2, \dots lie in the interval $-\pi < x < \pi$, $c_1 = 0$ and c_2, c_3, \dots do not exist.

Substituting $c_1 = 0$ in the equation for B_n ,

$$\pi \left(\frac{1}{2} - \frac{1}{2} \cos n\pi\right) = \cos n\pi \{f(-\pi+0) - f(\pi-0)\} + \{f(+0) - f(-0)\}.$$

Since this is true for all integral values of n ,

$$\frac{1}{2}\pi = \{f(+0) - f(-0)\} = \{f(\pi-0) - f(-\pi+0)\}.$$

Thus we see that, if the series is a Fourier's Series with ordinary discontinuities, $f(x)$ has discontinuities at $x=0$ and $x = \pm\pi$. Also, since $a_n' = b_n' = 0$, we should expect $f(x)$ to be a constant (say k) in the open interval $(0, \pi)$, and $-k$ in the open interval $(-\pi, 0)$.

As a matter of fact, the series is the Fourier's Series for $f(x)$, where

$$f(x) = \begin{cases} \frac{1}{4}\pi & \text{in } 0 < x < \pi, \\ -\frac{1}{4}\pi & \text{in } -\pi < x < 0. \end{cases}$$

106. Discussion of a case in which $f(x)$ has an infinity in $(-\pi, \pi)$. We have seen that Dirichlet's Conditions include the possibility of $f(x)$ having a certain number of points of infinite discontinuity in the interval, subject to the condition that the infinite integral $\int_{-\pi}^{\pi} f(x) dx$ is absolutely convergent.

Let us suppose that near the point x_0 , where $-\pi < x_0 < \pi$, the function $f(x)$ is such that

$$f(x) = \frac{\phi(x)}{(x-x_0)^\nu},$$

where $0 < \nu < 1$ and $\phi(x)$ is monotonic to the right and left of x_0 , while $\phi(x_0 \pm 0)$ do not both vanish.

In this case the condition for absolute convergence is satisfied.

Then, in determining a_n and b_n , where

$$\pi a_n = \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad \text{and} \quad \pi b_n = \int_{-\pi}^{\pi} f(x) \sin nx \, dx,$$

we break up the interval into

$$(-\pi, \alpha), \quad (\alpha, x_0), \quad (x_0, \beta), \quad \text{and} \quad (\beta, \pi),$$

where (α, β) is the interval in which $f(x)$ has the given form. In $(-\pi, \alpha)$ and (β, π) it is supposed that $f(x)$ is bounded and otherwise satisfies Dirichlet's Conditions, and we know from § 98 that these partial intervals give to a_n and b_n contributions of the order $1/n$.

The remainder of the integral, *e.g.* in a_n , is given by the sum of

$$\text{Lt}_{\delta \rightarrow 0} \int_{\alpha}^{x_0 - \delta} \frac{\phi(x)}{(x-x_0)^\nu} \cos nx \, dx \quad \text{and} \quad \text{Lt}_{\delta \rightarrow 0} \int_{x_0 + \delta}^{\beta} \frac{\phi(x)}{(x-x_0)^\nu} \cos nx \, dx,$$

these limits being known to exist.

We take the second of these integrals, and apply to it the Second Theorem of Mean Value.

Thus we have

$$\int_{x_0 + \delta}^{\beta} \frac{\phi(x)}{(x-x_0)^\nu} \cos nx \, dx = \phi(x_0 + \delta) \int_{x_0 + \delta}^{\xi} \frac{\cos nx}{(x-x_0)^\nu} \, dx + \phi(\beta) \int_{\xi}^{\beta} \frac{\cos nx}{(x-x_0)^\nu} \, dx,$$

where $x_0 + \delta \leq \xi \leq \beta$.

Putting $n(x-x_0) = y$, we obtain

$$\begin{aligned} \int_{x_0 + \delta}^{\beta} \frac{\phi(x)}{(x-x_0)^\nu} \cos nx \, dx &= \frac{\phi(x_0 + \delta)}{n^{1-\nu}} \int_{n\delta}^{n(\xi-x_0)} \frac{\cos(y+nx_0)}{y^\nu} \, dy \\ &\quad + \frac{\phi(\beta)}{n^{1-\nu}} \int_{n(\xi-x_0)}^{n\beta} \frac{\cos(y+nx_0)}{y^\nu} \, dy. \end{aligned}$$

$$\text{But} \quad \int_a^b \frac{\cos(y+nx_0)}{y^\nu} \, dy = \cos nx_0 \int_a^b \frac{\cos y}{y^\nu} \, dy - \sin nx_0 \int_a^b \frac{\sin y}{y^\nu} \, dy.$$

Also when a, b are positive,

$$\left| \int_a^b \frac{\cos y}{y^\nu} \, dy \right|, \quad \left| \int_a^b \frac{\sin y}{y^\nu} \, dy \right|$$

are both less than definite numbers independent of a and b , when $0 < \nu < 1$.

Thus, whatever positive integer n may be, and whatever value δ may have, subject to $0 < \delta < \beta - x_0$,

$$\left| \int_{x_0 + \delta}^{\beta} \frac{\phi(x)}{(x-x_0)^\nu} \cos nx \, dx \right| < K'/n^{1-\nu},$$

where K' is some positive number independent of n and ξ .

It follows that

$$\left| \operatorname{Lt}_{\delta \rightarrow 0} \int_{x_0+\delta}^{\beta} \frac{\phi(x)}{(x-x_0)^\nu} \cos nx \, dx \right| < K/n^{1-\nu}.$$

A similar argument applies to the integral

$$\int_a^{x_0-\delta} \frac{\phi(x)}{(x-x_0)^\nu} \cos nx \, dx.$$

It thus appears that the coefficient of $\cos nx$ in the Fourier's Series for the given function $f(x)$ is less in absolute value than $K/n^{1-\nu}$, where K is some positive number independent of n ; and a corresponding result holds with regard to the coefficient of $\sin nx$.

It is easy to modify the above argument so that it will apply to the case when the infinity occurs at $\pm \pi$.

107. The Uniform Convergence of Fourier's Series. We shall deal, first of all, with the case of the Fourier's Series for $f(x)$, when $f(x)$ is bounded in $(-\pi, \pi)$ and otherwise satisfies Dirichlet's Conditions. Later we shall discuss the case where a finite number of points of infinite discontinuity are admitted.

It is clear that the Fourier's Series for $f(x)$ cannot be uniformly convergent in any interval which contains a point of discontinuity; since uniform convergence, in the case of series whose terms are continuous, involves continuity in the sum.

Let $f(x)$ be bounded in the interval $(-\pi, \pi)$, and otherwise satisfy Dirichlet's Conditions in that interval. Then the Fourier's Series for $f(x)$ converges uniformly to $f(x)$ in any interval which contains neither in its interior nor at an end any point of discontinuity of the function.

As before the bounded function $f(x)$, satisfying Dirichlet's Conditions in $(-\pi, \pi)$, is defined outside that interval by the equation

$$f(x+2\pi) = f(x).$$

Then we can express $f(x)$ in any interval—e.g. $(-2\pi, 2\pi)$ —as the difference of two functions, which we shall denote by $F(x)$ and $G(x)$, where $F(x)$ and $G(x)$ are bounded, positive and monotonic increasing. They are also continuous at all points where $f(x)$ is continuous [§ 36].

Let $f(x)$ be continuous at a and b^* and at all points in $a < x < b$, where, to begin with, we shall assume $-\pi < a$ and $b < \pi$.

Also let x be any point in (a, b) .

* Thus $f(a+0) = f(a) = f(a-0)$ and $f(b+0) = f(b) = f(b-0)$.

Then with the notation of § 95,

$$S_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \frac{\sin \frac{1}{2}(2n+1)(x'-x)}{\sin \frac{1}{2}(x'-x)} dx'$$

$$= \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(x+2\alpha) \frac{\sin m\alpha}{\sin \alpha} d\alpha, \text{ where } m = 2n+1.*$$

Thus

$$S_n(x) = \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} F(x+2\alpha) \frac{\sin m\alpha}{\sin \alpha} d\alpha$$

$$- \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} G(x+2\alpha) \frac{\sin m\alpha}{\sin \alpha} d\alpha. \dots\dots\dots(1)$$

We shall now discuss the first integral in (1),

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} F(x+2\alpha) \frac{\sin m\alpha}{\sin \alpha} d\alpha,$$

i.e. $\int_0^{\frac{1}{2}\pi} F(x+2\alpha) \frac{\sin m\alpha}{\sin \alpha} d\alpha + \int_0^{\frac{1}{2}\pi} F(x-2\alpha) \frac{\sin m\alpha}{\sin \alpha} d\alpha.$

Let μ be any number such that $0 < \mu < \frac{1}{2}\pi$.

Then

$$\int_0^{\frac{1}{2}\pi} F(x+2\alpha) \frac{\sin m\alpha}{\sin \alpha} d\alpha = F(x+0) \int_0^{\frac{1}{2}\pi} \frac{\sin m\alpha}{\sin \alpha} d\alpha$$

$$+ \int_0^{\mu} \{F(x+2\alpha) - F(x+0)\} \frac{\sin m\alpha}{\sin \alpha} d\alpha$$

$$+ \int_{\mu}^{\frac{1}{2}\pi} \{F(x+2\alpha) - F(x+0)\} \frac{\sin m\alpha}{\sin \alpha} d\alpha$$

$$= I_1 + I_2 + I_3, \text{ say. } \dots\dots\dots(2)$$

We can replace $F(x+0)$ by $F(x)$, since $F(x)$ is continuous at x .

Now we know that $\int_0^{\frac{1}{2}\pi} \frac{\sin m\alpha}{\sin \alpha} d\alpha = \frac{1}{2}\pi,$

since m or $2n+1$ is an odd positive integer.

Thus $I_1 = \frac{1}{2}\pi F(x). \dots\dots\dots(3)$

Also $\{F(x+2\alpha) - F(x)\}$ is bounded, positive and monotonic increasing in any interval; and $\frac{\alpha}{\sin \alpha}$ is also bounded, positive and monotonic increasing in $0 < \alpha \leq \frac{1}{2}\pi$.

* We have replaced the limits $-\pi, \pi$ by $-\pi+x, \pi+x$ in the integral before changing the variable from x' to α by the substitution $x' = x+2\alpha$. Cf. § 101.

Therefore we can apply the Second Theorem of Mean Value to the integral

$$\int_0^\mu \phi(a) \frac{\sin ma}{a} da,$$

where $\phi(a) = \{F(x+2a) - F(x)\} \frac{a}{\sin a}$.

It follows that

$$I_2 = \{F(x+2\mu) - F(x)\} \frac{\mu}{\sin \mu} \int_\xi^\mu \frac{\sin ma}{a} da,$$

where $0 \leq \xi \leq \mu$.

But we know that

$$\left| \int_\xi^\mu \frac{\sin ma}{a} da \right| = \left| \int_{m\xi}^{m\mu} \frac{\sin a}{a} da \right| < \pi. \quad (\S 91.)$$

Therefore $|I_2| < \{F(x+2\mu) - F(x)\} \frac{\mu\pi}{\sin \mu}$(4)

Finally
$$I_3 = \{F(x+2\mu) - F(x)\} \int_\mu^{\xi'} \frac{\sin ma}{\sin a} da$$

$$+ \{F(x+\pi) - F(x)\} \int_{\xi'}^{\frac{1}{2}\pi} \frac{\sin ma}{\sin a} da,$$

where $\mu \leq \xi' \leq \frac{1}{2}\pi$.

But, if $0 < \theta < \phi \leq \frac{1}{2}\pi$,

$$\int_\theta^\phi \frac{\sin ma}{\sin a} da = \frac{1}{\sin \theta} \int_\theta^\chi \sin ma da + \frac{1}{\sin \phi} \int_\chi^\phi \sin ma da,$$

where $\theta \leq \chi \leq \phi$.

Therefore
$$\left| \int_\theta^\phi \frac{\sin ma}{\sin a} da \right| < \frac{2}{m} \{\operatorname{cosec} \theta + \operatorname{cosec} \phi\}$$

$$< \frac{4}{m} \operatorname{cosec} \theta.$$

It follows that

$$|I_3| < \frac{4}{m \sin \mu} [\{F(x+2\mu) - F(x)\} + \{F(x+\pi) - F(x)\}]$$

$$< \frac{4K}{m \sin \mu}, \quad \dots\dots\dots(5)$$

where K is some positive number, independent of m , and depending on the upper bound of $|f(x)|$ in $(-\pi, \pi)$.

Combining (3), (4) and (5), we see from (2) that

$$\left| \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} F(x+2a) \frac{\sin ma}{\sin a} da - \frac{1}{2} F(x) \right|$$

$$< \{F(x+2\mu) - F(x)\} \frac{\mu}{\sin \mu} + \frac{4K}{m\pi \sin \mu}. \quad \dots\dots(6)$$

A similar argument applies to the integral

$$\int_0^{\frac{1}{2}\pi} F(x-2a) \frac{\sin ma}{\sin a} da,$$

but in this case it has to be remembered that $F(x-2a)$ is monotonic decreasing as a increases from 0 to $\frac{1}{2}\pi$.

The corresponding result for this integral is that

$$\left| \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} F(x-2a) \frac{\sin ma}{\sin a} da - \frac{1}{2} F(x) \right| < |F(x-2\mu) - F(x)| \frac{\mu}{\sin \mu} + \frac{4K}{m\pi \sin \mu}, \dots\dots(7)$$

K as before being some positive number independent of m , and depending on the upper bound of $|f(x)|$ in $(-\pi, \pi)$.

Without loss of generality we can take K the same in (6) and (7).

From (6) and (7) we obtain at once

$$\left| \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} F(x+2a) \frac{\sin ma}{\sin a} da - F(x) \right| < \frac{\mu}{\sin \mu} \{ |F(x+2\mu) - F(x)| + |F(x-2\mu) - F(x)| \} + \frac{8K}{m\pi \sin \mu}. \quad (8)$$

Similarly we find that

$$\left| \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} G(x+2a) \frac{\sin ma}{\sin a} da - G(x) \right| < \frac{\mu}{\sin \mu} \{ |G(x+2\mu) - G(x)| + |G(x-2\mu) - G(x)| \} + \frac{8K'}{m\pi \sin \mu}. \quad (9)$$

Thus, from (1),

$$\begin{aligned} & \left| \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(x+2a) \frac{\sin ma}{\sin a} da - f(x) \right| \\ & < \frac{\mu}{\sin \mu} \{ |F(x+2\mu) - F(x)| + |F(x-2\mu) - F(x)| \\ & \quad + |G(x+2\mu) - G(x)| + |G(x-2\mu) - G(x)| \} \\ & \quad + \frac{16K}{m\pi \sin \mu}. \dots\dots\dots(10) \end{aligned}$$

Now $F(x)$ and $G(x)$ are continuous in $a < x < b$, and also when $x = a$ and $x = b$.

Thus, to the arbitrary positive number ϵ , there will correspond a positive number μ_0 (which can be taken less than $\frac{1}{2}\pi$) such that

$$|F(x+2\mu) - F(x)| < \epsilon, \quad |G(x+2\mu) - G(x)| < \epsilon,$$

when $|\mu| \leq \mu_0$, the same μ_0 serving for all values of x in $a \leq x \leq b$. [Cf. § 32.]

Also we know that $\mu \operatorname{cosec} \mu$ increases continuously from unity to $\frac{1}{2}\pi$ as μ passes from 0 to $\frac{1}{2}\pi$.

Choose μ_0 , as above, less than $\frac{1}{2}\pi$, and put $\mu = \mu_0$ in the argument of (1) to (10). This is allowable, as the only restriction upon μ was that it must lie between 0 and $\frac{1}{2}\pi$.

Then the terms on the right-hand side of (10), not including $16K/m\pi \sin \mu_0$, are together less than 8ϵ for all values of x in (a, b) .

So far nothing has been said about the number m , except that it is an odd positive integer $(2n+1)$.

