DISSIPATIVE DYNAMICAL SYSTEMS. I.

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1. As a first approximation to physical reality, the configuration of certain physical systems, such as: the motion of a set of point masses, considered by Rayleigh;[1] or the flow of electricity in ideal electric networks, as treated by Maxwell[2] and others, may be formulated analytically in terms of a system of linear differential equations of the second order. Many theoretical questions which arise from still more ideal dynamical systems which physically may be called conservative, have been investigated by Birkhoff[3] and others. It is the purpose of the present paper to define analytically the types of dynamical systems whose configuration is given by the previously mentioned set of equations, and to establish certain facts concerning the existence of algebraic variational principles for dissipative dynamical systems. In subsequent communications certain differential topological aspects of dissipative dynamical systems and their application to the theory of electrical networks will be considered.

2. In order to facilitate the precise definition of the types of dynamical systems that are to be considered, it will be convenient to review the formulation of a general dynamical system of point masses as given by Rayleigh.[1] The equations of motion of such a system are:

$$a_{ij}\ddot{Q}_j + b_{ij}\dot{Q}_j + c_{ij}Q_j = Q_i(t) \qquad i, j = 1, \ldots n \tag{2.1}$$

where a_{ij}, b_{ij}, c_{ij} are functions of class C' on a segment γ of the t axis, and where the following symmetric properties exist between the coefficients:

$$a_{ij} \equiv a_{ji},\ b_{ij} \equiv b_{ji},\ c_{ij} \equiv c_{ji}, \text{ and } |a_{ij}| \not\equiv 0^*$$

physically: a_{ij} are masses, b_{ij} dissipation coefficients, and c_{ij} elastic coefficients.

If equations 2.1 are multiplied by $\dot{Q}_i$ the so-called equations of activity[4] are obtained.

$$a_{ij}\ddot{Q}_j\dot{Q}_i + b_{ij}\dot{Q}_j\dot{Q}_i + c_{ij}Q_j\dot{Q}_i = Q_i\dot{Q}_i \tag{2.2}$$

where $i, j = 1, \ldots, n$.

If the following energy functions are defined:

* The usual summation convention will be hereinafter adopted; i.e., a repeated subscript means a finite summation with respect to that letter.

$$\text{Kinetic Energy:}\quad T = \int_0^t a_{ij}\ddot{Q}_j\dot{Q}_i\,dt$$

$$\text{Potential Energy: } U = \int_0^t c_{ij}Q_j\dot{Q}_i\,dt \qquad (2.3)$$

$$\text{Dissipation Function: } F = b_{ij}\dot{Q}_j\dot{Q}_i$$

equations 2.1 and 2.2 can be reduced to the following canonical forms, respectively:

$$\frac{d}{dt}\frac{\partial(T+U)}{\partial\dot{Q}_i} + \frac{\partial F}{\partial\dot{Q}_i} = Q_i \qquad (2.4)$$

$$\frac{d}{dt}(T+U) + F = Q_i\dot{Q}_i. \qquad (2.5)$$

Equation 2.5 can be considered as a formulation of the principle of conservation of energy, where the term $Q_i\dot{Q}_i$ may be regarded as the rate at which energy is supplied to the system from external sources.

3. Definition: *A linear dynamical system* has the property that its equations of motion and activity are given by the canonical forms 2.4 and 2.5. It is convenient to classify further linear dynamical systems as follows:

Type I. *A conservative linear dynamical system* has the property that $F \equiv 0$.

Type II. *A dissipative linear dynamical system* has the property that $F \not\equiv 0$.

4. Variational Principles:

Classically, it has been of formal interest to establish certain variational principles which give as a necessary consequence the equations of motion of linear dynamical systems. The question naturally arises, for what types of linear dynamical systems does such a principle exist? The answer to this question follows from certain recent results on the inverse problem of the calculus of variations by Davis[5] and Morse.[6]

The general form of a variational principle is to set up an integral of the form,

$$I = \int_{t_1}^{t_2} f_i(t, Q_i, Q_j, \dot{Q}_i, \dot{Q}_j)dt \qquad (4.1)$$

whose Euler equations are the equations of motion of the dynamical system under consideration. Usually the integrand, f_i, is expressed in terms of the energy functions, 2.3. Well-known principles of this type are the Principle of Least Action and Hamilton's Principle.

