The results of these large scale experiments show clearly that under uniform conditions the tubercle organism may be cultured for at least 14 years without deterioration in vigor of growth as evidenced by weight of the bacterial cells per unit of culture.

In this paper no attempt has been made to deal with bacterial dissociation or life cycles but rather to present certain observed facts which relate to the stability of different groups of bacteria when kept for years under suitable artificial conditions. To the author it appears that the supposition of gradual deterioration is not necessarily supported by facts. The results of the experiments here reported show that these forms may be kept in the laboratory for years and yet remain stable with respect to the properties tested. It is indeed significant that no matter how contrastive the artificial medium as compared with the natural environment to which the organisms were accustomed, many of the dominant physiological characteristics remain unchanged. The results of this study, therefore, support the fundamental concept of the stability of pure cultures of bacteria under suitable artificial conditions.

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*CONCERNING SEQUENCES OF HOMEOMORPHISMS**

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1. *Introduction.*—In his thesis, Knaster[1] defines a method of construction which he calls the method of bands, by means of which he is able to construct indecomposable continua of various types. These continua may also be thought of as examples to show that each member of a sequence of continua, each of which contains the following continuum of the sequence, may be homeomorphic with a rectangle, while the set of points common to all the continua of the sequence is not homeomorphic with a rectangle. The present paper is a contribution to the solution of the problem of determining conditions under which a homeomorphism

exists between the sets common to two sequences of point sets, the corresponding elements of which are homeomorphic. A certain principle of uniformity is used in this connection.

2. *Notation and Definitions.*—In the following, $[M_i]$ denotes a sequence of point sets: $M_1, M_2, M_3, \ldots$. For each i, there exists a homeomorphic transformation T_i of M_i, and we shall frequently denote $T_i(M_i)$ by N_i.

By saying that the sequence of points $[P_i]$ converges to the point P, we shall mean that the point P is the sequential limiting point of the sequence $[P_i]$.

A sequence of point sets $[T_i(M_i)]$ is said to be *uniformly homeomorphic* with respect to the sequence $[M_i]$, if given any positive number ϵ, there exist corresponding positive numbers δ and k, such that if $i > k$ and the distance between the points p and q of M_i is less than δ, then the distance between $T_i(p)$ and $T_i(q)$ is less than ϵ.

If the sequence $[M_i]$ is uniformly homeomorphic with respect to $[T_i(M_i)]$, and the sequence $[T_i(M_i)]$ is uniformly homeomorphic with respect to $[M_i]$, then the sequences $[M_i]$ and $[T_i(M_i)]$ are said to be *mutually uniformly homeomorphic.*

Note that if for every integer i we have $M_i = M_{i+1}$ and $T_i = T_{i+1}$, then the condition that the sequences be mutually uniformly homeomorphic is equivalent to the condition that a uniform homeomorphism exist between the sets M_i and $T_i(M_i)$, as previously defined by the present writer.[2]

3. *Homeomorphisms.*—We shall next give two sets of conditions under which a homeomorphism exists between the sets of common points of two sequences.

THEOREM 1. *In a metric space, let $[M_i]$ and $[N_i]$ be sequences of point sets such that for each value of i, M_i contains M_{i+1} and N_i contains N_{i+1}, and let M and N, respectively, denote the sets of points common to all the sets of the sequences $[M_i]$ and $[N_i]$. For each value of i, let T_i denote a homeomorphism such that $T_i(M_i) = N_i$. Let the sequences satisfy the further conditions:*

(1) *if p is any point of M, the sequence $[T_i(p)]$ converges to a point P of N;*

(2) *if Q is any point of N, the sequence $[T_i^{-1}(Q)]$ converges to a point q of M;*

(3) *the sequences $[M_i]$ and $[N_i]$ are mutually uniformly homeomorphic.*

Under these conditions there exists a uniform homeomorphism T such that $T(M) = N$.

Proof.—Let p be a given point of M. Then by (1) and (2), the sequence $[T_i(p)]$ converges to a point P of N and the sequence $[T_i^{-1}(P)]$ converges to a point p' of M. We shall show that $p' = p$.

Let ϵ denote an arbitary positive number, and let δ and k be positive numbers whose existence follows from (3), which are such that if $i > k$

and the distance between the points $T_i(p)$ and P of N_i is less than δ, then the distance between the points $T_i^{-1}T_i(p) = p$ and $T_i^{-1}(P)$ is less than ϵ. Since the sequence $[T_i(p)]$ converges to P, there are an infinite number of points of this sequence for which $i > k$ and whose distance from P is less than δ. Hence there are an infinite number of points of the sequence $T_i^{-1}(P)$ whose distance from the point p is less than the arbitrary number ϵ. It follows that the point p' to which the sequence $[T_i^{-1}(P)]$ converges is the point p.

