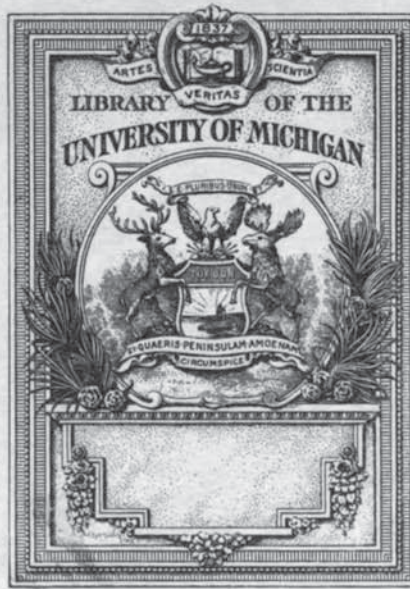


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Glaisher, James W. L.

On the law of facility of errors of observation and on the method of least squares.


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II. *Les Variations de la Pesanteur dans les Provinces Occidentales de l'Empire Russe.* Par A. SAWITSCH. (Communicated by the Astronomer Royal.)

Read January 12, 1872.

1. *Instruments employés, méthodes de Calcul et d'Observation.*

1. UN grand arc du méridien ayant été mesuré en Russie* avec toute la précision que comportent les nouveaux moyens d'observation, il était intéressant d'examiner les changements qu'éprouvent l'intensité de la pesanteur dans ces contrées et de comparer la marche de ces changements avec les variations qui se présentent dans les directions de la pesanteur, déterminées sur plusieurs points par les observations astronomiques et par les opérations géodésiques. C'est ce qui a engagé l'Académie des Sciences sur les diverses parties de l'arc qui s'étend de Tornea en Finlande jusqu'à Ismail en Moldavie, en choisissant les stations dont les positions géographiques et les élévations au-dessus du niveau de la mer sont connues par les travaux relatifs aux mesures des degrés du méridien. M. R. LENZ et moi nous avons fait les expériences durant l'été de l'année 1865 entre Tornea et St. Pétersbourg; pendant la même saison des années 1866 et 1868 les expériences ont été continuées par M. SMYSLOF et moi, de St. Pétersbourg à Ismail sur le Danube.

2. Nous nous sommes servis du pendule à reversion,—instrument qui a été employé avec tant de succès par les savants anglais. Deux pendules de ce genre, construits par M. REPSOLD, à Hambourg, étaient mis à notre

* *Arc du méridien de 25° 20' entre le Danube et la mer Glaciale*, ouvrage composé sur différents matériaux et rédigé par F. G. W. STRUVE.

disposition ; ils diffèrent des instrumens anglais par leurs dimensions et par les arrangements de leurs diverses parties. Ils ne battent qu'à-peu-près les $\frac{3}{4}$ d'une seconde du temps moyen et ne portent pas le curseur, ou petit poids mobile, destiné à rendre égales entre elles les durées d'oscillation autour de chacun des deux axes de rotation. En se conformant à la théorie de BESSEL, on a fait le pendule de telle sorte que sa figure est symétrique en-haut et en-bas par rapport à un plan horizontal passant par le milieu du pendule disposé verticalement ; mais les masses sont distribuées d'une manière inégale. Ainsi le pendule a deux lentilles, des mêmes formes et grandeurs : l'une est lourde et pleine, l'autre légère et intérieurement vide. Les axes de rotation sont des couteaux ou les tranchants de prismes en acier ; le support de l'axe est aussi en acier. Pendant les expériences le pendule et son trépied étaient renfermés dans une cage ou maisonnette, afin que les courants de l'air n'eussent pas d'influence sur les oscillations du pendule.

Un étalon et un comparateur, construits aussi par M. REPSOLD, servaient à évaluer la distance entre les deux couteaux du pendule ; M. SMYSLOF a vérifié cet étalon en le comparant à l'étalon normal, exécuté à Londres par MM. TROUGHTON et SIMMS, qui se trouve à l'Observatoire de Poulkova et qui a été examiné par M. BAILY.*

Un excellent mécanicien de St. Pétersbourg, M. BRAUER, nous a fourni deux appareils pour les vérifications de l'instrument. L'un sert à la détermination du centre de gravité du pendule, et l'autre est un niveau sur un patin convenable, qu'on peut placer sur le couteau inférieur quand l'autre couteau s'appuie en-haut sur le plan horizontal du support ; il est destiné à examiner le parallélisme des deux couteaux.

Sans entrer dans la description des détails, nous sommes heureux de dire que nos deux pendules ont été envoyés dans l'Inde et mis à la disposition de M. WALKER, pour servir à ses recherches scientifiques ; ils ne pouvaient pas être placés dans des mains plus habiles.

3. Pour trouver la durée des oscillations du pendule, nous avons employé la méthode de BORDA ; elle consiste à noter les temps des coin-

* *Memoirs of the Royal Astronomical Society*, Vol. ix., "Report of the New Standard Scale," by F. BAILY, p. 115, 1836.

cidences de notre pendule et de celui d'une horloge, dont la marche diurne se déduit des comparaisons de l'horloge aux chronomètres ; la marche de ces derniers en 24 heures du temps moyen a été vérifiée par les observations astronomiques, à l'aide d'un instrument des passages établi sur une colonne en briques.

Les amplitudes des oscillations et les températures de l'instrument ainsi que de l'air qui l'environne, sont données pour les 3 ou 4 premières oscillations, pour le milieu, et pour les 3 ou 4 dernières oscillations de chaque série d'expériences. Dans l'intérieur de la cage du pendule il y avait trois thermomètres centigrades, l'un placé vers le bout supérieur, l'autre vers le milieu, et le troisième en-bas du pendule. L'échelle de ces thermomètres est divisée de $\frac{1}{5}$ à $\frac{1}{5}$ degrés ; ces thermomètres ont été vérifiés à l'Observatoire physique de St. Pétersbourg. Il nous a paru suffisant d'inscrire les indications du baromètre au commencement et à la fin de chaque série d'expériences.

Les réductions de la durée des oscillations observées à celles qui correspondent à l'amplitude infiniment petite peuvent être calculées par la formule connue de BORDA, en admettant que les amplitudes décroissent en progression géométrique, quand le nombre d'oscillations s'augmente en progression arithmétique. Sans recourir à cette supposition, on peut facilement calculer les réductions par la méthode des quadratures.

Soit $2t$ l'intervalle du temps entre deux oscillations quelconques du pendule, soient de plus u' , u'' , u''' , les angles d'excursion du pendule au commencement, au milieu, et à la fin de l'intervalle $2t$; n le nombre d'oscillations observées pendant cet intervalle, et n_0 le nombre correspondant d'oscillations infiniment petites. Prenant t pour l'unité de temps et δt pour un élément infiniment petit de temps, on peut exprimer n_0 par l'équation

$$n_0 = n \int_{-1}^{+1} \left(1 + \frac{1}{16} \sin^2 u\right) d t.$$

Calculons d'abord les valeurs numériques :—

$$1 + \frac{1}{16} \sin^2 u' = U', \quad 1 + \frac{1}{16} \sin^2 u'' = U'', \quad 1 + \frac{1}{16} \sin^2 u''' = U''',$$

$$U''' - U'' - (U'' - U') = \Delta^{(2)},$$

nous aurons alors sans erreur sensible,

$$n_0 = n \left(U'' + \frac{1}{6} \Delta^{(3)} \right).$$

De la même manière on peut évaluer aussi l'intégrale logarithmique,

$$\int_{-1}^{+1} \text{Log} \left(1 + \frac{1}{16} \sin^2 u \right) \cdot dt.$$

4. Les durées d'oscillations de nos pendules ont été réduites à ce qu'elles seraient si la température était constante et égale à + 20° centigrades. Quant au coefficient linéaire de la dilatation des pendules par la chaleur, nous l'avons déduit des expériences sur la durée des oscillations dans les différentes températures, telles que + 20°·5 et + 20°, + 11°·7, + 20°, et + 30° centigrades. Soient N et N' les nombres d'oscillations infiniment petites correspondantes au même intervalle de temps et aux températures + θ et 20° centigrades, α le coefficient linéaire de la dilatation ; on a alors

$$N = N' \sqrt{\frac{1 + \alpha \cdot \theta}{1 + \alpha \cdot 20^\circ}} = N' \left(1 + \frac{1}{2} \alpha (\theta - 20^\circ) - \dots \right).$$

5. On n'a pas des règles bien certaines pour les réductions au niveau de la mer. Si n et n' sont les nombres d'oscillations du pendule pendant le même intervalle de temps au niveau de la mer et à une élévation h audessus de ce niveau, et si R est le rayon du globe terrestre, on peut admettre l'équation

$$n = n' \left(1 + \frac{h}{R} \right).$$

Dans ce cas on fait abstraction de l'attraction qu'exercent les couches terrestres comprises entre la station de l'observateur et le niveau de la mer. En considérant cette attraction, nous admettons, avec POISSON,

$$n = n' \left(1 + \frac{5}{8} \cdot \frac{h}{R} \right).$$

Comme les élévations de nos stations sont pour la plupart très petites, il est indifférent quelle formule serait choisie pour calculer les réductions au niveau de la mer.

6. Notre pendule à réversion est surtout remarquable par rapport aux corrections qui donnent la réduction au vide. La perte que le poids du pendule éprouve dans l'air, et le mouvement d'une certaine masse d'air, entraînée par le pendule, produisent une influence sensible sur la détermination de la longueur du pendule. BESSEL, POISSON, et M. STOKES, ont expliqué les principes de la réduction au vide, que M. BAILY a vérifiés par des nombreuses expériences. On peut voir dans l'ouvrage de M. BESSEL* sur ce sujet, qu'avec un instrument construit comme notre pendule à réversion on peut presque entièrement éliminer les incertitudes dans la réduction au vide.

Supposons que les durées d'oscillation du pendule soient déjà réduites au niveau de la mer, à une température constante et aux excursions infiniment petites ; soit dans de cas —

A, la durée d'oscillation quand le bout lourd du pendule est en-bas :
 a , la distance de l'axe de rotation au centre de gravité du pendule ;
 α , la densité de l'air lors des expériences qui déterminent A.

B, la durée d'oscillation quand le bout léger est en-bas ; b , la distance de l'axe de rotation au centre de gravité du pendule ; β , la densité de l'air pendant les expériences qui donnent B.

P, le poids du pendule, dans le vide ; p , le poids d'un volume d'air égal au volume du pendule ; m , la masse aérienne entraînée en mouvement par le pendule.

$m K, \left(\frac{a+b}{2}\right)^2$, l'augmentation du moment d'inertie du pendule à cause du mouvement de l'air entraîné par l'oscillation du pendule ; K est un coefficient constant pour un pendule ; il ne dépend que de la figure et des dimensions de ce dernier.

τ , la durée d'une oscillation du pendule simple, dont la longueur est $a + b$.

Admettant alors les principes de BESSEL, on parvient facilement à exprimer τ par l'équation suivante :

* *Untersuchungen ueber die Länge des einfachen Secundenpendels*, von F. W. BESSEL, Besonders abgedruckt aus den *Abhandlungen der Akademie zu Berlin für 1826*. Berlin, 1828, pp. 96-99, 126-129.

$$\tau^2 = \frac{1}{2}(A^2 + B^2) + \frac{1}{2} \left(\frac{a+b}{a-b} \right) \cdot \left\{ A^2 - B^2 - \frac{p}{P} (\alpha A^2 - \beta B^2) \right\} \\ - \frac{1}{4} \left(\frac{a+b}{a-b} \right) \cdot \frac{p}{P} \cdot K \cdot \tau^2 \cdot (a - \beta);$$

α et β sont ici exprimés en parties de la densité de l'air, qui servait à déterminer le poids p .

Les expériences qui donnent A et B se suivent les unes après les autres dans le court intervalle d'une heure; ainsi les densités α et β ne sauraient être bien différentes et le dernier terme de l'équation précédente, c. à. d. le terme qui dépend de K , se détruit à très peu près. On peut même éliminer ce terme d'une manière plus satisfaisante: pour cela on n'a qu'à faire les expériences dans cet ordre: 1. déterminer par ex. A ; 2. immédiatement après déterminer B ; et 3. de nouveau déterminer A ; ou, au contraire, chercher d'abord B , puis A , et enfin B . Le tout n'exigera que trois heures du temps ou un peu plus; les changements de la densité de l'air dans cet intervalle sont pour la plupart presque uniformes et proportionnels aux variations du temps. Ainsi la densité de l'air lors de la seconde série d'expériences est à-peu-près égale à la moyenne des densités de l'air dans les première et troisième séries. Donc en combinant B avec le terme moyen de deux déterminations A_1 et A_{11} correspondantes de A , on parvient à un résultat qui fixe τ d'une manière presque indépendante du coefficient inconnu K . En faisant dans ce cas $A = \frac{1}{2}(A_1 + A_{11})$ et en calculant le poids p de l'air pour les états moyens des thermomètres et du baromètre, relatifs aux trois séries d'expériences qui donnent A_1 , B et A_{11} , on trouve

$$\tau^2 = \frac{1}{2}(A^2 + B^2) + \frac{1}{2} \left(\frac{a+b}{a-b} \right) \cdot \left(1 - \frac{p}{P} \right) \cdot (A^2 - B^2).$$

Nos expériences ayant été faites dans l'ordre que nous venons d'expliquer, nous nous sommes servi de la formule précédente pour calculer τ . Comme A était toujours très peu différent de B , on pouvait même admettre sans erreur sensible l'équation approximative

$$\tau = \frac{1}{2}(A + B) + \frac{1}{2} \left(\frac{a+b}{a-b} \right) \left(1 - \frac{p}{P} \right) (A - B).$$

Pour faciliter les mesures des distances a et b , M. BRAUER a gravé sur

la verge cylindrique du pendule, des deux cotés du centre de gravité, quelques traits circulaires, parallèles entre eux et éloignés l'un de l'autre d'un demi-millimètre. A l'aide du cathétomètre on peut déterminer les distances de chaque couteau au trait moyen; un appareil, construit aussi par M. BRAUER, sert à trouver la position du centre de gravité du pendule par rapport au trait moyen.

Le poids du pendule a été obtenu par des pesées immédiates sur une balance; quant au volume du pendule nous avons mesuré les dimensions des diverses parties de nos instruments; leurs volumes se déduisent alors par le calcul.

7. Il nous reste encore à dire quelques mots sur l'influence de la figure des couteaux sur les recherches de la longueur du pendule simple, oscillant de la même manière que le pendule à réversion. Pour éliminer cette influence il est nécessaire de faire les expériences sur la durée des oscillations du pendule autour de chacun de ces deux axes de rotation, de chercher la distance entre les couteaux ainsi que la position du centre de gravité, et ayant fini toutes ces opérations de transposer les couteaux. Pour cela on détache ces derniers, on fixe au bout léger le couteau qui se trouvait auparavant au bout lourd, et on place l'autre couteau au bout lourd. Après cela on répète toutes les opérations qui ont été faites avant la transposition des couteaux.

Supposons qu'avant la transposition on ait trouvé: t , la durée d'une oscillation infiniment petite, réduite à la température + 20° centigrades et au niveau de la mer; a et b , les distances du centre de gravité du pendule au couteau qui est au bout lourd et au couteau qui est au bout léger. Désignons par t' , a' , b' , les mêmes choses trouvées après la transposition des couteaux; par Λ , la longueur du pendule simple à seconde, et par x la correction qui dépend de la figure des couteaux. Les valeurs numériques de Λ et de x se déduisent des équations

$$\begin{aligned}\Lambda &= \frac{a+b}{t^2} + \left(\frac{a+b}{a-b}\right) \cdot \frac{x}{t^2} \\ &= \frac{a'+b'}{t'^2} - \left(\frac{a'+b'}{a'-b'}\right) \cdot \frac{x}{t'^2}.\end{aligned}$$

Profitant de l'assistance bienveillante de M. RIKATSCHEF et de M. GROMADSKY nous avons fait à St. Pétersbourg les recherches nécessaires pour

obtenir les éléments $t, a, b, t', a', b', a + b$ et $a' + b'$ relatifs à chacun de nos pendules.

2. *Résultats des Expériences.*

Ces expériences se rapportent à 12 lieux différents, situés entre $60^{\circ} 51'$ et $42^{\circ} 20'$ de latitude boréale. La longueur du pendule à seconde a été trouvée à St. Pétersbourg égale à 440.958 lignes de Paris, sous la latitude de $59^{\circ} 56' 30''$.

Notre but n'était pas tant de chercher la longueur absolue du pendule que de rassembler des données nouvelles sur les variations de la pesanteur et de les comparer à ce qui a été trouvé dans les autres régions de la surface terrestre. On sait qu'à Londres la longueur du pendule simple à seconde a été déterminée avec une grande précision par M. le Capitaine KATER et par M. le Général SABINE; les mesures exécutées par M. BIOT dans la Grande Bretagne et par les savants anglais en France s'accordent parfaitement, dans le premier cas, avec ce qui a été obtenu par les savants anglais, et dans le second cas avec les résultats de M. BIOT. A ces travaux s'attachent aussi, comme à des points des départs, les observations entreprises par les voyageurs et par les marins dans les différentes parties du monde. Pour introduire nos expériences dans le système de ces recherches, sans solution de continuité, nous préférons à notre détermination directe de la longueur du pendule à St. Pétersbourg celle qui se déduit des oscillations d'un même pendule invariable, observées par M. le Comte LUETKE* à St. Pétersbourg et à l'Observatoire de Greenwich; la différence entre les longueurs du pendule à seconde à Greenwich et à Londres a été exactement fixée par le Général SABINE. Ainsi la longueur du pendule simple à seconde étant connue à Londres, on peut calculer sa longueur à St. Pétersbourg d'après le rapport des carrés des nombres d'oscillations infiniment petites que le pendule de comparaison, réduit au même température et au vide, a fait dans chacun de ces stations en un jour moyen. De cette manière le calcul donné pour la longueur du pendule simple à seconde à St. Pétersbourg 39.16975 pouces anglais ou 441.0319 lignes de Paris. Admettant cette longueur, nous avons déduit de nos expériences les résultats consignés dans le tableau suivant :

* *Mémoires de l'Académie des Sciences de St. Pétersbourg*, tome 3^{me}, 1^{ère} et 2^{de} livraisons. 1836. *Observations du pendule invariable par M. le Contre-Amiral LUETKE*, p. 51.

Lieux des Observations.	Latitude boréale.	Longitude à l'orient de Greenwich.	Longueur du pendule à seconde en lignes de Paris.
Tornea	65 ^o 50' 43"	1 ^h 36 ^m 54 ^s	441'2525
Nicolaïstadt	63 5 33	1 26 26	441'1293
St. Pétersbourg	59 56 30	2 1 14	441'0319
Réval	59 26 37	1 39 1	441'0190
Dorpat	58 22 47	1 46 54	440'9762
Jacobstadt	56 30 3	1 43 4	440'8900
Wilna	54 41 2	1 41 12	440'8353
Bélin	52 2 22	1 40 52	440'7268
Kréménetz	50 6 8	1 42 54	440'6533
Kaménetz-Podolsk	48 4 39	1 46 18	440'5844
Kischinef	47 1 30	1 55 18	440'5278
Ismail	45 20 34	1 55 16	440'4479

Désignons par Λ la longueur du pendule à seconde sous la latitude ϕ ; par z sa longueur à l'équateur et par y un coefficient constant; la théorie de l'attraction donne l'équation

$$\Lambda = z + y \sin^2 \phi.$$

Le tableau précédent fournit 12 valeurs de Λ et autant d'équations de condition, relatives à des stations différentes. Pour calculer z et y nous avons traité ces équations par la méthode des moindres carrés, et nous avons trouvé (en lignes de Paris) —

$$z = 439,2521; y = 2,3779.$$

D'après le théorème de CLAIRAUT l'aplatissement de l'ellipsoïde terrestre est égal à $\frac{5}{2} \cdot \frac{1}{289} - \frac{y}{z}$. Avec nos valeurs de y et de z , cet aplatissement est $\frac{1}{309}$, fraction un peu plus petite que l'aplatissement moyen ($\frac{1}{289}$) conclu de toutes les expériences du pendule, faites dans les différentes contrées. Cette discordance se concilie avec les recherches de M. BIOT; en combinant ses observations du pendule entre Unst et Formentera, ainsi que celles de

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MM. KATER et SABINE entre Paris et Unst, avec les observations de M. SABINE au Spitzberg et à Drontheim en Norvège, l'illustre physicien français trouve un affaiblissement successif du coefficient (γ) du carré du sinus de latitude à mesure que la latitude diminue. En même temps les longueurs du pendule à l'équateur, calculées au moyen de la formule analogue à la précédente, s'augmentent de plus en plus, et par conséquent l'aplatissement correspondant s'accroît. Ces résultats semblent prouver, comme le remarque M. BIOT, que les intensités de la pesanteur sur le continent de l'Europe écartent sensiblement des loix qu'elles devraient suivre sur la surface d'un ellipsoïde de révolution. C'est à cause de cette circonstance que l'on obtient pour l'aplatissement diverses déterminations, selon les contrées où l'on observe. Par beaucoup d'expériences du pendule entre l'équateur et le 45° de latitude, M. BIOT trouve l'aplatissement égal à $\frac{1}{276\frac{2}{3}}$, tandis qu'il n'est que $\frac{1}{306\frac{1}{4}}$ pour la zone entre 45° et 90° de latitude.

Pour examiner la sûreté de nos calculs nous avons comparé la longueur du pendule à chaque lieu particulier à ce qui donne la formule

$$\Delta = 439.2525 + 2.3779 \sin^2 \phi.$$

Nous nommons *écart* la différence : longueur observée—longueur calculée par la formule précédente. Voici les écarts pour nos 12 stations :

Lieux d'observation.	Ecarts.	Lieux d'observation.	Ecarts.
Tornea	+ 0.0200	Wilna	- 0.0001
Nicolaistadt	- 0.0141	Bélin	- 0.0035
St. Pétersbourg	- 0.0017	Kréménetz	+ 0.0017
Réval	+ 0.0033	Kaménetz-Podolsk	+ 0.0160
Dorpat	- 0.0002	Kischinef	+ 0.0030
Jacobstadt	- 0.0157	Ismail	- 0.0071

La somme des écarts positifs est + 0.0440
 La somme des écarts négatifs est - 0.0423

Ainsi la formule s'accorde bien avec les 12 équations de condition.

Quant aux écarts particuliers, ils dépendent des erreurs d'observations et d'anomalies dans les intensités de la pesanteur terrestre ; mais il serait difficile de découvrir dans ces écarts les traces certaines de ces anomalies et des causes locales qui les produisent.

Dans l'ouvrage de M. W. STRUVE sur les mesures de degrés du méridien on peut voir une discussion détaillée des latitudes des points principaux entre le Cap-nord et le Danube. Les différences en latitude, trouvés directement par les observations astronomiques, ne s'écartent des différences en latitude, déduites par le calcul des opérations géodésiques que de $\pm 1''\cdot 75$. Quoique ces écarts surpassent bien décidément les erreurs d'observation, ils ne sont pas aussi grands qu'on les rencontre dans les travaux relatifs à d'autres contrées. Nos stations sont dans le voisinage des points discutés par M. STRUVE : ainsi il paraît que dans les grandes plaines de la Russie occidentale les directions et les intensités de la pesanteur ne sont pas sujettes à des anomalies qui changent sensiblement d'une de nos stations à une autre.

III. *On the Geodesic Lines on an Ellipsoid.*

By Prof. CAYLEY.

Read January 13, 1871.

THE fundamental equations, in regard to the geodesic lines on an ellipsoid, were established by JACOBI, viz., representing by a, b, c , the squares of the semiaxes, that is, taking the ellipsoid to be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$$

(where $a > b > c$), if we introduce the elliptic co-ordinates h, k , and write

$$\frac{x^2}{a+h} + \frac{y^2}{b+h} + \frac{z^2}{c+h} = 1,$$

$$\frac{x^2}{a+k} + \frac{y^2}{b+k} + \frac{z^2}{c+k} = 1,$$

or, what is the same thing,

$$x^2 = \frac{a(a+h)(a+k)}{(a-b)(a-c)},$$

$$y^2 = \frac{b(b+h)(b+k)}{(b-c)(b-a)},$$

$$z^2 = \frac{c(c+h)(c+k)}{(c-a)(c-b)};$$

then, if β be an arbitrary constant, the differential equation of a geodesic line is

$$(1). \text{ const.} = \int dh \sqrt{\frac{h}{(a+h)(b+h)(c+h)(\beta+h)}} + \int dk \sqrt{\frac{h}{(a+k)(b+k)(c+k)(\beta+k)}}$$

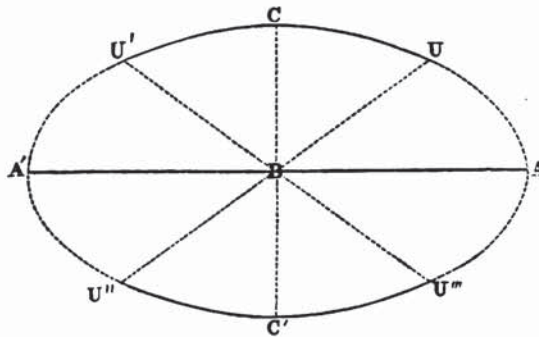
and the expression for the length of any arc of the curve is given by

$$(2). \quad s = \int d h \sqrt{\frac{h(\beta + h)}{(a + h)(b + h)(c + h)}} + \int d k \sqrt{\frac{k(\beta + k)}{(a + k)(b + k)(c + k)}}.$$

I propose in the present Memoir to develop the theory to the extent of showing how we can, by means of the first of these equations, explain the course of the geodesic lines; and for given numerical values of a, b, c , calculate, construct, and exhibit in a drawing the course of these lines: I attend more particularly to the series of geodesic lines through an umbilicus (which lines pass also through the opposite umbilicus), and to the case where the semiaxes are connected by the equation $ac - b^2 = 0$, a relation which simplifies the formulæ.

General Considerations as to the Course of the Lines.

1. It will be observed that h and k enter into the formulæ symmetrically: it will be convenient to distinguish between these co-ordinates by considering h as extending between the values $-a, -b$; and k as extending between the values $-b, -c$. Thus:—



$h = \text{const.}$ denotes a curve of curvature of the one kind, viz.—

$h = -a$, the principal section $A B A'$ (or major-mean section), $h = -b$, the curves $U U'$ and $U'' U'''$ (or portions of the umbilical section $A C A' C'$); similarly,

$k = \text{const.}$ denotes a curve

of curvature of the other kind, viz:—

$k = -c$, the principal section $C B C'$ (or minor-mean section), $k = -b$, the curves $U U'''$ and $U' U''$ (remaining portions of the umbilical section $A C A' C'$).

2. To any given (admissible) values of h, k , there correspond eight points, situate in the eight octants of the surface respectively; but, unless the contrary is expressed, it is assumed that the co-ordinates x, y, z , are positive, and that the point is situate in the octant ABC.

3. The constant β may have any value from $+a$ to $+c$; viz., if it has a value between a and b , or say, if $-\beta$ has an h -value, then the geodesic lines wholly between the two ovals of the curve of curvature $h = -\beta$ (being in general an indefinite undulating curve touching each oval an indefinite number of times). Similarly, if β has any value between b and c , or say, if $-\beta$ has a k -value, then the geodesic line lies wholly between the two ovals of the curve of curvature $k = -\beta$ (being in general an indefinite undulating curve touching each oval an indefinite number of times). The intermediate case is when $\beta = b$, or say when $-\beta$ has the umbilicar value: here the geodesic line is in general an indefinite undulating curve passing an infinite number of times through the opposite umbilici U, U'' , or U', U''' ; to fix the ideas, say through U, U'' .

Lines through an Umbilicus:

4. I attend in particular to the last-mentioned case, and thus write $\beta = b$. We may in the formula (1) fix at pleasure a limit of each integral; and writing for convenience

$$\Pi(h) = \int_{-a}^h \frac{-dh}{b+h} \sqrt{\frac{k}{(a+h)(c+h)}},$$

$$\Psi(k) = \int_k^{-c} \frac{dk}{b+k} \sqrt{\frac{k}{(a+k)(c+k)}},$$

the equation (1) becomes

$$\text{Const.} = \Pi(h) + \Psi(k).$$

5. It is to be observed, in regard to these integrals, that writing $h = -a + u$, we have

$$\Pi(h) = \int_0^u \frac{du}{a-b-u} \sqrt{\frac{a-u}{u(a-c-u)}},$$

which, for u small, is

$$= \frac{1}{a-b} \sqrt{\frac{a}{a-c}} \int_0^u \frac{du}{\sqrt{u}}, = \frac{2\sqrt{u}}{a-b} \sqrt{\frac{a}{a-c}}.$$

By the assistance of this formula the value of the integral may be calculated by quadratures; viz., the formula gives the integral for any small value of u , and we can then proceed by the method of quadratures. The integral becomes infinite for $h = -b$: suppose that we have by quadratures calculated it up to $h = -b - m$ (m small) then to calculate it up to any value $-b - m + u$ nearer to $-b$, we have

$$\begin{aligned} \Pi(h) &= \Pi(-b-m) + \int_0^u \frac{du}{m-u} \sqrt{\frac{b+m-u}{(a-b-m+u)(b-c+m-u)}} \\ &= \Pi(-b-m) + \sqrt{\frac{b}{(a-b)(b-c)}} \int_0^u \frac{du}{m-u} \\ &= \Pi(-b-m) - \sqrt{\frac{b}{(a-b)(b-c)}} \log\left(1 - \frac{u}{m}\right), \end{aligned}$$

where the second term is positive, and the value thus increases slowly with u , becoming as it should do $= \infty$ for $u = m$ or $h = -b$.

6. Similarly in the second integral writing $k = -c - v$, we have

$$\Psi(k) = \int_0^v \frac{dv}{b-c-v} \sqrt{\frac{c+v}{(a-c-v)v}},$$

* Except when the contrary is stated, the symbol "log" denotes throughout the hyperbolic logarithm.

which, for v small, is

$$= \frac{1}{b-c} \sqrt{\frac{c}{a-c}} \int \frac{dv}{\sqrt{v}}, = \frac{2\sqrt{v}}{b-c} \sqrt{\frac{c}{a-c}},$$

which is of the like assistance in regard to the calculation by quadratures. And if we have by quadratures calculated the integral up to $h = -b + n$ (n small), then, to calculate it up to any value $-b + n - v$ nearer to $-b$, we have

$$\begin{aligned} \Psi(k) &= \Psi(-b+n) + \int_0^v \frac{dv}{n-v} \sqrt{\frac{b-n+v}{(a-b+n-v)(b-c-n+v)}} \\ &= \Psi(-b+n) + \sqrt{\frac{b}{(a-b)(b-c)}} \int_0^v \frac{dv}{n-v} \\ &= \Psi(-b+n) - \sqrt{\frac{b}{(a-b)(b-c)}} \log\left(1 - \frac{v}{n}\right), \end{aligned}$$

where the second term is positive, and the value thus increases slowly with v , becoming as it should do $= \infty$ for $v = n$, or $k = -b$.