The following theorem due to Davis[5] will be used to establish the existence of a variational principle for the different types of linear dynamical systems:

If a given system of differential equations

$$H_i(t, Q_i, Q_j, \dot{Q}_i, \dot{Q}_j, \ddot{Q}_i, \ddot{Q}_j) = 0 \quad i, j = 1, \ldots, n$$

is to be the system of differential equations of the solutions of the problem of minimizing the integral, I, of the type indicated above, then their equations of variation must be a self-adjoint system along every curve $Q_i = Q_i(t)$.

Davis[5] and Morse[6] also show that necessary and sufficient conditions for the self-adjoint requirement are, using the system of equations 2.1:

$$\begin{aligned} a_{ij} &\equiv a_{ji} \\ b_{ij} + b_{ji} &\equiv 2\dot{a}_{ij} \\ \ddot{a}_{ij} - \dot{b}_{ij} &\equiv c_{ji} - c_{ij}. \end{aligned} \tag{4.2}$$

A linear dynamical system of type I will now be considered. In this case the requirement, that $F \equiv 0$, yields the result that $b_{ij} \equiv 0$. The relations 4.2 becomes in this case:

$$\begin{aligned} a_{ij} &\equiv a_{ji} \\ 0 &\equiv 2\dot{a}_{ij} \\ c_{ij} &\equiv c_{ji} \end{aligned} \tag{4.3}$$

From this result it is at once evident that a variational principle exists if and only if $a_{ij} \equiv$ constant. That variational principles for a system of type I exist is, of course, well known. The following theorem follows at once:

THEOREM I. *The equations of motion of a conservative linear dynamical system are given by a variational principle if and only if the masses of the system are constant.*

If a linear dynamical system of type II is considered, the relations 4.2 become by the use of the symmetric properties of the coefficients given in 2.1:

$$\begin{aligned} a_{ij} &\equiv a_{ji} \\ b_{ij} &\equiv \dot{a}_{ij}. \end{aligned} \tag{4.4}$$

The theorem which follows this is:

THEOREM II. *The equations of motion of a dissipative linear dynamical system are given by a variational principle if and only if the dissipation coefficients are identically equal to the rates of change of the corresponding masses.*

This result has the physical interpretation of connecting the mass of the system and the coefficient of dissipation of the system. If the mass of the system is a monotonically increasing function of time the dissipation function would be positive, and the system would lose energy. If the mass of the system decreases monotonically with time, the system would gain energy. If the mass of the system is an oscillatory function of time or a function of time whose first derivative is not always of one sign, several possibilities for the net gain or loss of energy by the system exist:

If a particular epoch, $t_1 \leq t \leq t_2$ be considered, the time-average rate of dissipation of energy will be given by:

$$I = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} F dt \tag{4.5}$$

If $I = 0$, the system will have no net energy loss during this epoch.

If $I > 0$ or $I < 0$ the system will have an average loss or gain of energy during this epoch.

A special case of a system of type II of particular physical importance is that in which the equations of motion 2.1 have constant coefficients. Now by relations 4.4, it is seen that

$$b_{ij} \equiv 0 \tag{4.6}$$

but this violates our hypothesis that $b_{ij} \not\equiv 0$.

Hence:

COROLLARY: *The equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle.*

[1] Rayleigh, *Theory of Sound*, Macmillan, **1**, 103 (1926).

[2] Maxwell, *A Treatise on Electricity and Magnetism*, Oxford, **1**, 309 (1892).

[3] Birkhoff, G. D., *Dynamical Systems*, Am. Math. Soc. Colloquium Pub., 1927.

[4] Carson, J. R., *Electric Circuit Theory and the Operational Calculus*, McGraw-Hill, 1926.

[5] Davis, D. R., *Trans. Am. Math. Soc.*, **30**, 711–713 (1928).

[6] Morse, Marston, *Math. Annalen*, **52**, 103 (1930).