In general the curve $p_1p_2p_3 = kP_\alpha$ is a curve of class three having $(\lambda_1, \lambda_2, \lambda_3)$ as foci and the joins $(\alpha_1\lambda, \alpha\lambda_2, \alpha\lambda_3)$ as tangents (the equation may be made homogeneous by using the quadratic relation between the perpendiculars from any three non-collinear points whose vanishing gives the circular points at ∞; the three perpendiculars are line coördinates corresponding to ordinary barycentric coördinates). If α coincides with any one of the points $(\lambda_1, \lambda_2, \lambda_3)$, λ_1 say, the boundary of the region sought for is found by stretching a string around the point λ_1 and the ellipse $p_2p_3 = k$.

In general, if the equation of a plane curve is given in the form $\phi(p_1, \ldots p_n) = 0$, its real foci are obtained by writing $2p_k = \frac{z-z_k}{t} + (\bar{z} - \bar{z}_k)t$ and letting $t \longrightarrow 0$ (for a tangent to the curve through a focus is of the form $z =$ const. or $\bar{z} =$ const. and in order that this may be obtainable from $2p = \frac{z}{t} + \bar{z}t$ we must either have $t = 0$ or $t = \infty$). It is clear then that in the general case the characteristic numbers of the matrix are the foci of the curve bounding the field of values of the matrix; for only the term of highest degree, i.e., $p_1p_2 \ldots . p_n$ occurs in determining the foci.

[1] F. Hausdorff, *Math. Zeits.*, **3**, 314–316 (1919).

REMARKS ON THE ERGODIC THEOREM OF BIRKHOFF

By Aurel Wintner

Department of Mathematics, The Johns Hopkins University

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In a recent paper[1] Birkhoff gave the proof of an extremely general recurrence theorem in dynamics which is of fundamental importance not only for stability questions connected with Celestial Mechanics, but also for the theory of adiabatic invariants and for Statistical Mechanics.[2] The object of the present note is to discuss the relation of Birkhoff's discovery with an essentially more special result, published by the writer a few years ago in connection with the theory of certain infinite matrices.[3] These matrices represent, in the sense of Frobenius, the groups defined by the almost periodic function of Bohr for which the conditionally periodic motions of Staude and Stäckel and the "grenzperiodisch" functions of H. Bohr, also occurring in dynamics,[4] are the simplest non-trivial examples. The method is, however, valid even if the recurrence character of the motion is weaker than almost-periodic (it holds for instance for functions representable only by Fourier *integrals*) and is connected in some points with several researches of N. Wiener.[5] The results which may be obtained in this manner do not yield, of course, the perfectly general theorem of Birkhoff inasmuch as a *certain* recurrence property, which will be designated as the Hadamard condition, is *presupposed*. On the other hand if this condition is fulfilled (as it is in the simplest concrete cases mentioned above), the results are not exactly identical with the theorem of Birkhoff. For instance, in the simplest cases where the Hadamard condition is fulfilled for all motions of the phase space the exceptional zero set not excluded by the general theorem of Birkhoff may be excluded. The ergodic system of Herglotz and Artin[6] shows that for general dynamical systems the exceptional zero set of Birkhoff may effectively exist.

Let $x(t)$ be a real function defined for all real values of t and continuous in any finite range. We suppose, for simplicity, that $x(t)$ is bounded in the infinite range $-\infty < t < +\infty$ and denote by a the lower limit, by b the upper limit of $x(t)$. Finally, we suppose that the average limit

$$\mu_v = \lim_{t=+\infty} \mu_v^t; \qquad \mu_v^t = (2t)^{-1} \int_{-t}^{t} v(\tau)\, d\tau,$$

introduced by Hadamard in his researches on Dirichlet series, exists for $v = [x(t)]^n$ where n is any positive integer. We then say that the function $x(t)$ is an H-function. On denoting by ξ an arbitrary real number and by $[x \leqq \xi]_t$ the set of the time points τ for which both conditions $|\tau| \leqq t$, $x(\tau) \leqq \xi$ are fulfilled, we put

$$\rho_x^t(\xi) = (2t)^{-1} . \text{ mes } [x \leqq \xi]_t \tag{1}$$

so that the relative frequency of $x(t)$ in the range $\xi_1 \leqq x < \xi_2$ during the time elapsed from $\tau = -t$ to $\tau = t$ is represented by $\rho_x^t(\xi_2) - \rho_x^t(\xi_1)$. We now have the following

Lemma.[7] If $x(t)$ is an H-function then the limit $\lim \rho_x^t(\xi)$ where $\lim t = +\infty$ exists save at an at most countable set of values ξ. In a more precise manner there exists a monotone function $\sigma_x(\xi)$; $-\infty < \xi < +\infty$ so that

$$\sigma_x(\xi) = \lim_{t=+\infty} \rho_x^t(\xi) \tag{2}$$

in all continuity points of the function $\sigma_x(\xi)$ which is unique under the normalizing condition $\sigma_x(\xi - 0) = \sigma_x(\xi)$, $-\infty < \xi < +\infty$. Obviously

$$\sigma_x(\xi) = 0 \text{ for } \xi < a, \text{ and } \sigma_x(\xi) = 1 \text{ for } \xi > b. \tag{3}$$

Finally we have

$$\int_{-\infty}^{+\infty} v^n\, d\sigma_x(v) = \lim_{t=+\infty} (2t)^{-1} \int_{-t}^{t} (x(\tau))^n d\tau \text{ for } n = 0, 1, 2, \ldots,$$

for instance

$$\int_{-\infty}^{+\infty} d\sigma_x(v) = 1, \int_{-\infty}^{+\infty} v\, d\sigma_x(v) = \mu_x, \tag{4}$$

so that the asympotic relative frequency of $x(t)$ in the range $x \leqq \xi$ is represented by $\sigma_x(\xi) = \int_{-\infty}^{\xi} d\sigma_x(v)$, provided ξ is a continuity point of σ_x.