Let n_0 be the smallest positive integer which satisfies the inequality

$$\frac{16K}{(2n_0+1)\pi \sin \mu_0} < \epsilon.$$

As K , μ_0 and ϵ are independent of x , so also is n_0 . We now choose m (*i.e.* $2n+1$) so that $n \geq n_0$.

Then it follows from (10) that

$$|S_n(x) - f(x)| < 9\epsilon, \text{ when } n \geq n_0,$$

the same n_0 serving for every x in $a \leq x \leq b$.

In other words, we have shown that the Fourier's Series converges uniformly to $f(x)$, under the given conditions, in the interval (a, b) .*

If $f(-\pi+0) = f(\pi-0)$, we can regard the points $\pm\pi, \pm 2\pi$, etc., as points at which $f(x)$, extended beyond $(-\pi, \pi)$ by the equation $f(x+2\pi) = f(x)$, is continuous, for we can give to $f(\pm\pi)$ the common value of $f(-\pi+0)$ and $f(\pi-0)$.

*It may help the reader to follow the argument of this section if we take a special case:

$$\left. \begin{aligned} \text{E.g. } f(x) &= 0, & -\pi \leq x \leq 0, \\ f(x) &= 1, & 0 < x \leq \pi. \end{aligned} \right\}$$

Then we have:

Interval.	$f(x)$	$F(x)$	$G(x)$
$-2\pi < x < -\pi$	1	1	0
$-\pi \leq x \leq 0$	0	1	1
$0 < x \leq \pi$	1	2	1
$\pi < x \leq 2\pi$	0	2	2

If $0 < a \leq x \leq b < \pi$, the interval (a, b) is an interval in which $f(x)$ is continuous.

The argument of the preceding section will then apply to the case in which $-\pi$ or π is an end-point of the interval (a, b) inside and at the ends of which $f(x)$ is continuous.

108. The Uniform Convergence of Fourier's Series (*continued*). By argument similar to that employed in the preceding section, it can be proved that when $f(x)$, bounded or not, satisfies Dirichlet's Conditions in the interval $(-\pi, \pi)$, and $f(x)$ is bounded in the interval (a', b') contained within $(-\pi, \pi)$, then the Fourier's Series for $f(x)$ converges uniformly in any interval (a, b) in the interior of (a', b') , provided $f(x)$ is continuous in (a, b) , including its end-points.

But, instead of developing the discussion on these lines, we shall now show how the question can be treated by Fejér's Arithmetic Means (cf. § 101), and we shall prove the following theorem :

Let $f(x)$, bounded or not, satisfy Dirichlet's Conditions in the interval $(-\pi, \pi)$, and let it be continuous at a and b and in (a, b) , where $-\pi \leq a$ and $b \leq \pi$. Then the Fourier's Series for $f(x)$ converges uniformly to $f(x)$ in any interval $(a + \delta, b - \delta)$ contained within (a, b) .

Without loss of generality we may assume $b - a \leq \pi$, for a greater interval could be treated as the sum of two such intervals.

$$\begin{aligned} \text{Let} \quad & 2\pi a_0' = \int_a^b f(x') dx', \\ & \left. \begin{aligned} \pi a_n' &= \int_a^b f(x') \cos nx' dx \\ \pi b_n' &= \int_a^b f(x') \sin nx' dx', \end{aligned} \right\} n \geq 1. \end{aligned}$$

and

Since $f(x)$ is continuous at a and b and in (a, b) , it is also bounded in (a, b) , and we can use the corollary to Fejér's Theorem (§ 101) and assert that the sequence of Arithmetic Means for the series

$$a_0' + \sum_1^{\infty} (a_n' \cos nx + b_n' \sin nx)$$

converges *uniformly* to $f(x)$ in (a, b) .

$$\text{Also} \quad |a_n' \cos nx + b_n' \sin nx| \leq (a_n'^2 + b_n'^2)^{\frac{1}{2}}.$$

But $f(x)$ is bounded in (a, b) and satisfies Dirichlet's Conditions therein.

$$\text{Thus we can write} \quad f(x) = F(x) - G(x),$$

where $F(x)$ and $G(x)$ are bounded, positive and monotonic increasing functions in (a, b) . It follows that we can apply the Second Theorem of Mean Value to the integrals

$$\int_a^b F(x') \frac{\cos nx'}{\sin nx'} dx', \quad \int_a^b G(x') \frac{\cos nx'}{\sin nx'} dx',$$

and we deduce at once that $(a_n'^2 + b_n'^2)^{\frac{1}{2}} < K/n$,

where K is some positive number depending on the upper bound of $|f(x)|$ in (a, b) .

Then we know, from the corollary to Theorem II. § 102, that the series

$$a_0' + \sum_1^{\infty} (a_n' \cos nx + b_n' \sin nx)$$

converges uniformly to $f(x)$ in (a, b) .

Let us now suppose x to be any point in the interval $(a + \delta, b - \delta)$ lying within the interval (a, b) .

With the usual notation

$$\left. \begin{aligned} 2\pi a_0 &= \int_{-\pi}^{\pi} f(x') dx', \\ \pi a_n &= \int_{-\pi}^{\pi} f(x') \cos nx' dx', \\ \pi b_n &= \int_{-\pi}^{\pi} f(x') \sin nx' dx'. \end{aligned} \right\}$$

It follows, as in § 103, that

$$(a_0 - a_0') + \sum_1^{\infty} \{(a_n - a_n') \cos nx + (b_n - b_n') \sin nx\}$$

is equal to

$$\frac{1}{\pi} \int_{\frac{1}{2}(x-a)}^{\frac{1}{2}\pi} f(x-2a) \frac{\sin(2n+1)a}{\sin a} da + \frac{1}{\pi} \int_{\frac{1}{2}(b-x)}^{\frac{1}{2}\pi} f(x+2a) \frac{\sin(2n+1)a}{\sin a} da,$$

$f(x)$ being defined outside the interval $(-\pi, \pi)$ by the equation

$$f(x+2\pi) = f(x).$$

Now $f(x)$ is supposed to have not more than a finite number of points (say m) of infinite discontinuity in $(-\pi, \pi)$, and $\int_{-\pi}^{\pi} |f(x')| dx'$ converges.

We can therefore take intervals $2\gamma_1, 2\gamma_2, \dots, 2\gamma_m$ enclosing these points, the intervals being so small that

$$\int_{2\gamma_r} |f(x')| dx' < 2\epsilon \sin \frac{1}{2}\delta, \quad [r = 1, 2, \dots, m]$$

ϵ being any given positive number.

Consider the integral

$$\int_{\frac{1}{2}(x-a)}^{\frac{1}{2}\pi} f(x-2a) \frac{\sin(2n+1)a}{\sin a} da,$$

x , as already stated, being a point in $(a + \delta, b - \delta)$.

As a passes from $\frac{1}{2}(x-a)$ to $\frac{1}{2}\pi$, we may meet some or all of the m points of discontinuity of the given function in $f(x-2a)$. Let these be taken as the centres of the corresponding intervals $\gamma_1, \gamma_2, \dots, \gamma_m$.

Also the smallest value of $(x-a)$ is δ .

$$\begin{aligned} \text{Thus} \quad \left| \int_{\gamma_r} f(x-2a) \frac{\sin(2n+1)a}{\sin a} da \right| &< \frac{1}{\sin \frac{1}{2}\delta} \int_{\gamma_r} |f(x-2a)| da \\ &< \frac{1}{2 \sin \frac{1}{2}\delta} \int_{2\gamma_r} |f(x')| dx' \\ &< \epsilon. \end{aligned}$$

When these intervals, such of them as occur, have been cut out, the integral

$$\int_{\frac{1}{2}(x-a)}^{\frac{1}{2}\pi} f(x-2a) \frac{\sin(2n+1)a}{\sin a} da$$

will at most consist of $(m+1)$ separate integrals (I_r) . [$r=1, 2, \dots, m+1$.]

In each of these integrals (I_r) we can take $f(x-2a)$ as the difference of two bounded, positive and monotonic increasing functions

$$F(x-2a) \quad \text{and} \quad G(x-2a).$$

Then, confining our attention to (I_r) , we see that

$$\begin{aligned} & \left| \int (F-G) \frac{\sin(2n+1)a}{\sin a} da \right| \\ &= \left| \int (F-G) \{ \operatorname{cosec} \delta/2 - (\operatorname{cosec} \delta/2 - \operatorname{cosec} a) \} \sin(2n+1)a da \right| \\ &\leq |J_1| + |J_2| + |J_3| + |J_4|, \end{aligned}$$

where

$$J_1 = \operatorname{cosec} \delta/2 \int F \sin(2n+1)a da,$$

$$J_2 = \operatorname{cosec} \delta/2 \int G \sin(2n+1)a da,$$

$$J_3 = \int F \{ \operatorname{cosec} \delta/2 - \operatorname{cosec} a \} \sin(2n+1)a da,$$

$$J_4 = \int G \{ \operatorname{cosec} \delta/2 - \operatorname{cosec} a \} \sin(2n+1)a da.$$

But we can apply the Second Theorem of Mean Value to each of these integrals, since the factor in each integrand which multiplies $\sin(2n+1)a$ is monotonic.

$$\text{It follows that} \quad |I_r| < \frac{K}{2n+1} \operatorname{cosec} \frac{1}{2}\delta,$$

where K is some positive number independent of n and x , and depending only on the values of $f(x)$ in $(-\pi, \pi)$, when the intervals $2\gamma_1, 2\gamma_2, \dots$ have been removed from that interval.

Thus

$$\begin{aligned} \left| \int_{\frac{1}{2}(x-a)}^{\frac{1}{2}\pi} f(x-2a) \frac{\sin(2n+1)a}{\sin a} da \right| &< \frac{(m+1)K}{(2n+1) \sin \frac{1}{2}\delta} + m\epsilon \\ &< (2m+1)\epsilon, \end{aligned}$$

when $(2n+1) > K \operatorname{cosec} \frac{1}{2}\delta$.

Since this choice of n is independent of x , the integral converges uniformly to zero as $n \rightarrow \infty$, when x lies in the interval $(a+\delta, b-\delta)$.

Similarly we find that

$$\int_{\frac{1}{2}(b-x)}^{\frac{1}{2}\pi} f(x+2a) \frac{\sin(2n+1)a}{\sin a} da$$

converges uniformly to zero when x lies in this interval.

Thus the series

$$(a_0 - a_0') + \sum_1^{\infty} \{ (a_n - a_n') \cos nx + (b_n - b_n') \sin nx \}$$

converges uniformly to zero in $(a+\delta, b-\delta)$.

But we have shown that the series

$$\alpha_0' + \sum_1^{\infty} (\alpha_n' \cos nx + b_n' \sin nx)$$

converges uniformly to $f(x)$ in (a, b) .

Since the sum of two uniformly convergent series converges uniformly, we see that

$$\alpha_0 + \sum_1^{\infty} (\alpha_n \cos nx + b_n \sin nx)$$

converges uniformly to $f(x)$ in $(a + \delta, b - \delta)$.

109. Differentiation and Integration of Fourier's Series. We have seen that, when $f(x)$ satisfies Dirichlet's Conditions, the Fourier's Series for $f(x)$ is uniformly convergent in the interior of any interval in which $f(x)$ is continuous. We may therefore integrate the series term by term within such an interval, and equate the result to the integral of $f(x)$ between the same limits. And this operation may be repeated any number of times.

With regard to term by term differentiation, a similar statement would obviously be untrue. We have shown that in certain general cases the order of the terms is $1/n$ or $1/n^2$. In the first alternative, it is clear that the series we would obtain by term by term differentiation would not converge; and the same remark applies to the other, when second differential coefficients are taken.

This difficulty does not occur in the application of Fourier's Series to the Conduction of Heat. In this case the terms are multiplied by a factor (e.g. $e^{-\kappa n^2 \pi^2 t/a^2}$), which may be called a convergency factor, as it increases the rapidity of the convergence of the series, and allows term by term differentiation both with regard to x and t .

If $f(x)$ is continuous in $(-\pi, \pi)$ and $f(-\pi) = f(\pi)$, while $f'(x)$ having only a finite number of points of infinite discontinuity is absolutely integrable, the Fourier's Series for $f'(x)$, whether it converges or not, is given by term to term differentiation of that corresponding to $f(x)$.

Let $\alpha_0, \alpha_1, b_1, \dots, \alpha_0', \alpha_1', b_1', \dots$ be the Fourier's Constants for $f(x)$ and $f'(x)$.

$$\begin{aligned} \text{Then} \quad 2\pi\alpha_0 &= \int_{-\pi}^{\pi} f(x) dx, & 2\pi\alpha_0' &= \int_{-\pi}^{\pi} f'(x) dx, \\ \pi\alpha_n &= \int_{-\pi}^{\pi} f(x) \cos nx dx, & \pi\alpha_n' &= \int_{-\pi}^{\pi} f'(x) \cos nx dx, \end{aligned}$$

etc.

Integrating the expression for α_n' by parts, we see that

$$\begin{aligned} \alpha_n' &= \frac{1}{\pi} \{f(\pi) - f(-\pi)\} \cos n\pi + \frac{n}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx' \\ &= nb_n, \end{aligned}$$

since we are given $f(\pi) = f(-\pi)$.

Also $\alpha_0' = 0$.

Similarly, starting with b_n' and integrating by parts, we see that

$$b_n' = -n\alpha_n.$$

Thus the Fourier's Series for $f'(x)$ is that which we obtain on differentiating the series for $f(x)$ term by term.

On the other hand, with the same assumptions, except that $f(\pi) \neq f(-\pi)$, the terms in the series for $f'(x)$ are given by

$$\left. \begin{aligned} a_0' &= \frac{1}{\pi} \{f(\pi) - f(-\pi)\}, \\ a_n' &= \frac{1}{\pi} \{f(\pi) - f(-\pi)\} \cos n\pi + nb_n, \\ b_n' &= -na_n. \end{aligned} \right\}$$

Thus the Fourier's Series for $f'(x)$ does not agree with that which we obtain on term by term differentiation of the series for $f(x)$.*

110. More General Theory of Fourier's Series. The questions treated in this chapter and the preceding may be looked at from another point of view.

We may start with the Fourier's Series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots, \dots\dots\dots(1)$$

where

$$\begin{aligned} 2\pi a_0 &= \int_{-\pi}^{\pi} f(x') dx', \\ \pi a_n &= \int_{-\pi}^{\pi} f(x') \cos nx' dx', \quad \pi b_n = \int_{-\pi}^{\pi} f(x') \sin nx' dx', \end{aligned}$$

the only condition imposed at this stage being that, if the arbitrary function is bounded, it shall be integrable in $(-\pi, \pi)$, and, if unbounded, the infinite integral $\int_{-\pi}^{\pi} f(x) dx$ shall be absolutely convergent.

We then proceed to examine under what conditions the series (1) converges for $x = x_0$.

It is not difficult to show that *the behaviour of the series at x_0 depends only on the nature of the function $f(x)$ in the neighbourhood of x_0 .*

Also a number of criteria have been obtained for the convergence of the series at x_0 to $\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)]$, when these limits exist.†

Further, it has been shown that, *if (a, b) be any interval contained in $(-\pi, \pi)$, such that $f(x)$ is continuous in (a, b) , including the end-points, the answer to the question whether the Fourier's Series converges uniformly in (a, b) , or not, depends only upon the nature of $f(x)$ in an interval (a', b') , which includes (a, b) in its interior, and exceeds it in length by an arbitrarily small amount.*

Again it can be shown that, *when $f(x)$ has only a finite number of points of infinite discontinuity, while $\int_{-\pi}^{\pi} f(x) dx$ converges absolutely, then $\int_a^{\beta} f(x) dx$, where $-\pi \leq a < \beta \leq \pi$, is represented by the convergent series obtained by*

* See also Gibson, *Edinburgh, Proc. Math. Soc.*, **12**, p. 47, et seq., 1894.

† Cf. de la Vallée Poussin, *loc. cit.*, T. II., §§ 137-143.

integrating the Fourier's Series (1) term by term, no assumption being made as regards the convergence of the original Fourier's Series.