7. It may be remarked that in $\Pi(h)$ and $\Psi(k)$ respectively the coefficient of the logarithmic term has in each case the same value $= \sqrt{\frac{b}{(a-b)(b-c)}}$. As regards the initial terms \sqrt{u} and \sqrt{v} , the coefficients are $\frac{1}{a-b} \sqrt{\frac{a}{a-c}}$ and $\frac{1}{b-c} \sqrt{\frac{c}{a-c}}$ respectively, which are equal if $\frac{a}{(a-b)^2} = \frac{c}{(b-c)^2}$, or $ac - b^2 = 0$.

8. We may consider the two geodesic lines $\Pi(h) \pm \Psi(k) = \text{const.}$; suppose that these each of them pass through the point P, co-ordinates (h_0, k_0) in the ABC octant of the ellipsoid; then for one of them we have $\Pi(h) - \Psi(k) = \Pi(h_0) - \Psi(k_0)$, and for the other of them we have $\Pi(h) + \Psi(k) = \Pi(h_0) + \Psi(k_0)$: I attend first to the former of these, say $\Pi(h) - \Psi(k) = C$ (where C is $= \Pi(h_0) - \Psi(k_0)$); and I say that this denotes the curve UPU". In fact, by reason of the equation $\Pi(h)$ and

$\Psi(k)$ must both increase or both diminish; they both increase as h passes from h_0 to $-b$, and as k passes from k_0 to $-b$: we may have $h = -b + u$, $k = -b + v$ where u and v are both indefinitely small, the functions Π and Ψ being then indefinitely large, but $\Pi - \Psi = C$; and we have thus a series of points nearer and nearer to the umbilicus U ; that is, we have the portion PU of the curve. Tracing the curve in the opposite direction, or considering h as passing from h_0 to $-a$, and k as passing from k_0 to $-c$, then if C be positive, k will attain the value $-c$, before h attains the value $-a$, say that we have simultaneously $h = h_1$, $k = -c$; the equation is $\Pi(h_1) - \Psi(-c) = C$, that is, $\Pi(h_1) = C$; and the geodesic line then arrives at a point P , on the arc CB of the minor-mean principal section. The function Ψ then changes its sign, viz., considering it as always positive, the equation is now $\Pi(h) + \Psi(k) = C$, k passing from the value $-c$ towards $-b$, that is, $\Psi(k)$ increasing, and therefore $\Pi(h)$ diminishing, or h passing from h_1 towards the value $-a$; until at last, say for $k = k_2$, we have $h = -a$, that is, $C = \Pi(-a) + \Psi(k_2)$, or $C = \Psi(k_2)$; the geodesic line here arrives at a point P_1 on the arc BA' of the major mean principal section. The function Π then changes its sign, viz., Π denoting a positive function as before, the equation is $-\Pi(h) + \Psi(k) = C$; h passes from $-a$ towards $-b$, that is $\Pi(h)$ increases, and therefore $\Psi(k)$ must also increase, or k pass from k_2 towards $-b$: we have at length $h = -b - u$, $k = -b + v$, u and v being each indefinitely small; and therefore Π and Ψ each indefinitely large (but $-\Pi + \Psi = C$); that is, we arrive at the umbilicus U'' , completing the geodesic line UPU'' .

9. If instead of $C = +$ we have $C = -$, everything is similar, but the geodesic line proceeding from U in the direction UP will first cut the arc BA of the major mean section at a point P_1 ; then the arc BC' of the minor mean section at a point P_2 ; and, finally, arrive as before at the umbilicus U''' .

10. The intermediate case is when $C = 0$, viz., we have here $\Pi(h) - \Psi(k) = 0$; the geodesic line here passes from U in the direction UP to B (extremity of the mean axis, $h = -a$, $k = -c$); Π and Ψ then each change their sign, so that, considering them as positive, the equation still is $\Pi(h) - \Psi(k) = 0$, and the geodesic line at last arrives at the

umbilicus U'' . It will be easily understood how in the like manner $\Pi(h) + \Psi(k) = C$ refers to the line $U' P U''$.

11. Reverting to the equation $\Pi(h) - \Psi(k) = C$, or as I will now write it

$$\Pi(h) - \Psi(k) = \Pi(h_0) - \Psi(k_0),$$

which belongs to the portion UP of the geodesic line $U P U''$, we require when h is $-b-u$, and $k = -b+v$ (u and v indefinitely small) to know the ratio of the increments u, v ; this in fact serves to determine the direction at U of the geodesic line through the given point (h_0, k_0) .

12. For this purpose writing $h = -b - u$, we find

$$\Pi(h) = \int_u^{a-b} \frac{du}{u} \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}},$$

which is

$$\begin{aligned} &= \int_u^{a-b} \frac{du}{u} \left\{ \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\} \\ &\quad + \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{a-b}{u}, \end{aligned}$$

and, when u is indefinitely small, this is

$$\Pi(h) = \int_0^{a-b} \frac{du}{u} \left\{ \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\} + \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{a-b}{u}.$$

Similarly, when $k = -b + v$, where v is indefinitely small

$$\Psi(k) = \int_0^{b-c} \frac{dv}{v} \left\{ \sqrt{\frac{b-v}{(a-b+v)(b-c-v)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\} + \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{b-c}{v}.$$

13. Each of the integrals is of the dimension $-\frac{1}{2}$ in a, b, c , and the difference of the integrals may be represented by

$$M \sqrt{\frac{b}{(a-b)(b-c)}};$$

we have therefore

$$\Pi(h) - \Psi(h) = \sqrt{\frac{b}{(a-b)(b-c)}} \left\{ M + \log \frac{a-b}{b-c} \frac{v}{u} \right\},$$

where

$$\begin{aligned} \sqrt{\frac{b}{(a-b)(b-c)}} M = & \int_0^{a-b} \frac{du}{u} \left\{ \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\} \\ & - \int_0^{b-c} \frac{dv}{v} \left\{ \sqrt{\frac{b-v}{(a-b+v)(b-c-v)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\}. \end{aligned}$$

14. Suppose the inferior limits replaced by the indefinitely small positive quantities $\varepsilon, \varepsilon'$ respectively; and for the variable in the second integral write $-u$; then

$$M = \int_{-(b-c)}^{a-b} \left\{ \frac{du}{u} \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\},$$

it being understood that the values $u = -\varepsilon'$ to $u = +\varepsilon$ are omitted from the integration: this is

$$= \int_{-(b-c)}^{a-b} \frac{du}{u} \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{a-b}{\varepsilon} \frac{\varepsilon'}{b-c}$$

with the same convention as to the integral; or if $\varepsilon' = \varepsilon$, then

$$M = M' - \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{a-b}{b-c},$$

where

$$\sqrt{\frac{b}{(a-b)(b-c)}} M' = \int_{-(b-c)}^{a-b} \frac{du}{u} \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} = \int_{-a}^{-c} \frac{dh}{b+h} \sqrt{\frac{h}{(a+h)(c+h)}},$$

the omitted elements being from $u = -\epsilon$ to $u = +\epsilon$; that is (in the language of CAUCHY) we take for the integral its *principal* value. And hence

$$\Pi(h) - \Psi(k) = \sqrt{\frac{b}{(a-b)(b-c)}} \left\{ M' + \log \frac{v}{u} \right\}.$$

15. By what precedes this is $= \Pi(h_0) - \Psi(k_0)$; or if we write simply (h, k) instead of (h_0, k_0) , that is, consider the geodesic line UP, which is drawn from the point P, co-ordinates (h, k) , to the umbilicus U, the co-ordinates of a point consecutive to the umbilicus are $-b-u, -b+v$, where u, v are connected by the last-mentioned equation, in which M' is a transcendental function depending on (a, b, c) but independent of the particular geodesic line.

16. If for the geodesic line through the point B, or say for the B-geodesic $\frac{v}{u} = \frac{v_0}{u_0}$, then $M' = -\log \frac{v_0}{u_0}$, and we have in general

$$\Pi(h) - \Psi(k) = \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{vu_0}{uv_0},$$

a result which I proceed to further transform as follows:

If x_0, y_0, z_0 refer to the umbilicus U, then considering first the consecutive point P on the geodesic line (co-ordinates $-b-u, -b+v$) and next the consecutive point Q on the umbilicar section, we have for these two points respectively,—

$$dx_0 = \frac{\frac{1}{2} \sqrt{a}(v-u)}{\sqrt{(a-b)(a-c)}},$$

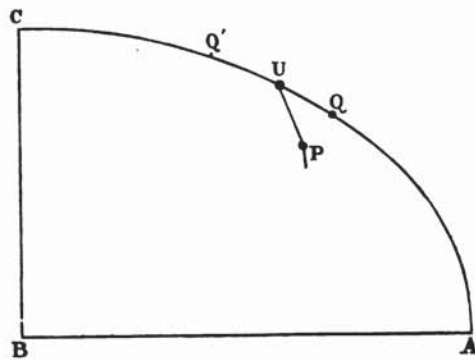
$$dy_0 = \frac{\sqrt{b} \sqrt{uv}}{\sqrt{(a-b)(b-c)}},$$

$$dz_0 = \frac{\frac{1}{2} \sqrt{c}(u-v)}{\sqrt{(b-c)(a-c)}},$$

$$\delta x_0 = \frac{\frac{1}{2} \sqrt{a}}{\sqrt{(a-b)(a-c)}},$$

$$\delta y_0 = c,$$

$$\delta z_0 = \frac{-\frac{1}{2} \sqrt{c}}{\sqrt{(b-c)(a-c)}},$$



say these are α, β, γ , and α', β', γ' ; and then

$$\begin{aligned} \alpha\alpha' + \beta\beta' + \gamma\gamma' &= \frac{1}{4} \left\{ \frac{a}{(a-b)(a-c)} + \frac{c}{(b-c)(a-c)} \right\} (v-u) = \frac{\frac{1}{4}(v-u)}{a-c} \left\{ \frac{a}{a-b} + \frac{c}{b-c} \right\} \\ &= \frac{\frac{1}{4}b(v-u)}{(a-b)(b-c)}, \end{aligned}$$

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= \left\{ \frac{a}{(a-b)(a-c)} + \frac{c}{(b-c)(a-c)} \right\} \cdot \frac{1}{4} (u-v)^2 + \frac{buv}{(a-b)(b-c)} \\ &= \frac{b}{(a-b)(b-c)} \cdot \frac{1}{4} (u+v)^2, \end{aligned}$$

$$\alpha'^2 + \beta'^2 + \gamma'^2 = \frac{b}{(a-b)(b-c)} \cdot \frac{1}{4},$$

whence

$$\sqrt{\alpha^2 + \beta^2 + \gamma^2} \sqrt{\alpha'^2 + \beta'^2 + \gamma'^2} = \frac{\frac{1}{4}(u+v)b}{(a-b)(b-c)},$$

and hence

$$\cos \phi = \frac{\alpha\alpha' + \beta\beta' + \gamma\gamma'}{\sqrt{\alpha^2 + \beta^2 + \gamma^2} \sqrt{\alpha'^2 + \beta'^2 + \gamma'^2}} = \frac{v-u}{v+u}. \text{ that is, } \cos(180^\circ - \phi) = \frac{u-v}{u+v},$$

or

$$\tan^2 \frac{1}{2} \phi = u,$$

where if U is the umbilicus, P the consecutive point $-b-u, -b+v$, and UQ the element of the umbilical principal section, $\phi = \angle P U Q$, $180^\circ - \phi = \angle P U Q'$. For the B-geodesic we have

$$2 \log \tan \frac{1}{2} \phi_0 = \log \frac{u_0}{v_0} = M'.$$

17. The foregoing equation for $\Pi(h) - \Psi(k)$ now becomes

$$\Pi(h) - \Psi(k) = \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{\tan^2 \frac{1}{2} \phi_0}{\tan^2 \frac{1}{2} \phi}.$$

viz., ϕ_0 is the south azimuth of the B-geodesic at the umbilicus, a mere

function of (a, b, c) and ϕ is the south azimuth at the umbilicus, of the geodesic line under consideration, so that we may consider the geodesic line to be determined by the south azimuth ϕ as its parameter.

Formulæ for the case $ac - b^2 = 0$.

18. I annex the following investigation in regard to the case $ac - b^2 = 0$.

We have in general

$$\begin{aligned} & \frac{1}{\sqrt{b(a-b)(b-c)}} \frac{d}{dx} \log \frac{\sqrt{-x(a-b)(b-c)} + \sqrt{-b(a+x)(c+x)}}{\sqrt{-x(a-b)(b-c)} + \sqrt{-b(a+x)(c+x)}} \\ &= -\frac{1}{b} \frac{1}{\sqrt{x(a+x)(c+x)}} \\ & \quad + \frac{1}{b} \frac{1}{b+x} \sqrt{\frac{x}{(a+x)(c+x)}} \\ & \quad + \frac{1}{bx+ac} \sqrt{\frac{x}{(a+x)c+x}}. \end{aligned}$$

In fact, denoting the logarithm by $\log \frac{P+Q}{P-Q}$, we have

$$\frac{d}{dx} \log \frac{P+Q}{P-Q} = \frac{2(PQ' - P'Q)}{P^2 - Q^2}$$

where

$$2(PQ' - P'Q) = 2PQ \left(\frac{Q'}{Q} - \frac{P'}{P} \right) = \sqrt{x(a+x)(c+x)} b(a-b)(b-c) \left\{ \frac{1}{a+x} + \frac{1}{c+x} - \frac{1}{x} \right\}$$

$$= \frac{\sqrt{b(a-b)(b-c)}}{\sqrt{x(a+x)(c+x)}} (x^2 - ab);$$

$$P^2 - Q^2 = -x(a-b)(b-c) + b(a+x)(c+x)$$

$$= (bx+ac)(b+x);$$

that is

$$\begin{aligned} \frac{2(P'Q' - P'Q)}{P^2 - Q^2} &= - \frac{\sqrt{b(a-b)(b-c)}}{\sqrt{x(a+x)(c+x)}} \frac{x^2 - ac}{(bx+ac)(b+x)} \\ &= \frac{\sqrt{b(a-b)(b-c)}}{\sqrt{x(a+x)(c+x)}} \left\{ -\frac{1}{b} + \frac{x}{b(b+x)} + \frac{x}{bx+ac} \right\}, \end{aligned}$$

which proves the theorem.

19. Hence in the particular case $ac = b^2$ we have

$$\begin{aligned} &\frac{1}{\sqrt{b(a-b)(b-c)}} \log \frac{\sqrt{-h(a-b)(b-c)} + \sqrt{-b(a+h)(c+h)}}{\sqrt{-h(a-b)(b-c)} - \sqrt{-b(a+h)(c+h)}} \\ &= -\frac{1}{b} \int_{-a}^h \frac{dh}{\sqrt{h(a+h)(c+h)}} \\ &\quad - \frac{2}{b} \int_{-a}^h \frac{dh}{b+h} \sqrt{\frac{h}{(a+h)(c+h)}} \quad \left(= +\frac{2}{b} \Pi(h) \right), \end{aligned}$$

that is

$$\Pi(h) = \frac{1}{2} \int_{-a}^h \frac{dh}{\sqrt{h(a+h)(c+h)}} + \frac{1}{2} \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{\sqrt{-h(a-b)(b-c)} + \sqrt{-b(a+h)(c+h)}}{\sqrt{-h(a-b)(b-c)} - \sqrt{-b(a+h)(c+h)}},$$

or say

$$= \frac{1}{2} \int_{-a}^h \frac{dh}{\sqrt{h(a+h)(c+h)}} + \frac{1}{2} \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{1+H}{1-H},$$

where

$$H^2 = \frac{b}{(a-b)(b-c)} \frac{(a+h)(c+h)}{h},$$

viz. we see that $\Pi(h)$ depends on the more simple integral

$$\int_{-a}^h \frac{dh}{\sqrt{h(a+h)(c+h)}}.$$

20. Similarly

$$\begin{aligned} & \frac{1}{\sqrt{b(a-b)(b-c)}} \log \frac{\sqrt{-k(a-b)(b-c)} + \sqrt{-b(a+k)(c+k)}}{\sqrt{-k(a-b)(b-c)} - \sqrt{-b(a+k)(c+k)}} \\ &= \frac{1}{b} \int_k^{-c} \frac{dk}{\sqrt{k(a+k)(c+k)}} \\ & \quad + \frac{2}{b} \int_k^{-c} \frac{dk}{b+k} \sqrt{\frac{b}{(a+k)(c+k)}} \left(= + \frac{2}{b} \Psi(k) \right), \end{aligned}$$

that is

$$\Psi(k) = -\frac{1}{2} \int_k^{-c} \frac{dk}{\sqrt{k(a+k)(c+k)}} + \frac{1}{2} \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{\sqrt{-k(a-b)(b-c)} + \sqrt{-b(a+k)(c+k)}}{\sqrt{-k(a-b)(b-c)} - \sqrt{-b(b+k)(c+k)}}$$

or say

$$\Psi(k) = -\frac{1}{2} \int_k^{-c} \frac{dk}{\sqrt{k(a+k)(c+k)}} + \frac{1}{2} \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{1+K}{1-K},$$

where

$$K^2 = \frac{b}{(a-b)(b-c)} \frac{(a+k)(c+k)}{k},$$

that is, $\Psi(k)$ depends on the more simple integral,

$$\int_k^{-c} \frac{dk}{\sqrt{k(a+k)(c+k)}}.$$

Write $h = -b - u$, $k = -b + v$, where u and v are indefinitely small, then

$$\Pi(h) - \Psi(k) = \frac{1}{2} \int_{-a}^{-c} \frac{dh}{\sqrt{h(a+h)(c+h)}} + \frac{1}{2} \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{1+U}{1-U} \frac{1-V}{1+V},$$

where

$$U^2 = \frac{\left(1 - \frac{u}{a-b}\right) \left(1 + \frac{u}{b-c}\right)}{1 + \frac{u}{b}} = 1 - \frac{b u^2}{(a-b)(b-c)(b+u)} \text{ (attending to } ac = b^2\text{),}$$

and

$$V^2 = \frac{\left(1 + \frac{v}{a-b}\right) \left(1 - \frac{v}{b-c}\right)}{1 - \frac{v}{b}} = 1 - \frac{b v^2}{(a-b)(b-c)(b-v)},$$

$$\Pi(h) - \Psi(k) = \frac{1}{2} \int_{-a}^{-c} \frac{dh}{\sqrt{h(a+h)(c+h)}} + \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{v}{u}.$$

21. Comparing with the result obtained for the general case the two agree, if only

$$\int_{-a}^{-c} \frac{dh}{b+h} \sqrt{\frac{h}{(a+h)(c+h)}} = \frac{1}{2} \int_{-a}^{-c} \frac{dh}{\sqrt{h(a+h)(c+h)}},$$

where on the left-hand side the integral has its principal value: a result which must therefore hold good when $ac = b^2$.

Calculation of the Umbilicar Geodesics for Ellipsoid $a:b:c = 4:2:1$.

22. As a specimen of the way in which we may, on a given ellipsoid, calculate the course of a geodesic line, I take the semiaxes to be as $2:\sqrt{2}:1$, or, for convenience, $a = 1000$, $b = 500$, $c = 250$; and, considering the geodesic lines through the umbilicus, I calculate by quadratures the functions

$$\Pi(h) = 100,000 \int_{-1000}^h \frac{-dh}{500+h} \sqrt{\frac{h}{(1000+h)(250+h)}},$$

$$\Pi(k) = 100,000 \int_k^{-250} \frac{dk}{500+k} \sqrt{\frac{k}{(1000+k)(250+k)}}.$$

The results do not pretend to minute accuracy: I have not attempted to estimate or correct for any error occasioned by the intervals (10 units) being too large; and there may possibly be accidental errors.

TABLE I.

$-h =$	Π'	$\Pi (h)$	$-h$	Π'	$\Pi (h)$	$-h$	Π'	$\Pi (h)$
1000	∞	0	840	27.6	6746	630	51.5	13972
999	231.4	462	830	27.8	7023	620	54.1	14499
998	164.0	659	820	27.9	7301	610	59.2	15066
997	134.2	809	810	28.1	7582	600	65.5	15689
996	116.5	934	800	28.4	7865	590	72.1	16377
995	104.4	1044	790	28.7	8151	580	80.9	17142
990	74.6	1492	780	29.2	8440	570	93.0	18011
980	54.0	2135	770	29.7	8735	560	106.8	19010
970	45.1	2630	760	30.3	9035	550	127.6	20183
960	40.1	3056	750	31.0	9341	540	159.1	21616
950	36.6	3439	740	31.8	9655	530	211.5	23469
940	34.2	3794	730	32.6	9977	520	316.7	26111
930	32.5	4127	720	33.6	10308	510	632.7	30858
920	31.2	4446	710	34.7	10650	505	1265.0	35602
910	30.2	4753	700	36.0	11004	504	1581.2	37014
900	29.4	5051	690	37.4	11371	503	2107.7	38834
890	28.8	5342	680	39.0	11754	502	3162.3	41398
880	28.4	5628	670	40.8	12153	501	6324.1	45792
870	28.1	5911	660	42.0	12567	500	∞	∞
860	27.9	6190	650	45.4	13005			
850	27.8	6469	640	48.2	13473			

TABLE II.

$-k$	Ψ'	$\Psi(k)$	$-k$	Ψ'	$\Psi(k)$	$-k$	Ψ'	$\Psi(k)$
250	∞	0	320	45.5	4655	440	107.1	12207
251	232.2	462	330	46.1	5114	450	127.9	13383
252	165.5	661	340	47.3	5581	460	159.2	14818
253	136.0	811	350	48.9	6062	470	211.6	16673
254	118.6	939	360	51.1	6562	480	316.7	19314
255	106.8	1051	370	53.8	7086	490	632.7	24062
260	78.1	1514	380	57.2	7641	495	1265.0	28806
270	60.5	2207	390	61.4	8235	496	1581.2	30218
280	51.7	2768	400	66.7	8875	497	2108.1	32037
290	48.2	3268	410	73.2	9575	498	3162.3	34602
300	46.3	3741	420	81.6	10349	499	6234.1	38995
310	45.5	4200	430	91.4	11214	500	∞	∞

23. But it is obviously convenient to revert these Tables so as to have for the common arguments a series of uniformly increasing values of Π or Ψ , viz., we obtain by interpolation the values of h and k belonging to the given values of Π or Ψ , and thus obtain the following Table. Here, in any line of the Table the values of h , k , are such that $\Pi(h) - \Psi(k) = 0$, viz., the values in question belong to successive points of the B-geodesic. And to obtain the values for any other geodesic line $\Pi(h) - \Psi(k) = \pm 500 m$, we have only to take each value of k from the line m lines above or below the line from which h is taken; and similarly the table gives at once the values belonging to a geodesic line $\Pi(h) + \Psi(k) = 500 m$.

TABLE III.

$\Pi = \Psi =$	h	D.	k	D.	$\Pi = \Psi =$	h	D.	k	D.
0	1000		250		13000	650.1		446.7	
500	998.8	1.2	251.4	1.4	14000	629.5	20.6	454.5	7.8
1000	995.4	3.4	254.5	3.1	15000	611.2	18.3	461.0	6.5
1500	989.9	5.5	259.7	5.2	16000	595.5	15.7	466.3	5.3
2000	982.1	7.8	267.0	7.3	17000	581.9	13.6	471.2	4.9
2500	972.6	9.5	275.2	8.2	18000	570.1	11.8	475.0	3.8
3000	961.1	11.5	284.6	9.4	19000	560.1	10.0	478.8	3.8
3500	948.3	12.8	294.9	10.3	20000	551.6	8.5	481.4	2.6
4000	933.8	14.5	305.6	10.7	21000	544.3	7.3	483.6	2.2
4500	918.2	15.6	316.6	11.0	22000	538.1	6.2	485.6	2.0
5000	901.7	16.5	327.5	10.9	23000	533.2	4.9	487.8	2.2
5500	884.5	17.2	338.3	10.8	24000	528.6	4.6	487.8	2.1
6000	866.8	17.7	348.7	10.4	26000	520.5	8.1	489.9	2.1
6500	848.9	17.9	358.8	10.1	28000	516.0	4.5	494.1	2.1
7000	830.8	18.1	368.3	9.5	30000	511.8	4.2	495.8	1.7
7500	812.9	17.9	377.5	9.2	32000	508.8	3.0	497.0	1.2
8000	795.3	17.6	386.1	8.6	34000	506.7	2.1	497.8	0.8
8500	778.0	17.3	394.1	8.0	36000	504.7	2.0	498.3	0.5
9000	761.2	16.8	401.8	7.7	38000	503.5	1.2	498.8	0.5
9500	744.9	16.3	408.9	7.1	39000	..	1.0	499.0	
10000	729.3	15.6	415.5	6.6	40000	502.5			
10500	714.4	14.9	421.7	6.2	42000	501.9	0.6		
11000	700.1	14.3	427.6	5.9	44000	501.4	0.5		
11500	686.6	13.5	432.9	5.3	45800	501.0	0.4		
12000	673.8	12.8	437.9	5.0	∞	500		500	
		23.7		8.8					

Graphical Construction: Projection on the Umbilicar Plane.

24. The most convenient mode of delineation of the geodesic lines is obtained by projecting them orthogonally on the umbilicar plane: the contour of the figure is here the umbilicar section, or ellipse $\frac{x^2}{a} + \frac{z^2}{c} = 1$; and the curves of curvature of each series are projected into elliptic arcs lying within the ellipse in question, the one set cutting at right angles the axes AA' , the other cutting at right angles the axes CC' ; the equations of the complete ellipses being

$$x^2 \frac{a-b}{a(a+h)} + z^2 \frac{c-b}{c(c+h)} - 1 = 0$$

and

$$x^2 \frac{(a-b)}{a(a+h)} + z^2 \frac{c-b}{c(c+h)} - 1 = 0.$$

25. I constructed, by means of the table, a drawing of this kind for the ellipsoid $a, b, c = 1000, 500, 250$, the lengths \sqrt{a} and \sqrt{c} being taken to be 12 inches and 6 inches respectively: the process consists in taking from the table for a series of values $\Pi = \Psi$ (say $\Pi = \Psi = 1000, = 2000$ &c.), the values of h and k , laying down for such values the elliptic arcs which represent the two curves of curvature respectively, thus dividing the bounding ellipse into a series of curvilinear rectangles, and then obtaining the geodesic lines by drawing the diagonals of these rectangles, and of course rounding of the corners so as to form continuous curves. The Plate shows on a reduced scale so much of the drawing as is comprised within a quadrant of the bounding ellipse (viz. it is a representation of an octant of the ellipsoid).

Elliptic-Function Formulæ.

26. I have in all that precedes abstained from the use of elliptic functions, since obviously the form $\sqrt{1-k^2 \sin^2 \phi}$ of the radical of an elliptic function is in nowise specially appropriate to the present question.

But (more particularly in the above-mentioned case $ac - b^2 = 0$, where the radical is $\sqrt{h(a+h)(c+h)}$ without any exterior factor $b+h$ in the denominator) the formulæ are expressible easily and elegantly by elliptic functions, and it is desirable to make the transformation. Reverting to the formulæ which, in the case in question (viz. when $ac - b^2 = 0$), give the values of $\Pi(h)$ and $\Psi(k)$; and writing therein $h = -a + (a-c) \sin^2 \varphi$, $k = -a + (a-c) \sin^2 \psi$, also

$$\kappa = \sqrt{1 - \frac{c}{a}}, \text{ or } \frac{c}{a} = 1 - \kappa^2, = \kappa'^2$$

we have

$$\int_{-a}^h \frac{dh}{\sqrt{h(a+h)(c+h)}} = 2 \int_0^\varphi \frac{d\varphi}{\sqrt{a - (a-c) \sin^2 \varphi}} = \frac{2}{\sqrt{a}} F(\kappa, \varphi)$$

$$\int_k^{-c} \frac{dk}{\sqrt{k(a+k)(c+k)}} = 2 \int_\psi^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{a - (a-c) \sin^2 \psi}} = \frac{2}{\sqrt{a}} \{F(\kappa) - F(\kappa, \psi)\},$$

27. Hence

$$\Pi(h) = \frac{1}{\sqrt{a}} F(\kappa, \varphi) + \frac{1}{2\sqrt{1-\kappa^2}} \log \frac{1+H}{1-H},$$

where

$$H = \frac{(1-\kappa') \sin \varphi \cos \varphi}{\sqrt{1-\kappa^2 \sin^2 \varphi}}.$$

(observe, as h passes from $-a$ to $-b$, φ passes from $\varphi=0$ to $\sin^2 \varphi = \frac{1}{1+\kappa}$ and H from $H=0$ to $H=1$.)

Similarly

$$\Psi(k) = \frac{1}{\sqrt{a}} \{F(\kappa) - (F(\kappa, \psi))\} + \frac{1}{2\sqrt{a(1-\kappa')}} \log \frac{1+K}{1-K},$$

where

$$k = \frac{(1+\kappa') \sin \psi \cos \psi}{\sqrt{1-\kappa^2 \sin^2 \psi}},$$

and as k passes from $-c$ to $-b$, ψ passes from $\frac{\pi}{2}$ to $\sin^2 \psi = \frac{1}{1+\kappa}$, and K from 0 to 1 .

28. The before-mentioned identical equation

$$\int_{-a}^{-c} \frac{dh}{b+h} \sqrt{\frac{h}{(a+h)(c+h)}} = \frac{1}{2} \int_{-a}^{-c} \frac{dh}{\sqrt{h(a+h)(c+h)}}$$

is by the same transformation converted into

$$\int_0^{\frac{\pi}{2}} \frac{1 - (1 - \kappa') \sin^2 \phi}{1 - (1 + \kappa') \sin^2 \phi} \frac{d\phi}{\sqrt{1 - \kappa'^2 \sin^2 \phi}} = 0;$$

To prove this, remark that the equation is

$$0 = \int_0^{\frac{\pi}{2}} d\phi \frac{\frac{1 - \kappa'}{1 + \kappa'} (1 - 1 + \kappa' \sin^2 \phi) + \frac{2\kappa'}{1 + \kappa'}}{1 - (1 + \kappa') \sin^2 \phi} \frac{1}{\Delta \phi},$$

viz. this is

$$0 = \frac{1 - \kappa'}{1 + \kappa'} F, + \frac{2\kappa}{1 + \kappa'} \Pi, (-1 - \kappa'),$$

or what is the same thing,

$$\Pi, (-1 - \kappa') = -\frac{1 - \kappa'}{2\kappa} F,$$

where $\Pi, (-1 - \kappa')$ denotes the principal value of the integral

$$\int_0^{\frac{\pi}{2}} d\phi \frac{1}{1 - (1 + \kappa') \sin^2 \phi} \frac{1}{\Delta \phi}.$$

Now (Leg. Fonct. Ellip. t. i. p. 71), we have

$$\Pi, (-\kappa^2 \sin^2 \theta) + \Pi, \left(-\frac{1}{\sin^2 \theta}\right) = F,$$

where, upon examination, it will appear that $\Pi, \left(-\frac{1}{\sin^2 \theta}\right)$, in fact, represents the principal value of the integral.

