PRINCIPAL DIRECTIONS IN A RIEMANNIAN SPACE

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1. A type of principal directions for a Riemannian space of N dimensions has been defined by Ricci (*Atti R. Ist. Veneto*, **63**, p. 1233). Four distinct types are defined in the present paper, of which one is identical with that of Ricci, although defined in a manner somewhat more simple.

We shall adopt the common convention of summation with respect to any index occurring twice in a product, except where the index is a capital letter. The manifold under consideration is of N dimensions; the small Roman indices imply a range or summation from 1 to N, the small Greek indices from 1 to $N-1$. The line element being given by

$$ds^2 = g_{mn}dx_m dx_n,$$

we define in the usual manner

$$G_{mn,st} = \frac{\partial}{\partial x_t}\begin{bmatrix} ms \\ n \end{bmatrix} - \frac{\partial}{\partial x_s}\begin{bmatrix} mt \\ n \end{bmatrix} + g^{ab}\left\{\begin{bmatrix} mt \\ a \end{bmatrix}\begin{bmatrix} ns \\ b \end{bmatrix} - \begin{bmatrix} ms \\ a \end{bmatrix}\begin{bmatrix} nt \\ b \end{bmatrix}\right\} \quad (1.1)$$

$$G_{ns} = g^{mt}\, G_{mn,st} \quad (1.2)$$

$$G = g^{ns}\, G_{ns}.$$

The word "surface" will be used to denote any $(N-1)$-space immersed in the given N-space.

2. Directions defined by invariant relations may be termed *principal.* Any invariant function of direction will, in general, yield such principal directions, corresponding to stationary values of the function. The following are types of principal directions:—

Type I: Consider the family of surfaces, $G =$ constant. Its orthogonal trajectories constitute principal directions; their equations are

$$\frac{\partial G}{\partial x_s} = \theta g_{st}\, \dot{x}_t \qquad (s=1,\ldots, N) \qquad (2.1)$$

where the point denotes differentiation with respect to the arc. Now, for any direction,

$$\dot{G}^2 = \frac{\dfrac{\partial G}{\partial x_s}\dfrac{\partial G}{\partial x_t}\, dx_s\, dx_t}{g_{st}\, dx_s\, dx_t};$$

therefore directions making $\dot{G}^2$ stationary satisfy

$$\frac{\partial G}{\partial x_s}\frac{\partial G}{\partial x_t}\, dx_t = \phi g_{st}\, dx_t \qquad (s = 1, \ldots, N)$$

or

$$\frac{\partial G}{\partial x_s} \dot{G} = \phi g_{st} \dot{x}_t \qquad (s = 1, \ldots, N).$$

But these are of the same form as (2.1), writing $\theta\dot{G} = \phi$, and hence the type of principal direction defined above is that for which $\dot{G}^2$, and consequently $\dot{G}$, is stationary.

Type II: Consider a point (P) and a geodesic passing through it. Along this geodesic

$$\ddot{G} = \left[\frac{\partial^2 G}{\partial x_s \partial x_t} - \begin{Bmatrix} st \\ m \end{Bmatrix} \frac{\partial G}{\partial x_m}\right] \dot{x}_s \dot{x}_t.$$

This quantity is a function of direction at P, and the principal directions, corresponding to stationary values, are given by

$$\left[\frac{\partial^2 G}{\partial x_s \partial x_t} - \begin{Bmatrix} st \\ m \end{Bmatrix} \frac{\partial G}{\partial x_m}\right] dx_t = \theta g_{st}\, dx_t \qquad (s = 1, \ldots, N). \quad (2.2)$$

Type III: The expression $G_{st}\dot{x}_s\dot{x}_t$ is invariant for any given direction. Principal directions, corresponding to stationary values, are given by

$$G_{st}\, dx_t = \theta g_{st} dx_t \qquad (s = 1, \ldots, N). \quad (2.3)$$

Eisenhart (*Proc. N. A. S.*, Vol. 8, No. 2, p. 24) has shown that the principal directions of Ricci (*loc. cit.*) may be expressed in this form. These directions may also be reached from other considerations. Any direction at a point (P) defines a surface consisting of all geodesics passing through P and perpendicular to the given direction. The curvature invariant $(\overline{G})$ of this surface at P depends only on the given direction. Those directions making $\overline{G}$ stationary are principal directions: they may be proved to be identical with those considered above.