The reader who wishes to pursue the study of Fourier's Series on these lines is referred to the treatises of Hobson and de la Vallée Poussin.

REFERENCES.

BÔCHER, *loc. cit.*

DE LA VALLÉE POUSSIN, *loc. cit.*, T. II. (2^e éd.), Ch. IV.

HOBSON, *loc. cit.*, Ch. VII.

LEBESGUE, *Léçons sur les séries trigonométriques*, Ch. III., Paris, 1906.

WHITTAKER AND WATSON, *loc. cit.*, Ch. IX.

Note.—The treatment of the uniform convergence of Fourier's Series in § 108 is founded on the proof given by Whittaker and Watson on pp. 169-170 of the 1915 edition of their *Modern Analysis*.

In the 1920 edition the authors have made considerable changes both in the proof of Fourier's Theorem (9. 42) and in the discussion of the uniform convergence of Fourier's Series (9. 44).

CHAPTER IX

THE APPROXIMATION CURVES AND GIBBS'S PHENOMENON IN FOURIER'S SERIES *

111. We have seen in § 104 that, when $f(x)$ is bounded and continuous, and otherwise satisfies Dirichlet's Conditions in $-\pi < x < \pi$, $f(\pi-0)$ being equal to $f(-\pi+0)$, and $f'(x)$ is bounded and otherwise satisfies Dirichlet's Conditions in the same interval, the coefficients in the Fourier's Series for $f(x)$ are of the order $1/n^2$, and the series is uniformly convergent in any interval.

In this case the approximation curves

$$y = S_n(x)$$

in the interval $-\pi < x < \pi$ will nearly coincide with

$$y = f(x),$$

when n is taken large enough.

As an example, let $f(x)$ be the odd function defined in $(-\pi, \pi)$ as follows:

$$\begin{aligned} f(x) &= -\frac{1}{4}\pi(\pi+x), & -\pi \leq x \leq -\frac{1}{2}\pi, \\ f(x) &= \frac{1}{4}\pi x, & -\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi, \\ f(x) &= \frac{1}{4}\pi(\pi-x), & \frac{1}{2}\pi \leq x \leq \pi. \end{aligned}$$

The Fourier's Series for $f(x)$ in this case is the Sine Series

$$\sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \dots,$$

which is uniformly convergent in any interval.

The approximation curves

$$y = \sin x,$$

$$y = \sin x - \frac{1}{3^2} \sin 3x,$$

$$y = \sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x,$$

* This chapter is founded on a paper by the author in *The American Journal of Mathematics*, **39**, p. 185, 1917.

are given in Fig. 30, along with $y = f(x)$, for the interval $(0, \pi)$, and it will be seen how closely the last of these three approaches the sum of the series right through the interval.

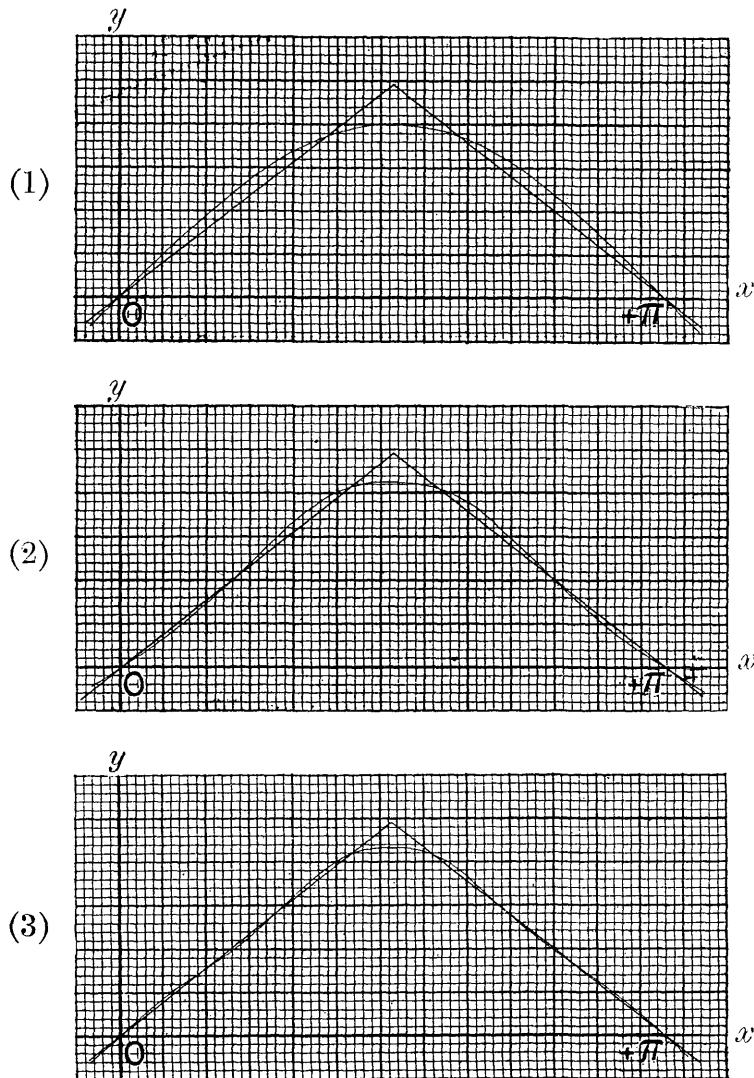


FIG. 30.

Again, let $f(x)$ be the corresponding even function :

$$f(x) = \frac{1}{4}\pi(\pi + x), \quad -\pi \leq x \leq -\frac{1}{2}\pi,$$

$$f(x) = -\frac{1}{4}\pi x, \quad -\frac{1}{2}\pi \leq x \leq 0,$$

$$f(x) = \frac{1}{4}\pi x, \quad 0 \leq x \leq \frac{1}{2}\pi,$$

$$f(x) = \frac{1}{4}\pi(\pi - x), \quad \frac{1}{2}\pi \leq x \leq \pi.$$

The Fourier's Series for $f(x)$ in this case is the Cosine Series

$$\frac{1}{16}\pi^2 - 2 \left\{ \frac{1}{2^2} \cos 2x + \frac{1}{6^2} \cos 6x + \frac{1}{10^2} \cos 10x + \dots \right\},$$

which is again uniformly convergent in any interval.

The approximation curves

$$y = \frac{1}{16}\pi^2 - \frac{1}{2} \cos 2x,$$

$$y = \frac{1}{16}\pi^2 - \frac{1}{2} \cos 2x - \frac{1}{18} \cos 6x,$$

are given in Fig. 31, along with $y = f(x)$, for the interval $(0, \pi)$, and again it will be seen how closely the second of these curves approaches $y = f(x)$ right through the interval.

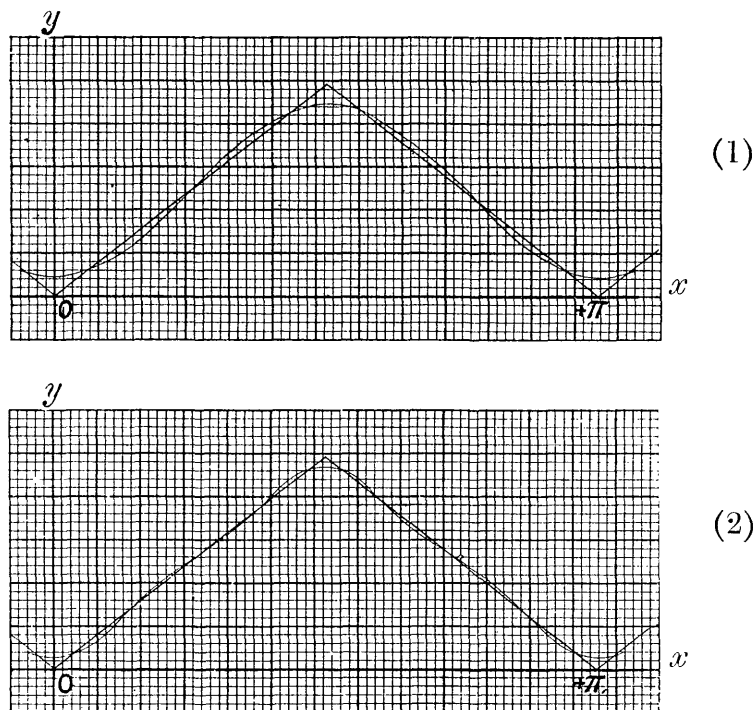


FIG. 31.

It will be noticed that in both these examples for large values of n the slope of $y = S_n(x)$ nearly agrees with that of $y = f(x)$, except at the corners, corresponding to $x = \pm \frac{1}{2}\pi$, where $f'(x)$ is discontinuous. This would lead us to expect that these series may be differentiated term by term, as in fact is the case.

112. When the function $f(x)$ is given by the equation

$$f(x) = x, \quad -\pi < x < \pi,$$

the corresponding Fourier's Series is the Sine Series

$$2\{\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots\}.$$

The sum of this series is x for all values in the open interval $-\pi < x < \pi$, and it is zero when $x = \pm \pi$.

This series converges uniformly in any interval $(-\pi + \delta, \pi - \delta)$ contained within $(-\pi, \pi)$ (cf. § 107), and in such an interval, by taking n large enough, we can make the approximation curves oscillate about $y = x$ as closely as we please.

Until 1899 it was wrongly believed that, for large values of n , each approximation curve passed at a steep gradient from the

point $(-\pi, 0)$ to a point near $(-\pi, -\pi)$, and then oscillated about $y=x$ till x approached the value π , when the curve passed at a steep gradient from a point near (π, π) to $(\pi, 0)$. And to those who did not properly understand what is meant by the sum of an infinite series, the difference between the approximation curves

$$y = S_n(x),$$

for large values of n , and the curve $y = \lim_{n \rightarrow \infty} S_n(x)$ offered considerable difficulty.

In Fig. 32, the line $y=x$ and the curve

$$y = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x\right)$$

are drawn, and the diagram might seem to confirm the above view of the matter—namely, that there will be a steep descent

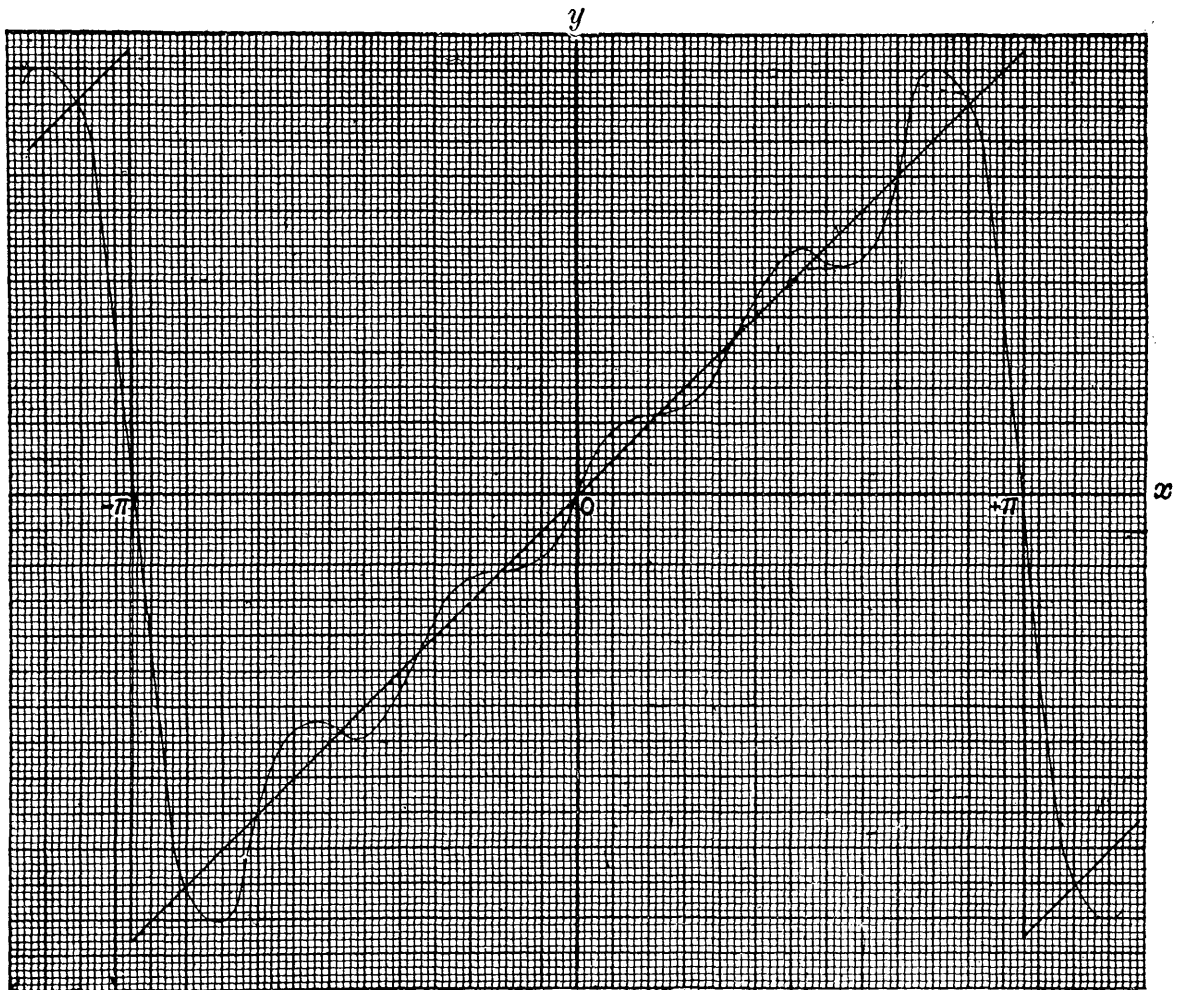


FIG. 32.

near one end of the line, from the point $(-\pi, 0)$ to near the point $(-\pi, -\pi)$, and a corresponding steep descent near the other end of the line. But it must be remembered that the convergence of this series is slow, and that $n=5$ would not count as a large value of n .

113. In 1899 Gibbs, in a letter to *Nature*,* pointed out that the approximation curves for this series do, in fact, behave in quite a different way at the points of discontinuity $\pm\pi$ in the sum. He stated, in effect, that the curve $y = S_n(x)$, for large values of n , falls from the point $(-\pi, 0)$ at a steep gradient to a point very nearly at a depth $2 \int_0^\pi \frac{\sin x}{x} dx$ below the axis of x , then oscillates above and below $y = x$ close to this line until x approaches π , when it rises to a point very nearly at a height $2 \int_0^\pi \frac{\sin x}{x} dx$ above the axis of x and then falls rapidly to $(\pi, 0)$.

The approximation curves, for large values of n , would thus approach closely to the continuous curve in Fig. 33, instead of the straight lines in Fig. 32, for the interval $(-\pi, \pi)$.

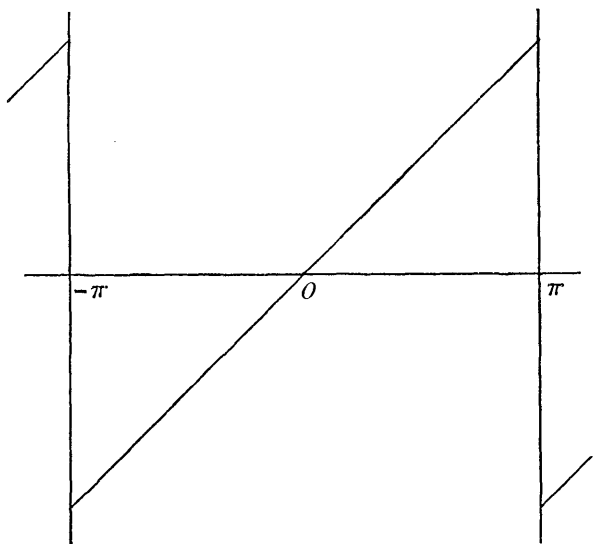


FIG. 33.

His statement was not accompanied by any proof. Though the remainder of the correspondence, of which his letter formed a part, attracted considerable attention, this remarkable observation passed practically unnoticed for several years. In 1906 Bôcher returned to the subject in a memoir † on Fourier's Series,

* *Nature*, 59, p. 606, 1899.