Thus we have shown that if p is a point of M and if $[T_i(p)]$ converges to the point P of N, then the sequence $[T_i^{-1}(P)]$ converges to the point p. Similarly, if Q is a point of N, and if $[T_i^{-1}(Q)]$ converges to the point q of M, then $[T_i(q)]$ converges to Q. Hence it follows that if p and q are distinct points of M, the points P and Q are distinct, and conversely. We can now define a $(1 - 1)$ reciprocal correspondence T between the points of M and the points of N, by making each point p of M correspond to the point of N to which the sequence $[T_i(p)]$ converges.

It remains to show that T is continuous. By (3) it follows that if ϵ is any arbitrary positive number, there exist positive numbers δ and k such that if $i > k$ and the distance between the points p and q of M is less than δ, then the distance between the points $T_i(p)$ and $T_i(q)$ is less than ϵ. But in that case the distance between the points P and Q to which the sequences $[T_i(p)]$ and $[T_i(q)]$ converge is not greater than ϵ. From this it follows that if the point p is a limit point of any subset of M, the point $T(p) = P$ is a limit point of the corresponding subset of N. Hence limit points are preserved under the transformation T and a similar argument shows that they are preserved under T^{-1}. Hence T is a homeomorphism.

The method of proof shows incidentally that T is uniformly continuous.[2] Hence T is a uniformly homeomorphic correspondence or a uniform homeomorphism.

THEOREM 2. *In a metric space, let $[M_i]$ and $[N_i]$ be sequences of compact point sets such that for each value of i, M_i contains M_{i+1} and N_i contains N_{i+1}. Let M and N, respectively, denote the sets of points common to all the sets of the sequences $[M_i]$ and $[N_i]$. For each value of i, let T_i denote a homeomorphism such that $T_i(M_i) = N_i$. Let the sequence satisfy the further conditions:*

(1) *if p is any point of M, the sequence $[T_i(p)]$ converges to a point P of N;*

(2) *if Q is any point of N, the set M contains every limit point of the set $\sum T_i^{-1}(Q)$;*

(3) *the sequences $[M_i]$ and $[N_i]$ are mutually uniformly homeomorphic.*

Under these conditions there exists a uniform homeomorphism T such that $T(M) = N$.

Proof.—Let Q be a point of N. Then since M_i is compact, the sequence $[T_i^{-1}(Q)]$ has limit points which are points of M, by (2). Suppose it has two limit points x and y. From (1) and (3), it follows that if ϵ is an arbitrary positive number, there exist positive numbers δ and k such that (a) the distance between the points $T_i(x)$ and X of N_i is less than ϵ for every $i > k$, (b) the distance between the points $T_i(y)$ and Y of N_i is less than ϵ for every $i > k$, (c) the distance between the points x and $T_j^{-1}(Q)$ of M_j is less than δ for some $j > k$, and hence the distance between the points $T_j(x)$ and Q of N_j is less than ϵ for some $j > k$, and (d) the distance between the points y and $T_h^{-1}(Q)$ of M_h is less than δ for some $h > k$, and hence the distance between the points $T_h(y)$ and Q of N_h is less than ϵ for some $h > k$. From (a) and (c) it follows that the distance between the points X and $T_j(x)$ of N_j is less than ϵ, and the distance between the points $T_j(x)$ and Q of N_j is less than ϵ, and hence the distance between the points X and Q of N is less than 2ϵ. Similarly from (b) and (d) it follows that the distance between the points Y and Q of N is less than 2ϵ. Therefore the distance between the points X and Y of N is less than 4ϵ, and since ϵ is arbitrary, it follows that $X = Y$.

But if $X = Y$, it follows from (a) and (b) that the distance between the points $T_i(x)$ and $T_i(y)$ of N_i is less than 2ϵ for every $i > k$. But then it is easily established by means of hypothesis (3) that the distance between the points x and y of M is less than α, where α is arbitrary. Hence $x = y$. In other words the sequence $[T_i^{-1}(Q)]$ has a single limit point to which it converges. Hence the hypotheses of Theorem 1 are fulfilled and the conclusion follows from Theorem 1.

4. *Homeomorphic Transformations.*—In this section it is shown that if the hypotheses of Theorems 1 and 2 are slightly weakened, we can no longer prove that M and N are homeomorphic, but can merely prove that there exists a continuous $(1 - 1)$ transformation of M into N, concerning whose inverse we know nothing except that it is $(1 - 1)$. We shall call such a transformation a *homeomorphic transformation.*

THEOREM 3. *In a metric space, let $[M_i]$ and $[N_i]$ be sequences of point sets such that for each value of i, M_i contains M_{i+1} and N_i contains N_{i+1}, and let M and N, respectively, denote the sets of points common to all the sets of the sequences $[M_i]$ and $[N_i]$. For each value of i, let T_i denote a homeomorphism such that $T_i(M_i) = N_i$. Let the sequences satisfy the further conditions:*

(1) *if p is any point of M, the sequence $[T_i(p)]$ converges to a point P of N, and if the points p and p' of M are distinct, than the points P and P' are distinct;*

(2) *if Q is any point of N, the sequence $[T_i^{-1}(Q)]$ converges to a point q of M;*

(3) *the sequence $[N_i]$ is uniformly homeomorphic with respect to the sequence $[M_i]$.*

Under these conditions there exists a homeomorphic transformation t, such that $t(M) = N$.