Type IV: The expression

$$g^{s_1t_1}\, g^{s_2t_2}\, g^{s_3t_3}\, G_{s_1s_2,s_3s}\, G_{t_1t_2,t_3t}\, \dot{x}_s\dot{x}_t \quad (2.4)$$

is invariant for any given direction. Principal directions, corresponding to stationary values, are given by

$$g^{s_1t_1}\, g^{s_2t_2}\, g^{s_3t_3}\, G_{s_1s_2,s_3s}\, G_{t_1t_2,t_3t}\, dx_t = \theta g_{st}\, dx_t \quad (s = 1, \ldots, N). \quad (2.5)$$

The four types defined above are not intended to be exhaustive of all types of principal directions. Type II, for example, will give principal directions if any other invariant function of position is substituted for G. Let us suppose that the expression (2.4) has the same value for all directions at any given point, but varies from point to point. The principal directions of Type IV are then indeterminate, but (2.4) is an invariant function of position and may therefore be substituted in (2.2) to yield principal directions.

3. In order to prove certain theorems concerning the principal directions defined above, we shall require particular coördinate systems. A system of coördinates will be said to be "O.T.(x_N)" if the parametric lines of x_N are the orthogonal trajectories of the family of surfaces x_N = constant. The necessary and sufficient conditions for an O.T.(x_N) system are easily seen to be

$$g_{N\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1). \tag{3.1}$$

A special type of O.T.(x_N) system is the "G.O.T.(x_N)" system for which the parametric lines of x_N are geodesics. The equations of a geodesic are

$$\ddot{x}_s + \left\{ {mn \atop s} \right\} \dot{x}_m \dot{x}_n = 0 \qquad (s = 1, \ldots, N)$$

or

$$g_{st}\ddot{x}_t + \left[{mn \atop s} \right] \dot{x}_m \dot{x}_n = 0 \qquad (s = 1, \ldots, N). \tag{3.2}$$

The coördinate system being O.T.(x_N), the parametric lines of x_N satisfy (3.2) if, and only if,

$$\left[{NN \atop \sigma} \right] = 0 \qquad (\sigma=1, \ldots, N-1)$$

and

$$g_{NN}\ddot{x}_N + \left[{NN \atop N} \right] \dot{x}_N^2 = 0.$$

The latter equation is always satisfied for the parametric lines of x_N, by virtue of the equation

$$g_{NN}\dot{x}_N^2 = 1;$$

the former are equivalent to

$$\frac{\partial g_{NN}}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1).$$

Hence we have, as necessary and sufficient conditions for a G.O.T. (x_N) system,

$$g_{N\sigma} = 0, \quad \frac{\partial g_{NN}}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1). \tag{3.3}$$

If we are given an ∞^1 family of surfaces, we can find an O.T.(x_N) system for which the family is given by x_N = constant. If we are given a single surface and draw the congruence of geodesics normal to it, it follows from the Calculus of Variations that this is a normal congruence, and that any two of the normal surfaces give equal intercepts on all the geodesics. Taking these geodesics as parametric lines of x_N and taking x_N as the distance measured along these geodesics from the given surface, we have a

G.O.T.(x_N) system for which the given surface has the equation $x_N = 0$, and $g_{NN} = 1$ throughout the manifold (cf. Bianchi, Vol. I, p. 336).

4. A "plane" surface—the "superficie geodetiche" of Ricci (*Atti R. Acc. Lincei*, Ser. 5, Vol. 12, p. 409)—may be defined as one whose geodesics are also geodesics of the containing manifold. Let us choose an O.T. (x_N) system for which a certain plane surface has the equation $x_N = 0$. The line element of the surface is given by

$$ds^2 = g_{\mu\nu}\,dx_\mu\,dx_\nu$$

and the equations of its geodesics are, in the form of (3.2),

$$g_{\sigma\tau}\ddot{x}_\tau + \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix}\dot{x}_\mu\dot{x}_\nu = 0 \qquad (4.1)$$

$$x_N = 0; \qquad (4.2)$$

for such a curve we find, for $\sigma = 1, \ldots, N-1$,

$$g_{\sigma t}\ddot{x}_t + \begin{bmatrix}mn\\ \sigma\end{bmatrix}\dot{x}_m\dot{x}_n = g_{\sigma\tau}\ddot{x}_\tau + \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix}\dot{x}_\mu\dot{x}_\nu, \quad \text{by (4.2)},$$

$$= 0, \quad \text{by (4.1)};$$

also

$$g_{Nt}\ddot{x}_t + \begin{bmatrix}mn\\ N\end{bmatrix}\dot{x}_m\dot{x}_n = \begin{bmatrix}\mu\nu\\ N\end{bmatrix}\dot{x}_\mu\dot{x}_\nu, \quad \text{by (3.1) and (4.2).}$$

$$= -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_N}\dot{x}_\mu\dot{x}_\nu, \quad \text{by (3.1).}$$