† *Ann. Math., Princeton, N.J.* (Ser. 2), 7, 1906.

See also a recent paper by Bôcher in *J. Math., Berlin*, 144, 1914, entitled "On Gibbs's Phenomenon."

Reference should also be made to Runge, *Theorie u. Praxis der Reihen*, pp. 170-180, Leipzig, 1904. A certain series is there discussed, and the nature of the jump in the approximation curves described; but no reference is made to Gibbs, and the example seems to have been regarded as quite an isolated one.

and greatly extended Gibbs's result. He showed, among other things, that the phenomenon which Gibbs had observed in the case of this particular Fourier's Series holds in general at ordinary points of discontinuity. To quote his own words:*

If $f(x)$ has the period 2π and in any finite interval has no discontinuities other than a finite number of finite jumps, and if it has a derivative which in any finite interval has no discontinuities other than a finite number of finite discontinuities, then as n becomes infinite the approximation curve $y = S_n(x)$ approaches uniformly the continuous curve made up of

(a) *the discontinuous curve $y = f(x)$,*

(b) *an infinite number of straight lines of finite lengths parallel to the axis of y and passing through the points a_1, a_2, \dots on the axis of x where the discontinuities of $f(x)$ occur. If a is any one of these points, the line in question extends between the two points whose ordinates are*

$$f(a-0) + \frac{DP_1}{\pi}, \quad f(a+0) - \frac{DP_1}{\pi},$$

where D is the magnitude of the jump in $f(x)$ at a ,† and

$$P_1 = \int_{\pi}^{\infty} \frac{\sin x}{x} dx = -0.2811.$$

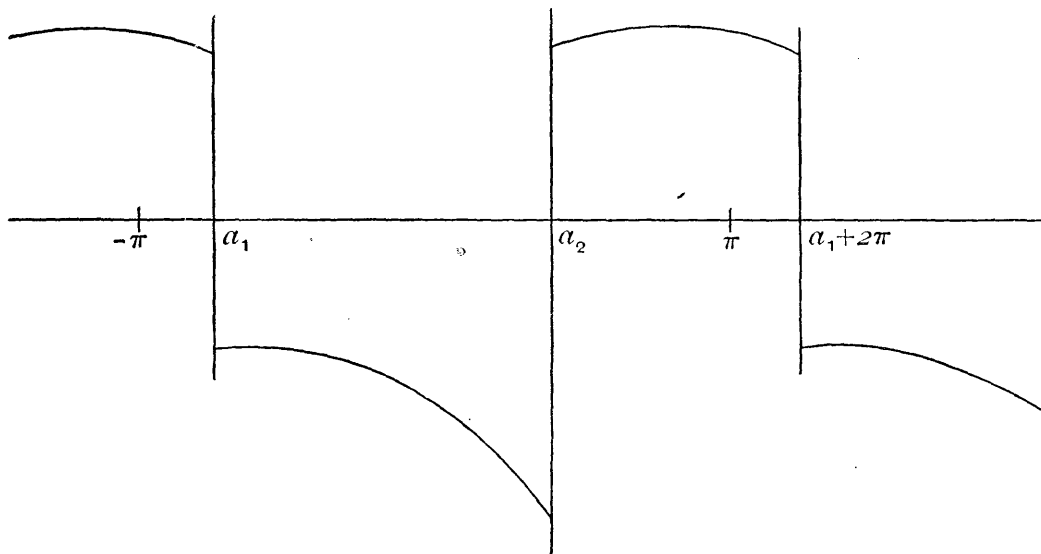


FIG. 34.

This theorem is illustrated in Fig. 34, where the amounts of the jumps at a_1, a_2 are respectively negative and positive. Until

* *loc. cit.*, p. 131.

† *i.e.* $D = f(a+0) - f(a-0)$.

Gibbs made his remark, it was supposed that the vertical lines extend merely between the points whose ordinates are $f(a \pm 0)$.

The series in the examples of Chapter VII. converge so slowly that the approximation curves of Figs. 21-23, 27, 28 are of little use in this connection.

114. The series on which Bôcher founded his demonstration of this and other extensions of Gibbs's theorem is

$$\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots,$$

which, in the interval $(0, 2\pi)$ represents the function $f(x)$ defined as follows:

$$\begin{aligned} f(0) &= f(2\pi) = 0, \\ f(x) &= \frac{1}{2}(\pi - x), \quad 0 < x < 2\pi. \end{aligned}$$

In this case

$$\begin{aligned} S_n(x) &= \sin x + \frac{1}{2} \sin 2x + \dots + \frac{1}{n} \sin nx \\ &= \int_0^x (\cos \alpha + \cos 2\alpha + \dots + \cos n\alpha) d\alpha \\ &= \frac{1}{2} \int_0^x \frac{\sin(n + \frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha - \frac{1}{2}x. \end{aligned}$$

The properties of the maxima and minima of $S_n(x)$ are not so easy to obtain,* nor are they so useful in the argument, as those of

$$\begin{aligned} R_n(x) &= \frac{1}{2}(\pi - x) - S_n(x) \\ &= \frac{1}{2}\pi - \frac{1}{2} \int_0^x \frac{\sin(n + \frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha. \end{aligned}$$

In his memoir Bôcher dealt with the maxima and minima of $R_n(x)$, and he called attention more than once to the fact that the height of a wave from the curve $y = f(x)$ was measured parallel to the axis of y . This point has been lost sight of in some expositions of his work.

In the discussion which follows, the series

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots)$$

is employed.

In the interval $(-\pi, \pi)$ it represents the function $f(x)$, defined as follows:

$$\begin{aligned} f(-\pi) &= f(0) = f(\pi) = 0, \\ f(x) &= -\frac{1}{2}\pi, \quad -\pi < x < 0, \\ f(x) &= \frac{1}{2}\pi, \quad 0 < x < \pi. \end{aligned}$$

* Gronwall has discussed the properties of $S_n(x)$ for this series, and deduced Gibbs's Phenomenon for the first wave, and some other results. Cf. *Math. Ann.*, Leipzig, **72**, 1912. Also Jackson, *Palermo, Rend. Circ. mat.*, **32**, 1911.

The sum of the first n terms of the series is denoted by $S_n(x)$, so that

$$S_n(x) = 2 \sum_1^n \frac{1}{(2r-1)} \sin(2r-1)x.$$

A number of interesting properties of the turning points of $y = S_n(x)$ are obtained by quite simple methods, and all the features of Gibbs's Phenomenon, as described by Bôcher, follow at once from these properties.

It is, of course, not an unusual thing that the approximation curves $y = S_n(x)$, for a series whose terms are continuous functions of x , differ considerably, even for large values of n , from the curve $y = \text{Lt}_{n \rightarrow \infty} S_n(x) = S(x)$. They certainly do so in the neighbourhood of a point where the sum of the series is discontinuous, and they also may do so in the neighbourhood of a point where the sum is continuous. This question we have already examined in Chapter III. But, in passing, it may be remarked that when $f(x)$ satisfies Dirichlet's Conditions, the approximation curves for the corresponding Fourier's Series will, for large values of n , differ only slightly from the curve $y = f(x)$ in the neighbourhood of points at which $f(x)$ is continuous, for we have shown that the Fourier's Series converges uniformly in any interval contained within an interval in which $f(x)$ is continuous.

The existence of maxima (or minima) of $S_n(x)$, the abscissae of which tend towards a as n increases (a being a value of x for which the sum of the series is discontinuous), while their ordinates remain at a finite distance from $f(a+0)$ and $f(a-0)$, is the chief feature of Gibbs's Phenomenon in Fourier's Series. And it is most remarkable that its occurrence in Fourier's Series remained undiscovered till so recent a date.*

THE TRIGONOMETRICAL SUM

$$S_n(x) = 2 \left(\sin x + \frac{1}{3} \sin 3x + \dots + \frac{1}{2n-1} \sin(2n-1)x \right).$$

115. If we define $f(x)$ by the equations

$$\left. \begin{aligned} f(-\pi) = f(0) = f(\pi) = 0, \\ f(x) = \frac{1}{2}\pi \dots, \quad 0 < x < \pi, \\ f(x) = -\frac{1}{2}\pi \dots, \quad -\pi < x < 0, \end{aligned} \right\}$$

* Cf. also Weyl: (i) "Die Gibbs'sche Erscheinung in der Theorie der Kugelfunktionen," *Palermo, Rend. Circ. mat.*, 29, 1910; (ii) "Über die Gibbs'sche Erscheinung und verwandte Konvergenzphänomene," *ibid.*, 30, 1910.

the Fourier's Series for $f(x)$ in the interval $-\pi \leq x \leq \pi$ is

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots).$$

As stated above, we denote by $S_n(x)$ this sum up to and including the term in $\sin(2n-1)x$. Then

$$S_n(x) = 2 \int_0^x (\cos x + \cos 3x + \dots + \cos(2n-1)x) dx = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha.$$

Since $S_n(x)$ is an *odd* function, we need only consider the interval $0 \leq x \leq \pi$.

We proceed to obtain the properties of the maxima and minima of this function $S_n(x)$.

I. Since, for any integer m ,

$$\sin(2m-1)(\frac{1}{2}\pi + x') = \sin(2m-1)(\frac{1}{2}\pi - x'),$$

it follows from the series that $S_n(x)$ is *symmetrical* about $x = \frac{1}{2}\pi$, and when $x=0$ and $x=\pi$ it is zero.

II. When $0 < x < \pi$, $S_n(x)$ is *positive*.

From (I) we need only consider $0 < x \leq \frac{1}{2}\pi$. We have

$$S_n(x) = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha = \frac{1}{2n} \int_0^{2nx} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha, \quad 0 < x \leq \frac{1}{2}\pi.$$

The denominator in the integrand is positive and continually increases in the interval of integration. By considering the successive waves in the graph of $\sin \alpha \operatorname{cosec} \frac{\alpha}{2n}$, the last of which may or may not be completed, it is clear that the integral is positive.

III. The turning points of $y = S_n(x)$ are given by

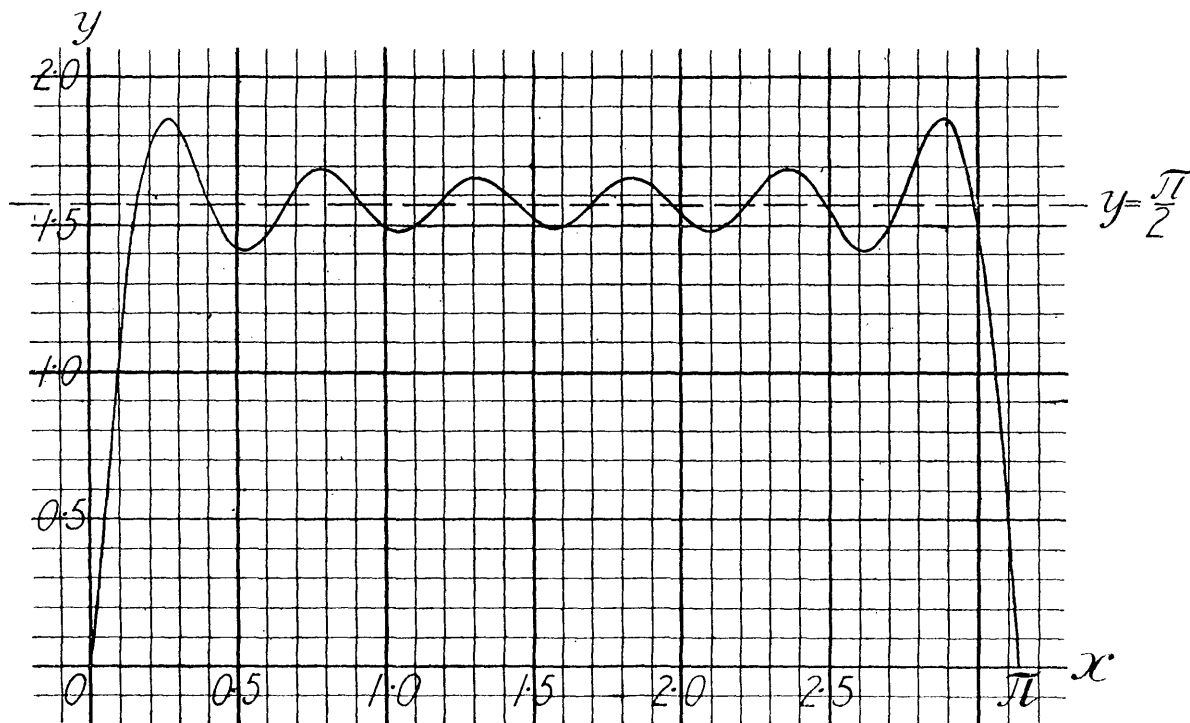
$$\begin{cases} x_1 = \frac{\pi}{2n}, & x_3 = \frac{3\pi}{2n}, \dots, & x_{2n-1} = \frac{2n-1}{2n}\pi \text{ (maxima),} \\ x_2 = \frac{\pi}{n}, & x_4 = \frac{2\pi}{n}, \dots, & x_{2(n-1)} = \frac{n-1}{n}\pi \text{ (minima).} \end{cases}$$

We have

$$y = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha, \quad \text{and} \quad \frac{dy}{dx} = \frac{\sin 2nx}{\sin x}.$$

The result follows at once.

IV. As we proceed from $x=0$ to $x=\frac{1}{2}\pi$, the heights of the maxima continually diminish, and the heights of the minima continually increase, n being kept fixed.



The curve $y=S_n(x)$, when $n=6$.

FIG. 35.

Consider two consecutive maxima in the interval $0 < x \leq \frac{1}{2}\pi$, namely, $S_n\left(\frac{2m-1}{2n}\pi\right)$ and $S_n\left(\frac{2m+1}{2n}\pi\right)$, m being a positive integer less than or equal to $\frac{1}{2}(n-1)$. We have

$$\begin{aligned} S_n\left(\frac{2m-1}{2n}\pi\right) - S_n\left(\frac{2m+1}{2n}\pi\right) &= \frac{1}{2n} \int_{(2m+1)\pi}^{(2m-1)\pi} \frac{\sin a}{\sin \frac{a}{2n}} da \\ &= -\frac{1}{2n} \left\{ \int_{(2m-1)\pi}^{2m\pi} \frac{\sin a}{\sin \frac{a}{2n}} da + \int_{2m\pi}^{(2m+1)\pi} \frac{\sin a}{\sin \frac{a}{2n}} da \right\}. \end{aligned}$$

The denominator in both integrands is positive and it continually increases in the interval $(2m-1)\pi \leq a \leq (2m+1)\pi$; also the numerator in the first is continually negative and in the second continually positive, the absolute values for elements at equal distances from $(2m-1)\pi$ and $2m\pi$ being the same.