Proof.—As in the proof of Theorem 1, we can show that if P is a point of N and if the sequence $[T_i^{-1}(P)]$ converges to the point p of M, then the sequence $[T_i(p)]$ converges to P.

Suppose that there is a point p of M such that $[T_i(p)]$ converges to the point P of N, but that $[T_i^{-1}(P)]$ converges to a point q of M, where $q \neq p$. From the preceding paragraph we see that since $[T_i^{-1}(P)]$ converges to q, the sequence $[T_i(q)]$ converges to P. But then we have both $[T_i(p)]$ and $[T_i(q)]$ converging to P, which is contrary to (1). Hence if p is any point of M and if $[T_i(p)]$ converges to the point P of N, then $[T_i^{-1}(P)]$ converges to p.

Thus we define a $(1 - 1)$ correspondence t between the points of M and the points of N by making each point p of M correspond to the point of N to which the sequence $T_i(p)$ converges. As in the proof of Theorem 1, we can show that if p is a limit point of any subset of M, then the corresponding point of N is a limit point of the corresponding subset of N. Hence we have defined a homeomorphic transformation t such that $t(M) = N$.

THEOREM 4. *In a metric space, let $[M_i]$ and $[N_i]$ be sequences of compact point sets such that for each value of i, M_i contains M_{i+1} and N_i contains N_{i+1}. Let M and N, respectively, denote the sets of points common to all the sets of the sequences $[M_i]$ and $[N_i]$. For each value of i, let T_i denote a homeomorphism such that $T_i(M_i) = N_i$. Let the sequences satisfy the further conditions:*

(1) *if p is any point of M, the sequence $[T_i(p)]$ converges to a point P of N, and if the points p and p′ of M are distinct, then the points P and P′ are distinct;*

(2) *if Q is any point of N, the set M contains every limit point of the set $\sum T_i^{-1}(Q)$;*

(3) *the sequence $[N_i]$ is uniformly homeomorphic with respect to the sequence $[M_i]$.*

Under these conditions there exists a homeomorphic transformation t, such that $t(M) = N$.

Proof.—As in the proof of Theorem 2, we show that if Q is any point of N, and the sequence $[T_i^{-1}(Q)]$ has two limit points x and y, the sequences $[T_i(x)]$ and $[T_i(y)]$ converge to the same point of N. But this is impossible by (1). Hence the sequence $T_i^{-1}(Q)$ converges to a single point of M. But in that case the hypotheses of Theorem 3 are satisfied, and the conclusion follows from that theorem.[3]

* Presented to the American Mathematical Society, October 31, 1931.

[1] B. Knaster, *Funda. Math.*, **3**, 247–286 (1921).

[2] H. M. Gehman, *Trans. Amer. Math. Soc.*, **29**, 553–568 (1927).

[3] In a recent paper by J. H. Roberts, *Trans. Amer. Math. Soc.*, **34**, 252–262 (1932), is given a theorem (Theorem 1) closely related to the theorems of this paper. While most of Roberts' results will hold true in a more general space than the one considered by him, the statement that "if π^{-1} is single-valued, it is continuous," will not be true in a general metric space.

AN IMPROVED EQUAL-FREQUENCY MAP OF THE NORMAL CORRELATION SURFACE, USING CIRCLES INSTEAD OF ELLIPSES

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In studying the correlation between two variables X and Y it is often important to compare a given scatter-diagram having observed values of $\bar{X}$, $\bar{Y}$, σ, τ, r, with a normal distribution having the same values of $\bar{X}$, $\bar{Y}$, σ, τ, r. (Here σ and τ denote the standard deviations of the X's and the Y's, respectively, and r denotes the coefficient of correlation.)

The usual method of making such a comparison is to draw the "50 per cent ellipse" on the diagram, and see whether 50 per cent of the dots lie within the ellipse. But this process is rather laborious, and becomes more so if one attempts to plot the "10% ellipse," the "20% ellipse," etc

The purpose of this paper is to show how this complicated family of ellipses can be replaced by a simple family of easily plotted concentric circles. These concentric circles, together with a family of equally spaced radial lines, will form a "cobweb map" which divides the plane into "townships of equal frequency" of any desired fineness of mesh (for example, percentiles, permilles, etc.).

This cobweb map makes possible a direct comparison between the observed distribution of dots and the theoretical distribution in the corresponding normal case.

The process of constructing the map is extremely simple, as follows.

Let figure 1 represent the given scatter-diagram, in which h is the class-interval for X, and k the class-interval for Y, and suppose that the means $\bar{X}$, $\bar{Y}$ (in the original units), the standard deviations σ, τ (in the original units), and the coefficient of correlation r (a pure number), have been computed in the usual way. Now re-draw the diagram in oblique form, as follows (see Fig. 2).

First, rotate the axis of Y through an angle ψ given by $\tan\psi = \dfrac{r}{\sqrt{1-r^2}}$.

Second, choose as the geometric unit of length to be used on the new