Writing herein $\sin^2 \theta = \frac{1}{1 - \kappa}$, and therefore $\cos^2 \theta = \frac{\kappa'}{1 + \kappa}$, or $\tan^2 \theta = \kappa'$, this is

$$\Pi, (-1 + \kappa') + \Pi, (-1 - \kappa') = F,$$

and the formula (p'), p. 141, attributing therein to θ the foregoing value, becomes

$$\Pi, (-1 - \kappa') = E, + \frac{1}{\kappa'} \{F, E(\theta) - E, F(\theta)\}.$$

But θ is the value for the bisection of the function $F,$ viz., we have

$$\begin{aligned} 2 F(\theta) &= F, \\ 2 E \theta &= E, + 1 - \kappa' \end{aligned}$$

whence

$$F, E(\theta) - E, F(\theta) = \frac{1}{2} (1 - \kappa') F,$$

or the formula in question gives

$$\Pi, (-1 + \kappa') = \frac{1 + \kappa'}{2 \kappa'} E,$$

whence

$$\Pi, (-1 - \kappa') = -\frac{1 - \kappa'}{2 \kappa'} F,$$

the result which was to be proved.

28. The value of M' (observing that $\frac{b}{(a-b)(b-c)} = \frac{1}{(\sqrt{a} - \sqrt{c})^2}$
 $= \frac{1}{a(1 - \kappa')^2}$) is

$$\frac{1}{\sqrt{a}(1 - \kappa')} M' = \frac{1}{2} \int_{-a}^{-c} \frac{dh}{\sqrt{h(a+h)(c+h)}}$$

which is

$$= \frac{1}{2} \frac{2}{\sqrt{2}} F'(\kappa),$$

that is, we have

$$M' = (1 - \kappa') F,(\kappa),$$

or, what is the same thing,

$$\log \tan \frac{1}{2} \phi_0 = \frac{1 - \kappa'}{2} F,(\kappa),$$

that is,

$$\log \tan \frac{1}{2} \varphi_0 = \frac{1 - \kappa'}{2} F_1(\kappa),$$

that is,

$$\tan \frac{1}{2} \varphi_0 = \left(\frac{1 - \kappa'}{2} F_1(\kappa) \right),$$

(φ_0 the South azimuth of the B-geodesic at the umbilicus).

29. I purposely calculated the Table by quadratures as being a method available where the equation $ac - b^2 = 0$ is not satisfied; but in the present case, where this equation is satisfied, the table might have been calculated from LEGENDRE'S Tables of Elliptic Integrals. Observe that $a = 1000$, $b = 500$, $c = 250$, gives $\kappa = \frac{\sqrt{3}}{2}$ or angle of modulus = 60° . As an instance of the comparison,* suppose $h = -800$, then $\sin^2 \varphi = \frac{200}{750} = \frac{4}{15}$, $\log \sin \varphi = 9.71298$, $\varphi = 31^\circ 5'$.

$$\sqrt{\frac{b}{(a-b)(b-c)}} = \sqrt{\frac{500}{500 \cdot 250}} = \frac{\sqrt{10}}{50} = .06326$$

$$H^c = \frac{500 \cdot 200 \cdot 550}{800 \cdot 500 \cdot 250} = \frac{110}{200} \quad \log. = \bar{1}.87018, \quad H = .7416$$

$$\frac{1 + H}{1 - H} = \frac{1.7416}{2.584} = 6.7582.$$

$$\Pi(h) = .03163 F(31^\circ 5') + .03163 \text{ h. l. } 6.7582$$

$$F 31^\circ = .56166$$

163

$$F 31^\circ 5' = .56329$$

$$\text{h. l. } 6.7852 = 1.91075$$

2.47404

$$\times \text{ by } .03163$$

.0782043

* In the present calculation, log. denotes an ordinary logarithm, the hyperbolic logarithm being distinguished as h. l.

or multiplying by 100,000 (factor introduced into my Table) this is = 7820.43. The value $\Pi (-800) = 7864$ given by my Table agrees sufficiently well with this, the correct value.

30. I calculate also the angle φ_0 , viz. we have

$$\begin{aligned} \text{h. l. } \tan \frac{1}{2} \varphi_0 &= \frac{1 - \kappa'}{2} F, \kappa = \frac{1}{4} F, (60^\circ). && \text{LEG. Vol. iii. Table viii.} \\ &= \frac{1}{4} 2.15651 = .53913 \end{aligned}$$

whence by LEG. Table iv.

$$\begin{aligned} \frac{1}{2} \varphi_0 &= 45^\circ + \frac{1}{2} \cdot 29^\circ 29'64 \\ &= 59^\circ 44'82 \end{aligned}$$

or

$$\varphi_0 = 119^\circ 29'64$$

This exceeds 90° , and since at the umbilicus the tangent plane is at right angles to the plane of projection, the B-geodesic should in the drawing proceed (as it in fact does) from U in the sense U C, touching the bounding ellipse at the point U.

IV. *The Second Part of a Memoir on the Development of the Disturbing Function in the Lunar and Planetary Theories.* By Prof. CAYLEY.

Read January 12, 1872.

THE present communication is a sequel to my paper, "The First Part of a Memoir on the Development of the Disturbing Function in the Lunar and Planetary Theories," *Memoirs R.A.S.*, vol. xxviii. (1859), pp. 187-215, and I have therefore entitled it as above, but it, in fact, relates only to the Planetary Theory. In the First Part, I gave in effect, but not explicitly, an expression for the general co-efficient $D(j, j')$ in terms of the co-efficients of the multiple cosines of θ in the expansions of the several powers $(r^2 + r'^2 - 2 r r' \cos \theta)^{-i-\frac{1}{2}}$, or say $(a^2 + a'^2 - 2 a a' \cos \theta)^{-i-\frac{1}{2}}$; viz., at the foot of page 208 I speak of the term involving $\cos(jU + j'U')$ as having a certain given value; the term in question is $D(j, j') \cos(jU + j'U')$; and consequently the expression for $D(j, j')$ is

$$D(j, j') = \Sigma \frac{\Pi_1(x - \frac{1}{2})}{\Pi x} \eta^{2x} \Sigma M_x^j R_x^{j'};$$

the omission was, however, a material one, inasmuch as this expression for the general co-efficient serves to connect my formulæ with LEVERRIER'S development, *Annales de l'Obs. de Paris*, t. I. (1855), pp. 275-330 and 358-383, and I resume the question for the purpose of supplying it.

Formula for the general Co-efficient $D(j, j')$.

In the First Part, the reciprocal of the distance of the two planets, or function

$$\{r^2 + r'^2 - 2 r r' (\cos U \cos U' + \sin U \sin U' \cos \Phi)\}^{-\frac{1}{2}}$$

is taken to be developed in multiple cosines of U , U' , the general term being

$$D(j, j') \cos(jU + j'U'),$$

where j, j' , have each of them any integer value from $-\infty$ to $+\infty$ (zero not excluded), but so that j, j' , are simultaneously even or simultaneously odd. We have $D(-j, -j') = D(j, j')$ and $D(j', j) = D(j, j')$; and it hence appears that the really distinct values of the co-efficient may be taken to be those for which j is not negative, and as regards absolute magnitude is not less than j' ; and for such values of j, j' we have the above-mentioned expression

$$D(j, j') = \Sigma \frac{\Pi_1(x - \frac{1}{2})}{\Pi x} \eta^{2x} \Sigma M_x^{\mathcal{S}} R_x^{\mathcal{S}},$$

which I proceed to explain and develop.

$\Pi_1(x - \frac{1}{2})$ and Πx (x being a positive integer) denote respectively $\frac{1}{2} \cdot \frac{3}{2} \cdot \dots \cdot (x - \frac{1}{2})$, and $1 \cdot 2 \cdot 3 \cdot \dots \cdot x$; in particular for $x = 0$, the value of each factorial is $= 1$.

η denotes $\sin \frac{1}{2} \Phi$.

The co-efficients $R_x^{\mathcal{S}}$ are those of the multiple cosines in certain developments, viz. we have

$$r^x r'^x \{r^2 + r'^2 - 2 r r' \cos(U - U')\}^{-x - \frac{1}{2}} = \Sigma R_x^i \cos i(U - U'),$$

where, as usual, i extends from $-\infty$ to ∞ and $R_x^{-i} = R_x^i$. Writing with LEVERRIER

$$(a^2 + a'^2 - 2 a a' \cos H)^{-\frac{1}{2}} = \frac{1}{2} \Sigma A^i \cos i H,$$

$$a a' (a^2 + a'^2 - 2 a a' \cos H)^{-\frac{1}{2}} = \frac{1}{2} \Sigma B^i \cos i H,$$

$$a^2 a'^2 (a^2 + a'^2 - 2 a a' \cos H)^{-\frac{1}{2}} = \frac{1}{2} \Sigma C^i \cos i H,$$

$$a^3 a'^3 (a^2 + a'^2 - 2 a a' \cos H)^{-\frac{1}{2}} = \frac{1}{2} \Sigma D^i \cos i H,$$

then $2 R_x^i$, $2 R_x^i$, $2 R_x^i$, $2 R_x^i$, are the same functions of r, r' , that A^i, B^i, C^i, D^i , respectively are of a, a' .

The expression of $M_x^{\mathcal{S}}$ is

$$M_x^{\mathcal{S}} = (-)^{x - \frac{1}{2}(j + j')} \frac{\Pi x}{\Pi \frac{1}{2}(x - j - \mathcal{S}) \Pi \frac{1}{2}(x + j' + \mathcal{S})} \frac{\Pi x}{\Pi \frac{1}{2}(x - j' + \mathcal{S}) \Pi \frac{1}{2}(x + j - \mathcal{S})};$$

and, finally, in the expression for $D(j, j')$, x has every integer value from

0 to ∞ , and, for any given value of x , \mathfrak{D} extends by steps of two units from the inferior value $-(x - j')$ to the superior value $x - j$.

It is convenient to write $x = \frac{1}{2}(j + j') + s$; we have then \mathfrak{D} extending from $-\frac{1}{2}(j - j') - s$ to $-\frac{1}{2}(j - j') + s$, or writing $\mathfrak{D} = -\frac{1}{2}(j - j') + \theta$, θ has the $s + 1$ values $s, s - 2, s - 4, \dots - s$, viz. for $s = 2p + 1$ the values are $\pm 1, \pm 3, \dots \pm (2p + 1)$, and for $s = 2p$, they are $0, \pm 2, \pm 4, \dots \pm 2p$.

Making these changes we have

$$D(j, j') = \Sigma \frac{\Pi_1 \{ \frac{1}{2}(j + j') + s - \frac{1}{2} \}}{\Pi \{ \frac{1}{2}(j + j') + s \}} \eta^{j+j'+2s} \Sigma M_{\frac{1}{2}(j+j')+s}^{-\frac{1}{2}(j-j')+\theta} R_{\frac{1}{2}(j+j')+s}^{-\frac{1}{2}(j-j')+\theta},$$

where

$$M_{\frac{1}{2}(j+j')+s}^{-\frac{1}{2}(j-j')+\theta} = (-)^s \frac{\Pi \{ \frac{1}{2}(j + j') + s \}}{\Pi \frac{1}{2}(s - \theta) \Pi \frac{1}{2}(j + j' + s + \theta)} \frac{\Pi \{ \frac{1}{2}(j + j') + s \}}{\Pi \frac{1}{2}(s + \theta) \Pi \frac{1}{2}(j + j' + s - \theta)},$$

viz. this is $(-)^s$ into the product of two binomial co-efficients, each belonging to the exponent $\frac{1}{2}(j + j') + s$.

Particular Cases, $j + j' = 0, 2, 4, 6$, being those required in the Planetary Theory.

Considering successively the cases $j + j' = 0, 2, 4, 6$, we have, first,

$$D(j, -j) = \Sigma \frac{\Pi_1 (s - \frac{1}{2})}{\Pi s} \eta^{2s} \Sigma (-)^s \left\{ \frac{\Pi s}{\Pi \frac{1}{2}(s - \theta) \Pi \frac{1}{2}(s + \theta)} \right\}^2 R_s^{-j+\theta}$$

which, developed as far as η^6 , is

$$\begin{aligned} (*) \quad D(j, -j) &= \frac{1}{2} A^{-j} \\ &\quad - \frac{1}{2} \eta^2 \frac{1}{2} (B^{-j+1} + B^{-j-1}) \\ &\quad + \frac{1 \cdot 3}{2 \cdot 4} \eta^4 \frac{1}{2} (C^{-j+2} + 4C^{-j} + C^{-j-2}) \\ &\quad - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \eta^6 \frac{1}{2} (D^{-j+3} + 9D^{-j+1} + 9D^{-j-1} + D^{-j-3}), \end{aligned}$$

where, and in what immediately follows A, B, C, D, are used to denote functions (not of (a, a') , but) of τ, τ' .

Secondly,

$$D(j, -j + 2) = \sum \frac{\Pi_1(s + \frac{1}{2})}{\Pi(s + 1)} \eta^s \sum \eta^{2s} \left\{ (-)^s \frac{\Pi(s + 1)}{\Pi \frac{1}{2}(s - \theta) \Pi \frac{1}{2}(s + \theta) + 1} \right. \\ \left. \times \frac{\Pi(s + 1)}{\Pi \frac{1}{2}(s + \theta) \Pi \frac{1}{2}(s - \theta) + 1} R_{s+1}^{-j+1+s} \right\},$$

which, developed to η^6 , is

$$(*) \quad D(j, -j + 2) = \eta^2 \left\{ \frac{1}{2} \cdot \frac{1}{2} B^{-j+1} \right. \\ \left. - \frac{1 \cdot 3}{2 \cdot 4} \eta^2 \cdot \frac{1}{2} (2 C^{-j+2} + 2 C^{-j}), \right. \\ \left. + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \eta^4 \cdot \frac{1}{2} (3 D^{-j+3} + 9 D^{-j+1} + 3 D^{-j-1}) \right\}.$$

Thirdly,

$$D(j, -j + 4) = \sum \frac{\Pi_1(s + \frac{3}{2})}{\Pi(s + 2)} \eta^s \cdot \sum \eta^{2s} (-)^s \left\{ \frac{\Pi(s + 2)}{\Pi \frac{1}{2}(s - \theta) \Pi \frac{1}{2}(s + \theta) + 2} \right. \\ \left. \times \frac{\Pi(s + 2)}{\Pi \frac{1}{2}(s + \theta) \Pi \frac{1}{2}(s - \theta) + 2} R_{s+2}^{-j+2+s} \right\},$$

which, developed to η^6 , is

$$(*) \quad D(j, -j + 4) = \eta^4 \left\{ \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{2} C^{-j+2} \right. \\ \left. - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \eta^2 \cdot \frac{1}{2} (3 D^{-j+3} + 3 D^{-j+1}) \right\};$$

and, fourthly,

$$D(j, -j + 6) = \sum \frac{\Pi_1(s + \frac{5}{2})}{\Pi(s + 3)} \eta^s \sum \eta^{2s} (-)^s \left\{ \frac{\Pi(s + 3)}{\Pi \frac{1}{2}(s - \theta) \Pi \frac{1}{2}(s + \theta) + 3} \right. \\ \left. \times \frac{\Pi(s + 3)}{\Pi \frac{1}{2}(s + \theta) \Pi \frac{1}{2}(s - \theta) + 3} R_{s+3}^{-j+3+s} \right\},$$

which, developed to η^6 , is simply

$$(*) \quad D(j, -j + 6) = \eta^6 \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{2} D^{-j+3}.$$

The foregoing formulæ, although obtained on the supposition $j = 0$, or positive, apply without alteration to the case $j = \text{negative}$, and the entire series of terms of an order not exceeding 6 as regards η may be written,

$$D(j, -j) \quad \cos(jU - jU) \\ + 2 D(j, -j + 2) \quad \cos(jU + (-j + 2)U) \\ + 2 D(j, -j + 4) \quad \cos(jU + (-j + 4)U) \\ + 2 D(j, -j + 6) \quad \cos(jU + (-j + 6)U),$$

where j has every integer value from $-\infty$ to $+\infty$.

Comparison with LEVERRIER.

This is in fact what LEVERRIER's expression becomes on putting therein $e = e' = 0$. To verify this, observe that LEVERRIER having defined his A^i, B^i, C^i, D^i , as above, writes further

$$\begin{aligned} E^i &= \frac{1}{2}(B^{i-1} + B^{i+1}), \\ G^i &= \frac{3}{8}(C^{i-2} + 4C^i + C^{i+2}), \\ H^i &= \frac{5}{16}(D^{i-3} + 9D^{i-1} + 9D^{i+1} + D^{i+3}), \\ L^i &= \frac{3}{4}(C^{i-2} + C^i), \\ S^i &= \frac{15}{16}(D^{i-3} + 3D^{i-1} + D^{i+1}), \\ T^i &= \frac{15}{16}(D^{i-3} + D^{i-1}). \end{aligned}$$

(Consequently $E^{-i} = E^i, G^{-i} = G^i, H^{-i} = H^i, L^{-i+2} = L^i, S^{-i+2} = S^i, T^{-i+4} = T^i$) and that the terms in question, putting in the coefficients $e = e' = 0$, are with him

$$\begin{aligned} &\{(1)^i + (11)^i \eta^2 + (17)^i \eta^4 + (20)^i \eta^6\} \cos(i\ell - i\lambda), \\ &\{(212)^i \eta^2 + (218)^i \eta^4 + (221)^i \eta^6\} \cos[i\ell - (i-2)\lambda - 2\tau], \\ &\{(372)^i \eta^4 + (375)^i \eta^6\} \cos[i\ell - (i-4)\lambda - 4\tau], \\ &\{(449)^i \eta^6\} \cos[i\ell - (i-6)\lambda - 6\tau], \end{aligned}$$

where, substituting for $(1)^i, (11)^i$, &c., their values, the coefficients are

$$\begin{aligned} &\frac{1}{2}A^i - \eta^2 \frac{1}{2}E^i + \eta^4 \cdot \frac{1}{2}G^i - \eta^6 \frac{1}{2}H^i, \\ &= \frac{1}{2}A^i - \eta^2 \cdot \frac{1}{4}(B^{i-1} + B^{i+1}) + \eta^4 \cdot \frac{3}{16}(C^{i-2} + 4C^i + C^{i+2}) \\ &\quad - \eta^6 \cdot \frac{5}{32}(D^{i-3} + 9D^{i-1} + 9D^{i+1} + D^{i+3}); \end{aligned}$$

$$\begin{aligned} & \eta^3 \cdot \frac{1}{2} B^{i-1} - \eta^4 \cdot L^i + \eta^6 S^i, \\ & = \eta^3 \cdot \frac{1}{2} B^{i+1} - \eta^4 \left(\frac{3}{4} C^{i-2} + C^i \right) + \eta^6 \cdot \frac{15}{16} (D^{i-3} + 3 D^{i-1} + D^{i+1}); \\ & \eta^4 \cdot \frac{3}{8} C^{i-2} - \eta^6 T^i, \\ & = \eta^4 \cdot \frac{3}{8} C^{i-2} - \eta^6 \cdot \frac{15}{16} (D^{i-3} + D^{i-1}); \end{aligned}$$

and

$$\eta^6 \cdot \frac{5}{16} D^{i-3}.$$

Writing herein j in place of i , and for $A^j, B^{j-1}, \&c.$, the equal values $A^{-j}, B^{-j+1}, \&c.$, we have precisely the foregoing coefficients $D(j, -j), \dots D(j, -j + 6)$.

The Development in Powers of e, e' .

The complete expression of the reciprocal of the distance is obtained from

$$\begin{aligned} & D(j, -j) \cos(jU - jU') \\ & + 2 D(j, -j + 2) \cos(jU + (-j + 2)U') \\ & + 2 D(j, -j + 4) \cos(jU + (-j + 4)U') \\ & + 2 D(j, -j + 6) \cos(jU + (-j + 6)U'), \end{aligned}$$

by writing therein for r, r', U, U' , instead of the circular, the elliptic values, that is, the values

$$\begin{aligned} r &= a \operatorname{elqr}(e, L - \Pi) & , & = a(1 + x), \\ r' &= a' \operatorname{elqr}(e', L' - \Pi') & , & = a'(1 + x'), \\ U &= \Pi - \Theta + \operatorname{elta}(e, L - \Pi), & = & \Pi - \Theta + f, \\ U' &= \Pi' - \Theta' + \operatorname{elta}(e', L' - \Pi'), & = & \Pi' - \Theta' + f'; \end{aligned}$$

L, Π, Θ the mean longitude in orbit, longitude of perihelion in orbit, and longitude of node; and the like for L', Π', Θ' ; "elqr" = elliptic quotient radius, "elta" = elliptic true anomaly; or what is the same thing, if we

write elta $(e, L - \Pi) = L - \Pi + \text{eltt}(e, L - \Pi)$, and the like for elta $(e', L' - \Pi')$, then

$$U = L - \Theta + \text{eltt}(e, L - \Pi), = L - \Theta + y,$$

$$U' = L' - \Theta' + \text{eltt}(e', L' - \Pi'), = L' - \Theta' + y'.$$

The process for doing this is explained, First Part, pp. 205-207, viz., writing $r = a(1 + x)$, $r' = a'(1 + x')$, and restoring j' instead of its value $(-j, \dots -j + 6$, as the case may be), we have a general term

$$\frac{1}{\Pi a \Pi a'} a^{\alpha} a'^{\alpha'} \left(\frac{d}{da}\right)^{\alpha} \left(\frac{d}{da'}\right)^{\alpha'} \cdot D(j, j') \cdot x^{\alpha} x'^{\alpha'} \cos [j(\Pi - \Theta + f) + j'(\Pi' - \Theta' + f')],$$

where $D(j, j')$ now denotes the value obtained by writing a, a' in place of r, r' , and f, f' are the true anomalies elta $(e, L - \Pi)$ and elta $(e', L' - \Pi')$. And the second factor, $x^{\alpha} x'^{\alpha'}$ into the cosine, is given as a series

$$\Sigma \Sigma ([\cos]^i + [\sin]^i) ([\cos]^{i'} + [\sin]^{i'}) \cos [i(L - \Pi) + i'(L' - \Pi') + j(\Pi - \Theta) - j'(\Pi' - \Theta')],$$

where $[\cos]^i, [\sin]^i$ are functions of e , $[\cos]^{i'}, [\sin]^{i'}$ functions of e' . Or, what is better, the term $x^{\alpha} x'^{\alpha'}$ into the cosine may be written $x^{\alpha} x'^{\alpha'} \cos [j(L - \Theta + y) + j'(L' - \Theta' + y')]$, and the expansion then is

$$\Sigma \Sigma ([\cos]^i + [\sin]^i) ([\cos]^{i'} + [\sin]^{i'}) \cos [i(L - \Pi) + i'(L' - \Pi') + j(L - \Theta) + j'(L' - \Theta')],$$

where as before $[\cos]^i, [\sin]^i$ are functions of e , $[\cos]^{i'}, [\sin]^{i'}$, are the same functions of e' , viz. the e -functions are those given in the two "datatables" $(x^0 \dots x^7) \cos jy$ and $(x^0 \dots x^7) \sin jy$, taken from LEVERRIER, which I have given in my "Tables of the Developments of Functions in the Theory of Elliptic Motion," *Memoirs R.A.S.* vol. xxix. (1861), pp. 191-306. In order to better show which are the symbols referred to, we may, instead of $[\cos]^i$, &c., write $[x^{\alpha} \cos jy]^i$, &c., the formula will then be

$$x^{\alpha} x'^{\alpha'} \cos [j(L - \Theta + y) + j'(L' - \Theta' + y')] =$$

$$\Sigma \Sigma ([x^{\alpha} \cos jy]^i + [x^{\alpha} \sin jy]^i) ([x'^{\alpha'} \cos j'y']^{i'} + [x'^{\alpha'} \sin j'y']^{i'})$$

$$\times \cos [i(L - \Pi) + i'(L' - \Pi') + j(L - \Theta) + j'(L' - \Theta')].$$

and if we attribute to i, i' any given values, that is, attend to any particular multiple cosine,

$$\cos [i(L - \Pi) + i'(L' - \Pi') + j(L - \Theta) + j'(L' - \Theta)],$$

the coefficient hereof will be

$$\sum \frac{1}{\Pi \alpha \Pi \alpha'} a^\alpha \left(\frac{d}{da}\right)^\alpha a'^{\alpha'} \left(\frac{d}{da}\right)^{\alpha'} D(j, j') \cdot ([x^\alpha \cos jy]^i + [x^\alpha \sin jy]^i) ([x^{\alpha'} \cos j'y']^{\alpha'} + [x^{\alpha'} \sin j'y']^{\alpha'}),$$

where α, α' each extend from zero to infinity, but to obtain the expression up to a given order p in e, e' , we take only the values up to $\alpha + \alpha' = p$.

Particular Case.

Thus, for instance, in $\cos [j(L - \Theta) - j'(L' - \Theta')]$ the terms independent of e' are

$$\begin{aligned} & D(j, -j) \{ [x^0 \cos jy]^0 + [x^0 \sin jy]^0 \}, \\ & + \frac{1}{1} a \left(\frac{d}{da}\right) D(j, -j) \{ [x^1 \cos jy]^0 + [x^1 \sin jy]^0 \}, \\ & + \frac{1}{1.2} a^2 \left(\frac{d}{da}\right)^2 D(j, -j) \{ [x^2 \cos jy]^0 + [x^2 \sin jy]^0 \}, \\ & + \&c. \end{aligned}$$

which, observing that in the present case the sine terms vanish, is

i	$\frac{e^i}{8}$	$\frac{e^i}{384}$	$\frac{e^i}{46080}$	
1	$-8j^2$	$+96j^4$ $-54j^2$	$-1280j^6$ $+3920j^4$ $-3440j^2$	1 . . . D(j, -j)
	+ 4	$-48j^2$	$-360j^2$	$\frac{1}{1} a \frac{d}{da}$. . . "
	+ 4	$-96j^2$	$+1920j^4$ $-1320j^2$	$\frac{1}{1.2} a^2 \left(\frac{d}{da}\right)^2$. . . "
		+ 144	$-2880j^2$	$\frac{1}{1.2.3} a^3 \left(\frac{d}{da}\right)^3$. . . "
		+ 144	$-5760j^2$	$\frac{1}{1.2.3.4} a^4 \left(\frac{d}{da}\right)^4$. . . "
			+ 14400	$\frac{1}{1.2..5} a^5 \left(\frac{d}{da}\right)^5$. . . "
			+ 14400	$\frac{1}{1.2..6} a^6 \left(\frac{d}{da}\right)^6$. . . "
			0	$\frac{1}{1.2..7} a^7 \left(\frac{d}{da}\right)^7$. . . "

viz. the term in e^i is

$$e^i \left\{ -j^2 + \frac{1}{2} a \frac{d}{da} + \frac{1}{4} \left(a \frac{d}{da} \right)^2 \right\} D(j, -j)$$

viz. writing $\eta = 0$, and therefore $D(j, -j) = \frac{1}{2} A^{-j}$, the term in e^i is

$$e^i \left\{ -j^2 + \frac{1}{2} a \frac{d}{da} + \frac{1}{4} a^2 \left(\frac{d}{da} \right)^2 \right\} \frac{1}{2} A^{-j}$$

which conformably with LEVERRIER'S subscript notation

$$A_1^i = \frac{1}{1} a \frac{d}{da} A^i, A_2^i = \frac{1}{1.2} a^2 \left(\frac{d}{da} \right)^2 A^i, \text{ \&c.,}$$

I write

$$\begin{aligned} & e^2 \left\{ -j^2 + \frac{1}{2} ()_1 + \frac{1}{4} 2 ()_2 \right\} \frac{1}{2} A^{-j} \\ &= e^2 \left\{ -\frac{1}{2} j^2 A^{-j} + \frac{1}{4} A_1^{-j} + \frac{1}{4} A_2^{-j} \right\}. \end{aligned}$$

The term in question is given by LEVERRIER as $(\frac{1}{2} e)^2 (2)^i = e^2 \cdot \frac{1}{4} (2)^i$, $h = i$ and $K^i = A^i = e^2 \cdot \frac{1}{4} (-2 i^2 A^i + A_1^i + A_2^i)$, which agrees.

Similarly the term in e^4 is

$$\begin{aligned} & \frac{e^4}{384} \left\{ 96j^4 - 54j^2 ()_1 - 96j^2 ()_2 + 144 ()_3 + 144 ()_4 \right\} \frac{1}{2} A^{-j}, \\ &= \frac{e^4}{768} \left\{ (96j^4 - 54j^2) A^{-j} - 48j^2 A_1^{-j} - 96j^2 A_2^{-j} + 144 A_3^{-j} + 144 A_4^{-j} \right\}. \end{aligned}$$

and the term in question is given by LEVERRIER as $(\frac{1}{2} e)^4 (4)^i = e^4 \cdot \frac{1}{16} (4)^i$, $h = i$ and $K^i = A^i$,

$$= e^4 \frac{1}{16} \left\{ \frac{1}{8} (-9 i^2 + 16 i^4) A^i - i^2 A_1^i - 2 i^2 A_2^i + 3 A_3^i + 3 A_4^i \right\},$$

which agrees. I have not made the comparison of any more terms.

LEVERRIER'S *Results expressed in terms of the Arguments,*
 $L' - \Theta', L' - \Pi', L - \Theta, L - \Pi.$

The angles which LEVERRIER uses in his arguments are $l', \lambda, \omega, \varpi'$, and τ' . viz. we have,

$$l' = \Theta' + (L' - \Theta'),$$

$$\lambda = \Theta' + (L - \Theta),$$

$$\varpi' = \Theta' + (\Pi' - \Theta'),$$

$$\omega = \Theta' + (\Pi - \Theta),$$

$$\tau' = \Theta'.$$

where L, Π, Θ , are the mean longitude of the planet m , its perihelion and the mutual node, all in the orbit of m ; and similarly L', Π', Θ' , are the

mean longitude of the planet m' , of its perihelion and of the mutual node, all in the orbit of m' . On substituting the foregoing values of l' , λ , &c., Θ' , as it should do, disappears, and the arguments are all of them linear functions of $L' - \Theta'$, $\Pi' - \Theta'$, $L - \Theta$, $\Pi - \Theta$; or, if we please, of $L' - \Theta'$, $L' - \Pi'$, $L - \Theta$, $L - \Pi$, that is of the distances of each planet from its own perihelion and from the mutual node. It is, I think, convenient to use these last angular distances, and accordingly I wrote in LEVERRIER'S arguments, write,

$$l' = \Theta' + (L' - \Theta'),$$

$$\lambda = \Theta' \dots \dots \dots + (L - \Theta),$$

$$\omega' = \Theta' + (L' - \Theta') - (L' - \Pi'),$$

$$\omega = \Theta' \dots \dots \dots + (L - \Theta) - (L - \Pi),$$

$$r' = \Theta',$$

and for the purpose of reference form as it were an Index to his result as follows:—

Reciprocal of Distance = as follows :

Terms of order zero : terms of orders 2, 4, 6, having the same arguments.