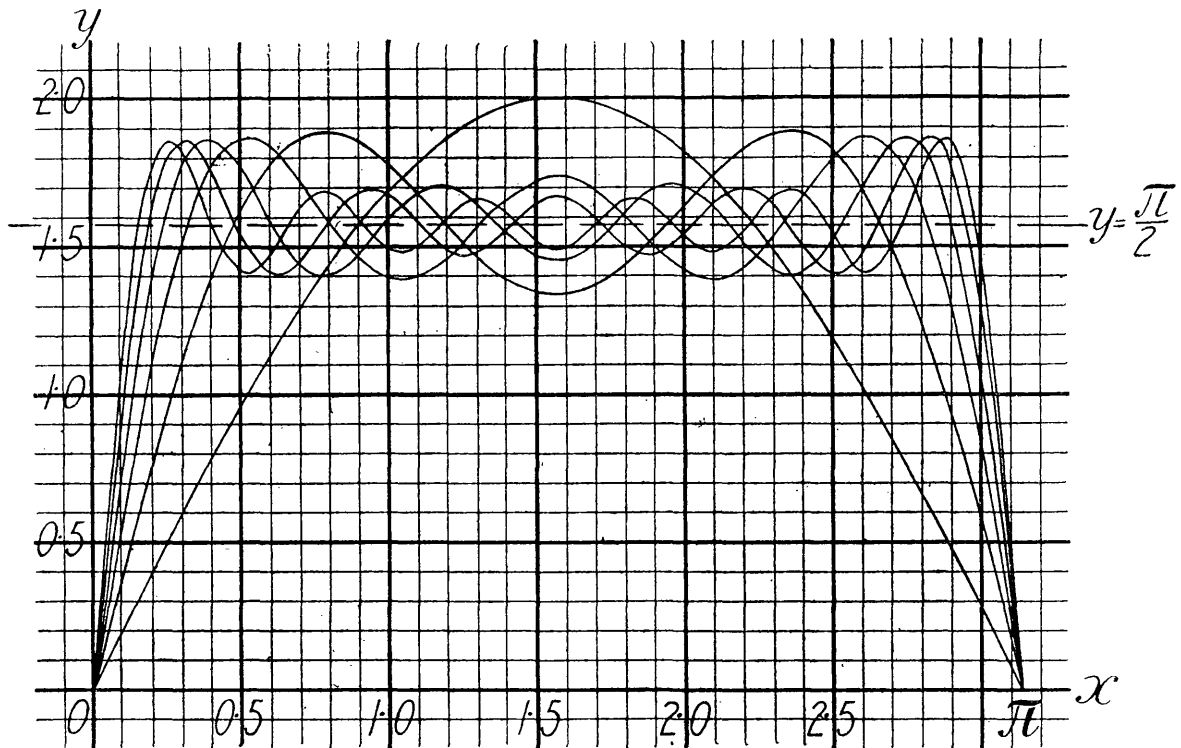
Thus the result follows. Similarly for the minima, we have to examine the sign of

$$S_n\left(\frac{m-1}{n}\pi\right) - S_n\left(\frac{m}{n}\pi\right),$$

where m is a positive integer less than or equal to $\frac{1}{2}n$.

V. *The first maximum to the right of $x=0$ is at $x=\frac{\pi}{2n}$ and its height continually diminishes as n increases. When n tends to infinity, its limit is*

$$\int_0^\pi \frac{\sin x}{x} dx.$$



The curves $y = S_n(x)$, when $n = 1, 2, 3, 4, 5$ and 6 .

FIG. 36.

We have

$$S_n\left(\frac{\pi}{2n}\right) = \int_0^{\frac{\pi}{2n}} \frac{\sin 2na}{\sin a} da = \frac{1}{2n} \int_0^\pi \sin a \operatorname{cosec} \frac{a}{2n} da.$$

Thus

$$\begin{aligned} & S_n\left(\frac{\pi}{2n}\right) - S_{n+1}\left(\frac{\pi}{2n+2}\right) \\ &= \int_0^\pi \sin a \left(\frac{1}{2n} \operatorname{cosec} \frac{a}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{a}{2n+2} \right) da. \end{aligned}$$

Since $a/\sin a$ continually increases from 1 to ∞ , as a passes from 0 to π , it is clear that in the interval with which we have to deal

$$\frac{1}{2n} \operatorname{cosec} \frac{a}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{a}{2n+2} > 0.$$

Thus
$$S_n\left(\frac{\pi}{2n}\right) - S_{n+1}\left(\frac{\pi}{2(n+1)}\right) > 0.$$

But, from (I), $S_n(x)$ is positive when $0 < x < \pi$.

It follows that $S_n\left(\frac{\pi}{2}\right)$ tends to a limit as n tends to infinity.

The value of this limit can be obtained by the method used by Bôcher for the integral $\int_0^x \frac{\sin(n + \frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha$.^{*} But it is readily obtained from the definition of the Definite Integral.

$$\begin{aligned} \text{For } S_n\left(\frac{\pi}{2n}\right) &= 2\left(\frac{\pi}{2n}\right) \left\{ \frac{2n}{\pi} \sin \frac{\pi}{2n} + \frac{2n}{3\pi} \sin \frac{3\pi}{2n} + \dots \right. \\ &\quad \left. + \frac{2n}{(2n-1)\pi} \sin \frac{2n-1}{2n} \pi \right\} \\ &= 2 \sum_{\substack{m=1 \\ 2nh=\pi}}^{2n} \left(\frac{\sin mh}{mh} h \right) - \sum_{\substack{m=1 \\ nh=\pi}}^n \left(\frac{\sin mh}{mh} h \right). \end{aligned}$$

Therefore

$$\text{Lt}_{n \rightarrow \infty} S_n\left(\frac{\pi}{2n}\right) = 2 \int_0^\pi \frac{\sin x}{x} dx - \int_0^\pi \frac{\sin x}{x} dx = \int_0^\pi \frac{\sin x}{x} dx.$$

VI. The result obtained in (V) for the first wave is a special case of the following:

The r^{th} maximum to the right of $x=0$ is at $x_{2r-1} = \frac{2r-1}{2n} \pi$, and its height continually diminishes as n increases, r being kept constant. When n tends to infinity, its limit is

$$\int_0^{(2r-1)\pi} \frac{\sin x}{x} dx,$$

which is greater than $\frac{1}{2}\pi$.

The r^{th} minimum to the right of $x=0$ is at $x_{2r} = \frac{r}{n} \pi$, and its height continually increases as n increases, r being kept constant.

When n tends to infinity, its limit is $\int_0^{2r\pi} \frac{\sin x}{x} dx$, which is less than $\frac{1}{2}\pi$.

To prove these theorems we consider first the integral

$$\int_0^{m\pi} \sin a \left(\frac{1}{2n} \operatorname{cosec} \frac{a}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{a}{2n+2} \right) da,$$

^{*} *Ann. Math., Princeton, N.J.* (Ser. 2), 7, p. 124. Also Hobson, *loc. cit.*, p. 649.

m being a positive integer less than or equal to $2n-1$, so that $0 < \frac{\alpha}{2n} < \pi$ in the interval of integration.

$$\text{Then } F(\alpha) = \frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} > 0$$

in this interval. (Cf. (V).)

Further,

$$\begin{aligned} F'(\alpha) &= \frac{1}{(2n+2)^2} \cos \frac{\alpha}{2n+2} \operatorname{cosec}^2 \frac{\alpha}{2n+2} - \frac{1}{(2n)^2} \cos \frac{\alpha}{2n} \operatorname{cosec}^2 \frac{\alpha}{2n} \\ &= \alpha^{-2} \{ \phi^2 \cos \phi \operatorname{cosec}^2 \phi - \psi^2 \cos \psi \operatorname{cosec}^2 \psi \}, \end{aligned}$$

where $\phi = \alpha/(2n+2)$ and $\psi = \alpha/2n$.

$$\begin{aligned} \text{But } \frac{d}{d\phi} (\phi^2 \cos \phi \operatorname{cosec}^2 \phi) \\ = -\phi \operatorname{cosec}^3 \phi [\phi(1 \mp \cos \phi)^2 \pm 2 \cos \phi (\phi \mp \sin \phi)]. \end{aligned}$$

And the right-hand side of the equation will be seen to be negative, choosing the upper signs for $0 < \phi < \frac{1}{2}\pi$ and the lower for $\frac{1}{2}\pi < \phi < \pi$.

Therefore $\phi^2 \cos \phi \operatorname{cosec}^2 \phi$ diminishes as ϕ increases from 0 to π .

It follows, from the expression for $F'(\alpha)$, that $F'(\alpha) > 0$, and $F(\alpha)$ increases with α in the interval of integration.

The curve

$$y = \sin x \left(\frac{1}{2n} \operatorname{cosec} \frac{x}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{x}{2n+2} \right), \dots, \quad 0 < x < m\pi,$$

thus consists of a succession of waves of length π , alternately above and below the axis, and the absolute values of the ordinates at points at the same distance from the beginning of each wave continually increase.

It follows that, when m is equal to 2, 4, ..., $2(n-1)$, the integral

$$\int_0^{m\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha$$

is negative; and, when m is equal to 1, 3, ..., $2n-1$, this integral is positive.

Returning to the maxima and minima, we have, for the r^{th} maximum to the right of $x=0$,

$$\begin{aligned}
 S_n(x_{2r-1}) - S_{n+1}(x_{2r-1}) &= \int_0^{\frac{2r-1}{2n}\pi} \frac{\sin 2na}{\sin a} da - \int_0^{\frac{2r-1}{2(n+1)}\pi} \frac{\sin 2(n+1)a}{\sin a} da \\
 &= \int_0^{(2r-1)\pi} \sin a \left(\frac{1}{2n} \operatorname{cosec} \frac{a}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{a}{2n+2} \right) da.
 \end{aligned}$$

Therefore, from the above argument, $S_n(x_{2r-1}) > S_{n+1}(x_{2r-1})$.

Also for the r^{th} minimum to the right of $x=0$, we have

$$S_n(x_{2r}) - S_{n+1}(x_{2r}) = \int_0^{2r\pi} \sin a \left(\frac{1}{2n} \operatorname{cosec} \frac{a}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{a}{2n+2} \right) da,$$

and $S_n(x_{2r}) < S_{n+1}(x_{2r})$.

By an argument similar to that at the close of (V) we have

$$\operatorname{Lt}_{n \rightarrow \infty} S_n(x_r) = \int_0^{r\pi} \frac{\sin x}{x} dx.*$$

It is clear that these limiting values are all greater than $\frac{1}{2}\pi$ for the maxima, and positive and less than $\frac{1}{2}\pi$ for the minima.

GIBBS'S PHENOMENON FOR THE SERIES

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots).$$

116. From the Theorems I.–VI. of § 115 all the features of Gibbs's Phenomenon for the series

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots) \dots, \quad -\pi \leq x \leq \pi,$$

follow immediately.

It is obvious that we need only examine the interval $0 \leq x \leq \pi$, and that a discontinuity occurs at $x=0$.

For large values of n , the curve

$$y = S_n(x),$$

where $S_n(x) = 2 \left(\sin x + \frac{1}{3} \sin 3x + \dots + \frac{1}{2n-1} \sin (2n-1)x \right)$, rises at a steep gradient from the origin to its first maximum, which is very near, but above, the point $\left(0, \int_0^\pi \frac{\sin x}{x} dx \right)$ (§ 115, V.). The curve, then, falls at a steep gradient, without reaching the axis of x (§ 115, II.), to its first minimum, which is very near, but below, the point $\left(0, \int_0^{2\pi} \frac{\sin x}{x} dx \right)$ (§ 115, VI.). It then oscillates

* For the values of $\int_0^{r\pi} \frac{\sin x}{x} dx$, see *Ann. Math., Princeton, N.J.* (Ser. 2), 7, p. 129.

above and below the line $y = \frac{1}{2}\pi$, the heights (and depths) of the waves continually diminishing as we proceed from $x=0$ to $x = \frac{1}{2}\pi$ (§ 115, IV.); and from $x = \frac{1}{2}\pi$ to $x = \pi$, the procedure is reversed, the curve in the interval $0 \leq x \leq \pi$ being symmetrical about $x = \frac{1}{2}\pi$ (§ 115, I.).

The highest (or lowest) point of the r^{th} wave to the right of $x=0$ will, for large values of n , be at a point whose abscissa is $\frac{r\pi}{2n}$ (§ 115, III.) and whose ordinate is very nearly

$$\int_0^{r\pi} \frac{\sin x}{x} dx \quad (\S 115, \text{VI}).$$

By increasing n the curve for $0 \leq x \leq \pi$ can be brought as close as we please to the lines

$$\left. \begin{array}{l} x=0, \quad 0 < y < \int_0^{\pi} \frac{\sin x}{x} dx, \\ 0 < x < \pi, \quad y = \frac{\pi}{2}, \\ x = \pi, \quad 0 < y < \int_0^{\pi} \frac{\sin x}{x} dx. \end{array} \right\}$$

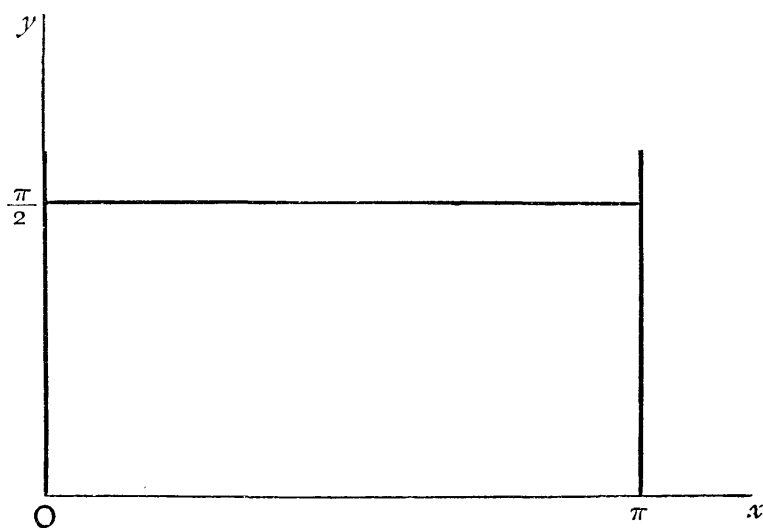


FIG. 37.

We may state these results more definitely as follows :

(i) If ϵ is any positive number, as small as we please, there is a positive integer ν' such that

$$|\frac{1}{2}\pi - S_n(x)| < \epsilon, \quad \text{when } n \geq \nu', \quad \epsilon \leq x \leq \frac{1}{2}\pi.$$

This follows from the uniform convergence of the Fourier's Series for $f(x)$, as defined in the beginning of this section, in an

interval which does not include, either in its interior or at an end, a discontinuity of $f(x)$ (cf. § 107).

(ii) Since the height of the first maximum to the right of $x=0$ tends from above to $\int_0^\pi \frac{\sin x}{x} dx$ as n tends to infinity, there is a positive integer ν'' such that

$$0 < S_n\left(\frac{\pi}{2n}\right) - \int_0^\pi \frac{\sin x}{x} dx < \epsilon, \text{ when } n \geq \nu''.$$

(iii) Let ν''' be the integer next greater than $\frac{\pi}{2\epsilon}$. Then the abscissa of the first maximum to the right of $x=0$, when $n \geq \nu'''$, is less than ϵ .

It follows from (i), (ii) and (iii) that, if ν is the greatest of the positive integers ν' , ν'' and ν''' , the curve $y = S_n(x)$, when $n \geq \nu$, behaves as follows:

It rises at a steep gradient from the origin to its first maximum, which is above $\int_0^\pi \frac{\sin x}{x} dx$ and within the rectangle

$$0 < x < \epsilon, \quad 0 < y < \int_0^\pi \frac{\sin x}{x} dx + \epsilon.$$

After leaving this rectangle, in which there may be many oscillations about $y = \frac{1}{2}\pi$, it remains within the rectangle

$$\epsilon < x < \pi - \epsilon, \quad \frac{1}{2}\pi - \epsilon < y < \frac{1}{2}\pi + \epsilon.$$

Finally, it enters the rectangle

$$\pi - \epsilon < x < n, \quad 0 < y < \int_0^\pi \frac{\sin x}{x} dx + \epsilon,$$

and the procedure in the first region is repeated.*

* The cosine series

$$\frac{4}{\pi} - \left[\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x + \dots \right],$$

which represents 0 in the interval $0 \leq x < \frac{\pi}{2}$ and $\frac{\pi}{2}$ in the interval $\frac{\pi}{2} < x \leq \pi$, can be treated in the same way as the series discussed in this article.

GIBBS'S PHENOMENON FOR FOURIER'S SERIES IN GENERAL.

117. Let $f(x)$ be a function with an ordinary discontinuity when $x = a$, which satisfies Dirichlet's Conditions in the interval $-\pi \leq x \leq \pi$.

