[1] For the properties of the generalized Laguerre polynomials see G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. II, Julius Springer, Berlin, 1925, pp. 94, 293 and 294. The notation used by these writers agrees with the one used in the present note.

[2] See a paper by Perron, O., *Arch. Math., Leipzig* (3) **22,** 1914 (329–340) where reference is given to Fejér's paper as well as a considerable extension of his results.

[3] Wigert, S., *Ark. Matem., Stockholm,* **15** (25), 1921 (1–22).

[4] See, e.g., G. N. Watson, *A Treatise on the Theory of Bessel Functions,* Cambridge Univ. Press, 1922, p. 395.

ON LAGUERRE'S SERIES. SECOND NOTE

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The summation theorem stated in the preceding note can be utilized in the convergence theory of Laguerre's series. In fact, suppose that $f(x)$ is continuous, and that the integral

$$\int_0^\infty e^{-at} t^\alpha \mid f(t) \mid dt \tag{J}$$

exists for every $a > \frac{1}{2}\cdot$ Then we know that the formal Fourier-Laguerre's series is summable Abel to $f(x)$ when $x \geqq 0$; thus the series will represent the function $f(x)$ whenever it is convergent. The expansion problem is consequently reduced to a simpler problem, namely that of finding sufficient conditions for the convergence of a given Laguerre's series. Some such conditions will be given in the following.

1. We begin with the following theorem:

I. *The series*

$$f(x) \sim \sum_{n=0}^\infty a_n^{(\alpha)} \quad L_n^{(\alpha)}(x) \tag{1}$$

converges when $x > 0$ *and represents* $f(x)$, *if* (i) $f(x)$ *is absolutely continuous on every finite interval* $0 < a < x < b$, (ii) $\lim_{x \to 0} x^{\alpha+1} f(x) = 0$, *and* (iii) *the integrals*

$$\int_0^\infty e^{-t} t^\alpha \mid f(t) \mid^2 dt \ \ and \ \int_0^\infty e^{-t} t^\alpha \mid t f'(t) \mid^2 dt \tag{2}$$

exist. The convergence of (1) *is uniform on* $0 < a \leqq x \leqq b < +\infty$.

PROOF:[1] Consider the integral

$$\int_{x_1}^{x_2} e^{-t} L_n^{(\alpha)}(t) t^{\alpha+1} f'(t)\, dt = I(x_1, x_2)$$

where $0 < x_1 < x_2 < +\infty$. In view of (i) $f'(x)$ exists almost everywhere on $(0, +\infty)$ and is summable over (x_1, x_2). Hence $I(x_1, x_2)$ exists. Further, $f(x)$ is an indefinite integral of $f'(x)$ on (x_1, x_2) and it is permitted to integrate by parts. Thus we obtain

$$I(x_1, x_2) = [e^{-t} L_n^{(\alpha)}(t)\, t^{\alpha+1} f(t)]_{x_1}^{x_2} - \int_{x_1}^{x_2} e^{-t} t^{\alpha} [(1+\alpha-t) L_n^{(\alpha)}(t) + t L'^{(\alpha)}_n(t)] f(t)\, dt. \quad (3)$$

Now let $x_1 \longrightarrow 0$, $x_2 \longrightarrow +\infty$. Then $I(x_1, x_2)$ tends to a finite limit and so does the integral on the right-hand side of (3). Denote for the moment the integrated part by $f_n(t)$. In view of (ii) $\lim_{t \to 0} f_n(t) = 0$. Hence $\lim_{t \to \infty} f_n(t)$ exists and equals zero, since $f_n(t)$ is integrable over $(0, +\infty)$.

Now it follows from formula (6) of the first note that

$$(1+\alpha-x) L_n^{(\alpha)}(x) + x L'^{(\alpha)}_n(x) = (n+1) L_{n+1}^{(\alpha)}(x) - n L_n^{(\alpha)}(x).$$

Hence, denoting the Fourier-Laguerre coefficients of $f(x) + xf'(x)$ by A_n, we have

$$A_n = (n+1)\, [a_n^{(\alpha)} - a_{n+1}^{(\alpha)}].$$

In view of the existence of the integral

$$\int_0^{\infty} e^{-t} t^{\alpha} |f(t) + tf'(t)|^2\, dt,$$

it follows from formula (5) of the first note that the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} |A_n|^2 = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} (n+1)^2 |a_n^{(\alpha)} - a_{n+1}^{(\alpha)}|^2 \quad (4)$$

is convergent. A simple calculation shows that

$$\sum_{n=1}^{\infty} | n^s a_n^{(\alpha)} - (n+1)^s a_{n+1}^{(\alpha)} | \quad (5)$$

converges, when $s < \frac{\alpha}{2} + \frac{1}{2}$. In proving this we have to use (4) together with (5) in the first note as well as well-known properties of the Γ-function.