There is of course the possibility that the *limit* function $\sigma_x(\xi)$ is constant in a subinterval $\alpha \leqq \xi \leqq \beta$ of the range $a \leqq \xi \leqq b$ of $x(t)$ [cf. (3)]. If this is *not* the case we designate $x(t)$ as a *proper* H-function. The conditionally periodic and "grenzperiodisch" functions are proper H-functions inasmuch as all almost-periodic functions are proper H-functions.[8]

It has been shown by Bohr[9] that the limit (2) does not necessarily exist at the discontinuity points ξ of the asympotic repartition function $\sigma_x(\xi)$ even if $x = x(t)$ is an almost periodic function.

Let now

$$\xi = x(t),\ \eta = y(t),\ \zeta = z(t),\ \ldots \tag{5}$$

be any sequence of N (not necessarily proper) H-functions so that (5) represents a curve in the N-dimensional phase space

$$-\infty < \xi < +\infty,\ -\infty < \eta < +\infty,\ -\infty < \zeta < +\infty, \ldots \tag{6}$$

(for a dynamical system with f degrees of freedom we have $N = 2f$). Let $R^t(\xi, \eta, \zeta, \ldots)$ denote the relative frequency of the curve (5) in the range

$$x \leqq \xi,\ y \leqq \eta,\ z \leqq \zeta,\ \ldots \tag{7}$$

of the phase space during the time elapsing from $\tau = -t$ to $\tau = t$ so that $R^t(\xi, \eta, \zeta, \ldots) = l/2t$ where l is the amount of time which the point spends during the time interval $-t \leqq \tau \leqq t$ in the range (7). The problem solved by Birkhoff affords the introduction of space averages and space densities (cf. Levi-Civita, loc. cit.) or asymptotic time averages instead of the time averages belonging to a finite range $-t \leqq \tau \leqq t$ and it concerns therefore the existence of the limit

$$S(\xi, \eta, \zeta, \ldots) = \lim_{t=+\infty} R^t(\xi, \eta, \zeta, \ldots) \tag{8}$$

which need not exist in the discontinuity points of the asymptotic repartition function[9] $S(\xi, \eta, \zeta, \ldots)$ [or even in a set of points $(\xi, \eta, \zeta, \ldots)$ having the N-dimensional measure zero]. The Lemma solves this limit problem for $N = 1$ only.

The compound relative frequency $R^t(\xi, \eta, \zeta, \ldots)$ is not simply the product of the partial relative frequencies $\rho_x^t(\xi)$, $\rho_y^t(\eta)$, $\rho_z^t(\zeta)$, ... [defined by (1)] inasmuch as these component probabilities are not independent of each other. However, if the functions are, for instance, almost-periodic then $R^t(\xi, \eta, \zeta, \ldots)$ may be represented[10] as a folding expression ("Faltung") of the functions $\rho^t_X(\xi)$, $\rho^t_Y(\eta)$, $\rho^t_Z(\zeta)$, ... where

$$X = x(t).\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}(\lambda t),\ Y = y(t).\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}(\lambda t),\ \ldots;\quad -\infty < \lambda < +\infty$$

and the existence of the Birkhoff limit (8) follows then by virtue of the theorems of Helly[11] on sequences of monotone functions.

If all N-functions (5) are almost-periodic functions or, in a more general manner, proper H-functions then the asymptotic repartition function $S(\xi, \eta, \zeta, \ldots)$ cannot be constant in an N-dimensional vicinity of a point of the phase space which coincides with a point of the curve (5). In other words the limit density of probability is then in the full range of the curve (5) different from zero.

In a subsequent note the Birkhoff theorem will be applied, in an explicit manner, to the Levi-Civita-Bohl differential equation[12]

$$dx/dt = f(x, t) \text{ where } f(x, t) = f(x + 1, t) = f(x, t + 1)$$

and to connected questions in Celestial Mechanics.

[1] G. D. Birkhoff, *Wash. Proc.*, **17,** 656–660 (1931).

[2] T. Levi-Civita, *Hamb. Abh.*, **6,** 330–348 (1928).

[3] A. Wintner, *Math. Zeit.*, **30,** 310–311 (1929).

[4] G. D. Birkhoff, *Acta. Math.*, **50,** 359–379 (1927).

[5] Cf., *Ibid.*, **55,** 257–258 (1931).

[6] E. Artin, *Hamb. Abh.*, **3,** 170–175 (1927).

[7] A. Wintner, loc. cit.

[8] A. Wintner, loc. cit., p. 312.

[9] H. Bohr, *Danske Videnskab. Selsk.*, **10,** Nr. 10 (1930).

[10] For almost periodic functions cf. A. Wintner, loc. cit., p. 318–319, for functions with a more general recurrence character the researches of N. Wiener.

[11] Cf., for instance, A. Wintner, *Spektraltheorie der unendlichen Matrizen*, Leipzig, 1929, Chapter II.

[12] H. Kneser, *Math. Annalen*, **91,** 154 (1924). For the astronomical literature of the repartition problem cf. A. Wintner, *Rend. Acc. Lincei*, **11** [6], 464 (1930) and H. Bohr, loc. cit.