Denote as usual by $f(a+0)$ and $f(a-0)$ the values towards which $f(x)$ tends as x approaches a from the right or left. It will be convenient to consider $f(a+0)$ as greater than $f(a-0)$ in the description of the curve, but this restriction is in no way necessary.

$$\text{Let } \phi(x-a) = 2 \sum_{r=1}^{\infty} \frac{1}{(2r-1)} \sin(2r-1)(x-a).$$

$$\left. \begin{aligned} \text{Then } \phi(x-a) &= \frac{1}{2}\pi, & \text{when } a < x < \pi+a, \\ \phi(x-a) &= -\frac{1}{2}\pi, & \text{when } -\pi+a < x < a, \\ \phi(+0) &= \frac{1}{2}\pi, & \phi(-0) = -\frac{1}{2}\pi, \\ \phi(0) &= 0 & \text{and } \phi(x) = \phi(x+2\pi). \end{aligned} \right\}$$

Now put

$$\psi(x) = f(x) - \frac{1}{2}\{f(a+0) + f(a-0)\} - \frac{1}{\pi}\{f(a+0) - f(a-0)\}\phi(x-a),$$

and let $f(x)$, when $x = a$, be defined as $\frac{1}{2}\{f(a+0) + f(a-0)\}$.

Then $\psi(a+0) = \psi(a-0) = \psi(a) = 0$, and $\psi(x)$ is continuous at $x = a$.

The following distinct steps in the argument are numbered for the sake of clearness :

(i) Since $\psi(x)$ is continuous at $x = a$ and $\psi(a) = 0$, if ϵ is a positive number, as small as we please, a number η exists such that

$$|\psi(x)| < \frac{\epsilon}{4}, \text{ when } |x-a| \leq \eta.$$

If η is not originally less than ϵ , we can choose this part of η for our interval.

(ii) $\psi(x)$ can be expanded in a Fourier's Series,* this series being uniformly convergent in an interval $\alpha \leq x \leq \beta$ contained within an interval which includes neither within it nor as an end-point any other discontinuity of $f(x)$ and $\phi(x-a)$ than $x = a$ (cf. § 108).

* If $f(x)$ satisfies Dirichlet's Conditions, it is clear from the definition of $\phi(x-a)$ that $\psi(x)$ does so also.

Let $s_n(x)$, $\phi_n(x-a)$ and $\sigma_n(x)$ be the sums of the terms up to and including those in $\sin nx$ and $\cos nx$ in the Fourier's Series for $f(x)$, $\phi(x-a)$ and $\psi(x)$. Then ϵ being the positive number of (i), as small as we please, there exists a positive integer ν' such that

$$|\sigma_n(x) - \psi(x)| < \frac{\epsilon}{4}, \text{ when } n \leq \nu',$$

the same ν' serving for every x in $a \leq x \leq \beta$.

$$\text{Also } |\sigma_n(x)| \leq |\sigma_n(x) - \psi(x)| + |\psi(x)| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},$$

in $|x-a| \leq \eta$, if $a < a-\eta < a < a+\eta < \beta$, and $n \geq \nu'$.

(iii) Now if n is even, the first maximum in $\phi_n(x-a)$ to the right of $x=a$ is at $a + \frac{\pi}{n}$; and if n is odd, it is at $a + \frac{\pi}{n+1}$. In either case there exists a positive integer ν'' such that the height of the first maximum lies between

$$\int_0^\pi \frac{\sin x}{x} dx \text{ and } \int_0^\pi \frac{\sin x}{x} dx + \frac{\pi\epsilon}{2\{f(a+0) - f(a-0)\}}, \text{ when } n \geq \nu''.$$

(iv) This first maximum will have its abscissa between a and $a+\eta$, provided that $\frac{\pi}{n} < \eta$.

Let ν''' be the first positive integer which satisfies this inequality.

(v) In the interval $a+\eta \leq x \leq \beta$, $s_n(x)$ converges uniformly to $f(x)$. Therefore a positive integer $\nu^{(iv)}$ exists such that, when $n \geq \nu^{(iv)}$,

$$|f(x) - s_n(x)| < \epsilon,$$

the same $\nu^{(iv)}$ serving for every x in this interval.

Now, from the equation defining $\psi(x)$,

$$s_n(x) = \frac{1}{2}(f(a+0) + f(a-0)) + \frac{f(a+0) - f(a-0)}{\pi} \phi_n(x-a) + \sigma_n(x).$$

It follows from (i)-(v) that if ν is the first positive integer greater than ν' , ν'' , ν''' and $\nu^{(iv)}$, the curve $y = s_n(x)$, when $n \geq \nu$, in the interval $a \leq x \leq \beta$, behaves as follows:

When $x=a$, it passes through a point whose ordinate is within $\frac{\epsilon}{4}$ of $\frac{1}{2}(f(a+0) + f(a-0))$, and ascends at a steep gradient to a point within ϵ of

$$\frac{1}{2}\{f(a+0) + f(a-0)\} + \frac{f(a+0) - f(a-0)}{\pi} \int_0^\pi \frac{\sin x}{x} dx.$$

This may be written

$$f(a+0) - \frac{f(a+0) - f(a-0)}{\pi} \int_{\pi}^{\infty} \frac{\sin x}{x} dx,$$

and, from Bôcher's table, referred to in § 115, we have

$$\int_{\pi}^{\infty} \frac{\sin x}{x} dx = -0.2811.$$

It then oscillates about $y = f(x)$ till x reaches $a + \eta$, the character of the waves being determined by the function $\phi_n(x-a)$, since the term $\sigma_n(x)$ only adds a quantity less than $\frac{\epsilon}{2}$ to the ordinate.

And on passing beyond $x = a + \eta$, the curve enters, and remains within, the strip of width 2ϵ enclosing $y = f(x)$ from $x = a + \eta$ to $x = \beta$.

On the other side of the point a a similar set of circumstances can be established.

Writing $D = f(a+0) - f(a-0)$, the crest (or hollow) of the first wave to the left and right of $x = a$ tends to a height

$$f(a-0) + \frac{DP_1}{\pi}, \quad f(a+0) - \frac{DP_1}{\pi},$$

where $P_1 = \int_{\pi}^{\infty} \frac{\sin x}{x} dx$.

REFERENCES.

BÔCHER, "Introduction to the Theory of Fourier's Series," *Ann. Math., Princeton, N.J.* (Ser. 2), 7, 1906.

GIBBS, *Scientific Papers*, Vol. II., p. 258, London, 1906.

And the other papers named in the text.

Note.—The approximation curves

$$y = \sum_1^n \frac{\sin(2r-1)x}{2r-1}$$

for $n=1, 6, 11$ and 16 can be obtained as lantern slides from the firm of Martin Schilling, Berlin. The last of these illustrates Gibb's Phenomenon even more clearly than Fig. 35 above.

CHAPTER X

FOURIER'S INTEGRALS

118. When the arbitrary function $f(x)$ satisfies Dirichlet's Conditions in the interval $(-l, l)$, we have seen in § 98 that the sum of the series

$$\frac{1}{2l} \int_{-l}^l f(x') dx' + \frac{1}{l} \sum_1^{\infty} \int_{-l}^l f(x') \cos \frac{n\pi}{l} (x' - x) dx' \dots\dots\dots(1)$$

is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point in $-l < x < l$ where $f(x+0)$ and $f(x-0)$ exist; and that at $x = \pm l$ its sum is $\frac{1}{2}[f(-l+0)+f(l-0)]$, when these limits exist.

Corresponding results were found for the series

$$\frac{1}{l} \int_0^l f(x') dx' + \frac{2}{l} \sum_1^{\infty} \cos \frac{n\pi}{l} x \int_0^l f(x') \cos \frac{n\pi}{l} x' dx', \dots\dots\dots(2)$$

and
$$\frac{2}{l} \sum_1^{\infty} \sin \frac{n\pi}{l} x \int_0^l f(x') \sin \frac{n\pi}{l} x' dx', \dots\dots\dots(3)$$

in the interval $(0, l)$.

Fourier's Integrals are definite integrals which represent the arbitrary function in an unlimited interval. They are suggested, but not established, by the forms these series appear to take as l tends to infinity.

If l is taken large enough, π/l may be made as small as we please, and we may neglect the first term in the series (1), assuming that $\int_{-\infty}^{\infty} f(x) dx$ is convergent. Then we may write

as
$$\frac{1}{l} \sum_1^{\infty} \int_{-l}^l f(x') \cos \frac{n\pi}{l} (x' - x) dx'$$

$$\frac{1}{\pi} \left[\Delta\alpha \int_{-l}^l f(x') \cos \Delta\alpha (x' - x) dx' + \Delta\alpha \int_{-l}^l \cos 2\Delta\alpha (x' - x) dx' + \dots \right],$$

where $\Delta\alpha = \pi/l$.

Assuming that this sum has a limit as $l \rightarrow \infty$, an assumption which, of course, would have to be defended if the proof were to be regarded as in any degree complete, its value would be

$$\frac{1}{\pi} \int_0^{\infty} d\alpha \int_{-\infty}^{\infty} f(x') \cos \alpha(x' - x) dx',$$

and it would follow that

$$\frac{1}{\pi} \int_0^{\infty} d\alpha \int_{-\infty}^{\infty} f(x') \cos \alpha(x' - x) dx' = \frac{1}{2} [f(x+0) + f(x-0)],$$

$$-\infty < x < \infty,$$

when these limits exist.

In the same way we are led to the Cosine Integral and Sine Integral corresponding to the Cosine Series and the Sine Series:

$$\left. \begin{aligned} \frac{2}{\pi} \int_0^{\infty} d\alpha \int_0^{\infty} f(x') \cos \alpha x \cos \alpha x' dx' \\ = \frac{1}{2} [f(x+0) + f(x-0)], \\ \frac{2}{\pi} \int_0^{\infty} d\alpha \int_0^{\infty} f(x') \sin \alpha x \sin \alpha x' dx' \\ = \frac{1}{2} [f(x+0) - f(x-0)], \end{aligned} \right\} 0 < x < \infty,$$

when these limits exist.

It must be remembered that the above argument is not a proof of any of these results. All that it does is to suggest the possibility of representing an arbitrary function $f(x)$, given for all values of x , or for all positive values of x , by these integrals.

We shall now show that this representation is possible, pointing out in our proof the limitation the discussion imposes upon the arbitrary function.

119. *Let the arbitrary function $f(x)$, defined for all values of x , satisfy Dirichlet's Conditions in any finite interval, and in addition let $\int_{-\infty}^{\infty} f(x) dx$ be absolutely convergent.*

Then

$$\frac{1}{\pi} \int_0^{\infty} d\alpha \int_{-\infty}^{\infty} f(x') \cos \alpha(x' - x) dx' = \frac{1}{2} [f(x+0) + f(x-0)],$$

at every point where $f(x+0)$ and $f(x-0)$ exist.

Having fixed upon the value of x for which we wish to evaluate the integral, we can choose a positive number a greater than x such that $f(x')$ is bounded in the interval $a \leq x' \leq b$,

where b is arbitrary, and $\int_a^\infty |f(x')| dx'$ converges, since with our definition of the infinite integral only a finite number of points of infinite discontinuity were admitted (§ 60).

It follows that $\int_a^\infty f(x') \cos a(x' - x) dx'$

converges uniformly for every a , so that this integral is a continuous function of a (§§ 83, 84).

Therefore $\int_0^q da \int_a^\infty f(x') \cos a(x' - x) dx'$ exists.

Also, by § 85,

$$\begin{aligned} \int_0^q da \int_a^\infty f(x') \cos a(x' - x) dx' &= \int_a^\infty dx' \int_0^q f(x') \cos a(x' - x) da \\ &= \int_a^\infty f(x') \frac{\sin q(x' - x)}{x' - x} dx'. \dots\dots(1) \end{aligned}$$

But $x' - x \geq a - x > 0$ in $x' \geq a$, since we have chosen $a > x$.

And $\int_a^\infty |f(x')| dx'$ converges.

Therefore $\int_a^\infty \frac{|f(x')|}{x' - x} dx'$ also converges.

It follows that $\int_a^\infty f(x') \frac{\sin q(x' - x)}{x' - x} dx'$

converges uniformly for every q .

Thus, to the arbitrary positive number ϵ there corresponds a positive number $A > a$, such that

$$\left| \int_{A'}^\infty f(x') \frac{\sin q(x' - x)}{x' - x} dx' \right| < \frac{\epsilon}{2}, \text{ when } A' \geq A > a, \dots\dots(2)$$

the same A serving for every value of q .

But we know from § 94 that

$$\begin{aligned} &\text{Lt}_{q \rightarrow \infty} \int_a^A f(x') \frac{\sin q(x' - x)}{x' - x} dx' \\ &= \text{Lt}_{q \rightarrow \infty} \int_{a-x}^{A-x} f(u+x) \frac{\sin qu}{u} du \\ &= 0, \end{aligned}$$

since $f(u+x)$ satisfies Dirichlet's Conditions in the interval $(a-x, A-x)$, and both these numbers are positive.

Thus we can choose the positive number Q so that

$$\left| \int_a^A f(x') \frac{\sin q(x' - x)}{x' - x} dx' \right| < \frac{\epsilon}{2}, \text{ when } q \geq Q \dots \dots \dots (3)$$

It follows from (2) and (3) that

$$\left| \int_a^\infty f(x') \frac{\sin q(x' - x)}{x' - x} dx' \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

when $q \geq Q$.

Thus
$$\text{Lt}_{q \rightarrow \infty} \int_a^\infty f(x') \frac{\sin q(x' - x)}{x' - x} dx' = 0,$$

and from (1),
$$\int_0^\infty da \int_a^\infty f(x') \cos a(x' - x) dx' = 0. \dots \dots \dots (4)$$

But, by § 87,

$$\begin{aligned} \int_0^q da \int_x^a f(x') \cos a(x' - x) dx' &= \int_x^a dx' \int_0^q f(x') \cos a(x' - x) da \\ &= \int_x^a f(x') \frac{\sin q(x' - x)}{x' - x} dx' \\ &= \int_0^{a-x} f(u + x) \frac{\sin qu}{u} du. \end{aligned}$$

Letting $q \rightarrow \infty$, we have

$$\int_0^\infty da \int_x^a f(x') \cos a(x' - x) dx' = \frac{\pi}{2} f(x + 0), \dots \dots \dots (5)$$

when $f(x + 0)$ exists.

Adding (4) and (5), we have

$$\int_0^\infty da \int_x^\infty f(x') \cos a(x' - x) dx' = \frac{\pi}{2} f(x + 0), \dots \dots \dots (6)$$

when $f(x + 0)$ exists.

Similarly, under the given conditions,

$$\int_0^\infty da \int_{-\infty}^x f(x') \cos a(x' - x) dx' = \frac{\pi}{2} f(x - 0), \dots \dots \dots (7)$$

when $f(x - 0)$ exists.

Adding (6) and (7), we obtain Fourier's Integral in the form

$$\frac{1}{\pi} \int_0^\infty da \int_{-\infty}^\infty f(x') \cos a(x' - x) dx' = \frac{1}{2} [f(x + 0) + f(x - 0)]$$

for every point in $-\infty < x < \infty$, where $f(x + 0)$ and $f(x - 0)$ exist.

120. More general conditions for $f(x)$. In this section we shall show that Fourier's Integral formula holds if the conditions in any finite interval remain as before, and the absolute convergence of the infinite integral

$\int_{-\infty}^{\infty} f(x) dx$ is replaced by the following conditions:

For some positive number and to the right of it, $f(x')$ is bounded and monotonic and $\text{Lt}_{x' \rightarrow \infty} f(x') = 0$; and for some negative number and to the left of it, $f(x')$ is bounded and monotonic and $\text{Lt}_{x' \rightarrow -\infty} f(x') = 0$.

Having fixed upon the value of x for which we wish to evaluate the integral, we can choose a positive number a greater than x , such that $f(x')$ is bounded and monotonic when $x' \geq a$ and $\text{Lt}_{x' \rightarrow \infty} f(x') = 0$.

Consider the integral $\int_a^{\infty} f(x') \cos \alpha(x' - x) dx'$.

By the Second Theorem of Mean Value,

$$\int_{A'}^{A''} f(x') \cos \alpha(x' - x) dx' = f(A') \int_{A'}^{\xi} \cos \alpha(x' - x) dx' + f(A'') \int_{\xi}^{A''} \cos \alpha(x' - x) dx',$$

where $a < A' \leq \xi \leq A''$.