From the definition of a plane surface and by (3.2), this latter quantity must vanish for all arbitrary directions in the surface. Therefore we must have, at all points of the surface,

$$\frac{\partial g_{\mu\nu}}{\partial x_N} = 0 \qquad (\mu,\nu = 1, \ldots, N-1). \qquad (4.3)$$

If we are given a plane surface and choose a G.O.T.(x_N) system for which the equation of the surface is $x_N = 0$, and $g_{NN} = 1$, then, at any point of the surface, (3.3) and (4.3) hold. Applying these conditions for the reduction of (1.1) and (1.2), we find that at any point of the surface

$$G_{N\nu,\sigma t} = 0 \qquad (\nu,\sigma,t = 1, \ldots, N-1) \qquad (4.4)$$

$$G_{N\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1) \qquad (4.5)$$

5. Theorem: *The direction of Type I is contained in those of Type II if, and only if, the lines of Type I are geodesic.* In order to establish this theorem we shall employ an O.T.(x_N) system for which the surface G = constant have the equations x_N = constant. The principal directions of Type I are then given by

$$dx_\sigma = 0 \qquad (\sigma = 1, \ldots, N-1); \qquad (5.1)$$

those of Type II by

$$\left.\begin{aligned}\left[\frac{\partial^2 G}{\partial x_\sigma \partial x_t} - \left\{\begin{matrix}\sigma t\\ m\end{matrix}\right\}\frac{\partial G}{\partial x_m}\right] dx_t &= \theta g_{\sigma t}\, dx_t \quad (\sigma = 1, \ldots, N-1)\\ \left[\frac{\partial^2 G}{\partial x_N \partial x_t} - \left\{\begin{matrix}Nt\\ m\end{matrix}\right\}\frac{\partial G}{\partial x_m}\right] dx_t &= \theta g_{Nt}\, dx_t.\end{aligned}\right\} \tag{5.2}$$

Applying the conditions,

$$g_{N\sigma} = 0, \qquad \frac{\partial G}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1)$$

to (5.2), we obtain

$$\left.\begin{aligned}-\left\{\begin{matrix}\sigma t\\ N\end{matrix}\right\}\frac{\partial G}{\partial x_N} dx_t &= \theta g_{\sigma\tau}\, dx_\tau \quad (\sigma = 1, \ldots, N-1)\\ \frac{\partial^2 G}{\partial x_N^2} dx_N - \left\{\begin{matrix}Nt\\ N\end{matrix}\right\}\frac{\partial G}{\partial x_N} dx_t &= \theta g_{NN}\, dx_N.\end{aligned}\right\} \tag{5.3}$$

It is easily seen that (3.1) imply $g^{N\sigma} = 0$ $(\sigma = 1, \ldots, N-1)$; employing these equations we find

$$\begin{aligned}\left\{\begin{matrix}\sigma_\tau\\ N\end{matrix}\right\} dx_t &= \frac{1}{2} g^{NN}\frac{\partial g_{NN}}{\partial x_\sigma} dx_N - \frac{1}{2} g^{NN}\frac{\partial g_{\sigma\tau}}{\partial x_N} dx_\tau \quad (\sigma = 1, \ldots, N-1)\\ \left\{\begin{matrix}Nt\\ N\end{matrix}\right\} dx_t &= \frac{1}{2} g^{NN}\frac{\partial g_{NN}}{\partial x_N} dx_N + \frac{1}{2} g^{NN}\frac{\partial g_{NN}}{\partial x_\tau} dx_\tau.\end{aligned}$$