				$L' - \Theta'$	$L' - \Pi'$	$L - \Theta$	$L - \Pi$
$(1)^i$	(1 .. 20)	..	cos	i	0	$-i$	0
$(21)^i (\frac{1}{2} e) (\frac{1}{2} e')$	(21 .. 30)	..	„	i	+ 1	$-i$	- 1
$(31)^i (\frac{1}{2} e)^2 (\frac{1}{2} e')^2$	(31 .. 34)	..	„	i	+ 2	$-i$	- 2
$(35)^i (\frac{1}{2} e)^3 (\frac{1}{2} e')^3$	(35 .. 35)	..	„	i	+ 3	$-i$	- 3
$(36)^i (\frac{1}{2} e)^2 \eta^2$	(36 .. 39)	..	„	i	0	$-i + 2$	- 2
$(40)^i (\frac{1}{2} e) (\frac{1}{2} e') \eta^2$	(40 .. 43)	..	„	i	- 1	$-i + 2$	- 1
$(44)^i (\frac{1}{2} e')^2 \eta^2$	(44 .. 47)	..	„	i	- 2	$-i + 2$	0
$(48)^i (\frac{1}{2} e)^3 (\frac{1}{2} e') \eta^2$	(48 .. 48)	..	„	i	+ 1	$-i + 2$	- 3
$(49)^i (\frac{1}{2} e) (\frac{1}{2} e')^3 \eta^2$	(49 .. 49)	..	„	i	- 3	$-i + 2$	+ 1

Terms of the first order : terms of orders 3, 5, 7, having the same arguments.

				L'-Θ'	L'-Π'	L-Θ	L-Π
(50) ⁴ $\frac{1}{2} e$	(50 .. 69)	..	cos	i	0	-i	+1
(70) ⁴ $\frac{1}{2} e'$	(70 .. 89)	..	"	i	+1	-i	0
(90) ⁴ $(\frac{1}{2} e)^2 (\frac{1}{2} e')$	(90 .. 99)	..	"	i	+1	-i	-2
(100) ⁴ $(\frac{1}{2} e) (\frac{1}{2} e')^2$	(100 .. 109)	..	"	i	+2	-i	-1
(110) ⁴ $(\frac{1}{2} e)^3 (\frac{1}{2} e')^2$	(110 .. 113)	..	"	i	+2	-i	-3
(114) ⁴ $(\frac{1}{2} e)^2 (\frac{1}{2} e')^3$	(114 .. 117)	..	"	i	+3	-i	-2
(118) ⁴ $(\frac{1}{2} e)^4 (\frac{1}{2} e')^3$	(118 .. 118)	..	"	i	+3	-i	-4
(119) ⁴ $(\frac{1}{2} e)^3 (\frac{1}{2} e')^4$	(119 .. 119)	..	"	i	+4	-i	-3
(120) ⁴ $(\frac{1}{2} e) \eta^2$	(120 .. 129)	..	"	i	0	-i+2	-1
(130) ⁴ $(\frac{1}{2} e') \eta^2$	(130 .. 139)	..	"	i	-1	-i+2	0
(140) ⁴ $(\frac{1}{2} e)^3 \eta^2$	(140 .. 143)	..	"	i	0	-i+2	-3
(144) ⁴ $(\frac{1}{2} e)^2 (\frac{1}{2} e') \eta^2$	(144 .. 147)	..	"	i	+1	-i+2	-2
(148) ⁴ $(\frac{1}{2} e)^3 (\frac{1}{2} e') \eta^2$	(148 .. 151)	..	"	i	-1	-i+2	-2
(152) ⁴ $(\frac{1}{2} e) (\frac{1}{2} e')^2 \eta^2$	(152 .. 155)	..	"	i	-2	-i+2	-1
(156) ⁴ $\frac{1}{2} e (\frac{1}{2} e')^2 \eta^2$	(156 .. 159)	..	"	i	-2	-i+2	+1
(160) ⁴ $(\frac{1}{2} e')^3 \eta^2$	(160 .. 163)	..	"	i	-3	-i+2	0
(164) ⁴ $(\frac{1}{2} e)^4 (\frac{1}{2} e') \eta^2$	(164 .. 164)	..	"	i	+1	-i+2	-4
(165) ⁴ $(\frac{1}{2} e)^3 (\frac{1}{2} e')^2 \eta^2$	(165 .. 165)	..	"	i	+2	-i+2	-3
(166) ⁴ $(\frac{1}{2} e)^2 (\frac{1}{2} e')^3 \eta^2$	(166 .. 166)	..	"	i	-3	-i+2	+2
(167) ⁴ $(\frac{1}{2} e) (\frac{1}{2} e')^4 \eta^2$	(167 .. 167)	..	"	i	-4	-i+2	+1
(168) ⁴ $(\frac{1}{2} e)^3 \eta^4$	(168 .. 168)	..	"	i	0	-i+2	-3
(169) ⁴ $(\frac{1}{2} e)^2 (\frac{1}{2} e') \eta^4$	(169 .. 169)	..	"	i	-1	-i+4	-2
(170) ⁴ $(\frac{1}{2} e) (\frac{1}{2} e')^2 \eta^4$	(170 .. 170)	..	"	i	-2	-i+4	-1
(171) ⁴ $(\frac{1}{2} e')^3 \eta^4$	(171 .. 171)	..	"	i	-3	-i+4	0

Terms of second order : terms of orders 4, 6, having the same arguments.

					L'-Θ'	L'-Π'	L-Θ	L-Π
(172) ⁴	($\frac{1}{2}e$) ²	(172 .. 181)	..	cos	i	0	-i	+2
(182) ⁴	($\frac{1}{2}e$) ($\frac{1}{2}e'$)	(182 .. 191)	..	"	i	+1	-i	+1
(192) ⁴	($\frac{1}{2}e'$) ²	(192 .. 201)	..	"	i	+2	-i	0
(202) ⁴	($\frac{1}{2}e$) ³ ($\frac{1}{2}e'$)	(202 .. 205)	..	"	i	+1	-i	-3
(206) ⁴	($\frac{1}{2}e$) ($\frac{1}{2}e'$) ³	(206 .. 209)	..	"	i	+3	-i	-1
(210) ⁴	($\frac{1}{2}e$) ⁴ ($\frac{1}{2}e'$) ²	(210 .. 210)	..	"	i	+2	-i	-4
(211) ⁴	($\frac{1}{2}e$) ² ($\frac{1}{2}e'$) ⁴	(211 .. 211)	..	"	i	+4	-i	-2
(212) ⁴	η ²	(212 .. 221)	..	"	i	0	-i+2	0
(222) ⁴	($\frac{1}{2}e$) ($\frac{1}{2}e'$) η ²	(222 .. 225)	..	"	i	+1	-i+2	-1
(226) ⁴	($\frac{1}{2}e$) ($\frac{1}{2}e'$) η ²	(226 .. 229)	..	"	i	-1	-i+2	+1
(230) ⁴	($\frac{1}{2}e'$) ⁴ η ²	(230 .. 230)	..	"	i	0	-i+2	-4
(231) ⁴	($\frac{1}{2}e$) ³ ($\frac{1}{2}e'$) η ²	(231 .. 231)	..	"	i	-1	-i+2	-3
(232) ⁴	($\frac{1}{2}e$) ² ($\frac{1}{2}e'$) ² η ²	(232 .. 232)	..	"	i	-2	-i+2	-2
(233) ⁴	($\frac{1}{2}e$) ($\frac{1}{2}e'$) ³ η ²	(233 .. 233)	..	"	i	-3	-i+2	-1
(234) ⁴	($\frac{1}{2}e'$) ⁴ η ²	(234 .. 234)	..	"	i	-4	-i+2	0
(235) ⁴	($\frac{1}{2}e$) ² ($\frac{1}{2}e'$) ² η ²	(235 .. 235)	..	"	i	+2	-i+2	-2
(236) ⁴	($\frac{1}{2}e$) ² ($\frac{1}{2}e'$) ² η ²	(236 .. 236)	..	"	i	-2	-i+2	+2
(237) ⁴	($\frac{1}{2}e$) ² η ⁴	(237 .. 237)	..	"	i	0	-i+4	-2
(238) ⁴	($\frac{1}{2}e$) ($\frac{1}{2}e'$) η ⁴	(238 .. 238)	..	"	i	-1	-i+4	-1
(239) ⁴	($\frac{1}{2}e'$) ² η ⁴	(239 .. 239)	..	"	i	-2	-i+4	0

Terms of third order : terms of orders 5, 7, having the same arguments.

					L'-Θ'	L'-Π'	L-Θ	L-Π
(240) ⁴	($\frac{1}{2}e$) ³	(240 .. 249)	..	cos	i	0	-i	+3
(250) ⁴	($\frac{1}{2}e$) ² ($\frac{1}{2}e'$)	(250 .. 259)	..	"	i	+1	-i	+2
(260) ⁴	($\frac{1}{2}e$) ($\frac{1}{2}e'$) ²	(260 .. 269)	..	"	i	+2	-i	+1
(270) ⁴	($\frac{1}{2}e'$) ³	(270 .. 279)	..	"	i	+3	-i	0

Terms of third order (concluded):—

					L'-θ'	L'-Π'	L-θ	L-Π
(280) ⁴	($\frac{1}{2}e$) ⁴	($\frac{1}{2}e'$)	(280 .. 283) ..	cos	i	+1	-i	-4
(284) ⁴	($\frac{1}{2}e$)	($\frac{1}{2}e'$) ⁴	(284 .. 287) ..	"	i	+4	-i	-1
(288) ⁴	($\frac{1}{2}e$) ³	($\frac{1}{2}e'$) ²	(288 .. 289) ..	"	i	+2	-i	-5
(290) ⁴	($\frac{1}{2}e$) ²	($\frac{1}{2}e'$) ³	(290 .. 299) ..	"	i	0	-i+2	+1
(300) ⁴	($\frac{1}{2}e'$)	η ²	(300 .. 309) ..	"	i	+1	-i+2	0
(310) ⁴	($\frac{1}{2}e$) ²	($\frac{1}{2}e'$) η ²	(310 .. 313) ..	"	i	-1	-i+2	+2
(314) ⁴	($\frac{1}{2}e$)	($\frac{1}{2}e'$) ² η ²	(314 .. 317) ..	"	i	+2	-i+2	-1
(318) ⁴	($\frac{1}{2}e$) ³	η ²	(318 .. 318) ..	"	i	0	-i+2	-5
(319) ⁴	($\frac{1}{2}e$) ⁴	($\frac{1}{2}e'$) η ²	(319 .. 319) ..	"	i	-1	-i+2	-4
(320) ⁴	($\frac{1}{2}e$) ³	($\frac{1}{2}e'$) ² η ²	(320 .. 320) ..	"	i	-2	-i+2	-3
(321) ⁴	($\frac{1}{2}e$) ²	($\frac{1}{2}e'$) ³ η ²	(321 .. 321) ..	"	i	-3	-i+2	-2
(322) ⁴	($\frac{1}{2}e$)	($\frac{1}{2}e'$) ⁴ η ²	(322 .. 322) ..	"	i	-4	-i+2	-1
(323) ⁴	($\frac{1}{2}e$) ⁵	η ²	(323 .. 323) ..	"	i	-5	-i+2	0
(324) ⁴	($\frac{1}{2}e$) ³	($\frac{1}{2}e'$) ² η ²	(324 .. 324) ..	"	i	-2	-i+2	+3
(325) ⁴	($\frac{1}{2}e$) ²	($\frac{1}{2}e'$) ³ η ²	(325 .. 325) ..	"	i	+3	-i+2	-2
(326) ⁴	($\frac{1}{2}e$)	η ⁴	(326 .. 329) ..	"	i	0	-i+4	-1
(330) ⁴	($\frac{1}{2}e'$)	η ⁴	(330 .. 333) ..	"	i	-1	-i+4	0
(334) ⁴	($\frac{1}{2}e$) ²	($\frac{1}{2}e'$) η ⁴	(334 .. 334) ..	"	i	+1	-i+4	-2
(335) ⁴	($\frac{1}{2}e$)	($\frac{1}{2}e'$) ² η ⁴	(335 .. 335) ..	"	i	-2	-i+4	+1

Terms of fourth order: terms of order 6, and of same argument.

					L'-θ'	L'-Π'	L-θ	L-Π
(336) ⁴	($\frac{1}{2}e$) ⁴		(336 .. 339) ..	cos	i	0	-i	+4
(340) ⁴	($\frac{1}{2}e$) ³	($\frac{1}{2}e'$)	(340 .. 343) ..	"	i	+1	-i	+3
(344) ⁴	($\frac{1}{2}e$) ²	($\frac{1}{2}e'$) ²	(344 .. 347) ..	"	i	+2	-i	+2
(348) ⁴	($\frac{1}{2}e$)	($\frac{1}{2}e'$) ³	(348 .. 351) ..	"	i	+3	-i	+1
(352) ⁴	($\frac{1}{2}e'$) ⁴		(352 .. 355) ..	"	i	+4	-i	0
(356) ⁴	($\frac{1}{2}e$) ⁵	($\frac{1}{2}e'$)	(356 .. 356) ..	"	i	+1	-i	-5
(357) ⁴	($\frac{1}{2}e$)	($\frac{1}{2}e'$) ⁵	(357 .. 357) ..	"	i	+5	-i-1	0

Terms of fourth order (concluded) :—

				L'—Θ'	L'—Π'	L—Θ	L—Π	
(358) ⁱ	($\frac{1}{2} e$) ² η ²	(358 .. 358)	..	cos	i	0	-i + 2	+ 2
(362) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) η ²	(362 .. 364)	..	"	i	+ 1	-i + 2	+ 1
(366) ⁱ	($\frac{1}{2} e'$) ² η ²	(366 .. 369)	..	"	i	+ 2	-i + 2	0
(370) ⁱ	($\frac{1}{2} e$) ³ ($\frac{1}{2} e'$) η ²	(370 .. 370)	..	"	i	- 1	-i + 2	+ 3
(371) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) ³ η ²	(371 .. 371)	..	"	i	+ 3	-i + 2	- 1
(372) ⁱ	η ⁴	(372 .. 375)	..	"	i	0	-i + 4	0
(376) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) η ⁴	(376 .. 376)	..	"	i	+ 1	-i + 4	- 1
(377) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) η ⁴	(377 .. 377)	..	"	i	- 1	-i + 4	+ 1

Terms of fifth order : terms of order 7 having the same arguments.

				L'—Θ'	L'—Π'	L—Θ	L—Π	
(378) ⁱ	($\frac{1}{2} e$) ⁵	(378 .. 381)	..	cos	i	0	-i	+ 5
(382) ⁱ	($\frac{1}{2} e$) ⁴ ($\frac{1}{2} e'$)	(382 .. 385)	..	"	i	+ 1	-i	+ 4
(386) ⁱ	($\frac{1}{2} e$) ³ ($\frac{1}{2} e'$) ²	(386 .. 389)	..	"	i	+ 2	-i	+ 3
(390) ⁱ	($\frac{1}{2} e$) ² ($\frac{1}{2} e'$) ³	(390 .. 393)	..	"	i	+ 3	-i	+ 2
(394) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) ⁴	(394 .. 397)	..	"	i	+ 4	-i	+ 1
(398) ⁱ	($\frac{1}{2} e'$) ⁵	(398 .. 401)	..	"	i	+ 5	-i	0
(402) ⁱ	($\frac{1}{2} e$) ⁶ ($\frac{1}{2} e'$)	(402 .. 402)	..	"	i	+ 1	-i	- 6
(403) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) ⁶	(403 .. 403)	..	"	i	+ 6	-i	- 1
(404) ⁱ	($\frac{1}{2} e$) ³ η ²	(404 .. 407)	..	"	i	0	-i + 2	+ 3
(408) ⁱ	($\frac{1}{2} e$) ² ($\frac{1}{2} e'$) η ²	(408 .. 411)	..	"	i	+ 1	-i + 2	+ 2
(412) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) ² η ²	(412 .. 415)	..	"	i	+ 2	-i + 2	+ 1
(416) ⁱ	($\frac{1}{2} e'$) ³ η ²	(416 .. 419)	..	"	i	+ 3	-i + 2	0
(420) ⁱ	($\frac{1}{2} e$) ⁴ ($\frac{1}{2} e'$) η ²	(420 .. 420)	..	"	i	- 1	-i + 2	+ 4
(421) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) ⁴ η ²	(421 .. 421)	..	"	i	+ 4	-i + 2	- 1
(422) ⁱ	($\frac{1}{2} e$) η ⁴	(422 .. 425)	..	"	i	0	-i + 4	+ 1
(426) ⁱ	($\frac{1}{2} e'$) η ⁴	(426 .. 429)	..	"	i	+ 1	-i + 4	0
(430) ⁱ	($\frac{1}{2} e$) ² ($\frac{1}{2} e'$) η ²	(430 .. 430)	..	"	i	- 1	-i + 4	+ 2
(431) ⁱ	($\frac{1}{2} e$) ($\frac{1}{2} e'$) ² η ²	(431 .. 431)	..	"	i	+ 2	-i + 4	- 1
(432) ⁱ	($\frac{1}{2} e$) η ⁶	(432 .. 432)	..	"	i	0	-i + 6	- 1
(433) ⁱ	($\frac{1}{2} e'$) η ⁴	(433 .. 433)	..	"	i	- 1	-i + 6	0

Terms of sixth order.

					L'-Θ'	L'-Π'	L-Θ	L-Π
(434) ^t	($\frac{1}{2}e$) ⁶	(434 .. 434)	..	cos	i	0	-i	+6
(435) ^t	($\frac{1}{2}e$) ⁵ ($\frac{1}{2}e'$)	(435 .. 435)	..	"	i	+1	-i	+5
(436) ^t	($\frac{1}{2}e$) ⁴ ($\frac{1}{2}e'$) ²	(436 .. 436)	..	"	i	+2	-i	+4
(437) ^t	($\frac{1}{2}e$) ³ ($\frac{1}{2}e'$) ³	(437 .. 437)	..	"	i	+3	-i	+3
(438) ^t	($\frac{1}{2}e$) ² ($\frac{1}{2}e'$) ⁴	(438 .. 438)	..	"	i	+4	-i	+2
(439) ^t	($\frac{1}{2}e$) ($\frac{1}{2}e'$) ⁵	(439 .. 439)	..	"	i	+5	-i	+1
(440) ^t	($\frac{1}{2}e'$) ⁶	(440 .. 440)	..	"	i	+6	-i	0
(441) ^t	($\frac{1}{2}e$) ⁴ η ²	(441 .. 441)	..	"	i	0	-i+2	+4
(442) ^t	($\frac{1}{2}e$) ³ ($\frac{1}{2}e'$) η ²	(442 .. 442)	..	"	i	+1	-i+2	+3
(443) ^t	($\frac{1}{2}e$) ² ($\frac{1}{2}e'$) ² η ²	(443 .. 443)	..	"	i	+2	-i+2	+2
(444) ^t	($\frac{1}{2}e$) ($\frac{1}{2}e'$) ³ η ²	(444 .. 444)	..	"	i	+3	-i+2	+1
(445) ^t	($\frac{1}{2}e'$) ⁴ η ²	(445 .. 445)	..	"	i	+4	-i+2	0
(446) ^t	($\frac{1}{2}e$) ² η ⁴	(446 .. 446)	..	"	i	0	-i+4	+2
(447) ^t	($\frac{1}{2}e$) ($\frac{1}{2}e'$) η ⁴	(447 .. 447)	..	"	i	+1	-i+4	+1
(448) ^t	($\frac{1}{2}e'$) ² η ⁴	(448 .. 448)	..	"	i	+2	-i+4	0
(449) ^t	η ⁶	(449 .. 449)	..	"	i	0	-i+6	0

Terms of seventh order.

					L'-Θ'	L'-Π'	L-Θ	L-Π
(450) ^t	($\frac{1}{2}e$) ⁷	(450 .. 450)	..	cos	i	0	-i	+7
(451) ^t	($\frac{1}{2}e$) ⁶ ($\frac{1}{2}e'$) ²	(451 .. 451)	..	"	i	1	-i	+6
(452) ^t	($\frac{1}{2}e$) ⁵ ($\frac{1}{2}e'$) ²	(452 .. 452)	..	"	i	2	-i	+5
(453) ^t	($\frac{1}{2}e$) ⁴ ($\frac{1}{2}e'$) ³	(453 .. 453)	..	"	i	3	-i	+4
(454) ^t	($\frac{1}{2}e$) ³ ($\frac{1}{2}e'$) ⁴	(454 .. 454)	..	"	i	4	-i	+3
(455) ^t	($\frac{1}{2}e$) ² ($\frac{1}{2}e'$) ⁵	(455 .. 455)	..	"	i	5	-i	+2
(456) ^t	($\frac{1}{2}e$) ($\frac{1}{2}e'$) ⁶	(456 .. 456)	..	"	i	6	-i	+1
(457) ^t	($\frac{1}{2}e'$) ⁷	(457 .. 457)	..	"	i	7	-i	0
(458) ^t	($\frac{1}{2}e$) ⁵ η ²	(458 .. 458)	..	"	i	0	-i+2	+5

Terms of seventh order (concluded).

			L'-O'	L'-II'	L-O	L-II
(459) ⁱ (½ e) ⁴ (½ e') η ² (459 .. 459)	..	cos	i	+ 1	- i + 2	+ 4
(460) ⁱ (½ e) ³ (½ e') ² η ² (460 .. 460)	..	"	i	+ 2	- i + 2	+ 3
(461) ⁱ (½ e) ² (½ e') ³ η ² (461 .. 461)	..	"	i	+ 3	- i + 2	+ 2
(462) ⁱ (½ e) (½ e') ⁴ η ² (462 .. 462)	..	"	i	+ 4	- i + 2	+ 1
(463) ⁱ (½ e') ⁵ η ² (463 .. 463)	..	"	i	+ 5	- i + 2	0
(464) ⁱ (½ e) ³ η ⁴ (464 .. 464)	..	"	i	0	- i + 4	+ 3
(465) ⁱ (½ e) ² (½ e') η ⁴ (465 .. 465)	..	"	i	+ 1	- i + 4	+ 2
(466) ⁱ (½ e) (½ e') ² η ⁴ (466 .. 466)	..	"	i	+ 2	- i + 4	+ 1
(467) ⁱ (½ e') ³ η ⁴ (467 .. 467)	..	"	i	+ 3	- i + 4	0
(468) ⁱ (½ e) η ⁶ (468 .. 468)	..	"	i	0	- i + 6	+ 1
(469) ⁱ (½ e') η ⁶ (469 .. 469)	..	"	i	+ 1	- i + 6	0

Here the several coefficients are ultimately given in terms of the before-mentioned quantities Aⁱ, Bⁱ, Cⁱ, Dⁱ, Eⁱ, Gⁱ, Hⁱ, Lⁱ, Sⁱ, Tⁱ (functions of *a*, *a'*), and their differential coefficients in regard to *a*

$$\left(A_1^i = \frac{1}{1} a \frac{d}{da} A^i, \quad A_2^i = \frac{1}{1.2} a \frac{d^2}{da^2} A^i, \text{ \&c.} \right),$$

as follows:—we have LEVERRIER, pp. 299–330, a list of functions (1), (2), . . . (154) of the form (1) = ½ Kⁱ, (2) = -2h² Kⁱ + K₁ⁱ + K₂ⁱ, (3) = -2i² Kⁱ + K₁ⁱ + K₂ⁱ, &c., involving *i*, *h*, and K_{*i*} and its derived functions K_{*i*}ⁱ, K_{*i*}ⁱ, &c. The coefficients of the several cosines are given by means of the functions in question, thus, first coefficient, above denoted as (1)ⁱ (1 . . . 20), is

$$= (1)^i + (2)^i \left(\frac{1}{2} e\right) + (3)^i \left(\frac{1}{2} e'\right) \dots + (20)^i \eta^6$$

where (1)ⁱ = (1), (2)ⁱ = (2) . . . writing in the functions (1), (2) . . . (10), *h* = *i*, and Kⁱ = Aⁱ;

$$\begin{aligned} (11)^i &= (1), \quad (12)^i = (2), \text{ \&c., writing } h = i \text{ and } K^i = -E^i, \\ (20)^i &= (1), \quad \text{writing } h = i \text{ and } K^i = -H^i, \end{aligned}$$

and so on for the various component coefficients (1)ⁱ, (2)ⁱ . . . (469)ⁱ.

But the resulting expressions, for the several integer values $i = -10$ to $+10$, are worked out in the Addition II. (*Numerical Tables for the Calculation of the Coefficients of the Development of the Disturbing Function*), pp. 358-383. And this Addition contains also, indicated by the letters δ and Δ respectively, the expressions of the terms which experience an alteration in passing from the development of the reciprocal of the distance to those of the disturbing functions m' upon m , and m upon m' respectively.

We have—

Disturbing Function m' upon m

$$= m' \left\{ -\frac{r \cos H}{r'^2} + \frac{1}{\rho} \right\}.$$

Disturbing Function m upon m'

$$= m \left\{ -\frac{r' \cos H}{r^2} + \frac{1}{\rho} \right\}.$$

The expressions of $-\frac{r \cos H}{r'}$ and $-\frac{r' \cos H}{r^2}$, developed to the third order in the excentricities and inclination, are given, LEVERRIER, pp. 272 and 274. Expressed in the terms of the foregoing arguments $L' - \Theta'$, &c., and in terms of a, a' in place of a and $a, = \frac{a}{a'}$, these are as follows:—

$-\frac{r \cos H}{r'^2} = \frac{a}{a'^2}$ into		$L' - \Theta'$	$L' - \Pi'$	$L - \Theta$	$L - \Pi$
$-1 + \frac{1}{2}(e^2 + e'^2) + \eta^2$	cos	1	0	-1	0
$-ee'$	"	+1	+1	-1	-1
$+\frac{3}{2}e - \frac{3}{2}ee'^2 - \frac{3}{2}e\eta^2$	"	+1	0	-1	+1
$-\frac{1}{2}e + \frac{1}{2}ee'^2 + \frac{3}{8}e^3 + \frac{1}{2}e\eta^2$	"	+1	0	-1	-1
$-2e' + e^2e' + \frac{3}{2}e'^3 + 2e'\eta^2$	"	+1	+1	-1	0
$-\frac{3}{2}e^2e'$	"	+1	+1	-1	-2
$+\frac{3}{16}ee'^2$	"	-1	+2	+1	-1
$-\frac{3}{16}ee'^2$	"	+1	+2	-1	-1
$+\frac{3}{2}e\eta^2$	"	+1	0	+1	-1

$-\frac{r \cos H}{r^2} = \frac{a}{a^2}$ into						L'-Θ'	L'-Π'	L-Θ	L-Π
$-\frac{1}{8}e^2$	COS	+ 1	0	- 1	+ 2
$-\frac{3}{8}e^2$	"	+ 1	0	- 1	- 2
$+ 3ee'$	"	+ 1	+ 1	- 1	+ 1
$-\frac{1}{8}e'^2$	"	- 1	+ 2	+ 1	0
$-\frac{2}{8}e'^2$	"	+ 1	- 2	- 1	0
$-\eta^2$	"	+ 1	0	+ 1	0
$-\frac{1}{4}e^3$	"	+ 1	0	- 1	+ 3
$-\frac{1}{3}e^3$	"	+ 1	0	- 1	- 3
$-\frac{1}{4}e^2e'$	"	+ 1	+ 1	- 1	+ 2
$-\frac{1}{8}ee'^2$	"	- 1	+ 2	+ 1	+ 1
$+\frac{1}{8}ee'^2$	"	+ 1	+ 2	- 1	+ 1
$-\frac{1}{6}e'^3$	"	- 1	+ 3	+ 1	0
$-\frac{1}{3}e'^3$	"	+ 1	+ 3	- 1	0
$-\frac{1}{2}e\eta^2$	"	+ 1	0	+ 1	+ 1
$- 2e'\eta^2$	"	+ 1	+ 1	0	+ 1

$-\frac{r' \cos H}{r^2} = \frac{a'}{a^2}$ into						L'-Θ'	L'-Π'	L-Θ	L-Π
$- 1 + \frac{1}{2}(e^2 + e'^2) + \eta^2$	COS	1	0	- 1	0
$- ee'$	"	+ 1	+ 1	- 1	- 1
$- 2e + ee'^2 + \frac{3}{2}e^3 + 2e\eta^2$	"	+ 1	0	- 1	- 1
$+\frac{3}{2}e' - \frac{3}{4}e^2e' - \frac{3}{2}e'\eta^2$	"	- 1	+ 1	+ 1	0
$-\frac{1}{2}e' + \frac{1}{4}e^2e' + \frac{3}{8}e'^3 + \frac{1}{2}e'\eta^2$	"	+ 1	+ 1	- 1	0
$+\frac{3}{8}e^2e'$	"	+ 2	- 1	- 2	+ 2
$-\frac{3}{16}e^2e'$	"	+ 1	+ 1	- 1	- 2
$-\frac{3}{4}ee'^2$	"	+ 1	+ 2	- 1	- 1
$+\frac{3}{2}e'\eta^2$	"	+ 1	- 1	+ 1	0
$-\frac{1}{8}e^2$	"	+ 1	0	- 1	+ 2
$-\frac{3}{8}e^2$	"	+ 1	0	- 1	- 2
$+ 3ee'$	"	- 1	+ 1	+ 1	+ 1
$-\frac{1}{8}e'^2$	"	- 1	+ 2	+ 1	0

$-\frac{r' \cos H}{r^2} = \frac{a}{a^2}$ into						L'-Θ'	L'-Π'	L-Θ	L-Π
$-\frac{3}{8} e'^2$	cos	+ 1	+ 2	- 1	0
$-\eta^2$	"	+ 1	0	+ 1	0
$-\frac{1}{6} e^3$	"	+ 1	0	- 1	+ 3
$-\frac{1}{3} e^3$	"	+ 1	0	- 1	- 3
$+\frac{8}{15} e^2 e'$	"	- 1	+ 1	+ 1	+ 2
$-\frac{1}{15} e^2 e'$	"	+ 1	+ 1	- 1	+ 2
$-\frac{1}{4} e e'^2$	"	- 1	+ 2	+ 1	+ 1
$-\frac{1}{24} e'^3$	"	- 1	+ 3	+ 1	0
$-\frac{1}{3} e'^3$	"	+ 1	+ 3	- 1	0
$-2 e \eta^2$	"	+ 1	0	+ 1	+ 1
$-\frac{1}{2} e' \eta^2$	"	+ 1	+ 1	+ 1	0

It is hardly necessary to observe that, to obtain the expressions of the Disturbing Functions, these additional terms are to be combined with the corresponding terms in the expression of the reciprocal of the distance: thus, in the Disturbing Function Ω (m' upon m), the entire term depending on $\cos [L' - \Theta' - (L - \Theta)]$ is

$$= m' \left\{ 2 (1, \dots 20)_{i=1} + \frac{a}{a^2} \left(-1 + \frac{1}{2} (e^2 + e'^2) + \eta^2 \right) \right\} \cos [(L' - \Theta') - (L - \Theta)],$$

where, however, the supplemental term is taken to the third order only.

Glaisher, James Whethead Lee

V. *On the Law of Facility of Errors of Observations, and on the Method of Least Squares.* By J. W. L. GLAISHER, B.A., F.R.A.S., F.C.P.S., Fellow of Trinity College, Cambridge.

Read April 12, 1872.