On the other hand, the Dirichlet's series

$$\sum_{n=1}^{\infty} \frac{L_n^{(\alpha)}(x)}{n^s} \tag{6}$$

converges when $s > \frac{\alpha}{2} + \frac{1}{4}$, in virtue of formula (9) of the first note, x being fixed positive. Further, the convergence is uniform with respect to x on any finite positive interval $0 < a \leqq x \leqq b$, if s is fixed $> \frac{\alpha}{2} + \frac{1}{4}$.

Now give s a fixed value, $\frac{\alpha}{2} + \frac{1}{4} < s < \frac{\alpha}{2} + \frac{1}{2}$, and consider the series

$$\sum_{n=1}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x) = \sum_{n=1}^{\infty} n^s a_n^{(\alpha)} \frac{L_n^{(\alpha)}(x)}{n^s} \tag{7}$$

For this particular value of s the series (5) and (6) are convergent, x being positive. Hence, (1) converges when $x > 0$ and uniformly with respect to x on $0 < a \leqq x \leqq b < +\infty$ in view of the uniform convergence of (6) with respect to x on the same interval. The series (1) may diverge or oscillate when $x = 0$, since the series $\sum_{n=0}^{\infty} \binom{n+\alpha}{n} a_n^{(\alpha)}$ is not necessarily convergent.

2. The expansion problem for Laguerre's series has been investigated previously in the special case $\alpha = 0$ by M^{me} V. Myller-Lebedeff[2] and in the general case by H. Weyl[3] who both base their analysis upon the theory of integral equations. The results of the latter writer are by far the more general ones. Expressed in terms of the polynomial series rather than the series of orthogonal functions and with due change of independent variable and parameter, the result of Weyl for the case $\alpha \geqq 0$ can be stated as follows:

The formal series (1) *represents* $f(x)$, *if* (i) $\lim_{x \to 0} x^\alpha f(x) = 0$, (ii) $\lim_{x \to \infty} e^{-x} x^\alpha f(x) = 0$, (iii) $G(x) = e^{-\frac{x}{2}} x^{\frac{\alpha-1}{2}} [(x-\alpha)f(x) - xf'(x)]$ *is continuous when* $x > 0$, *and* (iv) $\int_0^\infty G^2(x)dx$ *exists.*

The condition for the case $\alpha < 0$ is slightly different in form. Weyl remarks that the continuity of the derivative can be replaced by a less restrictive condition if Hellinger integrals are used.

3. To the preceding solution of the expansion problem we shall add a few conditions for the convergence of a Laguerre's series which involve restrictions of a different nature.

II. *If* $\lim n^{\frac{\alpha}{2} - \frac{1}{4}} a_n^{(\alpha)} = 0$, *and if* $F(x, r)$, *defined by formula* (12) *of the first note, is a holomorphic function of r at* $r = +1$ *when* $x = x_0$, *then* $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x_0)$ *is convergent.*

The condition of a_n implies that $\lim a_n^{(\alpha)} L_n^{(\alpha)}(x_0) = 0$; hence II is a consequence of the theorem of Fatou. If the series in question is the Fourier-Laguerre series of a function $f(x)$ which is continuous when $x = x_0$ and such that the integral (J) exists for all values of $a > \frac{1}{2}$, then $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x_0) = f(x_0)$. A similar remark applies to the theorems III and IV below. The condition $\lim n^{\frac{\alpha}{2} - \frac{1}{4}} a_n^{(\alpha)} = 0$ is necessary in order that $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x)$ shall converge on an interval or on a set of positive measure. This fact is readily seen from formula (7) of the first note in virtue of a theorem on trigonometric series, due to Cantor and extended by several authors in various directions.