Thus

$$\int_{A'}^{A''} f(x') \cos \alpha(x' - x) dx' = \frac{f(A')}{\alpha} \int_{\alpha(A' - x)}^{\alpha(\xi - x)} \cos u du + \frac{f(A'')}{\alpha} \int_{\alpha(\xi - x)}^{\alpha(A'' - x)} \cos u du.$$

Therefore

$$\left| \int_{A'}^{A''} f(x') \cos \alpha(x' - x) dx' \right| < \frac{4|f(A')|}{q_0} \text{ for } \alpha \geq q_0 > 0.$$

But we are given that $\text{Lt}_{x' \rightarrow \infty} f(x') = 0$.

It follows that $\int_a^{\infty} f(x') \cos \alpha(x' - x) dx'$

is uniformly convergent for $\alpha \geq q_0 > 0$, and this integral represents a continuous function of α in $\alpha \geq q_0$.*

Also, by § 85,

$$\begin{aligned} \int_{q_0}^q d\alpha \int_a^{\infty} f(x') \cos \alpha(x' - x) dx' &= \int_a^{\infty} dx' \int_{q_0}^q f(x') \cos \alpha(x' - x) d\alpha \\ &= \int_a^{\infty} f(x') \left\{ \frac{\sin q(x' - x)}{x' - x} - \frac{\sin q_0(x' - x)}{x' - x} \right\} dx'. \quad (1) \end{aligned}$$

But $x' - x \geq a - x > 0$ in the interval $x' \geq a$, since we have chosen a greater than x .

And $\int_a^{\infty} f(x') \frac{\sin q(x' - x)}{x' - x} dx$, $\int_a^{\infty} f(x') \frac{\sin q_0(x' - x)}{x' - x} dx$

both converge.

* These extensions are due to Pringsheim. Cf. *Math. Ann.*, Leipzig, **68**, p. 367, 1910, and **71**, p. 289, 1911. Reference should also be made to a paper by W. H. Young in *Edinburgh, Proc. R. Soc.*, **31**, p. 559, 1911.

Therefore, from (1),

$$\int_{q_0}^q d\alpha \int_a^\infty f(x') \cos \alpha(x' - x) dx' = \int_a^\infty f(x') \frac{\sin q(x' - x)}{x' - x} dx' - \int_a^\infty f(x') \frac{\sin q_0(x' - x)}{x' - x} dx'. \dots(2)$$

Now consider the integral

$$\int_a^\infty f(x') \frac{\sin q_0(x' - x)}{x' - x} dx'.$$

From the Second Theorem of Mean Value,

$$\int_{A'}^{A''} f(x') \frac{\sin q_0(x' - x)}{x' - x} dx' = f(A') \int_{A'}^\xi \frac{\sin q_0(x' - x)}{x' - x} dx' + f(A'') \int_\xi^{A''} \frac{\sin q_0(x' - x)}{x' - x} dx', \dots(3)$$

where $a < A' \leq \xi \leq A''$.

Also
$$\int_{A'}^\xi \frac{\sin q_0(x' - x)}{x' - x} dx' = \int_{q_0(A' - x)}^{q_0(\xi - x)} \frac{\sin u}{u} du,$$

the limits of the integral being both positive.

Therefore
$$\left| \int_{A'}^\xi \frac{\sin q_0(x' - x)}{x' - x} dx' \right| < \pi. \quad (\text{Cf. } \S 91.)$$

And similarly,
$$\left| \int_\xi^{A''} \frac{\sin q_0(x' - x)}{x' - x} dx' \right| < \pi.$$

Thus, from (3),
$$\left| \int_{A'}^{A''} f(x') \frac{\sin q_0(x' - x)}{x' - x} dx' \right| < 2\pi |f(A')|. \dots\dots\dots(4)$$

It follows that
$$\int_a^\infty f(x') \frac{\sin q_0(x' - x)}{x' - x} dx'$$

is uniformly convergent for $q_0 \geq 0$, and by § 84 it is continuous in this range.

Thus
$$\text{Lt}_{q_0 \rightarrow 0} \int_a^\infty f(x') \frac{\sin q_0(x' - x)}{x' - x} dx' = 0, \dots\dots\dots(5)$$

since the integral vanishes when $q_0 = 0$.

Also from (2),

$$\int_0^q d\alpha \int_a^\infty f(x') \cos \alpha(x' - x) dx' = \int_a^\infty f(x') \frac{\sin q(x' - x)}{x' - x} dx'. \dots\dots\dots(6)$$

But we have already shown that the integral on the right-hand side of (6) is uniformly convergent for $q \geq 0$.

Proceeding as in § 119, (2) and (3),* it follows that

$$\text{Lt}_{q \rightarrow \infty} \int_a^\infty f(x') \frac{\sin q(x' - x)}{x' - x} dx' = 0.$$

Thus, from (6),
$$\int_0^\infty d\alpha \int_a^\infty f(x') \cos \alpha(x' - x) dx' = 0. \dots\dots\dots(7)$$

* Or we might use the Second Theorem of Mean Value as proved in § 58 for the Infinite Integral.

But we know from § 119 that

$$\int_0^\infty d\alpha \int_x^\infty f(x') \cos \alpha(x' - x) dx' = \frac{\pi}{2} f(x+0), \dots\dots\dots(8)$$

when this limit exists.

Thus
$$\frac{1}{\pi} \int_0^\infty d\alpha \int_x^\infty f(x') \cos \alpha(x' - x) dx' = \frac{1}{2} f(x+0), \dots\dots\dots(9)$$

when this limit exists.

Similarly, under the given conditions, we find that

$$\frac{1}{\pi} \int_0^\infty d\alpha \int_{-\infty}^x f(x') \cos \alpha(x' - x) dx' = \frac{1}{2} f(x-0), \dots\dots\dots(10)$$

when this limit exists.

Adding (9) and (10), our formula is proved for every point at which $f(x \pm 0)$ exist.

121. Other conditions for f(x). We shall now show that Fourier's Integral formula also holds when the conditions at $\pm \infty$ of the previous section are replaced by the following :

(I.) *For some positive number and to the right of it, and for some negative number and to the left of it, f(x') is of the form g(x') cos(λx' + μ), where g(x') is bounded and monotonic in these intervals and has the limit zero as x' → ± ∞.*

Also, (II.)
$$\int_0^\infty \frac{g(x')}{x'} dx' \quad \text{and} \quad \int_{-\infty}^0 \frac{g(x')}{x'} dx' \quad \text{converge.}$$

We have shown in § 120 that when $g(x')$ satisfies the conditions named above in (I.), there will be a positive number a greater than x such that

$$\int_0^\infty d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' = 0.$$

But, if λ is any positive number,

$$\begin{aligned} & \int_0^\infty d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' \\ = & \int_0^\lambda d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' + \int_\lambda^\infty d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' \\ = & \frac{1}{2} \int_{-\lambda}^\lambda d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' + \int_\lambda^\infty d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' \\ = & \frac{1}{2} \int_{-\lambda}^\infty d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' + \frac{1}{2} \int_\lambda^\infty d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' \\ = & \frac{1}{2} \int_0^\infty d\alpha \int_a^\infty g(x') \cos (\alpha + \lambda)(x' - x) dx' + \frac{1}{2} \int_0^\infty d\alpha \int_a^\infty g(x') \cos (\alpha - \lambda)(x' - x) dx'. \end{aligned}$$

Therefore

$$\begin{aligned} & \int_0^\infty d\alpha \int_a^\infty g(x') \cos \alpha(x' - x) dx' \\ & = \int_0^\infty d\alpha \int_a^\infty g(x') \cos \lambda(x' - x) \cos \alpha(x' - x) dx' = 0 \dots\dots(1) \end{aligned}$$

Again we know, from § 120, that

$$\int_a^\infty g(x') \sin \alpha(x' - x) dx'$$

is uniformly convergent for $\alpha \geq \lambda_0 > 0$.

It follows that
$$\int_{\lambda_0}^{\lambda_1} d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx'$$

exists, for $\lambda_1 > \lambda_0 > 0$.

Also

$$\begin{aligned} \int_{\lambda_0}^{\lambda_1} d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx' &= \int_a^\infty dx' \int_{\lambda_0}^{\lambda_1} g(x') \sin \alpha(x' - x) d\alpha \quad (\text{by } \S 84) \\ &= \int_a^\infty g(x') \left\{ \frac{\cos \lambda_0(x' - x)}{x' - x} - \frac{\cos \lambda_1(x' - x)}{x' - x} \right\} dx' \\ &= \int_a^\infty g(x') \frac{\cos \lambda_0(x' - x)}{x' - x} dx' - \int_a^\infty g(x') \frac{\cos \lambda_1(x' - x)}{x' - x} dx', \dots(2) \end{aligned}$$

since both integrals converge.

But we are given that
$$\int_a^\infty \frac{g(x')}{x'} dx'$$

converges ; and it follows that
$$\int_a^\infty \frac{g(x')}{x' - x} dx'$$

also converges, so that we know that

$$\int_a^\infty g(x') \frac{\cos \lambda_0(x' - x)}{x' - x} dx'$$

is uniformly convergent for $\lambda_0 \geq 0$, and therefore continuous.

Thus
$$\text{Lt}_{\lambda_0 \rightarrow 0} \int_a^\infty g(x') \frac{\cos \lambda_0(x' - x)}{x' - x} dx' = \int_a^\infty \frac{g(x')}{x' - x} dx'.$$

It follows from (2) that, when $\lambda > 0$,

$$\int_0^\lambda d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx' = \int_a^\infty \frac{g(x')}{x' - x} dx' - \int_a^\infty g(x') \frac{\cos \lambda(x' - x)}{x' - x} dx'. \quad (3)$$

Also, as before, we find that, with the conditions imposed upon $g(x')$,*

$$\text{Lt}_{\lambda \rightarrow \infty} \int_a^\infty g(x') \frac{\cos \lambda(x' - x)}{x' - x} dx' = 0.$$

Therefore, from (2) and (3),

$$\int_\lambda^\infty d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx' = \int_a^\infty g(x') \frac{\cos \lambda(x' - x)}{x' - x} dx', \dots\dots\dots(4)$$

and
$$\int_0^\infty d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx' = \int_a^\infty \frac{g(x')}{x' - x} dx'. \dots\dots\dots(5)$$

* This can be obtained at once from the Second Theorem of Mean Value, as proved in § 58 for the Infinite Integral ; but it is easy to establish the result, as in § 119, without this theorem.

Again, since $\int_0^\lambda d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx'$
exists, we have $\int_{-\lambda}^\lambda d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx' = 0$(6)

From (6), and the convergence of $\int_{-\lambda}^\infty d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx'$, it follows that

$$\int_{-\lambda}^\infty d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx' - \int_\lambda^\infty d\alpha \int_a^\infty g(x') \sin \alpha(x' - x) dx' = 0.$$

Thus

$$\int_0^\infty d\alpha \int_a^\infty g(x') \sin (\alpha + \lambda)(x' - x) dx' - \int_0^\infty d\alpha \int_a^\infty g(x') \sin (\alpha - \lambda)(x' - x) dx' = 0.$$

Therefore $\int_0^\infty d\alpha \int_a^\infty g(x') \sin \lambda(x' - x) \cos \alpha(x' - x) dx' = 0$(7)

Multiply (1) by $\cos (\lambda x + \mu)$ and (7) by $\sin (\lambda x + \mu)$ and subtract.

It follows that

$$\int_0^\infty d\alpha \int_a^\infty g(x') \cos (\lambda x' + \mu) \cos \alpha(x' - x) dx' = 0. \dots\dots\dots(8)$$

And in the same way, with the conditions imposed upon $g(x')$, we have

$$\int_0^\infty d\alpha \int_{-\infty}^{-a'} g(x') \cos (\lambda x' + \mu) \cos \alpha(x' - x) dx' = 0. \dots\dots\dots(9)$$

These results, (8) and (9), may be written

$$\left. \begin{aligned} \int_0^\infty d\alpha \int_a^\infty f(x') \cos \alpha(x' - x) dx' &= 0, \\ \int_0^\infty d\alpha \int_{-\infty}^{-a'} f(x') \cos \alpha(x' - x) dx' &= 0, \end{aligned} \right\} \dots\dots\dots(10)$$

when $f(x') = g(x') \cos (\lambda x' + \mu)$ in (a, ∞) and $(-\infty, -a')$.

But we know that, when $f(x)$ satisfies Dirichlet's Conditions in $(-a', a)$,

$$\int_0^\infty d\alpha \int_{-a'}^a f(x') \cos \alpha(x' - x) dx' = \frac{\pi}{2} [f(x+0) + f(x-0)], \dots\dots\dots(11)$$

when these limits exist. [Cf. § 119 (5).]

Adding (10) and (11), we see that Fourier's Integral formula holds, when the arbitrary function satisfies the conditions imposed upon it in this section.

It is clear that the results just established still hold if we replace $\cos (\lambda x + \mu)$ in (I.) by the sum of a number of terms of the type

$$a_n \cos (\lambda_n x' + \mu_n).$$

It can be proved* that the theorem is also valid when this sum is replaced by an infinite series

$$\sum_1^\infty a_n \cos (\lambda_n x' + \mu_n),$$

when $\sum_1^\infty a_n$ converges absolutely and the constants λ_n , so far arbitrary, tend to infinity with n .

* Cf. Pringsheim, *Math. Ann.*, Leipzig, 68, p. 399.

122. Fourier's Cosine Integral and Sine Integral. In the case when $f(x)$ is given only for positive values of x , there are two forms of Fourier's Integral which correspond to the Cosine Series and Sine Series respectively.

I. In the first place, consider the result of §119, when $f(x)$ has the same values for negative values of x as for the corresponding positive values of x : i.e. $f(-x) = f(x)$, $x > 0$.

$$\begin{aligned} \text{Then } & \frac{1}{\pi} \int_0^{\infty} da \int_{-\infty}^{\infty} f(x') \cos a(x' - x) dx' \\ &= \frac{1}{\pi} \int_0^{\infty} da \int_0^{\infty} f(x') [\cos a(x' + x) + \cos a(x' - x)] dx' \\ &= \frac{2}{\pi} \int_0^{\infty} da \int_0^{\infty} f(x') \cos ax \cos ax' dx'. \end{aligned}$$

It follows from §119 that *when $f(x)$ is defined for positive values of x , and satisfies Dirichlet's Conditions in any finite interval, while $\int_0^{\infty} f(x) dx$ converges absolutely, then*

$$\frac{2}{\pi} \int_0^{\infty} da \int_0^{\infty} f(x') \cos ax \cos ax' dx' = \frac{1}{2} [f(x+0) + f(x-0)],$$

at every point where $f(x+0)$ and $f(x-0)$ exist, and when $x=0$ the value of the integral is $f(+0)$, if this limit exists.

Also it follows from §§120 and 121 that the condition at infinity may be replaced by either of the following:

(i) *For some positive number and to the right of it, $f(x')$ shall be bounded and monotonic and $\text{Lt}_{x' \rightarrow \infty} f(x') = 0$;*

or, (ii) *For some positive number and to the right of it, $f(x')$ shall be of the form $g(x') \cos(\lambda x' + \mu)$, where $g(x')$ is bounded and monotonic and $\text{Lt}_{x' \rightarrow \infty} g(x') = 0$. Also*

$$\int_0^{\infty} \frac{g(x')}{x'} dx' \text{ must converge.}$$

II. In the next place, by taking $f(-x) = f(x)$, $x > 0$, we see that, *when $f(x)$ is defined for positive values of x , and satisfies Dirichlet's Conditions in any finite interval, while $\int_0^{\infty} f(x) dx$ converges absolutely, then*

$$\frac{2}{\pi} \int_0^{\infty} da \int_0^{\infty} f(x') \sin ax \sin ax' dx' = \frac{1}{2} [f(x+0) + f(x-0)],$$

at every point where $f(x+0)$ and $f(x-0)$ exist, and when $x=0$ the integral is zero.

Also it follows from §§ 120 and 121 that the condition at infinity may be replaced by one or other of those given under I.