5-1462

THE *American Journal of Science and Arts* for June 1871 contains a historical note by Professor CLEVELAND ABBE, the object of which is to point out that Professor ROBERT ADRAIN, of New Brunswick, published the method of Least Squares in 1808, having been independently led to its discovery.

It is well known that all the proofs that have been given of the method of Least Squares contain, to say the least, some points of difficulty, and on this account any new investigation of the result is necessarily a matter of much interest. Although some of the investigations of the law of facility e^{-m^2} are far from rigorous, still there is not one that is not of some importance, as throwing additional light on the properties of this law; so that a fresh investigation, and one, moreover, by which the law, not previously known to the author, was discovered, might be expected to be a real addition to, or at all events confirmation of, the known processes. Dr. ADRAIN'S proof, however, seems to me much inferior, both in point of rigour and conclusiveness, to any of the usual investigations; but, for the reasons stated above, it appears worth while to notice the reasoning by which he obtained the law of facility.

Since the method of Least Squares was first proposed by LEGENDRE and GAUSS, there have been several demonstrations given, some of which have had for their object to prove the law of facility, while in others it was only sought to prove the method of combining linear equations, known as that of least squares.

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Some of the most important investigations were carefully analysed and improved by LESLIE ELLIS, in a memoir in the eighth volume of the *Cambridge Philosophical Transactions*, but with no special reference to the law of facility. Much has subsequently been written on the subject by HERSCHEL, ELLIS, BOOLE, DE MORGAN, &c. I propose, therefore, after noticing Prof. ADRAIN's proof, to give an account of the manner in which the law e^{-x^2} follows from the different assumptions that have been made with respect to the nature of errors, &c., and to examine how far these assumptions are consistent with one another; it will also be necessary to notice the manner in which GAUSS and ENCKE, LAPLACE, POISSON, ELLIS, DONKIN, &c., have considered the subject, and to discuss at some length the *à priori* evidence in favour of the assumption that the arithmetic mean is the most probable result of a number of presumably equally good discordant observations. The matter will be throughout considered chiefly with reference to its fundamental principles and assumptions, but in treating of LAPLACE's method, the peculiarity of which consists as much in the analysis as in the suppositions, it will be necessary to examine the mathematical part of the subject. The paper will conclude with a few remarks as to the most probable results, if the law of facility were known to be e^{-x^2} , a form, which, besides being *à priori* very natural, was, in fact, the one assumed by LAPLACE in one of his earlier memoirs.

It will be convenient to state Dr. ADRAIN's proof as briefly as possible and in differential (not fluxional) notation, and then to notice the points of interest contained in it. I fear that it will be thought that the comments on the investigation extend to a far greater length than the importance of the proof deserves; but several of the points suggested seem of sufficient interest to merit discussion, independently of the proof under consideration.

Dr. ADRAIN enunciates the question thus, "Supposing AB to be the true value of any quantity of which the measure by observation or experiment is $A b$, the error being $B b$; what is the expression of the probability that the error $B b$ happens in measuring AB ?" The investigation is then as follows: "Let $AB, BC, \&c.$, be successive distances of which the values by measure are $A b, b c, \&c.$, the whole error being $C c$; now, supposing the measures $A b, b c$, to be given, and also the whole error $C c$, we assume as a self-evident principle, that the most probable distances AB, BC , are proportional to measures $A b, b c$, and therefore the errors belonging to AB ,

BC, are proportional to their lengths, or to the measured values $A b, b c$. If, therefore, we represent the values of $A B, B C$, or of their measures $A b, b c$, by a, b , the whole error $C c$ by E ,* and the errors of the measures $A b, b c$, by x, y , we must for the greatest probability have the equation $\frac{x}{a} = \frac{y}{b}$."

Let $\phi(a, x)$ † express the probability that an error x occurs in measuring a distance a , the probability of the occurrence of errors x, y , in measuring distances a, b is $\phi(a, x) \cdot \phi(b, y)$. "If now we were to determine the values of x and y from the equations $x + y = E$ and $\phi(a, x) \cdot \phi(b, y) = \text{maximum}$, we ought evidently to arrive at the equations $\frac{x}{a} = \frac{y}{b}$; and since x and y are rational functions of the simplest order possible of a, b , and E , we ought to arrive at the equation $\frac{x}{a} = \frac{y}{b}$ without the intervention of roots; in other words, by simple equations; or, which amounts to the same thing in effect, if there be several forms of $\phi(a, x)$, and $\phi(b, y)$, that will fulfil the required condition, we must choose the simplest possible, as having the greatest possible degree of probability."

For $\phi(a, x) \cdot \phi(b, y)$, to be maximum subject to the condition $x + y = E$, we evidently have $\frac{\phi'(a, x)}{\phi(a, x)} dx + \frac{\phi'(b, y)}{\phi(b, y)} dy = 0, dx + dy = 0$, whence

$$\frac{\phi'(a, x)}{\phi(a, x)} = \frac{\phi'(b, y)}{\phi(b, y)}$$

"Now this equation ought to be equivalent to $\frac{x}{a} = \frac{y}{b}$; and this circumstance is effected in the simplest manner possible, by assuming $\frac{\phi'(a, x)}{\phi(a, x)} = \frac{m x}{a}$ and $\frac{\phi'(b, y)}{\phi(b, y)} = \frac{m y}{b}$; m being any fixed number which the question may require."

By integration, we have at once $\phi(a, x) = e^{a' + \frac{m x^2}{2a}}$. Dr. ADRAIN notices that m must be negative, and that "the probability that the errors $x, y, z \dots$ happen in the distances $a, b, c \dots$ is $e^{a' + b' + c' + \dots + \frac{m x^2}{2a} + \frac{m y^2}{2b} + \frac{m z^2}{2c} + \dots}$ &c." and the investigation concludes with the remark that "we have, on choosing the constants properly, $y = e^{j^2 - x^2}$ for the equation of the curve of probability,

* This is misprinted C in the paper.

† The notation is changed.

or putting $f^2 = 0$, $y = e^{-x^2}$, which is the simplest form of the equation expressing the nature of the curve of probability." Prof. ABBE extracts the account, which has in substance been given above from the *Analyst*, an American journal, for 1808, and adds that the general solution is followed by its application to four problems, whose enunciations only are given. Reference is also made to two papers by Dr. ADRAIN, in the *Transactions of the American Philosophical Society* for 1817, which Prof. ABBE states were written in 1808. The *Analyst* is not accessible to me, but, on referring to the *American Philosophical Transactions*, vol. i., new series, 1817, p. 119,* I see that Dr. ADRAIN has made use of the method of least squares, and, as he only refers to his own paper in 1808, it is to be inferred that he discovered this mode of treatment of observations, in ignorance of LEGENDRE'S and GAUSS'S writings. The manner of application is as follows:—Having a table of the lengths of the pendulums vibrating seconds in different latitudes, we ought, by CLAIRAUT'S theorem, to have the equation $r = x + y \sin^2 \lambda$ satisfied exactly, r being the length of the pendulum, λ the latitude, and x and y constants. This not being the case, Dr. ADRAIN determines x and y so as to make $(x + y \sin^2 \lambda_1 - r_1)^2 + (x + y \sin^2 \lambda_2 - r_2)^2 + \dots$ a minimum.†

Returning to Dr. ADRAIN'S reasoning, the assumption at the beginning is, that if we measure consecutive distances a and b , and know from other considerations that E is the total error committed, then the most probable distribution of the error over the two measurements is when $\frac{x}{a} = \frac{y}{b}$, x and y being the errors in a and b , so that $x + y = E$. This seems very far from being evident, not to say very far from being true, generally: it seems scarcely likely that the error should be directly proportional to the distance measured. In whatever manner the measurement is effected, one would expect a less relative error in a greater distance. If the distances were measured, as in the determination of a base line, the error ought certainly to be relatively less, the greater the length measured, to allow for the neutralisation of positive and negative errors, and $\frac{x}{\sqrt{a}} = \frac{y}{\sqrt{b}}$ would seem a more

* "Investigation of the Figure of the Earth and of the Gravity in different Latitudes."

† An identical application of the method of least squares is given by PUISSANT, *Traité de Géodésie*, vol. ii. p. 341, 1819.

natural assumption. If AB , BC , were angular distances in a horizontal plane measured by a theodolite, there would be no reason to anticipate a greater error in the measurement of a greater distance, and we should expect the most probable error to be independent of the distance measured. (This case agrees very well with the condition assumed in the question: If the angle between objects A and B were observed, and then travelling round the circle, the angle between B and A , the whole error would then be known as the difference between 360° and the sum of the observed values of AB , BA .) If, however, as seems likely by one of the examples to which he has applied his method, Dr. ADRAIN had in his mind the distances obtained in a field survey, the assumed relation might be as likely as any other, as though the error of the theodolite reading might be independent of the arc, a greater angle might in general be supposed to correspond to a greater distance. The majority of observations, however, are not of this class; they are generally readings of an instrument. On the whole, therefore, the assumption that we must for the greatest probability have the equations $\frac{x}{a} = \frac{y}{b}$ seems most arbitrary, whether it is intended as a result justified by experience or self-evident *à priori*. The only sort of reasoning by which a result of this kind could be justified would be somewhat similar to the following:—It is moderately clear that in general the most probable error will depend on the length measured, so that we have $\frac{x}{f(a)} = \frac{y}{f(b)}$. In some cases we see that $\frac{x}{a} = \frac{y}{b}$, $\frac{x}{\sqrt{a}} = \frac{y}{\sqrt{b}}$, &c., would give results agreeing pretty nearly with the error our judgment would lead us to assign. We are, however, utterly unable to assign any one form to f as more likely than another, unless we know something of the manner in which the measures are made. Knowing, therefore, nothing at all about f , we take, in order to deduce a rule, the form which will be most simple in appearance, anticipating that, by doing so, the succeeding analysis will be rendered more convenient. It is, however, to be remarked that, in the choice of the simplest, where several algebraical formulæ seem *in rerum naturâ* to be equally probable, we are really doing rather more than merely suiting our own convenience, as it is matter of experience that, as expressions of natural laws, less complicated formulæ occur more frequently than the more complex ones, just as simple equations are more common

than quadratics, &c., so that, in a hypothetical case, where no reason of any kind (such as an analogy, &c.) exists for preferring one form to another, we might be more or less justified by the Theory of Probability (regarded as drawing inferences from experience) in taking the simplest algebraical formula. This reasoning, however, is very slight, as we are thrown on all the fundamental difficulties of the subject of probability, viz., from what class the observed results are to be selected, so as to apply to the class under consideration, &c.; and it should only be admitted in the total absence of every reason which might indicate a likely form for the expression. It is thus clear how vague and arbitrary is the assumption of $\frac{x}{a} = \frac{y}{b}$, as the most probable equation, and, which is of more importance in inquiries of this kind, how difficult it is to determine the exact nature of the assumption made. Even assuming that the most probable errors are proportional to the lengths, one other point is worth notice. As positive and negative errors are (or clearly ought to be supposed) equally likely, we might take one negative, and write $\frac{x}{a} = \frac{y}{b}$ where $x - y = E$; this would, however, require x and y to be greater than in the case when $x + y = E$; and as the probability of errors decreases as their magnitude increases, the former case is the more likely of the two. It will appear further on, that the truth of the equation $\frac{x}{a} = \frac{y}{b}$ is not at all necessary for Dr. ADRAIN'S subsequent meaning, so that the above remarks, though bearing upon a statement in the proof, are not relevant to the ultimate result obtained.

In the second portion of the investigation we have to determine the form of ϕ from the sole condition that $\phi(a, x) \cdot \phi(b, y)$, subject to the condition $x + y = E$, shall be a maximum when $\frac{x}{a} = \frac{y}{b}$: the condition for the maximum we may write

$$\psi(a, x) = \psi(b, y),$$

(ϕ being put for $\frac{\phi'}{\phi}$), and this must be equivalent to

$$F\left(\frac{x}{a}\right) = F\left(\frac{y}{b}\right).$$

F being arbitrary: all that can be learned from this is the manner in which ψ involves a and x , viz., ψ must be a function of $\frac{x}{a}$; therefore,

$$\frac{\phi'(a, x)}{\phi(a, x)} = F\left(\frac{x}{a}\right) \quad \text{and} \quad \phi(a, x) = \left\{ \chi\left(\frac{x}{a}\right) \right\}^a,$$

the complete solution of the equation. It is important to observe that what is specially sought is, not the manner in which x and a enter into the equation, but the form of the facility function $\phi(x)$; and, by taking $\psi(x)$, proportional to x , the very form which it is required to find is assumed. As the deduction of the form $e^{-\lambda^2 x^2}$ is dependent simply on the proportionality of $\psi(x)$ and x , the result would have been the same if the equation had been $\frac{x}{f(a)} = \frac{y}{f(b)}$ instead of $\frac{x}{a} = \frac{y}{b}$. It is needless to comment on the arbitrary assumption that gives the form $e^{-\frac{m x^2}{2a}}$ to $\left\{ \chi\left(\frac{x}{a}\right) \right\}^a$; it cannot even be justified (in the manner previously mentioned), as the simplest form, for why make $\frac{\phi'}{\phi}$ simple in preference to ϕ ? If anything, the simplest and most suitable form should be given to $\left\{ \chi\left(\frac{x}{a}\right) \right\}^a$, and we have no means of doing this. These objections would apply with equal force if any number of distances $a, b, c \dots$ had been measured subject to $x + y + z + \dots = E$. It is curious to note that the author of the investigation seems not to have been aware that the probability of the occurrence of any given error must be infinitesimal; he continually speaks of the probability of an error x as being equal to $e^{a' + \frac{m x^2}{2a}}$. The same mistake has been made by others (including IVORY). This may explain why the relation between the constants is not determined by integration between $\pm \infty$, so as to find $\frac{h}{\sqrt{\pi}} e^{-\lambda^2 x^2}$ for the law of facility.

The very slight and inconclusive nature of Dr. ADRAIN'S reasoning would lead one to believe that he first remarked the convenience of treating equations by the method of least squares, and subsequently endeavoured to justify it, as above described, by the Theory of Probabilities. Whatever may be thought, however, of his reasoning, we must in Prof. ABBE'S words, "credit Dr. ADRAIN with the independent invention and application of the most valuable arithmetical process that has been invoked to aid the progress of the exact sciences."

I pass now to the other investigations of the law of facility and the method of least squares that have been given. The method, as is well

known, was first proposed in print by LEGENDRE, in his *Nouvelles Méthodes pour la Détermination des Orbites des Comètes* (Paris, 1806), as a convenient way of treating observations, without reference to the Theory of Chance. In his preface, LEGENDRE remarks, "La méthode qui me paroît la plus simple et la plus générale, consiste à rendre minimum la somme des quarrés des erreurs . . . et que j'appelle méthode des moindres quarrés . . ." In an Appendix (p. 72), in which the application of the method is explained, LEGENDRE'S words are, "De tous les principes qu'on peut proposer pour cet objet, je pense qu'il n'en est pas de plus général, de plus exact, ni d'une application plus facile que celui dont nous avons fait usage dans les recherches précédentes, et qui consiste à rendre minimum la somme des quarrés des erreurs." It may be noted that LEGENDRE recommends the rejection of observations which differ too much from the results obtained by the method.

Though first published by LEGENDRE, the method of least squares was applied by GAUSS, as he himself states,* as early as 1795, and the method is explained, and the usual law of facility for the first time found in the *Theoria Motus Corporum Cælestium* (Hamburg, 1809): this is the first investigation in which the theory is developed from the principles of Probability. In examining the proofs that have been given, it will be convenient to group them as follows:—

1. GAUSS'S original investigation, including ENCKE'S, DE MORGAN'S, and ELLIS'S remarks on the principle of the Arithmetic Mean.
2. LAPLACE'S method, including POISSON'S and ELLIS'S simplifications, and IVORY'S criticisms.
3. GAUSS'S second demonstration, and its connexion with LAPLACE'S.
4. Sir JOHN HERSCHEL'S proof, with ELLIS'S and BOOLE'S criticisms thereon.
5. Professor TAIT'S, and similar proofs.
6. DONKIN'S proof.

Besides the above, three demonstrations have been given by IVORY in Volumes LXV. and LXVIII.† (1825 and 1826) of the *Philosophical Magazine* (TILLOCH'S). These seem to have received quite as much atten-

* *Theoria Motus*, lib. ii. sect. iii. ch. 186.

† ELLIS incorrectly says lxvii. *Camb. Phil. Trans.*, vol. viii. p. 217.

tion as they deserve at the hands of ELLIS, who has pointed out very clearly their unsatisfactory character. IVORY endeavoured to establish the principle without recourse to the Theory of Probability, and, as a consequence, his reasoning is inconclusive. His first method depends on an analogy with the principle of the lever in mechanics, and the other two apparently involve confusion of thought. As they seem to have been sufficiently discussed by ELLIS, no formal examination will be made of them in this Paper, although they will be referred to incidentally. Demonstrations have also been given by BESSEL, Mr. CROFTON, &c., to which reference is made on p. 119.

GAUSS deduces the law of facility from the assumption, that if apparently equally good observations x_1, x_2, \dots, x_n are made of a certain quantity, then the most probable value of that quantity is the arithmetic mean of the observations, $\frac{x_1 + x_2 + \dots + x_n}{n}$. Suppose a is the true value of the quantity, then $x_1 - a, x_2 - a, \dots$ are the errors, and if $\varphi(x)$ be the law of facility the *a priori* probability of these errors is proportional to $\varphi(x_1 - a) \varphi(x_2 - a) \dots \varphi(x_n - a)$; whence it follows that, after the observations have been made, the probability that a was the true value is proportional to this same expression; which, therefore, in order to find the most probable value of a , must be made a maximum. Differentiating with regard to a , there results

$$\frac{\varphi'(x_1 - a)}{\varphi(x_1 - a)} + \frac{\varphi'(x_2 - a)}{\varphi(x_2 - a)} + \dots + \frac{\varphi'(x_n - a)}{\varphi(x_n - a)} = 0 \quad (1)$$

By hypothesis this must be satisfied by $a = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$, whatever integer value n may have. Writing then for brevity, $\psi(x)$ for $\frac{\varphi'(x)}{\varphi(x)}$, ψ is to be determined from this condition. Let $x_2 = x_3 = \dots = x_n = x_1 - n\alpha$, therefore $a = x_1 - (n - 1)\alpha$, whence

$$\psi\{(n - 1)\alpha\} + (n - 1)\psi(-\alpha) = 0$$

or

$$\frac{\psi\{(n - 1)\alpha\}}{(n - 1)\alpha} = \frac{\psi(-\alpha)}{-\alpha}, \text{ whatever } \alpha \text{ may be,}$$

therefore $\frac{\psi(x)}{x}$ is constant = m say, and $\varphi(x) = A e^{\frac{1}{2}mx^2}$; we must have m

negative ($= -h^2$) in order that (1) may give a maximum; and since the error must lie between $\pm \infty$, $\int_{-\infty}^{\infty} \phi(x) dx = 1$, and therefore, since

$$\int_0^{\infty} e^{-h^2 x^2} dx = \frac{\sqrt{\pi}}{2h},$$

$$\phi(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}. \quad (2)$$

We infer from this that the arithmetic mean is only the most probable result when the law of facility is given by (2), so that if it could be generally shown that the arithmetic mean possessed this property, we should have a demonstration that (2) was the only law of facility.

That we have no right to assume as an axiom that the arithmetic mean is the most probable result of every series of direct observations (presumed to be equally good) of the same quantity is quite clear; and if it were not clear *per se*, the above investigation which shows that this supposition requires the existence of a special law of facility proves that it cannot be true in all cases, as it is certainly possible to conceive observations subject to laws of facility different from (2).^{*} This reasoning is effective as showing that we cannot assert universally that in every series of observations the arithmetic mean is the most probable result, and that certainly this is not an axiom; but it does not prove that of all the methods that could be proposed of combining observations, this one will not in the long run give the most accurate results. To illustrate this point more fully: suppose one set of observations only is about to be made, it certainly will not follow that to take their arithmetic mean will be the best method of combining them; but it may be true that if 10,000 sets of observations of different kinds were made, the arithmetic means would on the whole be nearer to

* All this is to be understood on the hypothesis that no assumption whatever is made with regard to the nature of an error. Those who have asserted that the arithmetic mean is the most probable value and taken this as a basis for the treatment of observations, have done so on the supposition that we know nothing whatever of an error except that it is as likely to be positive as negative; and this is all the knowledge assumed in the text. It will appear farther on that the consideration of the manner in which an error no doubt does arise leads to GAUSS'S law; but this does not justify the above *assumption* that the arithmetic mean is the most probable result, but shows distinctly that this cannot be evident *per se* independently of the nature of an error.

the truth than any other series of results obtained by a uniform system of treatment, independent of the observations themselves.

In introducing his investigation, GAUSS makes no attempt to prove the principle of the arithmetic mean; his words are, "Quæ quoniam a priori definiri nequit, rem ab altera parte aggredientes inquiremus, cuinam functioni, tacite quasi pro basi acceptæ, proprie innixum sit principium trivium, cujus præstantia generaliter agnoscitur. Axiomatis scilicet loco haberi solet hypothesis, si quæ quantitas per plures observationes immediatas, sub æqualibus circumstantiis æqualique cura institutas, determinata fuerit, medium arithmeticum inter omnes valores observatos exhibere valorem maxime probabilem, si non absoluto rigore, tamen proxime saltem, ita ut semper tutissimum sit illi inhærere."* GAUSS'S view of the subject seems to have been as follows: we wish to find the best method of combining linear equations of the form

$$ax + by + cz + \dots = V \quad (3)$$

(V being directly observed), and judging of the relative precision of the results deduced for x, y, z, \dots . Now if the equations are of the form $x - V = 0$, it is known that to take the arithmetic mean and write $x = \frac{V_1 + V_2 + \dots + V_n}{n}$ gives a very good result, and we shall be contented if we can find as good a rule for determining x, y, z, \dots from equations of the form (3) as this is for determining x from equations of the form $x - V$. GAUSS therefore proceeds to find the law of facility (2) which makes the arithmetic mean the most probable result of a series of direct observations, and then makes use of it to find the most probable values of x, y, z, \dots from a system of linear equations. It follows from (2), taken in connexion with the reasoning by which it was established, that for the most probable values of x, y, z, \dots

$$e^{-h^2(a_1x + b_1y + \dots - V_1)^2 - h^2(a_2x + b_2y + \dots - V_2)^2 - \dots}$$

must be a maximum, whence

$$(a_1x + b_1y + \dots - V_1)^2 + (a_2x + b_2y + \dots - V_2)^2 + \dots$$

must be a minimum, the same result as LEGENDRE'S.

* *Theoria Motus*, lib. ii. sect. iii. ch. 177 (p. 212 of the Original Edition).

In the *Berliner Astronomisches Jahrbuch** for 1834, ENCKE has attempted to prove that the arithmetic mean of a series of direct observations is the most probable value of the quantity observed: the method, briefly stated, is as follows. If we have to determine x from two observations, a and b , we must take as the most probable value $x = \frac{1}{2}(a + b)$, as there is no reason why we should suppose x to have a value nearer to the result of one observation than to that of the other, positive and negative errors being equally probable. Suppose now, three observations a, b, c are made, then we must have $x =$ a symmetrical function of a, b, c .† The first two,—or generally two,—observations considered alone would have given, according to the arrangement taken, as the only results to be chosen

$$\frac{1}{2}(a + b), \quad \frac{1}{2}(a + c), \quad \frac{1}{2}(b + c);$$

to these the third observation adds c, b, a . Now we shall no longer be allowed to connect symmetrically both quantities in any single arrangement, as the one depends on two observations, the other on one. But there must unquestionably be some form for the connexion of both, which would also produce the result to be preferred from all three observations, and this form, which may be denoted by ψ , must be the same for all three. Therefore we have for x the three values

$$\begin{aligned} x &= \psi \left\{ \frac{1}{2}(a + b), c \right\} \\ &= \psi \left\{ \frac{1}{2}(a + c), b \right\} \\ &= \psi \left\{ \frac{1}{2}(b + c), a \right\} \end{aligned}$$

Writing $a + b + c = s$, $x = \psi \left\{ \frac{1}{2}(s - c), c \right\} = \psi(s, c)$, &c., but $\psi(s, c)$, $\psi(s, b)$, $\psi(s, a)$ cannot be identical unless $x = \psi(s)$ simply; then, putting $a = b = c$, $a = \psi(3a)$, whence $x = \frac{1}{3}(a + b + c)$. The method is extended by induction to the case when there are n observations.

The objection to this investigation is obvious. Why should the most

* "Ueber die Methode der Kleinsten Quadrate."

† *Jahrbuch*, p. 265. As the next twelve lines contain the critical portion of the proof they are translated quite literally.

probable result from a, b, c be a function of the most probable result from a and b , and from c ? What would be thought of the assumption that the position of the point equidistant from three points A, B, C, depended only on the position of the point equidistant from A and B, and on the position of C? Yet this seems quite as evident as the assertion made in the proof. ENCKE's investigation is reproduced by CHAUVENET in his Appendix on the Method of Least Squares with great confidence; he concludes with the following remark:* "The principle here demonstrated, that the arithmetical mean of a number of equally good observations is the most probable value of the observed quantity, is that which has been universally adopted as the most simple and obvious, and might well be received as axiomatic. The above demonstration is chiefly valuable as exhibiting somewhat more clearly the nature of the assumption that underlies the principle, which is that, under strictly similar circumstances, positive and negative errors of the same absolute amount are equally probable."

ENCKE has remarked that the principle of the arithmetic mean may either be regarded as proved by his mathematical demonstration, or by experience, as this method of combining simple observations has been universally adopted with uniform success; but it is almost needless to remark that though experience has shown that the principle gives very good results, it certainly cannot have shown that it gives the best possible, for the obvious reason that generally this method has been alone tried; while to establish that the results were the most probable it would be necessary to employ every possible method repeatedly, and even then the decision would be impossible. What was in effect GAUSS's view, viz., that the arithmetic mean is *practically* the best mode of combining simple observations, and that experience has justified its adoption by the accuracy of the results obtained, so that we shall be satisfied with an equally good method of treating linear equations, was quite reasonable and consistent, but he was very far from asserting, as a result deduced from the Theory of Probability, that the arithmetic mean is the most probable value of the quantity observed.

ELLIS † has pointed out that the rule of the arithmetic mean gives a

* CHAUVENET's *Practical and Spherical Astronomy*, vol. ii. p. 475.

† *Camb. Phil. Trans.*, vol. viii. p. 205.

result which certainly coincides with the true value when the number of observations is increased *sine limite*. Let a be the true value, x the observed value, ε the error, then

$$\begin{aligned}x_1 - a &= \varepsilon_1 \\x_2 - a &= \varepsilon_2 \\&\&c. = \&c.\end{aligned}$$

therefore

$$\Sigma(x - a) = \Sigma \varepsilon = 0$$

and

$$a = \frac{\Sigma x}{n},$$

since, in the long run, there being no permanent cause tending to make the sum of the positive errors differ from that of the negative errors, $\Sigma \varepsilon = 0$.* But exactly the same considerations, ELLIS proceeds, would show that $\Sigma f(\varepsilon) = 0$, f being any odd function, so that we should have $\Sigma f(x - a) = 0$.

There is no discrepancy between these results; both are true at the limit, and neither when n is finite, and "no satisfactory reason can be assigned why, setting aside mere convenience, the rule of the arithmetic mean should be singled out from the other rules which are included in the general equation $\Sigma f(x - a)^\dagger = 0$."

It might for a moment appear that, if the error was very small, the two results would always coincide, since $f(\varepsilon) = A\varepsilon + B\varepsilon^3 + \dots$ of which the first term need only be retained; but in the case of $A = 0$ we should have $\Sigma(x - a)^3 = 0$; or $\Sigma(x - a)^5 = 0$, &c., or we might even have fractional powers and take $\Sigma(x - a)^{\frac{1}{2}} = 0$, &c.

* We can scarcely say that $\Sigma \varepsilon$ must be zero when n is infinite, but it is clearly true that $\Sigma \varepsilon$ will be infinitesimal with regard to n , so that $\frac{1}{n} \Sigma \varepsilon$ is zero. This is all that is required for $a = \frac{1}{n} \Sigma x + \frac{1}{n} \Sigma \varepsilon = \frac{1}{n} \Sigma x$, and in the general case $\Sigma \varepsilon$ can be neglected, when n is infinite, compared to the terms involving a in $\Sigma f(x - a)$.

† An investigation of ELLIS's (loc. cit. p. 207) to show that it does not follow that, because the arithmetic mean is the most probable value when n is infinite, therefore it is so when n is finite has not been noticed in the text, as the conclusion seems sufficiently evident without it. It is only here alluded to in order to correct a misprint. The sentence "where k is that portion of K which is derived from observations of a_k " should be "where k_r is that portion of K which is derived from observations of a_r ."

To equate to zero the sums of any power of the errors except the first, would be practically out of the question, as the equations would not admit of convenient solution. Professor TAIT has given an excellent illustration of what an assumption for the sake of simplicity may amount to, in his remark that the principle of the arithmetic mean has been adopted from a multitude of others "just as we might suppose a calculator to insist on gravity varying as the direct distance instead of the inverse square on the ground that the problem of Three Bodies would then become as simple and its solution as exact, as they are now complicated and at best only approximate."*

ELLIS states that IVORY gave three demonstrations of the Method of Least Squares, but he really gave four; the reasoning of the second† one, however, is of so loose a character, that ELLIS may have passed it over intentionally, though IVORY undoubtedly regarded it as a demonstration. It seems to amount to no more than the following, and would be scarcely worth noticing if it were not that the concluding sentence expresses IVORY's opinion of the rule of the arithmetic mean. If a number of observations be made, the sum of the errors will approach zero, but the mean of the squares of the errors will approach more nearly to a determinate value as the observations are more numerous, and this limit may be taken as a measure of the precision of the observations. If, then, several sets of observations are made, that one deserves the preference, the mean of the squares of the errors of which is least. "Now, in solving a system of equations of condition in several different ways, the errors will acquire different magnitudes, just as happens in several sets of observations of unequal degrees of precision. . . . That mode of solution is therefore to be preferred in which the mean of the squares of the errors is the least," whence the method of least squares.

It is unnecessary to point out that one might as well assume the whole method at once, as assume that the mean of the sum of the squares is the measure of precision. IVORY does acknowledge that everything which has been said of the squares of the errors might have been said of any even powers of them, and the principle on which he justifies his choice of the second powers may in its simplest form be stated thus. For $(x_i - a)^{2n}$

* *Edinburgh Transactions*, vol. xxiv. p. 140.

† TILLOCH'S *Philosophical Magazine*, vol. lxxv. (1824), pp. 6 and 7.

+ $(x_2 - a)^{2n} + \dots$ to be a minimum, we must have

$$(x_1 - a)^{2n-1} + (x_2 - a)^{2n-1} + \dots = 0$$

or

$$\epsilon_1^{2n-1} + \epsilon_2^{2n-1} + \dots = 0$$

but "it is universally admitted that in a series of observations made in like circumstances, the simple sum of the errors is equal to zero, and not the sum of their cubes or fifth powers, or any other of their odd powers." Whether universally admitted or no, this is certainly not true.

