Finally, with the new experimental data we are now in a position to calculate for the first time the changes in the specific heat of a metal at high pressures. It appears that the specific heat at constant volume decreases under pressure by an amount of the same order of magnitude as the change of volume under the same pressure, but in most cases by a factor several fold greater.

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A FURTHER NOTE ON THE MATHEMATICAL THEORY OF POPULATION GROWTH[1]

BY RAYMOND PEARL AND LOWELL J. REED

In an earlier paper in these PROCEEDINGS we[2] showed that the expression

$$y = \frac{be^{ax}}{1 + ce^{ax}} \qquad \text{(i)}$$

gives an excellent fit to the known population growth of the United States since 1790. Since the first paper was published, we have extended and generalized our ideas on population growth with the results herein presented in skeleton outline. A more extended paper, giving a full discussion of our new results and of the pertinent literature is in press in Metron.

Considered generally, the curve

$$y = \frac{b}{e^{-ax} + c}$$

may be written

$$y = \frac{k}{1 + me^{ka'x}}, \qquad \text{(ii)}$$

where

$$k = b/c, \; m = 1/c, \text{ and } ka' = -a.$$

$a' = \frac{-a}{k}$

Now the rate of change of y with respect to x is given by

$$\frac{dy}{dx} = -a'y\ (k - y)$$

or

$$\frac{\frac{dy}{dx}}{y\,(k-y)} = -a'. \qquad \text{(iii)}$$

If y be the variable changing with time x (in our case population) equation (iii) amounts to the assumption that the time rate of change of y varies directly as y and as $(k-y)$, the constant k being the upper limit of growth, or, in other words, the value of the growing variable y at infinite time. Now since the rate of growth of y is dependent upon factors that vary with time, we may generalize (iii) by using $f\,(x)$ in place of $-a'$, $f\,(x)$ being some as yet undefined function of time.

Then

$$\frac{dy}{y\,(k-y)} = f(x)dx.$$

whence

$$\frac{k-y}{my} = e^{-k\int f(x)\,dx},$$

and

$$y = \frac{k}{1 + me^{-k\int f(x)dx}} = \frac{k}{1 + me^{F(x)}} \qquad \text{(iv)}$$

where

$$F(x) = -k\int f(x)dx.$$

Then the assumption that the rate of growth of the dependent variable varies as (a) that variable, (b) a constant minus that variable, and (c) an arbitrary function of time, leads to equation (iv), which is of the same form as (i), except that ax has been replaced by $F(x)$. If now we assume that $f(x)$ may be represented by a Taylor series, we have

$$y = \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots + a_nx^n}} \qquad \text{(v)}$$

If

$$a_2 = a_3 = a_4 = a_n = 0$$

then (v) becomes the same as (i).

If m becomes negative the curve becomes discontinuous at finite time. Since this cannot occur in the case of the growth of the organism or of populations, nor indeed so far as we are able to see, for any *phenomenal* changes with time, we shall restrict our further consideration of the equation to positive value only of m. Also since negative values of k would give negative values of y, which in the case of population or individual growth are unthinkable, we shall limit k to positive values.

With these limitations as to the values of m and k we have the following general facts as to the form of (v). y can never be negative, i.e., less than zero, nor greater than k. Thus the complete curve always falls between the x axis and a line parallel to it at a distance k above it. Further we have the following relations:

If

$$F(x) \doteq \infty \qquad y=0$$
$$F(x) \doteq -\infty \qquad y=k$$
$$F(x) \doteq -0 \qquad y=\frac{k}{1+m} \text{ from below}$$
$$F(x) \doteq +0 \qquad y=\frac{k}{1+m} \text{ from below.}$$

The maximum and minimum points of (v) occur where $\frac{dy}{dx}=0$.

But $\frac{dy}{dx} = y(k-y) \, . \, F'(x)$,
therefore we have maximum and minimum points wherever $F'(x) = 0$. The fact that $\frac{dy}{dx} = 0$ when either $y = 0$ or $y - k = 0$ shows that the curve passes off to infinity asymptotic to the lines $y = 0$ and $y = k$.

The points of inflection of (v) are determined by the intersections of (v) with the curve.

$$y = \frac{k}{2} - \frac{k}{2}\frac{F''(x)}{[F'(x)]^2} \qquad \text{(vi)}$$

Since we are seldom justified in using over five arbitrary constants in any practical problem, we may limit equation (v) still further by stopping at the third power of x. This gives the equation

$$y = \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3}}. \qquad \text{(vii)}$$

If a_n is positive the curve of equation (v) is reversed and becomes asymptotic to a line $A\,B$, at $x = -\infty$ and to the x axis at $x = +\infty$. Thus in equation (vii) a_3 negative is a case of growth, and a_3 positive is a case of decay.

Equation (vii) has several special forms that are of interest, among them being a form similar in shape to the autocatalytic curve (i.e., with no maximum or minimum points and only one point of inflection) except that it is free from the two restrictive features mentioned in our first paper, that is, location of the point of inflection in the middle and symmetry of the two limbs of the curve. Asymmetrical or skew curves of this sort can only arise when the equation, $F'(x) = 0$, has no real roots. While any odd value of n may yield this form of curve the simplest equation that will do it is that in which $n = 3$, so that the equation of this curve becomes that of (vii).

Having determined that the growth within any one epoch or cycle may be approximately represented by equation (i), or more accurately by (vii), the next question is that of treating several epochs or cycles. Theoretically, some form of (v) may be found by sufficient labor in the adjustment of constants so that one equation with say 5 or 7 constants would describe a long history of growth involving several cycles. Practically, however, we have found it easier and just as satisfactory in other respects to treat each cycle by itself. Since the cycles of any case of growth are additive, we may use for any single cycle the equation

$$y = d + \frac{k}{1 + me^{ka'x}} \qquad \text{(viii)}$$

or more generally

$$y = d + \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3}} \qquad \text{(ix)}$$

where in both of these forms d represents the total growth attained in all the previous cycles. The term d is therefore the lower asmptote of the cycle of growth under consideration and $d + k$ is its upper asymptote.

In treating any two adjacent cycles, it should be noted that the lower asymptote of the second cycle is frequently below the upper asymptote of the first cycle, due to the fact that the second cycle is often started before the first one has had time to reach its natural level. This for instance would be the case where a population entered upon an industrial era before the country had reached the limit of population possible under purely agricultural conditions.

The theory presented in this paper has been found to be entirely successful in fitting the population growth of many different countries, and in a subsequent publication this fact will be demonstrated with examples.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, The Johns Hopkins University, No. 81.

[2] Pearl, R. and Reed, L. J., "On the Rate of Growth of the Population of the United States Since 1790 and Its Mathematical Representation." PROC. NAT. ACAD. SCI., Vol. 6, pp. 275–288, 1920.