Hence (5.3) become

$$\left.\begin{aligned}-\frac{1}{2} g^{NN}\frac{\partial G}{\partial x_N}\left(\frac{\partial g_{NN}}{\partial x_\sigma} dx_N - \frac{\partial g_{\sigma\tau}}{\partial x_N} dx_\tau\right) &= \theta g_{\sigma\tau}\, dx_\tau \quad (\sigma = 1, \ldots, N-1)\\ \left(\frac{\partial^2 G}{\partial x_N^2} - \frac{1}{2} g^{NN}\frac{\partial G}{\partial x_N}\frac{\partial g_{NN}}{\partial x_N}\right) dx_N - \frac{1}{2} g^{NN}\frac{\partial G}{\partial x_N}\frac{\partial g_{NN}}{\partial x_\tau} dx_\tau &= \theta g_{NN}\, dx_N.\end{aligned}\right\} \tag{5.4}$$

Since we hypothesize that the Type I direction is determinate,

$$\frac{\partial G}{\partial x_N} \neq 0\,;$$

therefore (5.4) are satisfied by (5.1) if, and only if,

$$\frac{\partial g_{NN}}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1).$$

But, the coördinate system being O.T.(x_N), these conditions are necessary and sufficient that the parametric lines of x_N shall be geodesic, by (3.3). Therefore the theorem is established.

6. Theorem: *If there exists a plane surface, its normal direction is a principal direction of both Type III and Type IV.* Let us employ a G.O.T.

(x_N) system for which the equation of the plane surface is $x_N = 0$, and $g_{NN} = 1$. Since (3.3) and (4.5) hold at all points of the surface, the equations (2.3) for the directions of Type III become

$$\left.\begin{aligned} G_{\sigma\tau}\,dx_\tau &= \theta g_{\sigma\tau}\,dx_\tau \quad (\sigma = 1, \ldots, N-1) \\ G_{NN}\,dx_N &= \theta g_{NN}\,dx_N \end{aligned}\right\} \qquad (6.1)$$

These equations are satisfied by $dx_\sigma = 0$ $(\sigma = 1, \ldots, N-1)$, which is the direction normal to the plane surface, and thus the theorem is established for Type III. This part of the theorem has also been proved by Ricci (*Atti R. Ist. Ven. loc. cit.*).

By (3.3) the equations (2.5) for the directions of Type IV become

$$g^{s_1t_1} g^{s_2t_2} g^{s_3t_3} G_{s_1s_2,s_3\sigma} G_{t_1t_2,t_3t}\,dx_t = \theta g_{\sigma\tau} dx_\tau \quad (\sigma = 1, \ldots, N-1) \qquad (6.2)$$

$$g^{s_1t_1} g^{s_2t_2} g^{s_3t_3} G_{s_1s_2,s_3N} G_{t_1t_2,t_3t}\,dx_t = \theta g_{NN} dx_N. \qquad (6.3)$$

Let us consider the surviving terms in the left hand sides of these equations, the relations effecting reductions being from (3.3),

$$g^{N\sigma} = 0 \quad (\sigma = 1, \ldots, N-1);$$

from the well known properties of the tensor-components,

$$G_{NN,st} = 0, \quad G_{st,NN} = 0 \quad (s,t = 1, \ldots, N);$$

while from (4.4) we see that any tensor-component vanishes if one and only one of its indices is N. In (6.2), if $t = N$, then either t_1 or t_2 must be N. Therefore either s_1 or s_2 must be N; therefore s_3 must be N. Hence $t_3 = N$, and the second tensor vanishes. Therefore there are no surviving terms in (6.2) for which $t = N$, and (6.2) may be written

$$A_{\sigma\tau}\,dx_\tau = \theta g_{\sigma\tau}\,dx_\tau \quad (\sigma = 1, \ldots, N-1). \qquad (6.4)$$

In (6.3) either s_1 or s_2 must be N. Therefore either t_1 or t_2 must be N; therefore either t_3 or t must be N. But if $t_3 = N$, then $s_3 = N$ and the term vanishes. Therefore $t_3 \pm N$, and the only surviving term is that for which $t = N$. Thus (6.3) may be written

$$B dx_N = \theta g_{NN}\,dx_N. \qquad (6.5)$$

But (6.4) and (6.5) are satisfied simultaneously by $dx_\sigma = 0$ $(\sigma = 1, \ldots, N-1)$ and thus the theorem is proved.

In the case $N = 2$, the directions of Types III and IV become indeterminate, since through any point we can draw a "plane surface" (in this case a geodesic curve) so that its normal may have any arbitrarily assigned direction.