Perhaps the most important contribution that has been made to the theory of the arithmetic mean is contained in a Paper by DE MORGAN *On the Theory of Errors of Observation*.* It is there pointed out that the mean value of all the given values is also the *mean supposition of all possible suppositions* as to the mode of obtaining value. The investigation is too long to reproduce, but the result may be stated thus. If $\phi(x_1, x_2, \dots, x_n)$ is the most probable result of the discordant observations, x_1, x_2, \dots, x_n then

$$\begin{aligned} \phi(x_1, x_2, \dots, x_n) = & \frac{x_1 + x_2 + \dots + x_n}{n} + Q \Sigma \epsilon_i^2 + R \Sigma \epsilon_i \epsilon_j + S \Sigma \epsilon_i^3 \\ & + T \Sigma \epsilon_i^2 \epsilon_j + U \Sigma \epsilon_i \epsilon_j \epsilon_k + \dots \end{aligned}$$

the ϵ 's being small quantities, and Q, R, S . . . dependent on ϕ ; "hence $\phi(x_1, x_2, \dots, x_n)$ is $\frac{1}{n} \Sigma x$ augmented by terms of which we have no knowledge whatever, either as to sign or value, and no means of getting any: we are therefore wholly without reason for supposing that the value of $\phi(x_1, \dots)$ lies on one side of the average rather than the other, and must take this average as the most probable value *à priori*."

It is not perhaps very easy to estimate at its true value the above result; it shows that $\frac{1}{n}(x_1 + x_2 + \dots + x_n)$ is the mean value of different forms of ϕ , but scarcely that it is the most probable value. For instance, to recur to the simpler case, we might say $\Sigma(x - a) = \Sigma \epsilon$, and if we know nothing and can find nothing about $\Sigma \epsilon$, there is no reason why $\Sigma(x - a)$ should

* *Cambridge Philosophical Transactions*, vol. x. pp. 416, &c.

lie on one side of zero rather than the other, therefore we must take $\Sigma (x-a) = 0$; but similar reasoning would lead us take $\Sigma (x-a)^3 = 0$ &c. In this case indeed we do know something more about $\Sigma \epsilon$, viz., that it is $\epsilon_1 + \epsilon_2 \dots + \epsilon_n$; and it can be shown that the most probable value of this sum is equal to zero, positive and negative values of the errors being equally likely.* In the case considered by DE MORGAN (denoting all the terms involving ϵ 's by A) the evidence that $A = 0$ is the most probable value of A is not so strong. It is perhaps as well to take the extreme case and notice that if $x = p + A$, where we know nothing about A and have no means of knowing anything, then because we have not the slightest reason for supposing x to be greater than p rather than less than p , it does not at all follow that $x = p$ is the most probable value of x , any more than it follows that if we are told that a point lies somewhere on the line between two given points α and β , the middle point of $\alpha \beta$ is a more probable position of the point than any other. This fallacy does not, I think, enter into DE MORGAN'S theorem, though at first sight it certainly seems to do so. Before leaving the subject of the arithmetic mean, a remark of ENCKE'S,* taken from LAMBERT'S *Photometry*, should be noticed. If we regard the lengths of the perimeters of the inscribed and circumscribed polygons of n sides as two observations of the length of the perimeter of the circle, the arithmetic mean is not the most probable value, as it is a better approximation to add to the perimeter of the inscribed figure the third part of the

* That is to say, that $\iiint \dots d\epsilon_1 \dots d\epsilon_n$, subject to the condition $\epsilon_1 + \epsilon_2 \dots + \epsilon_n > l$ and $< l + h$ (h being an infinitesimal constant) is a maximum when $l = 0$; if the errors are subject to laws of facility $\phi(\epsilon)$, such that $\phi(\epsilon)$ decreases with increase of ϵ , the probability of the zero value is increased. This is clear on general grounds: it follows analytically from the consideration that if the integral be denoted by $\psi(l)$, then $\psi(l) = \psi(-l)$, so that $\psi(0)$ must be either a maximum or a minimum, and as it cannot be the latter, it must be the former; or more fully thus

$$\psi(l) = \frac{h}{\pi} \int_0^\infty \left\{ \int_{-\infty}^\infty \phi_1(\epsilon) \cos \epsilon \theta d\epsilon \right\} \dots \left\{ \int_{-\infty}^\infty \phi_n(\epsilon) \cos \epsilon \theta d\epsilon \right\} \cos l \theta d\theta$$

and since $\cos l \theta$ passes over its period more rapidly as l increases, the positive and negative portions of the integral cancel each other more nearly as l increases, and $\psi(l)$ is a maximum when $l = 0$. See pp. 98, 110, and *Phil. Mag.*, March, 1872.

† *Berliner Astronomisches Jahrbuch*, 1834, p. 263.

difference of the perimeters. The answer is obvious, viz., that the lengths in question are not of the nature of observations;* other reasons could also be given.

If there are only two observations, the arithmetic mean is, as is evident from general considerations, the most probable value; this also follows from the theory, for in this case, $\phi(\varepsilon)$ being the law of facility,

$$\frac{\phi'(x_1 - a)}{\phi(x_1 - a)} + \frac{\phi'(x_2 - a)}{\phi(x_2 - a)} = 0 \quad (4)$$

and since ϕ is an even function, ϕ' is an odd one, therefore $\frac{\phi'}{\phi}$ is an odd function, and (4) becomes on writing $\frac{x_1 + x_2}{2}$ for a ,

$$\frac{\phi'(\frac{x_1 - x_2}{2})}{\phi(\frac{x_1 - x_2}{2})} - \frac{\phi'(\frac{x_1 - x_2}{2})}{\phi(\frac{x_1 - x_2}{2})} = 0,$$

which is true.

Although the principle of the arithmetic mean regarded as a postulate from which to deduce the law of facility, will not be considered again in this Paper; still all the remarks applicable to the method of least squares will have reference to it as a particular case of that general rule.

Whatever may be thought of the rigour of the method by which GAUSS was first led to assume the general truth of the law of facility $e^{-h^2x^2}$, it is undoubtedly to him that the treatment of observations by the method of least squares (including under that title the determination of the mean error, &c.) is due, but whatever strictly philosophical basis the subject has must be attributed to LAPLACE, who proved the usual theorems for the combination of observations (supposed numerous) without assuming any special law of facility for the individual observations.

LAPLACE'S investigations are rendered difficult, partly by want of fullness of explanation, and partly by the peculiar character of the notation used and the novelty of the methods, which, though great examples of analytical skill, appear now so different from the way in which it is natural to consider the subject that it is not easy to feel any great confidence at first in the results as stated by LAPLACE. This difficulty has produced com-

* This will appear even more clearly from the discussion of the nature of an error on pp. 105, 120.

mentators who have rendered portions of LAPLACE'S work far easier to follow. A great part of DE MORGAN'S "Treatise on Probability" in the *Encyclopædia Metropolitana*, is translated and adapted from LAPLACE, enriched by comments. The manner in which LAPLACE'S principles lead to the method of least squares, when there are more than two unknowns, and the relations of GAUSS'S second demonstration to LAPLACE'S occupy a conspicuous place in ELLIS'S Paper, already referred to, and a commentary on the whole work is supplied by TODHUNTER'S *History of the Theory of Probability* (Cambridge and London, 1865).

LAPLACE only proves the method of least squares in the case where there are but two unknown quantities to be determined, and remarks that the same mode of proof will apply, whatever be the number. The necessary extension of the analysis was made by ELLIS, who, however, showed that the results obtained were the same as those found by making the sum of the squares of the residuals a minimum, only by an *à posteriori* verification. This slight blemish can easily be removed by the aid of determinants, and it seems worth while to restate LAPLACE'S principles for the treatment of linear equations, following ELLIS generally, but making this alteration, and keeping the reasoning distinct from the analysis on which the law of facility of the aggregate of the errors depends.

Let the equations of condition be

$$\left. \begin{array}{l} a_1 x + b_1 y + c_1 z + \dots - V_1 = \epsilon_1 \\ a_2 x + b_2 y + c_2 z + \dots - V_2 = \epsilon_2 \\ \quad \quad \quad \&c. \quad \quad \quad \&c. \\ a_n x + b_n y + c_n z + \dots - V_n = \epsilon_n \end{array} \right\}$$

n equations, involving p unknowns $x, y, z \dots$, (n being $> p$). $a_1, b_1 \dots a_n, b_n \dots$ are known quantities given by theory for each observation, and calculable if necessary before the observations are made; V_1, V_2, \dots are quantities determined by direct observation; and $\epsilon_1, \epsilon_2, \dots$ are the errors committed in observing V_1, V_2, \dots and are of course unknown.

To solve these equations multiply by $\mu_1, \mu_2 \dots \mu_n$ and add, thus

$$\begin{aligned} (\mu_1 a_1 + \mu_2 a_2 \dots + \mu_n a_n) x + (\mu_1 b_1 \dots + \mu_n b_n) y + \dots \\ = (\mu_1 V_1 \dots + \mu_n V_n) + (\mu_1 \epsilon_1 \dots + \mu_n \epsilon_n); \end{aligned}$$

choose μ_1, μ_2, \dots so that

$$\left. \begin{aligned} \mu_1 a_1 + \mu_2 a_2 \dots + \mu_n a_n &= 1 \\ \mu_1 b_1 + \mu_2 b_2 \dots + \mu_n b_n &= 0 \\ \mu_1 c_1 + \mu_2 c_2 \dots + \mu_n c_n &= 0 \\ \text{\&c.} & \qquad \qquad \qquad \text{\&c.} \end{aligned} \right\} \quad (5)$$

(p conditions between n quantities); we then have

$$x = [\mu V]^* + [\mu \epsilon]$$

Suppose that the laws of facility† for the observations $V_1, V_2 \dots$ were $\phi_1(x), \phi_2(x) \dots$ and let us calculate the chance that $\mu_1 \epsilon_1 + \mu_2 \epsilon_2 \dots + \mu_n \epsilon_n$ shall lie between the limits l and $-l$. This, of course, cannot in general be done unless $\phi_1(x), \phi_2(x) \dots$ are given; but it is found‡ that if the number (n) of equations be very large, the probability in question will take a certain limiting form, whatever functions $\phi_1, \phi_2 \dots$ may be, and when n is large the probability that $[\mu \epsilon]$ lies between $-l$ and l is equal to

$$\frac{2}{\sqrt{\pi}} \int_0^l \frac{e^{-x^2}}{2\sqrt{[\mu^2 k^2]}} dx \quad (6)$$

Now (5) gives p conditions between the n quantities $\mu_1, \mu_2 \dots \mu_n$: let therefore the $n - p$ conditions to which we may still subject them be used in making (6) a minimum, so that we shall have the greatest probability that $[\mu \epsilon]$, the error of x , may lie between l and $-l$.

For (6) to be a maximum $[\mu^2 k^2]$, must be a minimum, viz., $k_1^2 \mu_1^2 + k_2^2 \mu_2^2 \dots + k_n^2 \mu_n^2$ must be a minimum, this and the equations (5) give

$$\left. \begin{aligned} k_1^2 \mu_1 d\mu_1 + k_2^2 \mu_2 d\mu_2 \dots + k_n^2 \mu_n d\mu_n &= 0 \\ a_1 d\mu_1 + a_2 d\mu_2 \dots + a_n d\mu_n &= 0 \\ b_1 d\mu_1 + b_2 d\mu_2 \dots + \mu_n d\mu_n &= 0 \\ \text{\&c.} & \qquad \qquad \qquad \text{\&c.} \end{aligned} \right\}$$

* $[\mu V]$ is written for $\mu_1 V_1 + \mu_2 V_2 \dots + \mu_n V_n$, a notation introduced by GAUSS, and generally adopted in the *Theory of Errors*. $[ab]$ is clearer than $\Sigma(ab)$.

† If the observed value of a quantity be x , and the chance of an error between ϵ and $\epsilon + d\epsilon$ be $\phi(\epsilon) d\epsilon$, then $\phi(x)$ will be called indifferently the law of facility of the observation x , or of the error ϵ . Also since it is only the form of ϕ with which we are concerned, $A\phi(x)$ A being any constant will be called the law of facility indifferently with $\phi(x)$; the constant being always determinable by the condition $A \int_{-\infty}^{\infty} \phi(x) dx = 1$

‡ See pp. 96, &c.; $k^2 = \int_0^{\infty} x^2 \phi_1(x) dx$.

Multiply the first of these equations by -1 , and the second, third, &c., by $\lambda_1, \lambda_2, \&c.$, and add; therefore

$$\left. \begin{aligned} k_1^2 \mu_1 &= a_1 \lambda_1 + b_1 \lambda_2 + \dots \\ k_2^2 \mu_2 &= a_2 \lambda_1 + b_2 \lambda_2 + \dots \\ &\&c. \qquad \qquad \&c. \end{aligned} \right\} \quad (7)$$

Multiply by $\frac{a_1}{k_1^2}, \frac{a_2}{k_2^2}, \&c.$, and add, therefore

$$1 = \left[\frac{a^2}{k^2} \right] \lambda_1 + \left[\frac{ab}{k^2} \right] \lambda_2 + \dots$$

Similarly by multiplying by $\frac{b_1}{k_1^2}, \frac{b_2}{k_2^2} \&c.$, and adding,

$$0 = \left[\frac{ab}{k^2} \right] \lambda_1 + \left[\frac{b^2}{k^2} \right] \lambda_2 + \dots;$$

$\&c. \qquad \qquad \&c.$

write these equations

$$\begin{aligned} \alpha_1 \lambda_1 + \alpha_2 \lambda_2 + \dots &= 1 \\ \beta_1 \lambda_1 + \beta_2 \lambda_2 + \dots &= 0 \\ \gamma_1 \lambda_1 + \gamma_2 \lambda_2 + \dots &= 0 \\ &\&c. \qquad \qquad \&c. \end{aligned}$$

so that $\alpha_2 = \beta_1, \alpha_3 = \gamma_1, \&c.$, then solving for λ_1 ,

$$\begin{aligned} M \lambda_1 &= A_1 \\ M \lambda_2 &= A_2 \&c. \end{aligned}$$

$$\text{where } M = \begin{vmatrix} \alpha_1, \alpha_2, \alpha_3 \dots \\ \beta_1, \beta_2, \beta_3 \dots \\ \gamma_1, \gamma_2, \gamma_3 \dots \\ \dots \end{vmatrix}$$

and $A_1, A_2, \&c.$, are the minors of $\alpha_1, \alpha_2, \&c.$

Therefore, from (7)

$$M \mu_1 = \frac{a_1}{k_1^2} A_1 + \frac{b_1}{k_1^2} A_2 + \dots$$

and since $x = \mu_1 V_1 + \mu_2 V_2 + \dots$

$$M x = \frac{a_1}{k_1^2} V_1 A_1 + \frac{b_1}{k_1^2} V_1 A_2 + \dots$$

$$\begin{aligned}
 & + \frac{a_2}{k_2^2} V_2 A_1 + \frac{b_2}{k_2^2} V_2 A_2 + \dots \\
 & + \dots \dots \dots \\
 & = \left[\frac{a}{k^2} V \right] A_1 + \left[\frac{b}{k^2} V \right] A_2 + \dots
 \end{aligned}$$

which is the value of x we should obtain from the system of equations

$$\begin{aligned}
 \alpha_1 x + \alpha_2 y + \alpha_3 z + \dots &= \left[\frac{a}{k^2} V \right] \\
 \beta_1 x + \beta_2 y + \beta_3 z + \dots &= \left[\frac{b}{k^2} V \right], \\
 \&c. \qquad \qquad \qquad \&c.
 \end{aligned}$$

or, writing for $\alpha_1, \alpha_2, \&c.$; their values,

$$\begin{aligned}
 \left[\frac{a^2}{k^2} \right] x + \left[\frac{ab}{k^2} \right] y + \dots &= \left[\frac{a}{k^2} V \right] \\
 \left[\frac{ab}{k^2} \right] x + \left[\frac{b^2}{k^2} \right] y + \dots &= \left[\frac{b}{k^2} V \right] \\
 \&c. \qquad \qquad \qquad \&c.
 \end{aligned}$$

the same equations as result from rendering a minimum the sum of the squares of the errors multiplied by the weights $\frac{1}{k_1^2}, \frac{1}{k_2^2} \&c.$, viz.

$$\frac{1}{k_1^2} (a_1 x + b_1 y + \dots - V_1)^2 + \frac{1}{k_2^2} (a_2 x + b_2 y + \dots - V_2)^2 + \dots$$

A similar treatment of $y, z \&c.$, would lead to the same result, whence the method of least squares follows. If $\phi_1 = \phi_2 = \&c.$ (as LAPLACE supposes the case) $k_1^2 = k_2^2 = \&c.$

The above is the method of LAPLACE, who, however, as previously remarked, contented himself with giving the process when there were but two unknowns x and y ; the extension to any number is due to ELLIS. I proceed now to the analysis requisite to the determination of the probability that $\mu_1 \varepsilon_1 + \dots \mu_n \varepsilon_n$ lies between $\pm l$. This LAPLACE effects by considering the coefficient of $e^{l \cdot w}$ in the expansion of

$$\left\{ \phi \left(\frac{n}{n} \right) e^{-n \cdot w} + \phi \left(\frac{n-1}{n} \right) e^{-(n-1) \cdot w} \dots + \phi \left(\frac{0}{n} \right) \dots + \phi \left(\frac{n}{n} \right) e^{n \cdot w} \right\} s^*$$

* LAPLACE'S *Théorie des Probabilités*, pp. 333, &c. All the references will be made to the National Edition (t. vii. of the *Œuvres de Laplace*), 1847.

and obtaining its value by means of his well-known theorem for approximating to the values of integrals involving quantities raised to high powers. ELLIS was the first to replace the algebraic determination of this coefficient by a multiple integral which he evaluated by means of the double integral of FOURIER'S Theorem. LAPLACE investigates several different causes, viz., firstly, when the law of facility is a constant, so that errors of all amounts are equally likely; secondly, when the law of facility at each observation = $\phi(x)$, positive and negative errors being equally likely; and lastly, when the law of facility is the same for all the observations, but positive and negative errors need not be equally likely or have the same ranges. POISSON, however, in the *Connaissance des Temps*, 1827 (p. 284 *et seq.*), has proved a more general case which includes all of these; the investigation is reproduced by TODHUNTER, pp. 561 to 566 of his *History of Probability*. The analysis resembles that of LAPLACE in this respect, viz., that the errors are treated as integer multiples of a certain quantity ω which is ultimately made infinitesimal. This seems rather to complicate the matter, as it renders the investigation, in effect, an evaluation of a multiple integral, quite from first principles. In the *Philosophical Magazine*, for March, 1872, I somewhat simplified ELLIS'S analysis by replacing FOURIER'S double integral by a single integral first made use of for the evaluation of definite integrals by LEJEUNE-DIRICHLET, and I now propose to prove POISSON'S general theorem by means of the same principle.

It is required to find the value of

$$\iint \dots \phi_1(\epsilon_1) \phi_2(\epsilon_2) \dots \phi_n(\epsilon_n) d\epsilon_1 d\epsilon_2 \dots d\epsilon_n \quad (8)$$

where $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ may have all values subject to the condition that $\mu_1 \epsilon_1 + \mu_2 \epsilon_2 \dots + \mu_n \epsilon_n$ lies between $c - \eta$ and $c + \eta$. The facility functions $\phi_1(\epsilon_1), \dots, \phi_n(\epsilon_n)$ may be discontinuous to any extent; (to take a case that probably very nearly represents the physical fact, $\phi_i(\epsilon_i)$ may be equal to $f_i(\epsilon_i)$ from $\epsilon_i = 0$ to $\epsilon_i = a_i$, and to $F_i(\epsilon_i)$ from $\epsilon_i = 0$ to $\epsilon_i = -b_i$, and vanish except between the limits a_i and $-b_i$. This corresponds to the case when positive and negative errors are not equally probable and have not equal ranges, the law of facility being different for each observation.)

The integral LEJEUNE-DIRICHLET made use of is

$$\frac{2}{\pi} \int_0^\infty \frac{\sin \theta}{\theta} \cos \gamma \theta d\theta$$

which = 1, if γ lies between -1 and 1 , and = 0 if γ transcends these limits.

If then we multiply (1) by

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin \theta}{\theta} \cos \left(\frac{\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n - c}{\eta} \theta \right) d\theta$$

or, which is the same integral transformed by assuming $\theta = \eta \theta'$, by

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin \eta \theta}{\theta} \cos (\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n - c) \theta d\theta,$$

we may take all the integrals in (8) with limits $\pm \infty$, as this factor will leave (8) unaltered unless $\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n - c$ transcends the limits $\pm \eta$, when it reduces it to zero, (8) thus becomes

$$\int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \phi(\varepsilon_1) \dots \phi(\varepsilon_n) \cos (\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n - c) \theta \frac{\sin \eta \theta}{\theta} d\theta d\varepsilon_1 \dots d\varepsilon_n$$

which is equal to the real part of

$$\int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \phi(\varepsilon_1) e^{\mu_1 \varepsilon_1 \theta i} \dots \phi(\varepsilon_n) e^{\mu_n \varepsilon_n \theta i} e^{-c \theta i} \frac{\sin \eta \theta}{\theta} d\theta d\varepsilon_1 \dots d\varepsilon_n \quad (9)$$

Let

$$\int_{-\infty}^{\infty} \phi(\varepsilon_m) \cos \mu_m \varepsilon_m \theta d\varepsilon_m = \rho_m \cos r_m, \quad \int_{-\infty}^{\infty} \phi(\varepsilon_m) \mu_m \varepsilon_m d\varepsilon_m = \rho_m \sin r_m$$

therefore

$$\int_{-\infty}^{\infty} \phi(\varepsilon_m) e^{\mu_m \varepsilon_m \theta i} d\varepsilon_m = \rho_m e^{r_m i},$$

whence (9) becomes

$$\int_0^{\infty} \rho_1 \rho_2 \dots \rho_n e^{(r_1 + r_2 + \dots + r_n - c) i} \frac{\sin \eta \theta}{\theta} d\theta$$

the real part of which is

$$\int_0^{\infty} \rho_1 \rho_2 \dots \rho_n \cos (r_1 + r_2 + \dots + r_n - c \theta) \frac{\sin \eta \theta}{\theta} d\theta$$

a result arrived at by POISSON in the course of his investigation. The proof, therefore proceeds from this point exactly as he has given it; that is to say, it is shown that $\xi_1, \xi_2 \dots \xi_n$ in the above integral may be expressed approximately by $e^{-x^2 \rho}$ and $r_1 + \dots + r_n$ by $l \theta, x^2$ standing for $[\mu^2 h^2]$ and l for $[\mu k]$, h_i^2 and k_i being certain constants, depending on the laws of facility; thus we have ultimately

$$\begin{aligned} & \frac{2}{\pi} \int_0^\infty e^{-x^2 \rho} \cos(l-c)\theta \frac{\sin \eta \theta}{\theta} d\theta & (10) \\ & = \frac{2}{\pi} \int_0^\infty \left\{ \int_{-n}^n \cos(l-c+t)\theta dt \right\} e^{-x^2 \rho} d\theta \\ & = \frac{1}{2x\sqrt{\pi}} \int_{-n}^n e^{-\frac{(l-c+t)^2}{4x^2}} dt^* & (11) \end{aligned}$$

If the laws of facility are such that positive and negative errors are equally probable, $l = 0, c = 0$, and is the quantity previously denoted by l , so that (11) becomes

$$\frac{1}{2x\sqrt{\pi}} \int_{-n}^n e^{-\frac{t^2}{4x^2}} dt = \frac{2}{\sqrt{\pi}} \int_0^{\frac{n}{2x}} e^{-t^2} dt$$

which since $x^2 = [\mu^2 h^2]$ agrees with (6).

The investigation of the method of least squares, when (11) instead of (6) is used, is very similar to the above. The result is somewhat different, as the final conclusion is that $[\frac{1}{h^2}(\epsilon - k)^2]$ instead of $[\frac{\epsilon^2}{h^2}]$ must be a minimum,† so that the usual treatment of equations, by rendering a minimum

* In this investigation TODHUNTER's notation has been followed, with only trivial exceptions. In one part of the work it was more convenient to write ρ_m &c., for ρ , &c., as i was used in the same expressions for $\sqrt{-1}$. TODHUNTER uses γ_i for μ_i . From (10) we see at once that the result may be written $\frac{1}{\sqrt{\pi}} \left\{ \text{Erfc} \frac{(l-c+\eta)}{2x} - \text{Erfc} \frac{(l-c-\eta)}{2x} \right\}$ (*Phil. Mag.* Dec. 1871. p. 421), a form easily seen to be identical with (11).

† Additional theorems on the method of least squares not included in the scope of this paper are given by LAPLACE in his First Supplement. LAPLACE's method, as there described, has been extended by TODHUNTER in vol. xi. of the *Cambridge Transactions*.

the sum of the squares of the residuals, not only assumes equal weights for the observations, but also that positive and negative errors are equally probable.

It has been pointed out by ELLIS and TODHUNTER that LAPLACE'S proof does not show the method of least squares to give the most probable values of the elements to be determined. This LAPLACE himself draws attention to: he usually speaks of the method as the most advantageous. The concluding words of the fourth chapter of the *Théorie Analytique des Probabilités* are "cette méthode . . . donne . . . les corrections les plus précises, du moins lorsqu'on ne veut employer que des équations finales qui soient linéaires, condition indispensable, lorsqu'on considère à la fois un grand nombre d'observations; autrement l'élimination des inconnues et leur détermination seraient impraticables."

IVORY has attempted to show that LAPLACE'S investigation tacitly assumes the law of facility $\phi(x)$ to be $e^{-h^2x^2}$, and that, "whatever merit it may have in other respects, it is neither more nor less general than the other solutions of the problem."*

As IVORY'S criticism bears directly on the subject of this paper, and has not been discussed by ELLIS, it is necessary to examine it here, although every reader of LAPLACE'S analysis must feel perfectly certain that no assumption with regard to the form of $\phi(x)$ is either openly or tacitly made. IVORY does not attempt to point out where in LAPLACE'S work the alleged restriction is introduced, or even to examine the analysis of the latter at all, but he justifies his statement by an investigation the object of which is to show that it follows from LAPLACE'S results that the arithmetic mean is the most probable of all results, whence, since this is only true when the law of facility is $e^{-h^2x^2}$, it follows that LAPLACE must have implicitly assumed this form for $\phi(x)$. $V_1 V_2 \dots$ being the observed values of a quantity x , and $\epsilon_1 \epsilon_2 \dots$ being the errors,

$$\begin{aligned} x - V_1 &= \epsilon_1 \\ x - V_2 &= \epsilon_2 \\ &\dots \\ x - V_n &= \epsilon_n \end{aligned}$$

Multiply these equations by $\mu_1 \mu_2 \dots$ and add, $\mu_1 \mu_2 \dots$ being chosen so that $[\mu] = 1$, and we have

* *Philosophical Magazine* (TILLOCH'S), vol. lxx. (1825), p. 165.

$$x = [\mu V] + [\mu \epsilon]$$

LAPLACE'S investigation shows that the chance that $[\mu \epsilon]$, n being very great, lies between u and $u + du$ is

$$\frac{h e^{-\frac{h^2 u^2}{4[\mu^2]}}}{\sqrt{\{\pi[\mu^2]\}}} du \quad (12)$$

in which

$$\frac{1}{h^2} = \int_0^{\infty} x^2 \phi(x) dx$$

Suppose, for clearness' sake, that we regard a quantity as accurately determined if the error lies between zero and a fixed infinitesimal quantity k , then the chance taht by taking the system of factors $\mu_1 \mu_2 \dots$ the correct result is obtained, is

$$\frac{h k}{\sqrt{\pi}} \frac{1}{\sqrt{[\mu^2]}} \quad (13)$$

The system of factors, therefore, that renders the probability of the result obtained by means of them greater than the probability of a result obtained by means of any other system of factors is that given by making $[\mu^2]$ a minimum. This condition, taken together with $[\mu] = 1$, leads to $\mu_1 = \mu_2 = \&c.$, whence the arithmetical mean is the most probable value of x .

Now, as was proved by GAUSS and also by LAPLACE himself, the arithmetic mean is only the most probable result if the law of facility for each observation is $e^{-h^2 x^2}$, whence it follows that LAPLACE must have implicitly assumed this to be the law.

The fallacy in this reasoning is to assume that, because LAPLACE'S result is the most probable obtainable by linear combination of the equations, therefore it is the most probable result of all. It is necessary that the law should be $e^{-h^2 x^2}$ only if the arithmetic mean is the most probable of all values; LAPLACE merely shows the arithmetic mean to be the most probable among values obtained by means of different systems of factors. Another point ignored is the fact that LAPLACE'S analysis requires n to be very great, an assumption of which no use is made in the above reasoning. IVORY does, however, allude to this in his remark "that it [LAPLACE'S inves-

tigation] is confined to the case of a great number of errors, in order to render the calculations practicable."* That is to say, that

$$\int \int \dots \phi(\epsilon_1) \dots \phi(\epsilon_n) d\epsilon_1 \dots d\epsilon_n$$

subject to the condition that $[\mu \epsilon]$ lies between x and $x + dx$ is always of the form $A e^{-h^2 x^2} dx$, whatever n may be, although, owing to the imperfections of our analysis, this is only capable of proof when n is great: a statement palpably untrue, for suppose n were equal to 2 or 3.

I propose now to examine the exact sense in which LAPLACE'S factors, and by implication the method of least squares, give the most probable or the most advantageous results.

Resuming IVORY'S reasoning, we have the chance of obtaining the correct result by means of the factors μ_1, μ_2, \dots given by (13). If, therefore, we call that system of factors the most probable by the use of which the highest probability of an accurate result is obtained, then LAPLACE'S is the most probable system of factors, and among linear combinations the arithmetic mean is the most probable result.

On the other hand, it is perfectly clear that, even among linear combinations, the arithmetic mean a of n presumably equal good observations V_1, V_2, \dots cannot be the most probable result; for since $a - V_1, a - V_2, \dots$ are not identical, we must suppose that an observation for which this difference is small was better than one for which it was larger, so that the result contradicts the assumption that all the observations were equally good. In fact, a is but a first approximation, and for a second we ought to weight the observations as suggested by the values of $a - V_1, a - V_2, \dots$; and so on.

The consistency of these two different results is apparent on consideration. If we, with LAPLACE, assume the same law of facility for each observation (viz. assume that the law of facility is *known* to be $\phi(x)$, and that it is *known* to be the same at each observation), then, whatever form $\phi(x)$ has, so long as n is large, the arithmetic mean is the most probable value.

* This remark shows conclusively, I think, that IVORY failed to understand properly LAPLACE'S investigation, as it certainly appears from it not only that $A e^{-h^2 x^2} dx$ is the limiting form of the above integral when n is very great, but also that this is not generally the form unless such is the case.