It should be noticed that, when we express the arbitrary function $f(x)$ by any of Fourier's Integrals, we must first decide for what value of x we wish the value of the integral, and that this value of x must be inserted in the integrand before the integrations indicated are carried out (cf. § 62).

123. Fourier's Integrals. Sommerfeld's Discussion. In many of the problems of Applied Mathematics in the solution of which Fourier's Integrals occur, they appear in a slightly different form, with an exponential factor (e.g. $e^{-\kappa\alpha^2 t}$) added. In these cases we are concerned with the *limiting value* as $t \rightarrow 0$ of the integral

$$\phi(t) = \frac{1}{\pi} \int_0^\infty d\alpha \int_a^b f(x') \cos \alpha(x' - x) e^{-\kappa\alpha^2 t} dx',$$

and, so far as the actual physical problem is concerned, the value of the integral for $t=0$ is not required.

It was shown, first of all by Sommerfeld,* that, when the limit on the right-hand side exists,

$$\begin{aligned} \text{Lt}_{t \rightarrow 0} \frac{1}{\pi} \int_0^\infty d\alpha \int_a^b f(x') \cos \alpha(x' - x) e^{-\kappa\alpha^2 t} dx' &= \frac{1}{2} [f(x+0) + f(x-0)], \\ &\text{when } a < x < b, \\ &= \frac{1}{2} f(a+0), \text{ when } x = a, \\ &= \frac{1}{2} f(b-0), \text{ when } x = b, \end{aligned}$$

the result holding in the case of any integrable function given in the interval (a, b) .

The case when the interval is infinite was also treated by Sommerfeld, but it has been recently examined in much greater detail by Young.† It will be sufficient in this place to state that, when the arbitrary function satisfies the conditions at infinity imposed in §§ 120-122, Sommerfeld's result still holds for an infinite interval.

However, it should be noticed that we cannot deduce the value of the integral

$$\frac{1}{\pi} \int_0^\infty d\alpha \int_a^b f(x') \cos \alpha(x' - x) dx'$$

from the above results. This would require the continuity of the function

$$\phi(t) = \frac{1}{\pi} \int_0^\infty d\alpha \int_a^b f(x') \cos \alpha(x' - x) e^{-\kappa\alpha^2 t} dx'$$

for $t=0$.

We have come across the same point in the discussion of Poisson's treatment of Fourier's Series. [Cf. § 99.]

* Sommerfeld, *Die willkürlichen Functionen in der mathematischen Physik*, Diss. Königsberg, 1891.

† W. H. Young, *loc. cit.*, *Edinburgh, Proc. R. Soc.*, **31**, 1911.

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And the papers named in the text.

EXAMPLES ON CHAPTER X.

1. Taking $f(x)$ as 1 in Fourier's Cosine Integral when $0 < x < 1$, and as zero when $1 < x$, show that

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha, \quad (0 < x < 1)$$

$$\frac{1}{2} = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha, \quad (x = 1)$$

$$0 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha. \quad (1 < x).$$

2. By considering Fourier's Sine Integral for $e^{-\beta x}$ ($\beta > 0$), prove that

$$\int_0^{\infty} \frac{\alpha \sin \alpha x}{\alpha^2 + \beta^2} d\alpha = \frac{\pi}{2} e^{-\beta x},$$

and in the same way, from the Cosine Integral, prove that

$$\int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + \beta^2} d\alpha = \frac{\pi}{2\beta} e^{-\beta x}.$$

3. Show that the expression

$$\frac{2\alpha^2}{\pi} \int_0^{\infty} \cos\left(\frac{\theta x}{\alpha}\right) \frac{d\theta}{\theta^3} \int_0^{\theta} v^2 \cos v dv$$

is equal to x^2 when $0 \leq x < a$, and to zero when $x > a$.

4. Show that $\frac{2}{\pi} \int_0^{\infty} \sin qx \left\{ \frac{h}{q} + \tan \alpha \frac{\sin qb - \sin qa}{q^2} \right\} dq$

is the ordinate of a broken line running parallel to the axis of x from $x=0$ to $x=a$, and from $x=b$ to $x=\infty$, and inclined to the axis of x at an angle α between $x=a$ and $x=b$.

5. Show that $f(x) = \frac{1}{\sqrt{|x|}}$ satisfies the conditions of § 120 for Fourier's Integral, and verify independently that

$$\frac{2}{\pi} \int_0^{\infty} d\alpha \int_0^{\infty} \cos \alpha x \cos \alpha x' \frac{dx'}{\sqrt{x'}} = \frac{1}{\sqrt{x}} \text{ when } x > 0.$$

6. Show that $f(x) = \frac{\sin x}{x}$ satisfies the conditions of § 121 for Fourier's Integral, and verify independently that

$$\frac{1}{\pi} \int_0^{\infty} d\alpha \int_{-\infty}^{\infty} \sin x' \cos \alpha(x' - x) \frac{dx'}{x'} = \frac{\sin x}{x}.$$

APPENDIX I

PRACTICAL HARMONIC ANALYSIS AND PERIODOGRAM ANALYSIS

1. Let $y=f(x)$ be a given periodic function, with period 2π . We have seen that, for a very general class of functions, we may represent $f(x)$ by its Fourier's Series

$$\left. \begin{aligned} a_0 + a_1 \cos x + a_2 \cos 2x + \dots \\ + b_1 \sin x + b_2 \sin 2x + \dots \end{aligned} \right\},$$

where $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are the Fourier's Constants for $f(x)$. We may suppose the range of x to be $0 \leq x \leq 2\pi$. If the period is a , instead of 2π , the terms $\frac{\cos}{\sin} nx$ are replaced by $\frac{\cos}{\sin} 2n\pi x/a$, and the range becomes $0 \leq x \leq a$.

However, in many practical applications, y is not known analytically as a function of x , but the relation between the dependent and independent variables is given in the form of a curve obtained by continuous observations. Or again, we may only be given the values of y corresponding to isolated values of x , the observations having been made at definite intervals. In the latter case we may suppose that a continuous curve is drawn through the isolated points in the plane of x, y . And in both cases the Fourier's Constants for the function can be obtained by mechanical means. One of the best known machines for the purpose is Kelvin's Harmonic Analyser.*

2. The practical questions referred to above can also be treated

* Such mechanical methods are described in the handbook entitled *Modern Instruments and Methods of Calculation*, published by Bell & Sons in connection with the Napier Tercentenary Meeting of 1914.

Similarly, we find that

$$\frac{1}{2}(2n+1)b_r = \sum_{s=1}^{2n} y_s \sin rsa.$$

A trigonometrical series of $(2n+1)$ terms has thus been formed, whose sum takes the required values at the points

$$0, a, 2a, \dots, 2na, \quad \text{where } (2n+1)a = 2\pi.$$

It will be observed that as $n \rightarrow \infty$ the values of a_0, a_1, \dots and b_1, b_2, \dots reduce to the integral forms for the coefficients, but as remarked in § 90, p. 199, this passage from a finite number of equations to an infinite number requires more careful handling if the proof is to be made rigorous.

3. For purposes of calculation, there are advantages in taking an even number of equidistant points instead of an odd number. Suppose that to the points

$$0, a, 2a, \dots, (2n-1)a, \quad \text{where } na = \pi,$$

we have the corresponding values of y ,

$$y_0, y_1, y_2, \dots, y_{2n-1}.$$

In this case we can obtain the values of the $2n$ constants in the expression

$$\left. \begin{aligned} a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_{n-1} \cos (n-1)x + a_n \cos nx \\ + b_1 \sin x + b_2 \sin 2x + \dots + b_{n-1} \sin (n-1)x \end{aligned} \right\}$$

so that the sum shall take the values $y_0, y_1, \dots, y_{2n-1}$ at these $2n$ points in $(0, 2\pi)$.

It will be found that

$$\left. \begin{aligned} a_0 &= \frac{1}{2n} \sum_{s=0}^{2n-1} y_s \\ a_r &= \frac{1}{n} \sum_{s=0}^{2n-1} y_s \cos rsa, \quad \text{if } r \neq n \\ a_n &= \frac{1}{2n} \sum_{s=0}^{2n-1} y_s \cos s\pi \\ b_r &= \frac{1}{n} \sum_{s=0}^{2n-1} y_s \sin rsa \end{aligned} \right\} a = \pi/n.$$

Runge* gave a convenient scheme for evaluating these constants in the case of 12 equidistant points. This and a similar table devised by Whittaker for the case of 24 equidistant points

* *Zs. Math., Leipzig*, 48, 1903, and 52, 1905. Also *Theorie u. Praxis der Reihen*, pp. 147-164, Leipzig, 1904.

will be found in Tract No. 4 of the Edinburgh Mathematical Tracts.*

4. This question may be looked at from another point of view. Suppose we are given the values of y , viz.

$$y_0, y_1, y_2, \dots, y_{m-1},$$

corresponding to the points

$$0, a, 2a, \dots, (m-1)a, \quad \text{where } ma = 2\pi.$$

Denote these values of x , as before, by

$$x_0, x_1, x_2, \dots, x_{m-1}.$$

$$\text{Let } S_n(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx \\ + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx.$$

For a given value of n , on the understanding that $m > 2n + 1$,† the $2n + 1$ constants $a_0, a_1, \dots, a_n, b_1, \dots, b_n$ are to be determined so that $S_n(x)$ shall approximate as closely as possible to y_0, y_1, \dots, y_{m-1} at x_0, x_1, \dots, x_{m-1} .

The Theory of Least Squares shows that the closest approximation will be obtained by making the function

$$\sum_{s=0}^{m-1} (y_s - S_n(x_s))^2,$$

regarded as a function of $a_0, a_1, \dots, a_n, b_1, \dots, b_n$, a minimum.

The conditions for a minimum, in this case, are :

$$\left. \begin{aligned} \sum_{s=0}^{m-1} (y_s - S_n(x_s)) &= 0 \\ \sum_{s=0}^{m-1} (y_s - S_n(x_s)) \cos px_s &= 0 \\ \sum_{s=0}^{m-1} (y_s - S_n(x_s)) \sin px_s &= 0 \end{aligned} \right\} p = 1, 2, \dots, n.$$

It will be found, as in § 2 above, that these equations lead to the following values for the coefficients :

* Carse and Shearer, *A Course in Fourier's Analysis and Periodogram Analysis*, Edinburgh, 1915.

† If $m < 2n + 1$, we can choose the constants in any number of ways so that $S_n(x)$ shall be equal to y_0, y_1, \dots, y_{m-1} at x_0, x_1, \dots, x_{m-1} , for there are more constants than equations. And if $m = 2n + 1$, we can choose the constants in one way so that this condition is satisfied.

$$\begin{aligned}
 ma_0 &= \sum_{s=0}^{m-1} y_s, \\
 \left. \begin{aligned}
 \frac{1}{2}ma_r &= \sum_{s=0}^{m-1} y_s \cos r s \alpha \\
 \frac{1}{2}mb_r &= \sum_{s=0}^{m-1} y_s \sin r s \alpha
 \end{aligned} \right\} \begin{array}{l} r = 1, 2 \dots n, \\ m \text{ odd.} \end{array}
 \end{aligned}$$

But if m is even, the coefficient a_r (when $r = \frac{1}{2}m$) is given by

$$ma_{\frac{1}{2}m} = \sum_{s=0}^{m-1} y_s \cos s\pi,$$

the others remaining as above.

In some cases, it is sufficient to find the terms up to $\cos x$ and $\sin x$, viz.

$$a_0 + a_1 \cos x + b_1 \sin x.$$

The values of a_0 , a_1 and b_1 , which will make this expression approximate most closely to

$$\begin{array}{l}
 y_0, y_1, y_2, \dots y_{m-1} \\
 \text{at} \quad 0, \alpha, 2\alpha, \dots (m-1)\alpha, \quad \text{when } m\alpha = 2\pi,
 \end{array}$$

are then given by :

$$\begin{aligned}
 ma_0 &= \sum_{s=0}^{m-1} y_s, \\
 \frac{1}{2}ma_1 &= \sum_{s=0}^{m-1} y_s \cos sa, \\
 \frac{1}{2}mb_1 &= \sum_{s=0}^{m-1} y_s \sin sa.
 \end{aligned}$$

Tables for evaluating the coefficients in such cases have been constructed by Turner.*

5. In the preceding sections we have been dealing with a set of observations known to have a definite period. The graph for the observations would repeat itself exactly after the lapse of the period; and the function thus defined could be decomposed into simple undulations by the methods just described.

But when the graph of the observations is not periodic, the function may yet be represented by a sum of periodic terms whose periods are incommensurable with each other. The gravitational attractions of the heavenly bodies, to which the tides are due, are made up of components whose periods are not commensurable. But in the tidal graph of a port the component

* *Tables for Harmonic Analysis*, Oxford University Press, 1913.

due to each would be resolvable into simple undulations. A method of extracting these trains of simple waves from the record would allow the schedule of the tidal oscillations at the port to be constructed for the future so far as these components are concerned.

The usual method of extracting from a graph of length L , a part that repeats itself periodically in equal lengths λ is to cut up the graph into segments of this length, and superpose them by addition or mechanically. If there are enough segments, the sum thus obtained, divided by the number of the segments, approximates to the periodic part sought; the other oscillations of different periods may be expected to contribute a constant to the sum, namely the sum of the mean part of each.

6. The principle of this method is also used in searching for hidden periodicities in a set of observations taken over a considerable time. Suppose that a period T occurs in these observations and that they are taken at equal intervals, there being n observations in the period T .

Arrange the numbers in rows thus :

$$\begin{array}{cccccc}
 u_0, & u_1, & u_2, & \dots & u_{n-2}, & u_{n-1}. \\
 u_n, & u_{n+1}, & u_{n+2}, & \dots & u_{2n-2}, & u_{2n-1}. \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 u_{(m-1)n}, & u_{(m-1)n+1}, & u_{(m-1)n+2}, & \dots & u_{mn-2}, & u_{mn-1}.
 \end{array}$$

Add the vertical columns, and let the sums be

$$U_0, \quad U_1, \quad U_2, \quad \dots \quad U_{n-2}, \quad U_{n-1}.$$

In the sequence $U_0, U_1, U_2, \dots, U_{n-1}, U_n$ the component of period T will be multiplied m -fold, and the variable parts of the other components may be expected to disappear, as these will enter with different phases into the horizontal rows, and the rows are supposed to be numerous. The difference between the greatest and least of the numbers $U_0, U_1, U_2, \dots, U_{n-2}, U_{n-1}$ furnishes a rough indication of the amplitude of the component of period T , if such exists; and the presence of such a period is indicated by this difference being large.

Let y denote the difference between the greatest and least of the numbers $U_0, U_1, U_2, \dots, U_{n-2}, U_{n-1}$ corresponding to the trial period x . If y is plotted as a function of x , we obtain a "curve

of periods." This curve will have peaks at the values of x corresponding to the periodicities which really exist. When the presence of such periods is indicated by the curve, the statistics are then analysed by the methods above described.

This method was devised by Whittaker for the discussion of the periodicities entering into the variation of variable stars.* It is a modification of Schuster's work, applied by him to the discussion of the statistics of sunspots and other cosmical phenomena.† To Schuster, the term "*periodogram analysis*" is due, but the "curve of periods" referred to above is not identical with that finally adopted by Schuster and termed *periodograph* (or *periodogram*).

For numerical examples, and for descriptions of other methods of attacking this problem, reference may be made to the Edinburgh Mathematical Tract, No. 4, already cited, and to Schuster's papers.

* *Monthly Notices, R.A.S.*, **71**, p. 686, 1911.

See also a paper by Gibb, "The Periodogram Analysis of the Variation of SS Cygni," *ibid.*, **74**, p. 678, 1914.

† The following papers may be mentioned :

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APPENDIX II

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$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a) da, \quad a_p = \frac{1}{\pi} \int_{-\pi}^{\pi} f(a) \cos pa da, \quad b_p = \frac{1}{\pi} \int_{-\pi}^{\pi} f(a) \sin pa da$$

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