III. *If* $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x_0)$, $x_0 > 0$, *is summable Abel and if* $n^{\frac{\alpha}{2} + \frac{3}{4}} a_n^{(\alpha)}$ *is bounded, the series is actually convergent.*

Since $| a_n^{(\alpha)} L_n^{(\alpha)}(x) | < C/n$, the theorem is a consequence of a well-known Tauberian theorem due to Hardy and Littlewood.

IV. *The series* $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x)$ *converges when* $x > 0$, *if either* (i) $-1 < \alpha < -\frac{1}{2}$ *and* $\sum_{n=0}^{\infty} | a_n^{(\alpha)} - a_{n+1}^{(\alpha)} |$ *is convergent, or* (ii) $-1 < \alpha < +\frac{1}{2}$ *and* $\sum_{n=0}^{\infty} | a_n^{(\alpha)} + a_{n+1}^{(\alpha)} |$ *is convergent.*

The proof of this theorem is based upon §2 of the first note. When $-1 < \alpha < -\frac{1}{2}$, the series $\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)$ is convergent. The convergence of this series together with that of $\sum_{n=0}^{\infty} | a_n^{(\alpha)} - a_{n+1}^{(\alpha)} |$ implies the convergence of $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x)$. When $-1 < \alpha < +\frac{1}{2}$, the series $\sum_{n=0}^{\infty} (-1)^n L_n^{(\alpha)}(x)$ is convergent. The convergence of this series together with that of $\sum_{n=0}^{\infty} | a_n^{(\alpha)} + a_{n+1}^{(\alpha)} |$ implies the convergence of $\sum_{n=0}^{\infty} (-1)^n a_n^{(\alpha)} (-1)^n L_n^{(\alpha)}(x)$. So far we have tacitly assumed $x > 0$. In the second case we can, however, permit x to be zero, provided $\alpha < 0$. If $\alpha > 0$, we require some additional condition to ensure convergence when $x = 0$. If $\alpha = 0$, such a condition is given by $\lim a_n^{(0)} = 0$; this condition together with the convergence of $\sum_{u=0}^{\infty} | a_n^{(0)} + a_{n+1}^{(0)} |$ implies the convergence of $\sum_{n=0}^{\infty} a_n^{(0)} = \sum_{n=0}^{\infty} a_n^{(0)} \ L_n^{(0)}(0)$.

[1] It is sufficient to prove the convergence of the series. It follows from (2) with the aid of Schwarz's inequality that the integral (J) exists for every $a > \frac{1}{2}$.

[2] *Math. Annalen,* **64,** 1907 (388–416).

[3] *Ibid,* **66,** 1909 (273–324).

GROUPS CONTAINING A RELATIVELY SMALL NUMBER OF SYLOW SUBGROUPS

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If p^m is the highest power of the prime number p which divides the order g of a given group G then G contains $1 + kp$, k being a positive integer or zero, subgroups of order p^m and its subgroups of this order are known as Sylow subgroups. The object of the present paper is to consider some relative properties of the subgroups of order p^m contained in G when their number is less than $2p^2 + 1$. In particular, we shall prove that when G contains more than one but less than $p^2 + 1$ such subgroups then it must involve an invariant subgroup of order p^{m-1} and when it contains more than one but less than $2p^2 + 1$ such subgroups then it must involve either an invariant subgroup of order p^{m-2} or an invariant subgroup order p^{m-1}. It is clear that the former of these theorems is useful only when m exceeds unity while the latter is useful only when m exceeds 2. Moreover, it is obvious that if in $g = p^m r$ and the number r is fixed, it is always possible to make m sufficiently large so that G involves an invariant subgroup of order p^α, $\alpha > 0$, for all larger values of m.

To prove the former of the theorems noted above it may first be observed than when G involves less than $p^2 + 1$ Sylow subgroups of order p^m then every pair of these subgroups must have p^{m-1} operators in common, for if two such subgroups have $p^{m-\alpha}$ operators in common then one of them has p^α conjugates under the other. Hence it may be assumed in the proof of this theorem that if K_1 and K_2 represent two subgroups of order p^m contained in G then their cross-cut K_0 is of order p^{m-1} and is invariant under K_1 and also under K_2. If K_0 were not invariant under G there would be a subgroup K_3 of order p^m contained in G which would not involve K_0. Each of the two distinct subgroups of order p^{m-1} which K_3