Next suppose that the law of facility is known to be $\phi_1(x)$ for the first observation, $\phi_2(x)$ for the second, and so on, then it follows from ELLIS'S analysis that the most probable system of factors are $\frac{1}{k_1^2}, \frac{1}{k_2^2}, \dots$ where

$$k_i^2 = \int_0^{\infty} x^2 \phi_i(x) dx.$$

Now the case that actually occurs is this: n observations are made, and V_1, V_2, \dots, V_n are obtained, but there is no information given as to $\phi_1(x), \phi_2(x), \dots$ except such as the observations themselves afford. Thus the actual case does not coincide with that supposed in the analytical investigation. We have, as it were, not only to determine the most probable system of factors, but also the most probable values of h, h_2, \dots or, which amounts to the same thing, the most probable weights. If there exists no reason to prefer one observation to another, we assume, by way of getting an approximate result, that all the observations follow the same law of facility, this being of course a very different thing from knowing that they do so; we are then in a position to make an approximate estimate of the relative value of the observations. DE MORGAN has described such a process. "Assuming the weights as nearly as they can be found, ascertain the most probable result, from which find the weights of the equations. If these agree with the assumed weight the process is finished; if not, repeat the process with the new weights, and so on, until a result is obtained for which the assumed and deduced weights of the equations are sufficiently near to equality." (*Encyc. Metropol.* "Theory of Probability," p. 456.)

Thus, if the law of facility of each observation were given, the method of least squares (weighted) leads to the most probable results, but, practically, as it is part of the question to find the weights also, the result is only approximate.

The above distinction between the two cases, viz. (1) when the law of facility is known to be the same for each observation, and (2) when such is presumed to be the case, simply because we have no *a priori* reason to prefer one observation to another, though it must have been clearly apprehended by LAPLACE (as witness his 'method of situation,' described in his *Second Supplement*), I have not been able to find stated explicitly anywhere. ELLIS seems to allude to the same idea in his illustration to show "that the

results given by what LAPLACE called the most advantageous system of factors are not strictly speaking the most probable of all possible results ;" he points out that since LAPLACE'S system of factors μ_1, μ_2, \dots are independent of V_1, V_2, \dots , the conclusion is wholly irrespective of the value of V_1, V_2, \dots so that the comparison is merely one of methods ; while when V_1, V_2, \dots are given we can compare *results*.* If by the values LAPLACE assigns to μ_1, μ_2, \dots ELLIS meant the values which LAPLACE absolutely did assign to μ_1, μ_2, \dots the objection is valid, though hardly worth stating, as others more important could be urged. But if, as seems more probable,† ELLIS meant the factors which he himself assigned, extending LAPLACE'S principle to the case when the laws of facilities are different, the matter seems not quite accurately stated (since k_i depends on $\phi_i(x)$, for all knowledge of which we are indebted to V_1, V_2, \dots). The method does give the best results if we use the system of factors $\frac{1}{k_1^2}, \frac{1}{k_2^2}, \dots$, but unfortunately k_1, k_2, \dots in any series of observations are unknown, and at best only admit of probable determination, so that the question is not, Given k_1, k_2, \dots find μ_1, μ_2, \dots , but, Given V_1, V_2, \dots find μ_1, μ_2, \dots and k_1, k_2, \dots . The usual practice is to take $k_1 = k_2 = \dots$ (unless the observer assigns an arbitrary weight to an observation, viz. guesses a value for k_i), and the result is not the best obtainable.‡ I do not think that in any place LAPLACE has assumed, except by way of illustration, that the law of facility of a single error is $e^{-h^2x^2}$; he proves that the law of facility for $\mu_1 x_1 + \mu_2 x_2 + \dots + \mu_n x_n$ (n very great) must be of the form $e^{-h^2x^2}$, whatever be the laws of x_1, x_2, \dots , if positive and negative errors be equally probable ; but nowhere does he assume that if one observation only is made its law of facility is $e^{-h^2x^2}$.

It follows from LAPLACE'S analysis at once that the law of facility of the arithmetical (or weighted arithmetical) mean of any large number of observations follows the law $e^{-h^2x^2}$, so that, if we were to regard any given observation as if it were the arithmetic mean (or the result of the linear combination) of a large number of observations, we should be justified in taking this to be the law. Or, we shall obtain the same law $e^{-h^2x^2}$ if we

* The reasoning is developed at much greater length by ELLIS.

† ELLIS is not likely to have enlarged on defects he had remedied himself.

‡ In all the above explanation one quantity only is supposed to be determined from the observations, but the remarks are of general application.

regard each actual error as formed by the linear combination of a large number of errors due to different independent sources. This latter point of view of the nature of an error seems most natural and true. In any observation, where great care is taken, so that no large errors can occur, we can see that its accuracy is influenced by a great many circumstances which ultimately depend on independent causes; the state of the observer's eye and his physiological condition in general, the state of the atmosphere, of the different parts of the instrument, &c., evidently depend on a great number of causes, which each contributes to the actual error. The above supposition not only seems to be a true one, but also to include all that can be asserted with anything approaching to certainty of the nature of an error. It will be observed that the law follows, whether the errors from the independent sources are small or not, provided that the actual error $\varepsilon = \mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 \dots + \mu_n \varepsilon_n$, ε_i being an elementary error supposed subject to the law $\phi_i(x)$; but, unless the errors $\varepsilon_1, \varepsilon_2 \dots$ are very small, we are not entitled to replace

$$\varepsilon = f(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$$

by

$$\varepsilon = \mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 \dots + \mu_n \varepsilon_n.$$

We thus have as a consequence of what appears to be a true conception of an error the form $e^{-h^2 x^2}$ for the law of facility, and the great accuracy with which the errors in a set of observations made apparently under similar circumstances agree with this law, strongly confirms the hypothesis.* It has sometimes been thought that, owing to the indeterminateness of h , the above law might be assimilated approximately to any other law, but the general closeness of the approximation seems to negative such a conclusion, and the preceding reasoning seems to place it beyond doubt that this function does *in rerum naturá* represent the law of facility; and then, of course, the rule of least squares, &c. follows. The difficulty of the determination of h still remains, as it is impossible for the observer to say whether the circumstances of two observations are the same or no. As a first approximation, they are, of course, assumed to be so, if there appears no reason to the contrary, and the result generally justifies the supposition

* It is for this reason, no doubt, that the numbers in so many statistical tables follow this law; the deviations from the average being due to the combined action of a very great number of causes.

very nearly; then, in a second approximation, we should deduce values of the weights, or of h, h_2, \dots , and so on. It does not even follow that h need be the same in a set of observations made at one time, as the physical state of the observer may change from fatigue, &c. As before remarked, LAPLACE does not seem to have regarded an error in this light, although one would have thought his analysis would have suggested it to him; it has sometimes been supposed that this was the method of proving the rule of least squares adopted by LAPLACE.

Reverting to LAPLACE'S investigations, one very important property of the law $e^{-h^2x^2}$, viz., its continual reproduction of itself, follows as a consequence of its being a limiting form. If the law of facility of X be $e^{-h^2x^2}$, and that of Y , $e^{-k^2x^2}$, then that of $X + Y$ will be $e^{-\frac{h^2+k^2}{2}x^2}$ which is of the standard form. This is a consequence of LAPLACE'S Theorem, for if the value of X be regarded as formed of the combination of m observations,* subject to any laws of facility, so that the error, $\mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 \dots + \mu_m \varepsilon_m$ would be subject to the law $e^{-h^2x^2}$, and the error of Y , viz., $\mu'_1 \varepsilon'_1 \dots + \mu'_n \varepsilon'_n$ be subject to the law $e^{-k^2x^2}$, then the error of $X + Y$, viz., $\mu_1 \varepsilon_1 \dots + \mu'_1 \varepsilon'_1 \dots$, will be subject to the law $e^{-l^2x^2}$.† And, since from (6),

$$\frac{1}{4h^2} = \mu_1^2 \int_0^\infty x^2 \phi_1(x) dx + \dots + \mu_m^2 \int_0^\infty x^2 \phi_m(x) dx$$

$$\frac{1}{4k^2} = \mu_1'^2 \int_0^\infty x^2 \psi_1(x) dx + \dots + \mu_n'^2 \int_0^\infty x^2 \psi_n(x) dx$$

while

$$\frac{1}{4l^2} = \mu_1^2 \int_0^\infty x^2 \phi_1(x) dx + \dots + \mu_1'^2 \int_0^\infty x^2 \psi_1(x) dx + \dots$$

we have

$$\frac{1}{l^2} = \frac{1}{h^2} + \frac{1}{k^2}.$$

The fact that the assumption of the law $e^{-l^2x^2}$ for a single observation also leads to the same law for every linear combination of observations, has

* Or, if we take the view of the nature of an error which has just been described, and which seems undoubtedly correct.

† LAPLACE himself has devoted an article to the discussion of a case, when "the method of least squares becomes necessary," in which reasoning of the same kind as the above is made use of. *Théorie*, pp. 373, &c.

often been regarded as confirmatory of its truth, and the above reasoning affords, directly from fundamental principles, a complete explanation of this remarkable property.

It also results from being a limiting form, that, if one law is to be adopted, this must be it; for, admitting the necessity of combining results linearly, the ultimate results must follow the law.

After explaining the manner of treating equations, described on pp. 93-96, LAPLACE proceeds to view the matter in a different light; we have for the error of the result, $\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n = u$, say. Regarding this as a loss, the "disadvantage" (used in this sense opposite to advantage) is

$$\frac{\int_0^{\infty} u \phi(u) du}{\int_0^{\infty} \phi(u) du},$$

$\phi(u)$ being the law of facility of u . This quantity is equal to twice what LAPLACE calls the mean error to be apprehended (*l'erreur moyenne à craindre*), and since

$$\phi(u) = A e^{-\frac{u^2}{4[\mu^2 k^2]}},$$

its value is $\frac{2}{\sqrt{\pi}} \sqrt{[\mu^2 k^2]}$, which is a minimum when $[\mu^2 k^2]$ is so, and leads to the same conclusion as before.

GAUSS,* adopting this view of the matter, remarked that it involved the postulate that the detriment to which the error $[\mu \varepsilon]$ gave rise, was proportional to $[\mu \varepsilon]$. Now, strictly speaking, the detriment does not admit of arithmetical evaluation at all, and we may just as well suppose it represented by $[\mu \varepsilon]^2$. So long as the detriment is represented by a function of $[\mu \varepsilon]$ such that both vanish together, one supposition is not more arbitrary than another. If then the detriment = $[\mu \varepsilon]^2$, then the mean value

$$= \left[\mu_i^2 \int_{-\infty}^{\infty} \varepsilon^2 \phi_i(\varepsilon) d\varepsilon \right] + \left[\left[\mu_i \mu_j \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon \varepsilon' \phi_i(\varepsilon) \phi_j(\varepsilon') d\varepsilon d\varepsilon' \right] \right] \\ = 2 [\mu^2 k^2].$$

* In this brief explanation of GAUSS's second demonstration, which was published in the *Theoria Combinationis Observationum Erroribus minimis obnoxia* (1823), I have followed ELLIS closely, whose account differs but slightly from GAUSS's own.

The second series of integrals vanishes since the positive and negative portions of them cancel one another, and the result is the same as LAPLACE'S.

Of this investigation, ELLIS says, "Nothing can be simpler or more satisfactory than this demonstration. It is free from all analytical difficulty, and applicable, whatever be the number of observations, whereas that of LAPLACE requires the number to be very large." With this remark, I cannot at all agree. There seems no great difference in principle between assuming that the mean value of $(\mu_1 \varepsilon_1 \dots + \mu_n \varepsilon_n)^2$ must be a minimum, and assuming the whole method at once in the shape of making $(a_1 x + \dots V_1)^2 + (a_2 x + \dots - V_2)^2 + \dots$ a minimum. To take the square of an error as a measure of its importance is as arbitrary as to take the sum of the squares of the errors as a measure of the precision of the observations. Both would give results very near the truth, but nothing further can be said *à priori*. LAPLACE'S first method of treating the subject appears far preferable to his second; and this is also clear analytically, for, making use of the value obtained for $\varphi(u)$ in the first method, the same result, viz. that $[\mu^2 k^2]$ must be a minimum, will follow, if we assume the detriment caused by an error ε to vary as ε^m , or even as $A \varepsilon^p + B \varepsilon^q + C \varepsilon^r + \dots$, the coefficients A, B, C... being positive, while $m, p, q, r \dots$ may be integral or fractional, but must be positive, which last condition of course we should expect, since the detriment cannot diminish for an increasing error. To prove this, write, for simplicity,

$$\alpha = \frac{1}{4 [\mu^2 k^2]},$$

then

$$\varphi(u) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha u^2},$$

and

$$\begin{aligned} \frac{\int_0^\infty u^m \varphi(u) du}{\int_0^\infty \varphi(u) du} &= 2 \sqrt{\frac{\alpha}{\pi}} \int_0^\infty u^m e^{-\alpha u^2} du \\ &= \sqrt{\frac{\alpha}{\pi}} \int_0^\infty v^{\frac{m-1}{2}} e^{-\alpha v} dv \\ &= \sqrt{\frac{\alpha}{\pi}} \cdot \frac{1}{\alpha^{\frac{m+1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \\ &= \frac{2^m}{\sqrt{\pi}} \Gamma\left(\frac{m+1}{2}\right) \left\{ [\mu^2 k^2] \right\}^{\frac{m}{2}} \end{aligned} \quad (14)$$

which, so long as n is positive, is a minimum when $[\mu^2 k^2]$ is so. Similarly, for $A \varepsilon^p + B \varepsilon^q + \dots$.

The two modes of performing the analysis adopted by LAPLACE and GAUSS give rise to a theorem in multiple integrals, which it is of some importance to verify, in order to place beyond all doubt the identity of the principles involved in the two cases.

The theorem is that if

$$\int \dots \int \phi_1(\varepsilon_1) \dots \phi_n(\varepsilon_n) d\varepsilon_1 \dots d\varepsilon_n \quad (15)$$

subject to the condition $\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n > u$ and $< u + du$ be denoted by $\phi(u) du$, then

$$\int_{-\infty}^{\infty} u^m \phi(u) du = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n)^m \phi_1(\varepsilon_1) \dots \phi_n(\varepsilon_n) d\varepsilon_1 \dots d\varepsilon_n \quad (16)$$

m being any quantity, positive or negative, integral or fractional, and ϕ_1, ϕ_2, \dots being such that

$$\int_{-\infty}^{\infty} \phi_1(\varepsilon) d\varepsilon = 1, \int_{-\infty}^{\infty} \phi_2(\varepsilon) d\varepsilon = 1, \dots$$

The integration on the right-hand side of (16) can always be performed if m be integral; thus if $m = 2$, it = $2 [\mu^2 k^2]$, if $m = 4$, it = $2 [\mu^4 k^4] + 24 [[\mu_i^2 \mu_j^2 k_i^2 k_j^2]]$, &c. To prove the above proportion, write, for brevity, p for $\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n$, then, since

$$\frac{2}{\pi} \int_0^{\infty} \cos p \theta \frac{\sin u \theta}{\theta} d\theta = 0, \text{ unless } p \text{ lies between } \pm u, \text{ when it} = 1$$

therefore, differentiating, $\frac{2}{\pi} du \int_0^{\infty} \cos p \theta \cos u \theta d\theta = 0$, unless p lies between u and $u + du$, or between $-u$ and $-(u + du)$, when it is equal to unity.

By reasoning of a similar kind, $\frac{2}{\pi} du \int_0^{\infty} \sin p \theta \sin u \theta d\theta = 0$, unless p lies between u and $u + du$ when it = 1, or p lies between $-u$ and $-(u + du)$, when it = -1.

Adding these results, therefore, $\frac{1}{\pi} du \int_0^{\infty} \cos(p - u) \theta d\theta^* = 0$, unless

* The advantage of this form over $\frac{2}{\pi} du \int_0^{\infty} \cos p \theta \cos u \theta d\theta$ is that the latter = 1 both when p is intermediate to u and $u + du$, and 0 u and $-(u + du)$, while we are in search of a function which = 1 only when p is intermediate to u and $u + du$.

p lies between u and $u + du$ when it = 1. This same result may also be obtained perhaps more clearly thus: we know that

$$\begin{aligned} \frac{1}{\pi} \int_0^{\infty} \frac{\sin(p-u)\theta}{\theta} d\theta &= \frac{1}{2} \text{ if } p > u \\ &= 0 \text{ if } p = 0 \\ &= -\frac{1}{2} \text{ if } p < u \end{aligned}$$

therefore

$$\begin{aligned} \frac{1}{\pi} \int_0^{\infty} \frac{\sin(p-u-du)\theta}{\theta} d\theta - \frac{1}{\pi} \int_0^{\infty} \frac{\sin(p-u)\theta}{\theta} d\theta \\ &= 0 \text{ if } p > u + du \\ &= -1 \text{ if } p > u \text{ and } < u + du \\ &= 0 \text{ if } p < u \\ \text{and} \\ &= \frac{1}{2} \text{ if } p = u \text{ or } = u + du \end{aligned}$$

whence the integral previously obtained is found by expanding the first of the two integrals last written in powers of du and retaining the first term.

Making use of this result, in (15)

$$\varphi(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_0^{\infty} \phi_1(\varepsilon_1) \dots \phi_n(\varepsilon_n) \cos(p-u)\theta d\varepsilon_1 \dots d\varepsilon_n d\theta$$

and the left-hand side of (16) becomes

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \phi_1(\varepsilon_1) d\varepsilon_1 \dots \int_{-\infty}^{\infty} \phi_n(\varepsilon_n) d\varepsilon_n \int_0^{\infty} \int_{-\infty}^{\infty} u^m \cos(p-u)\theta d\theta du \quad (17)$$

But by FOURIER'S Theorem

$$\frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} u^m \cos(p-u)\theta d\theta du = p^m \quad (18)$$

whence (17) is equal to the right-hand side of (16), and the proposition is proved. It is easy to see also from the nature of the proof that the theorem would still be true if $\psi(u)$ and $\psi(p)$ were written for u^m and p^m .

The above demonstration, although one of a kind that experience has justified, so that no one who has examined similar investigations would feel

any doubt of the accuracy of the result, is of course open to objection; as for instance, $\int_0^\infty \cos p \theta \cos u \theta d \theta$ must be indeterminate, and the integral in (17) involving the quantity u^m (m positive) and having infinite limits must be infinite. Such difficulties are common in investigations relating to FOURIER'S Theorem, Definite Integrals, &c.; some of the present defects may be remedied by introducing throughout a factor $e^{-k \theta}$ so that we have ultimately *

$$\frac{d u}{\pi} \int_0^\infty e^{-k \theta} \cos (p-u) \theta d \theta$$

which = 0, unless p lies between u and $u + d u$, when it = 1, k being a zero of such a grade that the ∞ of the limit of the integral is of a higher grade than k^{-1} ; (17) then becomes

$$\frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty e^{-k \theta} u^m \cos (p-u) \theta d \theta d u = p^m$$

the form in which BOOLE has proved FOURIER'S Theorem.† The matter is still not quite free from some difficulty, which, however, this is not the proper place to discuss as the result is undoubtedly true. ELLIS, on pp. 212, 216, of his memoir, has in effect verified a particular case of the above theorem, viz., that of $m = 2$; his analysis involves some slight ambiguities which seem inseparable from the subject.

It is worth while to verify that when n is great GAUSS'S results agree with LAPLACE'S. Take the case of $m = 4$; then GAUSS'S method gives from (16) mean of

$$\{[\mu \epsilon]\}^4 = 2 [\mu^4 \kappa^4] + 24 [[\mu_i^2 \mu_j^2 k_i^2 k_j^2]] \quad (19)$$

$$\kappa_i^4 \text{ being } \int_0^\infty \epsilon^4 \phi_i(\epsilon) d \epsilon, \text{ and } k_i^2 \text{ as before}$$

while LAPLACE'S result is from (14)

$$\frac{16}{\sqrt{\pi}} \Gamma\left(\frac{5}{2}\right) \{[\mu^2 k^2]\}^2 = 12 \{[\mu^2 k^2]\}^2 \quad (20)$$

* *Messenger of Mathematics*, vol. v. 1871, pp. 239, 242. The exponential factor renders the differentiation with regard to u permissible, i.e., justifies the neglect of $(d u)^2$ &c.

† *Irish Transactions*, vol. xxi. p. 124.

Now the terms in $[\mu^4 \kappa^4]$ are n in number, while those in the second term of (19) are n^2 in number; neglecting therefore the former term compared to the latter, n being large, (19) becomes $= 24 [[\mu_i^2 \mu_j^2 k_i^2 k_j^2]]$ which is the value assumed by (20) when the terms in $12 [\mu^4 \kappa^4]$ are neglected for the same reason. It is clear that the coincidence of the results for any value of m could be shown in the same manner.

The next demonstration to be noticed is that given by Sir JOHN HERSCHEL in the *Edinburgh Review* for July, 1850, in a notice of QUETELET'S *Lettres sur la Théorie des Probabilités*. It depends on the independence of the x and y deflections in the deviation of a stone let fall on a plane. The proof was originally stated in a somewhat popular form, but was translated into analytical language by ELLIS in a Paper in the *Philosophical Magazine*, for November, 1850, and it is so well known that it is unnecessary to reproduce it here. It is also needless to point out in detail the unwarrantable character of the assumption of equally probable x and y deflections, as this has already been done by ELLIS, who has remarked that unless it can be shown that a deviation y occurs with the same comparative frequency when x has one value as when it has another, we are not entitled to say that the probability of the occurrence of two deviations, x and y , is the product of their probabilities; it should be noticed that the axes of x and y are any arbitrary system of rectangular axes intersecting at the point vertically under the point from which the stone is dropped. HERSCHEL in the *Review* asserted that if shots were fired at a wafer on a wall, and the wafer was subsequently removed, the centre of gravity of the shot-marks would be the most probable position of the wafer. ELLIS, near the conclusion of his Letter, attempted to show that on the assumption of independent x and y deflections this would not be the case. The substance of his reasoning is, that since $\frac{h^2}{\pi} e^{-h^2(x^2+y^2)} dx dy$ is the chance of hitting the area $dx dy$, therefore the chance that the total deflection will lie between r and $r + dr$ (i.e.,

* "Remarks on an alleged Proof of the Method of Least Squares, contained in a late number of the *Edinburgh Review*. In a letter addressed to Professor J. D. FORBES." The proof is also given by BOOLE, *Edinburgh Transactions*, vol. xxi. p. 628, *Finite Differences* (Cambridge, 1860), p. 228, FALL DE BRUNO, *Calcul des Erreurs*, p. 43, by THOMSON and TAIT, *Natural Philosophy*, p. 313, &c., where it is spoken of as "simple and apparently satisfactory." From the German translation of this last work it is extracted by SCHLÖMILCH, who regards it as "einfache und anschauliche," in the *Zeitschrift für Mathematik und Physik*, vol. xvii. p. 89 (Jan. 1872).

the chance of an error between r and $r + dr$ is $2 h^2 e^{-h^2 r^2} r dr$ and then "the centre of gravity of the shot-marks is not the most probable position of the wafer." The assumption is that the chance of hitting $dx dy$ is $\frac{h^2}{\pi} e^{-h^2(x^2 + y^2)}$, so that if before a shot is fired an infinitesimal area α distant r from the wafer is fixed upon, the chance of hitting this area is $\frac{h^2}{\pi} e^{-h^2 r^2} \alpha$; also the chance of hitting an annular element, radii r and $r + dr$ is $2 h^2 e^{-h^2 r^2} r dr$. Now suppose the shots have been fired and the wafer removed: it is required, Given the shot-marks, to find the most probable position of the wafer. To fix the ideas, suppose there are three shot-marks A, B, C. ELLIS'S reasoning would make the problem as follows: suppose any point O was the wafer, and $OA = r_1$, $OB = r_2$, $OC = r_3$, then the *a priori* chance of a shot falling between distances r and $r + dr$ from O is $2 h^2 e^{-h^2 r^2} r dr$, therefore the probability of the observed event is

$$8 h^6 e^{-h^2(r_1^2 + r_2^2 + r_3^2)} r_1 r_2 r_3 dr_1 dr_2 dr_3$$

and therefore the most probable position of O *a posteriori* is that which makes

$$e^{-h^2(r_1^2 + r_2^2 + r_3^2)} r_1 r_2 r_3$$

a maximum. But the correct solution certainly is: Trace round A, B, C, small areas each equal to α ; these areas when α is indefinitely diminished are the points A, B, C. Then the *a priori* chance of the observed event (viz., that A, B, C, are hit) is

$$\frac{h^6}{\pi^3} e^{-h^2(r_1^2 + r_2^2 + r_3^2)} \alpha^3$$

and the *a posteriori* chance that O was the point, A, B, C having been hit, is proportional to $e^{-h^2(r_1^2 + r_2^2 + r_3^2)}$ which is a maximum when $r_1^2 + r_2^2 + r_3^2$ is a minimum, i. e., when O is the centre of gravity of the points A, B, C.

In the correct solution we have the *a priori* question, 1. Given the position of the wafer, find the chance that the target will be struck in A, B, C; and the inverse question, 2. Given that the shots have struck A, B, C, find the most probable position of the wafer. But in ELLIS'S mode of solution the *a priori* question is, Given the position of the wafer, find the chance that the target will be struck at any point on the circumferences

of the three circles through A, B, C; but there is no corresponding inverse question. Given three errors, find the most probable position of O is unmeaning.*

It will be noticed that the law $2 h^2 e^{-h^2 r^2} r$ gives the facility of errors, that is to say, if round O a series of circles radii $k, 2k \dots$ (k infinitesimal) be drawn, then the chance of the shot hitting the r th ring so formed is proportional to $r e^{-h^2 r^2}$, and as should clearly be the case the innermost ring is the most unlikely of all to be struck owing to its small size; the ring that will receive most shots is the $\left(\frac{1}{h k \sqrt{2}}\right)$ th, beyond which the smallness of the chance of hitting an annulus so far from O more than counterbalances its increased size. Thus among points (all points being of course infinitesimal areas of the same size), O is the most likely to be hit; but among annuli (all annuli being of the same breadth) it is the least likely. Another portion of the same letter is devoted to an investigation, the deduction

* It was only after the above was written, that I found that in the next number of the *Philosophical Magazine* (December, 1850), ELLIS had himself stated that his solution was incorrect, and that the centre of gravity of the shot-marks was the most probable position of the wafer. As, however, he has given no proof, and as the matter is of interest for its own sake, I have not altered the remarks in the text. ELLIS adds to his correction the remark, "I thus not only omitted to notice that the reviewer's conclusion would not follow from his own hypothesis, but by this omission was led to introduce an error of my own." This does not seem to me fair. In the original statement that the centre of gravity was the most probable position of the wafer, and throughout his proof (excepting alone the assumption of independent x and y deflections), HERSCHEL was correct. On p. 21, however, a confusion between the laws $e^{-h^2 r^2}$ and $r e^{-h^2 r^2}$ arises, and the number of the shot-marks in the rings (107, 213, &c.) are incorrect, having been calculated, as is evident from an inspection of ENCKE's table of $\int_0^t e^{-t^2} dt$, by the principle that the chance of the shot striking the annulus, radii r and $r + dr$, is $\frac{h}{\sqrt{\pi}} e^{-h^2 r^2} dr$; but this is the only error connected with this part of the subject that HERSCHEL can be fairly charged with. It should be stated, however, that he clearly was not conversant with LAPLACE's researches; he speaks, for example, in two places (pp. 18, 30), of QUETELET as having given LAPLACE's analysis "stripped of all superfluous difficulties and reduced to the most simple and elementary form." It is true that the problem given by QUETELET (*Lettres*, pp. 380-386) is to be found in LAPLACE (pp. 301-304), but no use of it is made by its author in connexion with the Theory of Errors. It is the same as that subsequently given by Prof. TAIT, and noticed in pp. 116-118 of this Paper.

drawn from which is that the equation for determining the form of f results from a tacit predetermination of that function, so that the statement that the chance of the element $dx dy$ being struck is $f(x^2)f(y^2) dx dy$, involves either "a simple mistake or a *petitio principii*." ELLIS'S analysis does not appear to give any information that is not quite evident from general considerations. Since the fact contained in

$$f(x^2)f(y^2) = f(x^2 + y^2)f(0)$$

is identical with that contained in

$$f(x^2) = A e^{-\lambda^2 x^2},$$

any supposition that leads to the equation rests on a tacit predetermination of the form of the function, and in general any hypothesis involves the conclusions that follow from it. In the above case, however, the assumption and conclusion are so closely connected, that one would expect the one to be pretty nearly as evident as the other. Reversing then the question, it appears that e^{-r} is a very natural law of facility, and in this case the x and y deflections are not independent, and this alone would suggest the illegitimacy of the assumption, the arbitrary nature of which is apparent on consideration. It is, however, undeniable that the independence of the x and y deflections, whatever be the (rectangular) axes, is a most remarkable property of the law of facility $e^{-\lambda^2 x^2}$.

In the *Edinburgh Transactions* (vol. xxi. pp. 628, &c.), BOOLE has reproduced HERSCHEL'S proof, and made some remarks on ELLIS'S Letter. Assuming HERSCHEL'S principle, we have $f(x^2) = A e^{-\lambda^2 x^2}$, and since the stone must fall at some point for which x lies between $\pm \infty$, $\int_{-\infty}^{\infty} A e^{-\lambda^2 x^2} = 1$, and the law is $\frac{h}{\sqrt{\pi}} e^{-\lambda^2 x^2}$, so that the probability of its falling on $dx dy$ is $\frac{h^2}{\pi} e^{-\lambda^2(x^2+y^2)} dx dy$. BOOLE then proceeds, "This result admits of a remarkable confirmation. For it is manifest that the probability that the ball will fall somewhere between the distances x and $x + \delta x$ from the axis of y ought to be equal to the above expression integrated with respect to y between the limits $-\infty$ and ∞ . But that probability has been determined to be $\frac{h}{\sqrt{\pi}} e^{-\lambda^2 x^2} \delta x$; we ought therefore to have

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$$\int_{-\infty}^{\infty} \left(\frac{h^2}{\pi} e^{-h^2(x^2+y^2)} \delta x \right) dy = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2} \delta x$$

an equation which is actually true."

We assume the law of facility along the axis of x to be $f(x^2)$, the constant being chosen so that $\int_{-\infty}^{\infty} f(x^2) dx = 1$, it can then scarcely be called a remarkable confirmation that

$$\int_{-\infty}^{\infty} \{f(x^2) dx\} f(y^2) dy = f(x^2) dx$$

in fact the equation $a = 1$ is assumed, and $a b = b$, deduced from it, is called a confirmation.

Against ELLIS's reasoning to show that HERSCHEL's principle involved a mistake, or *petitio principii*, BOOLE urges that consistency of results can never be a proof of mistake, and that alone it offers no adequate ground for the suspicion of a *petitio principii*. This, of course, is true, and, as before remarked, ELLIS's analysis appears superfluous; but the proof involves no *petitio principii*, it merely makes an unauthorised assumption, viz., assumes as evident a result as much in need of proof as is the ultimate result itself.

With the part of the letter, in which ELLIS objects to HERSCHEL's assumption of the law $e^{-h^2 x^2}$, which "is nothing more than the expression of our complete ignorance of the causes of error and their mode of action," I entirely agree. Knowing nothing of an error, we can prove nothing; but, in point of fact, we do know something of an error, viz. that it is the accumulation of many smaller errors, and this knowledge is sufficient to establish the law $e^{-h^2 x^2}$, which is actually verified by observation. It is most remarkable that so general and important a theorem should follow from such meagre premises; and, regarding the matter in a purely analytical point of view, that the multiple integral on page 97, should have a constant limiting form for all forms of $\phi_1, \phi_2, \&c.$ is certainly a striking result.

Professor TAIT's paper, "On the Law of Frequency of Error," is printed in vol. xxiv. of the *Edinburgh Transactions*. The principle on which the investigation rests seems to be, that an error arising from any source may be compared to the deviation from the most probable result of the number of white or black balls obtained by a great number of drawings from a bag containing white and black balls in a known ratio. Suppose a bag to

contain white and black balls whose numbers are as p to q , where $p + q = 1$, then, if $n (= \alpha + \beta)$, drawings are made (the balls being replaced), the most probable result is, that there will be α white and β black balls drawn where $\alpha : \beta = p : q$, so that $\alpha = pn$, $\beta = qn$. Now regard $\alpha - pn$, the deviation from the most probable result, as measuring the error, so that $\alpha - pn = mx$, x , being the error; then it can be shown that x being very large, the chance of the error x is proportional to $e^{-\frac{m^2 x^2}{2pqn}}$, or $e^{-\mu x^2}$. If, therefore, we assume an analogy between the result of a cause producing an error and the above drawing, the chance of an error from one cause is $A e^{-\mu x^2}$, from another $A_1 e^{-\mu x^2}$, &c. so that the actual error follows the law $e^{-\lambda^2 x^2}$.

The above is in substance Professor TAIT'S process, and the result follows sufficiently easily, if we admit the similarity between an error due to one cause, and the deviation from the most probable value of the result of a large number of drawings of black and white balls. This analogy, however, appears very vague, and I can see no justification for assimilating two cases, which seem quite distinct. From Mr. CROFTON'S remarks,* I infer that he attached no great force to the analogy; he points out also that "the proof only applies to the combination of a number of elementary errors, each of which follows that law. But it is quite certain that many simple errors do not follow that law; hence the method is altogether deficient in generality." The last objection is important: it seems quite clear that we can never prove that the error from each source follows one law.

Professor TAIT'S analysis is not new, nor is its application to the theory of errors new. It forms the first proposition of LAPLACE'S third chapter,† and is given by DE MORGAN in the *Encyclopædia Metropolitana*; but, although in the works of both these writers the method of least squares is subsequently developed, neither refers to it as suggestive or confirmatory of the law $e^{-\lambda^2 x^2}$; and this fact seems to show plainly that in their opinion it was not available for the purpose. What is in effect the same reasoning is applied to determine the law of facility, as in Professor TAIT'S memoir, in the notes to QUETELET'S *Lettres* (p. 380, &c.), in LIAGRE'S *Calcul des Probabilités* (pp. 61, 67, &c.), and in FAA DE BRUNO'S *Calcul des Erreurs* (p. 42).

* "On the Proof of the Law of Errors of Observation." *Phil. Trans.* 1870. p. 177.

† *Théorie Analytique des Probabilités*, p. 301, &c.

In the introductory remarks to Professor TAIT's paper, there are some statements that may convey a false expression of the nature of LAPLACE's methods. After the description contained in this paper of the researches of the latter, it is only necessary to allude to one sentence, viz. that in which Professor TAIT states that "he assumes at starting, that these separate contributions are as likely to be of one magnitude as another," &c., an assumption which he refutes. In point of fact, LAPLACE made no such assumption; he supposed each "contribution" to be subject to the law of facility $\phi(x)$, and on this general case all his deductions rest; the particular case of $\phi(x) = \text{constant}$ was only proved before the general theorem, I imagine, in order to exhibit first, on a simple case, the nature of the analysis. The sufficiency of LAPLACE's general theorem, Professor TAIT subsequently admits.

It may be mentioned that the common origin of the form e^{-x^2} in the two cases, is LAPLACE's rule for approximating to the values of integrals containing quantities raised to higher exponents by the assumption $y = Y e^{-\sigma}$ (which contains STIRLING's theorem, $\Gamma(x+1) = \sqrt{2\pi x} x^x e^{-x} (1 + \frac{1}{12x} + \dots)$ as a particular case).

The last demonstration that will be referred to is that given by DONKIN, in the first volume of the *Quarterly Journal of Mathematics*. DONKIN observes that, if two observations of an unknown quantity x give $x = a$ and $x = b$, and we have no reason for putting more confidence in one than the other, then the most probable value of x is $\frac{1}{2}(a+b)$, but that we cannot, without making some further assumption, extend this to the case of three observations. Suppose that, arising from the first observation $\phi(x-a) dx$ is the probability that the true value of x is between x and $x+dx$, then $C \phi(x-a) \phi(x-b) dx$ is the probability arising from both observations combined that such is the case. "On the other hand, since the most probable value of x arising from the combined observations must be $\frac{1}{2}(a+b)$, it appears a natural and obvious assumption (though I do not pretend that it is not an assumption) that the probability that x is between x and $x+dx$ must be expressible in the form

$$\psi \left(x - \frac{a+b}{2} \right) dx,$$

so that we shall have

$$C \phi(x-a) \phi(x-b) = \psi\left(x - \frac{a+b}{2}\right) **$$

The result that $\phi(x) = A e^{-ax+bx^2}$ then follows by the solution of this functional equation. Here DONKIN himself has pointed out the assumption made, which, of course, could not be justified rigorously or even rendered so probable that the reader could feel any conviction of the validity of the result obtained by means of it. As a basis from which to derive the law of facility such assumptions as DONKIN'S and HERSCHEL'S seem out of the question, but they are interesting, as giving properties of the law, and showing that, to say the least, it is most natural. It may be mentioned that DONKIN has developed (without introducing a law of facility) the usual methods of combining weighted observations, by means of the analogy between thought and mechanics, as expressed by the equilibrium of belief and the equilibrium of matter, in the fifteenth volume of LIOUVILLE'S *Journal*. One remark that DONKIN makes (*Quarterly Journal*, vol. i. p. 160) ought not to be passed over. Speaking of his own assumption, he says that "the utmost that any such process can pretend to establish is not that the unknown law of facility of error *is* expressed by a function of this form (which would manifestly be an absurd pretension), but that *the law being unknown* the most probable result is to be obtained by proceeding *as if* it were known to have the form in question." It does not seem to me to establish the one more than the other.

This concludes the series of demonstrations, the principles of which it was the main object of this paper to examine and compare. It ought to be mentioned that there are two memoirs, containing different investigations that I have not noticed, viz. BESSEL'S "Untersuchungen über die Wahrscheinlichkeit der Beobachtungsfehler," which occupies numbers 358 and 359 (1838) of the *Astronomische Nachrichten*, and Mr. CROFTON'S paper, in the *Philosophical Transactions* (1870), "On the Proof of the Law of Errors of Observations." The latter regarding an error as formed of smaller errors due to numerous sources finds the law $e^{-k^2(x-a)^2}$, but the analysis is quite different to LAPLACE'S or POISSON'S. The introductory remarks prefixed to the memoir concerning the nature of errors appear very true and valuable. HAGEN'S *Gründzüge der Wahrscheinlichkeitsrechnung* I have not seen.†

* Loc. cit., p. 159.

† At the end of FAL DE BRUNO'S *Calcul des Erreurs* there is a list of writers on the

As the result of the examination of all the proofs described in this paper it seems to me that the only sound philosophical basis on which the law of facility $e^{-h^2x^2}$ rests, is the supposition that an actual error is formed by the accumulation of a great number of small errors due to different independent sources, and subject to the arbitrary laws of facility $\phi_1(x), \phi_2(x) \dots$; and as I think it is clear that this is the way in which an error really does arise, the law $e^{-h^2x^2}$, for an individual error, is, in my opinion, proved by LAPLACE. Two points, however, must be noticed: (1) that the law is a limiting form, strictly true only if the number of sources is absolutely infinite, and (2) that it assumes that $\phi_1, \phi_2 \dots$ are such that $\phi_1(x) = \phi_1(-x), \dots$. If this is not the case (and we have no right to assume it to be so), the resulting law is $e^{-h^2(x+a)^2}$. Except for this second objection (which is not important, since $a = \left[\mu \int_{-\infty}^{\infty} x \phi(x) dx \right]^*$, and will be very small, or otherwise, since it is evident that one observation being made the observed value is the most probable one of the quantity), the method of least squares follows immediately, as in GAUSS'S method (p. 85). Granting the necessity of combining our equations linearly, LAPLACE'S analysis gives a double reason for the method of least squares when the number of observations is large. The method described on pp. 103-104, of returning to the observations again after finding the most probable result by the rule of least squares, and assigning weights, the process being repeated as often as necessary, distinctly appears to be the proper and philosophical method of treating observations; and, as a consequence, Professor PIERCE'S criterion† for the rejection of doubtful observations seems to me to be destitute of scientific precision. If an observation has been made as carefully as the rest, it ought on no account to be neglected entirely. It may be, and no doubt is, true that in many cases it is better to reject it than to retain it, giving it an equal weight with the best observation, but the true principle

subject of Least Squares, with the titles of their works, which is extremely imperfect. The names of BESSEL, BOOLE, DONKIN, ELLIS, ENCKE, HERSCHEL, IVORY, LEGENDRE, TAIT, and TODHUNTER (all of whose writings were published prior to 1869, the date of the work), do not appear, nor does the list contain GAUSS'S *Theoria Motus*.

* This follows from POISSON'S Theorem, p. 99.

† See GOULD'S (American) *Astronomical Journal*, vol. ii. p. 161. The criterion is also explained nearly in Professor PIERCE'S words in CHAUVENET'S *Astronomy*, vol. ii. p. 558.

is to weight the observation as the method itself indicates, when an abnormal observation would receive a very small weight. It appears quite evident that under no circumstances have we a right to say an observation has no weight, though it may be better to give it none than to give it as much as the best. The fact of such a criterion having been proposed is a strong argument in favour of the necessity of the completion of the method of least squares as indicated; attention to this necessity was first insisted on by DE MORGAN.

In one of his earliest memoirs LAPLACE supposed the law of facility to be e^{-mx} ; the reasoning* by which he justifies this assumption is of the most trivial kind, but still the law is an extremely natural one, and is of the form which, I think, any one, without the aid of analysis, would be inclined to adopt as satisfying in the simplest way the condition of rapidly decreasing, with an increase of x , and having the axis of x as an asymptote.† It seems, therefore, worth while to investigate briefly one or two results that follow from this law.

As e^{-mx} is to remain unaltered when $-x$ is written for x , it is convenient to write it in the form $e^{-m\sqrt{x^2}}$. (Some writers have assumed that because $\varphi(x) = -\varphi(-x)$, $\varphi(x)$ denoting the law of facility, therefore we might assume $\varphi(x) = \psi(x^2)$, and have evidently not regarded $\sqrt{x^2}$ as being of this form.)

Suppose two observations are made of a quantity, and that the values a and b are found, then the chance that x is the real value of the quantity, is proportional to

$$e^{-m \{ \sqrt{(x-a)^2} + \sqrt{(x-b)^2} \}}$$

* *Mémoires de Mathématique et de Physique . . . par divers Savans*, t. vi. 1774. TODHUNTER speaks of the reasons adduced as "very slight." (*History of Probability*, p. 469.)

† The curve $y = e^{-mx}$ cuts the axis of y at an angle $\tan^{-1} \frac{1}{m}$, and, therefore, slopes downwards from it at an acute angle. The curve $y = e^{-h^2 x^2}$ cuts it at right angles (and is of a dome-shaped form at the vertex); while such a curve as $y = e^{-m\sqrt{x}}$ has a cusp at its highest point; this shows that the true curve of facility $y = e^{-h^2 x^2}$ better agrees with what we should expect; for it is natural to anticipate that any value differing slightly from that observed is very nearly as likely to be the true one as the observed value itself; so that (restricting ourselves to integer powers of x) $y = e^{-h^2 x^2}$ contains the lowest power of x in its exponent that satisfies this condition.

The most probable value of x is that which makes this a maximum, viz. that which makes $\sqrt{(x-a)^2} + \sqrt{(x-b)^2}$ a minimum. The form of the curve $y = \sqrt{(x-a)^2} + \sqrt{(x-b)^2}$ is shown in figure (1), in which $OA = a$ and $OB = b$, which is supposed greater than a ; that is to say,

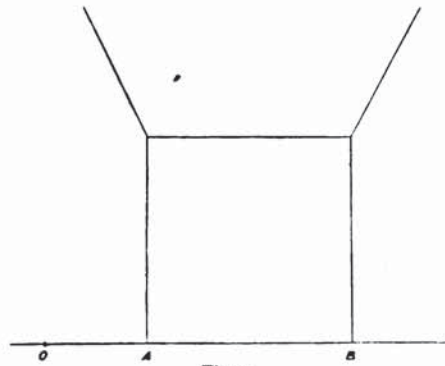


Fig. 1.

that if $x > b$, $y = 2x - a - b$, if $x < b$ and $> a$, $y = b - a$, and if $x < a$, $y = -2x + a + b$.

Every value of x , therefore, between $x = a$ and $x = b$ is equally probable, and is equally entitled to be considered the most probable value of x .

Suppose three observations a, b, c are made of x , then the most probable value of x is that which makes

$\sqrt{(x-a)^2} + \sqrt{(x-b)^2} + \sqrt{(x-c)^2}$ a minimum. The curve $y = \sqrt{(x-a)^2} + \sqrt{(x-b)^2} + \sqrt{(x-c)^2}$ is drawn in Fig. 2, in which

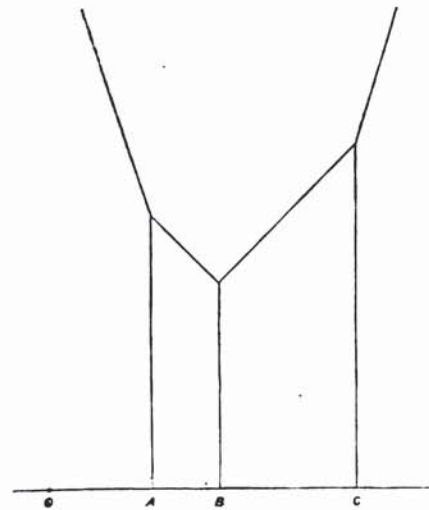


Fig. 2.

$OA = a$, $OB = b$, $OC = c$, and a, b, c , are supposed in ascending order of magnitude: if $x > c$, $y = 3x - a - b - c$; if $x < c$ and $> b$, $y = x + c - b - a$; if $x < b$ and $> a$, $y = -x + b + c - a$, and if $x < a$, $y = -3x + a + b + c$.

The most probable value of x , therefore, is $x = b$. A little consideration shows that these results are true generally, and we have the remarkable conclusion that if $2n + 1$ direct observations $V_1, V_2, \dots, V_{2n+1}$ are made of a quantity x , then V_n (the middle one) is the most probable value of x , and if $2n$ observations V_1, V_2, \dots, V_{2n} are

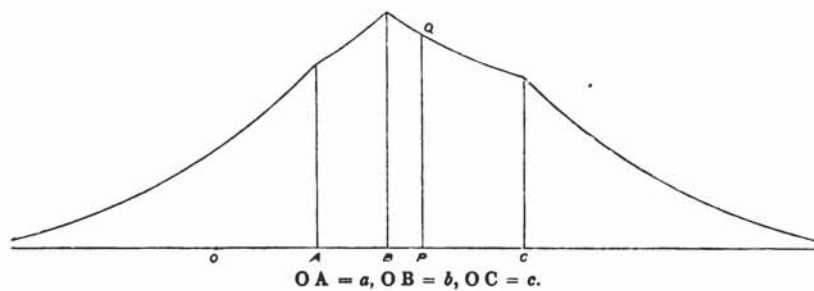
made, then any value of x between V_n and V_{n+1} (the two middle ones) has an equal right to be called the most probable value of x .

The method adopted by LAPLACE in his memoir of 1774 (in which only three observations are supposed to be made) is in effect as follows. Draw the curve

$$y = e^{-m \{ \sqrt{(x-a)^2} + \sqrt{(x-b)^2} + \sqrt{(x-c)^2} \}}$$

which is as in Fig. 3, and take as the mean value of x from a, b, c , the

Fig. 3.



abscissa corresponding to the ordinate PQ , which divides the area of the curve into two equal portions, so that the chance of an error greater than x is equal to the chance of an error less than x .

It may be remarked in conclusion that the Theory of Errors by no means originated with GAUSS. SIMPSON in 1757 and LAGRANGE in 1773 wrote on the subject, and in 1778 DANIEL BERNOULLI assumed the law of facility to be $\sqrt{(a^2 - x^2)}$, so that the curve of facility was a circle. The writings of these mathematicians are described by TODHUNTER on pp. 211, 236, 237, 307-309, of his *History of the Theory of Probability*, the work to which I am indebted for references to them.

Trinity College, Cambridge,
April 5, 1872.

POSTSCRIPT.

In the discussion of the principle of the arithmetic mean, a reference has been accidentally omitted to a memoir of BOOLE contained in Vol. XXI. of the *Edinburgh Transactions*. This is the same memoir as that which is referred to on pp. 115 and 116, but the investigation in question is given in the earlier portion of the memoir, while the part noticed in this Paper occurs by way of illustration near the end. The result of BOOLE'S investigation is that if n observations $p_1, p_2, \dots p_n$ be made of the same quantity, then the most probable value of that quantity is a certain linear function of $p_1, p_2, \dots p_n$; this BOOLE demonstrates by his Calculus of Logic, and the analysis is of so peculiar a character, that although I have devoted some time to the memoir, I feel scarcely qualified to express a decided opinion on its merits. It is sufficient to state here that the coefficients of $p_1, p_2, \dots p_n$ in the final result involve two sets of constants $a_1, a_2, \dots a_n$ and $c_1, c_2, \dots c_n$, the former being the probabilities, before the observations are made that they will be such as they prove to be, and the latter the *à posteriori* probabilities that, when made, they are correct; and that when $a_1, a_2, \dots a_n$ are all taken equal to one another, and also $c_1, c_2, \dots c_n$, the result takes the form of the arithmetic mean, $\frac{1}{n}(p_1 + p_2 + \dots + p_n)$.

August 26, 1872.

A LIST OF PERSONS
TO WHOM
THE MEDALS OR TESTIMONIALS OF THE SOCIETY
HAVE BEEN ADJUDGED.

1823.
June 13. CHARLES BABBAGE, Esq.
The Gold Medal.— For his Invention of an Engine for computing and printing Mathematical Tables.
- Professor JOHANN FRIEDRICH ENCKE
The Gold Medal.— For his Investigations relative to the Comet which bears his name.
- CHARLES RUMKER, Esq.
The Silver Medal.— For his Re-discovery of ENCKE'S Comet in 1822.
- M. JEAN LOUIS PONS.
The Silver Medal.— For his Discovery of two Comets in 1822.
1826.
Feb. 7. J. F. W. HERSCHEL, Esq. and JAMES SOUTH, Esq.
The Gold Medal, each.— For their important Researches on the subject of Multiple Stars.
- Feb. 10. Professor STRUVE.
The Gold Medal.— For his important Researches on the subject of Multiple Stars.
1827.
Feb. 2. FRANCIS BAILY, Esq.
The Gold Medal.— For his "New Tables for determining the places of 2881 Stars."
- WILLIAM SAMUEL STRATFORD, Esq.
The Silver Medal.— For his Superintendence of the Computation of "New Tables for determining the places of 2881 Stars."
- Feb. 5. Colonel MARK BEAUFOY.
The Silver Medal.— For his valuable Collection of Observations, particularly those of the Eclipses of *Jupiter's* Satellites.

126 *List of Persons to whom Medals or Testimonials have been adjudged.*

1828.
Jan. 11. Sir THOMAS MACDOUGALL BRISBANE, K.C.B.
The Gold Medal. — For his Establishment of an Observatory, and for an important series of Observations made at Paramatta.
- JAMES DUNLOP, Esq.
The Gold Medal. — For his Observations of the Nebulæ of the Southern Hemisphere.
- Feb. 4. Miss CAROLINE HERSCHEL.
The Gold Medal. — For her recent Reduction to January 1800, of the Nebulæ discovered by Sir WILLIAM HERSCHEL.
1829.
Jan. 9. Rev. WILLIAM PEARSON.
The Gold Medal. — For his work, entitled "An Introduction to Practical Astronomy."
Professor BESSEL.
The Gold Medal. — For his Zone Observations.
Professor SCHUMACHER.
The Gold Medal. — For the Publication of his various Astronomical Tables, and the "Astronomische Nachrichten."
1830.
Jan. 8. Mr. WILLIAM RICHARDSON.
The Gold Medal. — For his Investigation of the Constant of Aberration.
Professor ENCKE.
The Gold Medal. — For the New Berlin Ephemeris.
1831.
Jan. 14. Captain KATER.
The Gold Medal. — For his Invention of the Vertical Floating Collimator.
Baron DAMOISEAU.
The Gold Medal. — For his Memoir upon the Theory of the Moon, and for his Lunar Tables.
1833.
Jan. 11. Professor AIRY.
The Gold Medal. — For his Discovery of the long Inequality of *Venus* and the Earth.
1835.
Jan. 9. Lieutenant JOHNSON.
The Gold Medal. — For his Catalogue of 606 Southern Stars.
1836.
Jan. 8. Sir JOHN F. W. HERSCHEL.
The Gold Medal. — For his Catalogue of Nebulæ, printed in the "Philosophical Transactions" for 1833.

List of Persons to whom Medals or Testimonials have been adjudged. 127

1837.
Jan. 13. PROFESSOR ROSENBERGER.
The Gold Medal.—For his Investigations relative to HALLEY'S Comet.
1839.
Jan. 11. HON. JOHN WROTTESELEY.
The Gold Medal.—For his Catalogue of the Right Ascensions of 1318 Stars.
1840.
Jan. 10. M. JEAN PLANA.
The Gold Medal.—For his work, entitled "Théorie du Mouvement de la Lune."
1841.
Jan. 8. PROFESSOR BESSEL.
The Gold Medal.—For his Observations and Researches on the Parallax of 61 Cygni.
1842.
Jan. 14. M. HANSEN.
The Gold Medal.—For his Researches in Physical Astronomy.
1843.
Jan. 13. FRANCIS BAILY, Esq.
The Gold Medal.—For his Experiments to determine the Mean Density of the Earth in repetition of what is generally termed the "Cavendish Experiment."
1845.
Jan. 10. Captain WILLIAM HENRY SMYTH, R.N.
The Gold Medal.—For his "Bedford Catalogue," forming the second part of his work entitled "Celestial Cycle."
1846.
Jan. 9. GEORGE BIDDELL AIRY, Esq. Astronomer Royal.
The Gold Medal.—For his Reduction of the Observations of Planets made at the Royal Observatory, Greenwich, from 1750 to 1830.
1848.
Jan. 14. *Testimonials were awarded to,*
GEORGE BIDDELL AIRY, Esq. Astronomer Royal.
For the Lunar Reductions recently made at Greenwich.
JOHN COUCH ADAMS, Esq.
For his Researches in the Problem of Inverse Perturbations leading to the Discovery of the Planet *Neptune*.
PROFESSOR ARGELANDER.
For his Catalogue of Stars.

128 *List of Persons to whom Medals or Testimonials have been adjudged.*

1848.

Jan. 14. GEORGE BISHOP, Esq.
For the Foundation of an Observatory leading to various Astronomical Discoveries.

Lieut.-Col. GEORGE EVEREST.
For his Measurement of the Indian Arc.

Sir JOHN F. W. HERSCHEL.
For his Work on the Southern Hemisphere.

Professor P. A. HANSEN.
For his Lunar Theory and Computation of Perturbations.

M. HENCKE.
For his Discovery of two Planets, *Astræa* and *Hebe*.

JOHN RUSSELL HIND, Esq.
For his Discovery of two Planets, *Iris* and *Flora*.

M. U. J. LE VERRIER.
For his Researches in the Problem of Inverse Perturbations leading to the Discovery of the Planet *Neptune*.

Sir JOHN LUBBOCK.
For his Researches in the Theory of Perturbations.

M. M. WEISSE.
For his Catalogue of Stars in BESSEL'S ZONES.

1849.

Feb. 9. WILLIAM LASSELL, Esq.
The Gold Medal. — For the Construction of his Equatoreal Instrument and for the Discoveries made with it.

1850.

Feb. 8. M. OTTO VON STRUVE.
The Gold Medal. — For his Paper on the Constant of Precession.

1851.

Feb. 15. DR. ANNIBALE DE GASPARIS.
The Gold Medal. — For the Discovery of three Planets, *Hygeia*, *Parthenope*, and *Egeria*.

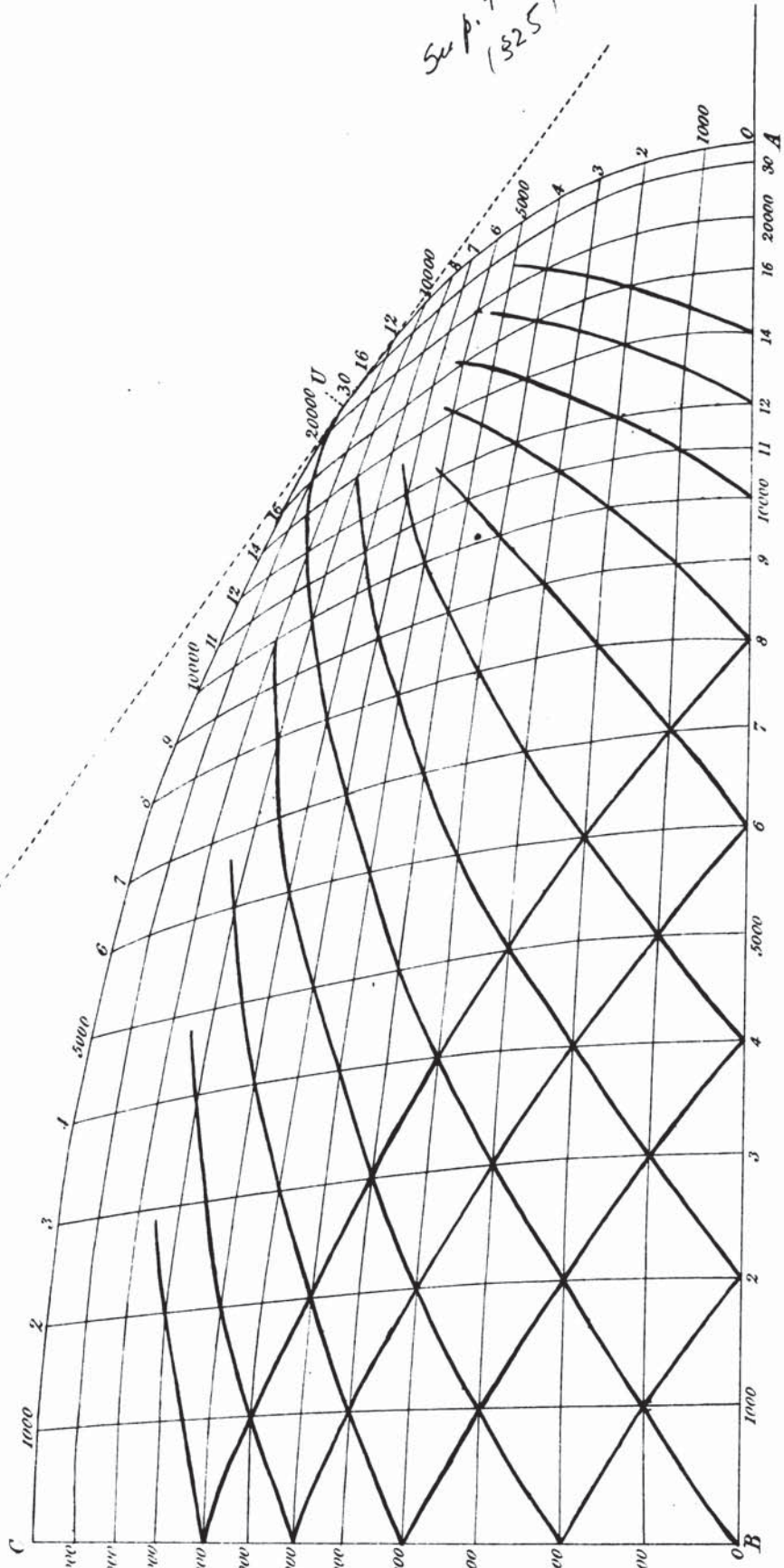
List of Persons to whom Medals or Testimonials have been adjudged. 129

1852.
Feb. 13. Dr. C. A. F. PETERS.
The Gold Medal. — For his Papers on the Parallax of the Fixed Stars,
and on the Constant of Nutation.
1853.
Feb. 11. JOHN RUSSELL HIND, Esq.
The Gold Medal. — For the Discovery of eight Planets, and other
Astronomical Discoveries.
1854.
Feb. 10. M. CHARLES RUMKER.
The Gold Medal. — For his Catalogue of 12,000 Stars, and for other
Astronomical Services.
1855.
Feb. 9. Rev. W. R. DAWES.
The Gold Medal. — For his Astronomical Labours generally.
1856.
Feb. 8. ROBERT GRANT, Esq. M.A.
The Gold Medal. — For his "History of Physical Astronomy."
1857.
Feb. 13. M. SCHWABE.
The Gold Medal. — For his Discovery of the Periodicity of the Solar
Spots.
1858.
Feb. 12. Rev. ROBERT MAIN, M.A.
The Gold Medal. — For his various Contributions to the *Memoirs* of the
Society.
1859.
Feb. 11. R. C. CARRINGTON, Esq.
The Gold Medal. — For his "Redhill Catalogue of 3735 Circumpolar
Stars."
1860.
Feb. 10. Professor P. A. HANSEN.
The Gold Medal. — For his Lunar Tables.
1861.
Feb. 8. M. HERMANN GOLDSCHMIDT.
The Gold Medal. — For his Discovery of thirteen of the Minor Planets,
and other Astronomical Discoveries.

128 *List of Persons to whom Medals or Testimonials have been adjudged.*

1862.
Feb. 14. WARREN DE LA RUE, Esq.
The Gold Medal.— For his Astronomical Researches, and especially for his Application of Photography.
1863.
Feb. 13. Professor ARGELANDER.
The Gold Medal.— For his Survey of the Northern Heavens.
1865.
Feb. 10. Professor G. P. BOND.
The Gold Medal.— For his work on the Comet of DONATI, and other Astronomical Researches.
1866.
Feb. 9. Professor ADAMS.
The Gold Medal.— For his Contributions to the Development of the Lunar Theory.
1867.
Feb. 8. W. HUGGINS, Esq. and Professor MILLER.
The Gold Medal.— For their Researches in Astronomical Physics.
1868.
Feb. 14. M. LEVERRIER.
The Gold Medal.— For his Planetary Tables.
1869.
Feb. 12. E. J. STONE, Esq.
The Gold Medal.— For his Rediscussion of the Transit of *Venus* in 1769, and his other contributions to Astronomy.
1870.
Feb. 11. M. DELAUNAY.
The Gold Medal.— For his "Théorie de la Lune."
1872.
Feb. 9. Signor SCHIAPARELLI.
The Gold Medal.— For his Researches on the Connexion between the Orbits of Comets and Meteors.

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